A bound on judicious bipartitions of directed graphs

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Abstract   Judicious partitioning problems on graphs ask for partitions that bound several quantities simultaneously, which have received much attention lately. Scott (2005) asked the following natural question: What is the maximum constant c_d such that every directed graph D with m arcs and minimum outdegree d admits a bipartition $V(D) = V_1 \cup V_2$ satisfying $\min\{e(V_1, V_2), e(V_2, V_1)\} \geqslant c_d m$? Here, for $i = 1, 2$, $e(V_i, V_{3-i})$ denotes the number of arcs in D from V_i to V_{3-i}. Lee et al. (2016) conjectured that every directed graph D with m arcs and minimum outdegree at least $d \geqslant 2$ admits a bipartition $V(D) = V_1 \cup V_2$ such that

$$\min\{e(V_1, V_2), e(V_2, V_1)\} \geqslant \left(\frac{d-1}{2(2d-1)} + o(1) \right) m.$$

In this paper, we show that this conjecture holds under the additional natural condition that the minimum indegree is also at least d.

Keywords   directed graph, partition, outdegree, indegree, tight component

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1   Introduction

The Max-Cut problem asks for a cut with maximum weight in a weighted graph, which has been studied extensively from both combinatorial and computational perspectives. It is NP-hard even when restricted to the class of triangle-free cubic graphs [25]. A simple calculation shows that every graph with m edges has a cut with at least $m/2$ edges. Answering a question of Erdős, Edwards [7, 8] improved this lower bound to $m/2 + (\sqrt{2m + 1/4} - 1/2)/4$, which is tight for complete graphs with odd order. Moreover, for a certain range of m, Alon [1] gave an additive improvement of order $m^{1/4}$. For special classes of graphs, such as subcubic graphs [23, 26], the main term in the Edwards' bound may be improved. For more references concerning partitions of graphs or hypergraphs, see [9, 10, 13, 15, 19, 24].

For a directed graph (digraph for short) D and $S, T \subseteq V(D)$, we use $e(S, T)$ to denote the number of arcs of D directed from S to T. A directed (unweighted) version of the Max-Cut problem is to find

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a partition $V(D) = V_1 \cup V_2$ of a given digraph D that maximizes $e(V_1, V_2)$. It is easy to see from the above bound of Edwards that every digraph D with m arcs has a partition $V(D) = V_1 \cup V_2$ with $e(V_1, V_2) \geqslant m/4 + (\sqrt{2m + 1/4} - 1/2)/8$, and that any regular orientation of a complete graph of odd order shows that this bound is tight. Alon et al. [2] considered partitions of acyclic digraphs and proved that every acyclic digraph D with m arcs has a partition $V(D) = V_1 \cup V_2$ such that $e(V_1, V_2) \geqslant m/4 + \Omega(m^{2/3})$. For positive integers k and l, let $\mathcal{D}(k, l)$ denote the family of digraphs such that every vertex has indegree at most k or outdegree at most l. It is proved in [2] that every digraph $D \in \mathcal{D}(k, l)$ with m arcs has a partition $V(D) = V_1 \cup V_2$ such that $e(V_1, V_2) \geqslant (k + l + 2)m/(4k + 4l + 6)$. For digraphs $D \in \mathcal{D}(1, 1)$ with m arcs, Chen et al. [6] showed that D has a partition $V(D) = V_1 \cup V_2$ such that $e(V_1, V_2) \geqslant 3(m - 1)/8$.

In practice, one often needs to find a partition of a given graph or digraph that simultaneously bounds several quantities. Such problems are called *Judicious Partitioning Problems* by Bollobás and Scott [5]. In [22], Scott asked the following natural question: What is the maximum constant c_d such that every digraph D with m arcs and minimum outdegree d admits a bipartition $V(D) = V_1 \cup V_2$ satisfying

$$\min\{e(V_1, V_2), e(V_2, V_1)\} \geqslant c_d m?$$

For $d = 1$, the graph, obtained from $K_{1, n-1}$ by adding a single edge inside the part of size $n - 1$, admits an orientation in which the minimum outdegree is 1 and $\min\{e(V_1, V_2), e(V_2, V_1)\} \leqslant 1$ for every partition $V(D) = V_1 \cup V_2$. Hence, $c_1 = 0$. Lee et al. [18] studied Scott's question and made the following conjecture:

Conjecture 1.1 (See [18]).   Let d be an integer satisfying $d \geqslant 2$. Every digraph D with m arcs and minimum outdegree at least d admits a bipartition $V(D) = V_1 \cup V_2$ with

$$\min\{e(V_1, V_2), e(V_2, V_1)\} \geqslant \left(\frac{d - 1}{2(2d - 1)} + o(1)\right)m.$$

In the same paper, Lee et al. verified Conjecture 1.1 for $d = 2, 3$, and noted that their method is not adequate for $d \geqslant 4$.

It is not clear to us whether or not the minimum outdegree condition alone is sufficient for Conjecture 1.1 with large d. However, we show in this paper that Conjecture 1.1 holds under the natural (additional) assumption that the minimum indegree of D is at least d.

Theorem 1.2.   *Let D be a digraph with m arcs and assume that both the minimum outdegree and the minimum indegree of D are at least $d \geqslant 2$. Then D admits a bipartition $V(D) = V_1 \cup V_2$ with*

$$\min\{e(V_1, V_2), e(V_2, V_1)\} \geqslant \left(\frac{d - 1}{2(2d - 1)} + o(1)\right)m.$$

In the remainder of this section, we describe notation and terminology used in the proof of Theorem 1.2. In Section 2, we collect previous results (as well as a concentration inequality) that we will need. The proof of Theorem 1.2 will be given in Section 3. In Section 4, we offer some concluding remarks.

All digraphs considered in this paper are finite with no loops and no parallel arcs. For a digraph D, we denote by $V(D)$ and $A(D)$ the *vertex set* and the *arc set* of D, respectively. Let $|D| := |V(D)|$ be the *order* of D, and $e(D) := |A(D)|$ be the *size* of D. The *underlying graph* of D is obtained from D by ignoring arc orientations and removing redundant parallel edges.

Given two vertices x, y in a digraph D, we write xy for the arc directed from x to y. Let $N_D^+(x) := \{z : xz \in A(D)\}$ and $N_D^-(x) := \{z : zx \in A(D)\}$. Then $d_D^+(x) := |N_D^+(x)|$ and $d_D^-(x) := |N_D^-(x)|$ are the outdegree and indegree of x, respectively. The minimum outdegree of D is $\delta^+(D) := \min\{d_D^+(x) : x \in V(D)\}$ and the minimum indegree of D is $\delta^-(D) := \min\{d_D^-(x) : x \in V(D)\}$. Let $\delta^0(D) := \min\{\delta^+(D), \delta^-(D)\}$. The *degree* of $x \in V(D)$ is defined as $d_D(x) := d_D^+(x) + d_D^-(x)$. We use $\delta(D) := \min\{d_D(x) : x \in V(D)\}$ and $\Delta(D) := \max\{d_D(x) : x \in V(D)\}$ to denote the *minimum degree* and *maximum degree* of D, respectively.

Let D be a digraph. For $X \subseteq V(D)$, the subgraph of D *induced* by X is denoted by $D[X]$. Let $e_D(X)$ denote the number of arcs in $D[X]$, and let $D - X$ denote the digraph obtained from D by deleting X and all arcs incident with X.

Throughout this paper, we drop the reference to D in the above notations if there is no danger of confusion.

2   Lemmas

In this section, we list previous results needed in our proof of Theorem 1.2. Lee et al. [17] introduced the notion of a tight component in a graph when analyzing bisections of graphs. A connected graph T is *tight* if

- for every vertex $v \in V(T)$, $T - v$ contains a perfect matching, and
- for every vertex $v \in V(T)$ and every perfect matching M of $T - v$, no edge in M has exactly one end adjacent to v.

Note that if T is tight, then the order of T is odd and each vertex in T has even degree. Lu et al. [20] gave the following characterization of tight graphs.

Lemma 2.1 (See [20]).   *A connected graph G is tight if and only if every block of G is an odd clique.*

The following result bounds the number of tight components in a graph, which is an easy consequence of Lemma 2.1.

Lemma 2.2 (See [17]).   *For each integer $i \geqslant 0$, let d_i denote the number of vertices with degree i in a graph G. Then the number of tight components τ of G satisfies*

$$\tau \leqslant \frac{d_0}{1} + \frac{d_2}{3} + \frac{d_4}{5} + \cdots.$$

We also need a result from [12], which provides a bound on $d_0/1 + d_2/3 + \cdots$ (and, hence, on τ) under additional constraints.

Lemma 2.3 (See [12]).   *Let n, ρ, α, δ be fixed nonnegative integers with $\delta - \alpha \geqslant 1$, and let d_i be a real number with $0 \leqslant d_i \leqslant n$ for $i \in \{0, 1, \ldots, n-1\}$ such that*

$$\sum_{i=0}^{n-1} d_i \leqslant n \quad and \quad (\delta - \alpha)d_0 + (\delta - \alpha - 1)d_1 + \cdots + d_{\delta - \alpha - 1} \leqslant \rho.$$

Then

$$\frac{d_0}{1} + \frac{d_2}{3} + \cdots \leqslant \frac{n + \rho}{\delta - \alpha + 1}.$$

We now turn to lemmas on partitions for digraphs, first of which concerns dense digraphs.

Lemma 2.4 (See [18]).   *Let D be a digraph with n vertices and m arcs. For any $\varepsilon > 0$, if $m \geqslant 8n/\varepsilon^2$ or $\Delta(D) \leqslant \varepsilon^2 m/4$, then D admits a bipartition $V(D) = V_1 \cup V_2$ with $\min\{e(V_1, V_2), e(V_2, V_1)\} \geqslant m/4 - \varepsilon m$.*

One approach for finding a "good" bipartition of a digraph D is to first partition $V(D)$ into two sets X and Y (with X consisting of certain high degree vertices), then partition X into two sets X_1 and X_2 with a certain property, and finally apply a randomized algorithm to partition Y. The following lemma will be used to identify a useful partition of X.

Lemma 2.5 (See [14]).   *Let D be a digraph and $V(D) = X \cup Y$ be a partition of D with $e(X) = 0$. For any given partition $X = X_1 \cup X_2$, define its gap to be*

$$\theta(X_1, X_2) = (e(X_1, Y) + e(Y, X_2)) - (e(X_2, Y) + e(Y, X_1)).$$

If $X_1 \cup X_2$ is a partition of X which minimizes $|\theta(X_1, X_2)|$, then $|\theta(X_1, X_2)| \leqslant |Y|$.

Note that $|\theta(X_1, X_2)|$ is the absolute value of $\theta(X_1, X_2)$, while $|Y|$ is the cardinality of Y. We remark that $\theta(X_1, X_2)$ was introduced by Lee et al. in [18].

The next lemma allows us to extend a partial bipartition X_1, X_2 of a digraph to a "good" bipartition of the entire digraph. This will be used several times in the proof of Theorem 1.2.

Lemma 2.6 (See [14]).   *For any real constants C, $\varepsilon > 0$, there exist γ, $n_0 > 0$ for which the following holds. Let D be a digraph with $n \geqslant n_0$ vertices and at most Cn arcs. Suppose $X \subseteq V(D)$ is a set of at most γn vertices which have been partitioned into $X_1 \cup X_2$. Let $Y = V(D) \setminus X$ and τ be the number of tight components in $G[Y]$, where G is the underlying graph of D. If every vertex in Y has degree at most γn in G, then there is a bipartition $V(D) = V_1 \cup V_2$ with $X_i \subseteq V_i$ for $i = 1, 2$ such that*

$$e(V_1, V_2) \geqslant e(X_1, X_2) + \frac{e(X_1, Y) + e(Y, X_2)}{2} + \frac{e(Y)}{4} + \frac{n - \tau}{8} - \varepsilon n,$$

$$e(V_2, V_1) \geqslant e(X_2, X_1) + \frac{e(X_2, Y) + e(Y, X_1)}{2} + \frac{e(Y)}{4} + \frac{n - \tau}{8} - \varepsilon n.$$

For the step to partition Y, we will need the following version of the Azuma-Hoeffding inequality [3, 11]; see Corollary 2.27 in [16].

Lemma 2.7 (See [3, 11]).   *Let $Z_1, \ldots, Z_n$ be independent random variables taking values in $\{1, \ldots, k\}$, let $Z := (Z_1, \ldots, Z_n)$, and let $f : \{1, \ldots, k\}^n \to \mathbb{N}$ such that $|f(Y) - f(Y')| \leqslant c_i$ for any $Y, Y' \in \{1, \ldots, k\}^n$ which differ only in the i-th coordinate. Then for any $z > 0$,*

$$\mathrm{P}(f(Z) \geqslant \mathrm{E}(f(Z)) + z) \leqslant \exp\left(\frac{-z^2}{2\sum_{i=1}^n c_i^2}\right),$$

$$\mathrm{P}(f(Z) \leqslant \mathrm{E}(f(Z)) - z) \leqslant \exp\left(\frac{-z^2}{2\sum_{i=1}^n c_i^2}\right).$$

3   Proof of Theorem 1.2

In this section, we prove Theorem 1.2. Our approach is similar to that used by Lee et al. [18], which was used first by Bollobás and Scott [4] and again by Ma et al. [21]. However, additional ideas are needed to make this approach work.

Let $d \geqslant 2$ be an integer, let D be an n-vertex digraph with m arcs, and assume $\delta^0(D) := d \geqslant 2$ (i.e., both minimum outdegree $\delta^+(D)$ and minimum indegree $\delta^-(D)$ are at least d). Then, since $\delta^+(D) \geqslant d$,

$$m = \sum_{v \in V(D)} d^+(v) \geqslant dn.$$

We need to show that D admits a bipartition $V(D) = V_1 \cup V_2$ with

$$\min\{e(V_1, V_2), e(V_2, V_1)\} \geqslant \left(\frac{d-1}{2(2d-1)} + o(1)\right)m,$$

and we will reduce this to proving (3.3).

We may assume that

$$d \geqslant 4,$$

as Lee et al. [18] showed that, for $d \leqslant 3$, such a bipartition exists. We may also assume that

$$m \leqslant 128(2d-1)^2 n \quad \text{and} \quad \Delta(D) \geqslant \frac{m}{64(2d-1)^2}, \tag{3.1}$$

for, otherwise, applying Lemma 2.4 with $\varepsilon = 1/(8d-4)$ gives a partition $V(D) = V_1 \cup V_2$ such that

$$\min\{e(V_1, V_2), e(V_2, V_1)\} \geqslant \left(\frac{1}{4} - \frac{1}{8d-4}\right)m = \frac{d-1}{2(2d-1)}m.$$

Thus, we may assume that

$$n \text{ is sufficiently large.}$$

For clarity of presentation, we divide the remainder of the proof into nine steps.

Step 1.   Identifying the set X, and partitioning X to $X_1 \cup X_2$.

Let X be the set of vertices of D with degree at least $n^{3/4}$, and let $Y = V(D) - X$ and $n' := |Y|$. Then

$$2m \geqslant \sum_{v \in X} d(v) \geqslant |X| n^{3/4}.$$

It follows from (3.1) that

$$|X| \leqslant 256(2d-1)^2 n^{1/4} = O(n^{1/4}). \tag{3.2}$$

Thus, $e(X) \leqslant |X|^2 = O(n^{1/2})$. Therefore, for the sake of simplicity, we remove all the arcs of D within X and update m to be the new total number of arcs in the digraph D. In terms of this new m, it suffices to show that for any sufficiently small real $\varepsilon > 0$, D admits a bipartition $V(D) = V_1 \cup V_2$ with

$$\min\{e(V_1, V_2), e(V_2, V_1)\} \geqslant \left(\frac{d-1}{2(2d-1)} - \varepsilon \right) m.$$

Let $m_1 = e(X, Y) + e(Y, X)$ and $m_2 = e(Y)$. Then $m = m_1 + m_2$. Given a partition $X = X_1 \cup X_2$, let

$$\theta(X_1, X_2) = (e(X_1, Y) + e(Y, X_2)) - (e(X_2, Y) + e(Y, X_1))$$

be its *gap*.

We choose the partition $X_1 \cup X_2$ such that

$$|\theta(X_1, X_2)| \text{ is minimum.}$$

By the symmetry between X_1 and X_2, we may assume

$$\theta(X_1, X_2) \geqslant 0.$$

Throughout the proof, we write $\theta := \theta(X_1, X_2)$, unless it concerns a different partition of X.

Step 2.   Extending $X_1 \cup X_2$ to a partition $W_1 \cup W_2$ of $V(D)$.

Let G be the underlying graph of D, and let τ be the number of tight components in $G[Y]$. As n is sufficiently large, we may apply Lemma 2.6 and conclude that D admits a bipartition $V(D) = W_1 \cup W_2$ such that $X_i \subseteq W_i$ for $i = 1, 2$, and

$$\min\{e(W_1, W_2), e(W_2, W_1)\}$$
$$\geqslant \frac{1}{2} \min\{e(X_1, Y) + e(Y, X_2), e(X_2, Y) + e(Y, X_1)\} + \frac{e(Y)}{4} + \frac{n - \tau}{8} - \varepsilon n$$
$$= \frac{m_1 - \theta}{4} + \frac{m_2}{4} + \frac{n - \tau}{8} - \varepsilon n$$
$$= \frac{m - \theta}{4} + \frac{n - \tau}{8} - \varepsilon n$$
$$\geqslant \left(\frac{d-1}{2(2d-1)} - \varepsilon \right) m + \frac{1}{4} \left(\frac{n}{2} + \frac{m}{2d-1} - \theta - \frac{\tau}{2} \right) \quad \text{(since } m \geqslant dn\text{)}.$$

Thus, to prove Theorem 1.2, it suffices to show

$$\frac{n}{2} + \frac{m}{2d-1} \geqslant \theta + \frac{\tau}{2}. \tag{3.3}$$

Step 3.   Bounding θ, m and m_2.

Since $\tau \leqslant n$, (3.3) holds if $\theta \leqslant \frac{m}{2d-1}$. Thus, we may assume

$$\theta > \frac{m}{2d-1}. \tag{3.4}$$

Therefore, since $\theta \leqslant n' = |Y|$ (by Lemma 2.5),

$$dn \leqslant m < (2d-1)n'. \tag{3.5}$$

Note that we may assume

$$m_2 > \frac{d+1}{2(2d-1)}m. \tag{3.6}$$

For, otherwise, taking the sum of outdegrees (respectively, indegrees) of all vertices in Y, we have

$$\min\{e(X,Y), e(Y,X)\} = \min\left\{\sum_{v\in Y} d^+(v), \sum_{v\in Y} d^-(v)\right\} - m_2 \geqslant dn' - m_2 > \frac{d-1}{2(2d-1)}m,$$

where the last inequality holds since $m < (2d-1)n'$ (by (3.5)). Thus, $V(D) = X \cup Y$ gives the desired partition of D.

Step 4. Analyzing small tight components.

We need to analyze tight components with small order to obtain better bounds on m and τ. For $i \geqslant 1$, let $\mathcal{T}_i$ be the collections of tight components of order i in $G[Y]$, and let A_i be the set of those vertices of D each of which is contained in some member of $\mathcal{T}_i$. Note that by Lemma 2.1, each $T \in \mathcal{T}_1$ is an isolated vertex in $G[Y]$ and each $T \in \mathcal{T}_3$ is a triangle in $G[Y]$.

We bound the number of arcs between A_i and X in both directions for $i = 1, 3$. For each $x \in A_1$, $\min\{e(x, X), e(X, x)\} \geqslant d$ since $\delta^0(D) \geqslant d$. Thus,

$$\min\{e(A_1, X), e(X, A_1)\} \geqslant d|A_1|. \tag{3.7}$$

For each triangle $T \in \mathcal{T}_3$, $D[V(T)]$ has at most 6 arcs as D has no parallel arcs. Hence,

$$\begin{aligned} &\min\{e(V(T), X), e(X, V(T))\} \\ &= \min\left\{\sum_{x\in V(T)} d^+(x), \sum_{x\in V(T)} d^-(x)\right\} - e(D[V(T)]) \\ &\geqslant (d-2)|T|. \end{aligned}$$

Note that the last inequality holds as $\delta^0(D) \geqslant d$ and $|T| = 3$. So

$$\min\{e(A_3, X), e(X, A_3)\} \geqslant \sum_{T\in\mathcal{T}_3} (d-2)|T| = (d-2)|A_3|. \tag{3.8}$$

Step 5. Better bounds on τ and m.

First, we find a good partition of $D' := D[Y] - (A_1 \cup A_3)$. Let $V(D') = U_1 \cup U_2$ be a (random) partition of $V(D')$ obtained by placing each $v \in V(D')$ into U_1 or U_2, independently, with probability $1/2$.

For each arc $e = uv \in A(D')$, let I_e be the indicator random variable of the event that $u \in U_1$ and $v \in U_2$. Then, by the linearity of expectation,

$$\mathrm{E}[e_{D'}(U_1, U_2)] = \sum_{e\in A(D')} \mathrm{E}[I_e] = \sum_{e\in A(D')} (\mathrm{P}[u\in U_1]\mathrm{P}[v\in U_2]) = \frac{1}{4}e(D').$$

Similarly,

$$\mathrm{E}[e_{D'}(U_2, U_1)] = \frac{1}{4}e(D').$$

Note that changing the placement of each $v \in V(D')$ cannot affect $e_{D'}(U_1, U_2)$ or $e_{D'}(U_2, U_1)$ by more than $d_{D'}(v) \leqslant n^{3/4}$, so we may apply Lemma 2.7. Note

$$L := \sum_{v\in V(D')} d^2_{D'}(v) \leqslant n^{3/4} \sum_{v\in V(D')} d_{D'}(v) = 2n^{3/4}e(D').$$

Let $z = 2n^{3/8}(e(D'))^{1/2}$. Then by Lemma 2.7,

$$\mathrm{P}(e_{D'}(U_1, U_2) \leqslant \mathrm{E}[e_{D'}(U_1, U_2)] - z) \leqslant \exp\left(-\frac{z^2}{2L}\right) = \mathrm{e}^{-1},$$

and (by symmetry)
$$\mathrm{P}(e_{D'}(U_2, U_1) \leqslant \mathrm{E}[e_{D'}(U_2, U_1)] - z) \leqslant \mathrm{e}^{-1}.$$

Thus, with positive probability, there exists a partition $V(D') = U_1 \cup U_2$ such that

$$\min\{e_{D'}(U_1, U_2), e_{D'}(U_2, U_1)\} \geqslant \frac{1}{4}e(D') - 2n^{3/8}(e(D'))^{1/2} = \frac{1}{4}e(D') - O(m^{7/8}), \qquad (3.9)$$

where the equality holds since $dn \leqslant m \leqslant 128(2d-1)^2 n$.

We now derive better bounds on m and τ, by working with the following partition of D: $V_1 = U_1 \cup A_1 \cup A_3$ and $V_2 = X \cup U_2$. Note that $e(A_i, V(D')) = e(V(D'), A_i) = 0$ for $i = 1, 3$. Also note that, for each $T \in \mathcal{T}_3$, $e(D[V(T)]) \leqslant 6$; so $e(D') \geqslant m_2 - 2|A_3|$. Hence,

$$\begin{aligned}
e(V_1, V_2) &\geqslant e(A_1, X) + e(A_3, X) + e_{D'}(U_1, U_2) \\
&\geqslant d|A_1| + (d-2)|A_3| + \frac{1}{4}e(D') - O(m^{7/8}) \quad \text{(by (3.7), (3.8) and (3.9))} \\
&\geqslant \frac{1}{4}m_2 + d|A_1| + \left(d - \frac{5}{2}\right)|A_3| - O(m^{7/8}) \quad \text{(since } e(D') \geqslant m_2 - 2|A_3|).
\end{aligned}$$

Similarly,

$$\begin{aligned}
e(V_2, V_1) &\geqslant e(X, A_1) + e(X, A_3) + e_{D'}(U_2, U_1) \\
&\geqslant \frac{1}{4}m_2 + d|A_1| + \left(d - \frac{5}{2}\right)|A_3| - O(m^{7/8}).
\end{aligned}$$

We may assume that

$$\frac{1}{4}m_2 + d|A_1| + \left(d - \frac{5}{2}\right)|A_3| \leqslant \frac{d-1}{2(2d-1)}m, \qquad (3.10)$$

since, otherwise, $V(D) = V_1 \cup V_2$ gives the desired bipartition for D (as n and, hence, m are sufficiently large). Thus,

$$|A_1| \leqslant \frac{1}{d}\left(\frac{d-1}{2(2d-1)}m - \frac{1}{4}m_2 - \left(d - \frac{5}{2}\right)|A_3|\right) < \frac{3d-5}{8d(2d-1)}m - \left(1 - \frac{5}{2d}\right)|A_3|,$$

since $m_2 > \frac{d+1}{2(2d-1)}m$ (by (3.6)). Therefore,

$$\begin{aligned}
\tau &\leqslant |A_1| + \frac{|A_3|}{3} + \frac{n' - |A_1| - |A_3|}{5} \\
&= \frac{n'}{5} + \frac{4}{5}|A_1| + \frac{2}{15}|A_3| \\
&\leqslant \frac{n'}{5} + \frac{4}{5}\left(\frac{3d-5}{8d(2d-1)}m - \left(1 - \frac{5}{2d}\right)|A_3|\right) + \frac{2}{15}|A_3| \\
&= \frac{1}{5}n' + \frac{3d-5}{10d(2d-1)}m - \left(\frac{2}{3} - \frac{2}{d}\right)|A_3| \\
&< \frac{1}{5}n' + \frac{3d-5}{10d(2d-1)}m \quad \text{(as } d \geqslant 4). \qquad (3.11)
\end{aligned}$$

Then

$$\begin{aligned}
\frac{n}{2} + \frac{m}{2d-1} - \theta - \frac{\tau}{2} &\geqslant \frac{n'}{2} + \frac{m}{2d-1} - \theta - \frac{1}{2}\left(\frac{1}{5}n' + \frac{3d-5}{10d(2d-1)}m\right) \quad \text{(by (3.11))} \\
&= \frac{2}{5}n' + \frac{17d+5}{20d(2d-1)}m - \theta \\
&\geqslant \frac{17d+5}{20d(2d-1)}m - \frac{3}{5}n' \quad \text{(since } \theta \leqslant n').
\end{aligned}$$

Thus, if $m \geqslant \frac{12d(2d-1)}{17d+5}n'$, then (3.3) holds and, hence, $W_1 \cup W_2$ gives the desired partition for D. So we may assume that

$$m < \frac{12(2d-1)}{17d+5}dn'. \tag{3.12}$$

Step 6.   Bounding m_1.

To establish an upper bound on m_1, we sum up the outdegrees (respectively, indegrees) of all vertices in Y. Thus, by $\delta^0(D) \geqslant d$, $\min\{e(X,Y), e(Y,X)\} \geqslant dn' - m_2$; so by (3.12),

$$\min\{e(X,Y), e(Y,X)\} \geqslant \frac{17d+5}{12(2d-1)}m - m_2.$$

If $m_2 \leqslant \frac{11d+11}{12(2d-1)}m$, then

$$\min\{e(X,Y), e(Y,X)\} \geqslant \frac{d-1}{2(2d-1)}m,$$

and $X \cup Y$ forms the desired partition for D.

So we may assume $m_2 > \frac{11d+11}{12(2d-1)}m$. Then by (3.12), we have

$$m_1 = m - m_2 < \frac{13d-23}{12(2d-1)}m < \frac{13d-23}{17d+5}dn'. \tag{3.13}$$

Step 7.   Bounding α, the number of huge vertices.

For a vertex $v \in X$, let $s^-(v) = d^-(v) - d^+(v)$, $s^+(v) = d^+(v) - d^-(v)$, and $s(v) = \max\{s^+(v), s^-(v)\}$. As in [18], we call a vertex $v \in X$ *huge* if $s(v) \geqslant \theta$. For the partition $X = X_1 \cup X_2$ (previously chosen to minimize θ) and a vertex $v \in X$, v is *forward* if either $v \in X_1$ and $s^+(v) > 0$, or $v \in X_2$ and $s^-(v) > 0$. Since $\theta > 0$ (by (3.4)), X has at least one forward vertex.

We claim that all forward vertices in X are huge. Let v be a forward vertex in X, and let the partition $X_1' \cup X_2'$ of X be obtained from $X_1 \cup X_2$ by switching the side of v. Then

$$\begin{aligned}
\theta(X_1', X_2') &= (e(X_1', Y) + e(Y, X_2')) - (e(X_2', Y) + e(Y, X_1')) \\
&= (e(X_1, Y) + e(Y, X_2) - s(v)) - (e(X_2, Y) + e(Y, X_1) + s(v)) \\
&= \theta - 2s(v).
\end{aligned}$$

Since $X = X_1 \cup X_2$ is a partition of X minimizing $\theta = |\theta(X_1, X_2)|$, $|\theta(X_1', X_2')| \geqslant \theta$. Therefore, $\theta - 2s(v) \leqslant -\theta$; so $s(v) \geqslant \theta$ and, hence, v is huge.

Let X' be the set of huge vertices in X and $X'' = X - X'$. Let $\alpha := |X'|$ and $\rho := \sum_{v \in X''} d(v)$. Then $\alpha \geqslant 1$, and

$$m_1 = \sum_{v \in X} d(v) = \sum_{v \in X'} d(v) + \sum_{v \in X''} d(v) \geqslant \alpha\theta + \rho. \tag{3.14}$$

Since $\theta > m/(2d-1)$ (by (3.4)), it follows from (3.13) and (3.14) that

$$\frac{\alpha}{2d-1}m \leqslant \frac{13d-23}{12(2d-1)}m,$$

which implies

$$1 - \frac{\alpha}{2(2d-1)} > 0. \tag{3.15}$$

To derive a better upper bound on α, we sum up the degrees of all vertices in Y. So

$$2dn' \leqslant \sum_{v \in Y} d(v) = e(X,Y) + e(Y,X) + 2e(Y) = m_1 + 2m_2,$$

which, together with (3.14), gives

$$m = m_1 + m_2 \geqslant \frac{2dn' + m_1}{2} \geqslant \frac{2dn' + \alpha\theta + \rho}{2}. \tag{3.16}$$

By (3.11) and (3.12), we have

$$\tau \leqslant \frac{1}{5}n' + \frac{3d-5}{10d(2d-1)}\frac{12d(2d-1)}{17d+5}n' = \frac{7d-5}{17d+5}n';$$

so

$$
\begin{aligned}
&\frac{n}{2} + \frac{m}{2d-1} - \theta - \frac{\tau}{2} \\
&\qquad \geqslant \frac{n}{2} + \frac{m}{2d-1} - \theta - \frac{7d-5}{2(17d+5)}n' \\
&\qquad \geqslant \frac{n'}{2} + \frac{1}{2(2d-1)}(2dn' + \alpha\theta + \rho) - \theta - \frac{7d-5}{2(17d+5)}n' \quad \text{(by (3.16))} \\
&\qquad \geqslant \left(\frac{1}{2} + \frac{2d}{2(2d-1)} - \frac{7d-5}{2(17d+5)}\right)n' - \left(1 - \frac{\alpha}{2(2d-1)}\right)\theta \\
&\qquad \geqslant \left(\frac{1}{2} + \frac{2d}{2(2d-1)} - \frac{7d-5}{2(17d+5)}\right)n' - \left(1 - \frac{\alpha}{2(2d-1)}\right)n' \quad \text{(by (3.15) and } \theta \leqslant n') \\
&\qquad = \left(\frac{\alpha+1}{2(2d-1)} - \frac{7d-5}{2(17d+5)}\right)n'.
\end{aligned}
$$

Hence, if $\alpha \geqslant \frac{2d(7d-17)}{17d+5}$, then

$$\frac{n}{2} + \frac{m}{2d-1} - \theta - \frac{\tau}{2} \geqslant \left(\frac{\alpha+1}{2(2d-1)} - \frac{7d-5}{2(17d+5)}\right)n' > 0;$$

so (3.3) holds, and $W_1 \cup W_2$ gives the desired partition. We may thus assume that

$$\alpha < \frac{2d(7d-17)}{17d+5} < d-1. \tag{3.17}$$

Step 8.   Better bounds on τ.

We bound τ in terms of n' and ρ. Let d_i denote the number of vertices with degree i in $G[Y]$. Then

$$\sum_{i \geqslant 0} d_i = n'.$$

For $0 \leqslant i \leqslant d - \alpha - 1$, each vertex $v \in Y$ with degree i in $G[Y]$ has degree at most $2i$ in $D[Y]$, which, together with the fact that $\delta^0(D) \geqslant d$, shows $e(v, X) + e(X, v) \geqslant 2d - 2i$. Thus, the number of arcs of D incident with v and counted in ρ is at least $2d - 2i - 2\alpha$. Consequently,

$$(2d - 2\alpha)d_0 + (2d - 2\alpha - 2)d_1 + \cdots + 2d_{d-\alpha-1} \leqslant \rho,$$

i.e.,

$$(d - \alpha)d_0 + (d - \alpha - 1)d_1 + \cdots + d_{d-\alpha-1} \leqslant \rho/2.$$

Therefore, by Lemmas 2.2 and 2.3, we have

$$\tau \leqslant \frac{n' + \rho/2}{d - \alpha + 1}. \tag{3.18}$$

Step 9.   Completing the proof by considering $\alpha = 1$ and $\alpha \geqslant 2$.

For convenience, let $m_1 = \beta dn'$. Then, by (3.13), we have

$$\beta \leqslant \frac{13d - 23}{17d + 5}. \tag{3.19}$$

A simple calculation shows that

$$(1 - \beta)d - \frac{3}{2} > 0. \tag{3.20}$$

If $\alpha = 1$ then

$$
\frac{n}{2} + \frac{m}{2d-1} - \theta - \frac{\tau}{2}
$$

$$
\geqslant \frac{n'}{2} + \frac{1}{2(2d-1)}(2dn' + \theta + \rho) - \theta - \frac{n' + \rho/2}{2d} \quad \text{(by (3.16) and (3.18))}
$$

$$
= \left(\frac{1}{2} + \frac{2d}{2(2d-1)} - \frac{1}{2d} \right)n' - \left(1 - \frac{1}{2(2d-1)} \right)\theta + \left(\frac{1}{2(2d-1)} - \frac{1}{4d} \right)\rho
$$

$$
\geqslant \left(\frac{1}{2} + \frac{2d}{2(2d-1)} - \frac{1}{2d} \right)n' - \left(1 - \frac{1}{2(2d-1)} \right)n' \quad \text{(since } \theta \leqslant n' \text{ by Lemma 2.5)}
$$

$$
= \left(\frac{1}{2d-1} - \frac{1}{2d} \right)n'
$$

$$
> 0.
$$

Hence, (3.3) holds, and $W_1 \cup W_2$ gives the desired partition.

So we may assume $\alpha \geqslant 2$ and, hence,

$$
\frac{1}{4(d-\alpha+1)} > \frac{1}{2(2d-1)}. \tag{3.21}
$$

Note that

$$
\frac{n}{2} + \frac{m}{2d-1} - \theta - \frac{\tau}{2}
$$

$$
\geqslant \frac{n'}{2} + \frac{1}{2(2d-1)}(2dn' + \alpha\theta + \rho) - \theta - \frac{n' + \rho/2}{2(d-\alpha+1)} \quad \text{(by (3.16) and (3.18))}
$$

$$
= \left(\frac{1}{2} + \frac{2d}{2(2d-1)} - \frac{1}{2(d-\alpha+1)} \right)n' - \left(1 - \frac{\alpha}{2(2d-1)} \right)\theta
$$

$$
\quad - \left(\frac{1}{4(d-\alpha+1)} - \frac{1}{2(2d-1)} \right)\rho
$$

$$
> \left(\frac{4d-1}{2(2d-1)} - \frac{1}{2(d-\alpha+1)} \right)n' - \left(1 - \frac{\alpha}{2(2d-1)} \right)\theta
$$

$$
\quad - \left(\frac{1}{4(d-\alpha+1)} - \frac{1}{2(2d-1)} \right)(\beta dn' - \alpha\theta) \quad \text{(by (3.14), (3.21) and since } m_1 = \beta dn')
$$

$$
= \left(\frac{(4+\beta)d-1}{2(2d-1)} - \frac{\beta d+2}{4(d-\alpha+1)} \right)n' - \left(1 - \frac{\alpha}{4(d-\alpha+1)} \right)\theta.
$$

First, suppose $1 - \frac{\alpha}{4(d-\alpha+1)} \geqslant 0$. Then

$$
\frac{n}{2} + \frac{m}{2d-1} - \theta - \frac{\tau}{2}
$$

$$
\geqslant \left(\frac{(4+\beta)d-1}{2(2d-1)} - \frac{\beta d+2}{4(d-\alpha+1)} \right)n' - \left(1 - \frac{\alpha}{4(d-\alpha+1)} \right)n' \quad \text{(since } \theta \leqslant n')
$$

$$
= \frac{(2d-1)(\alpha-1) - (\beta d+1)(2\alpha-3)}{4(2d-1)(d-\alpha+1)}n'
$$

$$
> \frac{(2\alpha-3)((1-\beta)d-3/2)}{4(2d-1)(d-\alpha+1)}n'
$$

$$
> 0 \quad \text{(since } \alpha \geqslant 2 \text{ and by (3.20)).}
$$

So (3.3) holds, and $W_1 \cup W_2$ gives the desired partition.

Thus, we may assume $1 - \frac{\alpha}{4(d-\alpha+1)} < 0$, so $\alpha > 4d/5$. Note that

$$
\frac{n}{2} + \frac{m}{2d-1} - \theta - \frac{\tau}{2}
$$

$$> \left(\frac{(4+\beta)d-1}{2(2d-1)} - \frac{\beta d+2}{4(d-\alpha+1)} \right) n' - \left(1 - \frac{\alpha}{4(d-\alpha+1)} \right) \frac{m}{2d-1} \quad \text{(by (3.4))}$$

$$\geqslant \left(\frac{(4+\beta)d-1}{2(2d-1)} - \frac{\beta d+2}{4(d-\alpha+1)} \right) n' - \left(1 - \frac{\alpha}{4(d-\alpha+1)} \right) \frac{d}{2d-1} n \quad \text{(by the fact } m \geqslant dn)$$

$$= \left(\frac{(2+\beta)d-1}{2(2d-1)} - \frac{\beta d+2}{4(d-\alpha+1)} + \frac{\alpha d}{4(d-\alpha+1)(2d-1)} \right) n' - O(n^{1/4}) \quad \text{(as } n' = n - O(n^{1/4}))$$

$$= \left(\frac{4d^2 - 3\alpha d - 2\alpha\beta d - 2d + 2\alpha + 3\beta d}{4(d-\alpha+1)(2d-1)} \right) n' - O(n^{1/4})$$

$$> \left(\frac{4d - 3\alpha - 2\alpha\beta - 2/5}{4(d-\alpha+1)(2d-1)} \right) dn' - O(n^{1/4}) \quad \text{(since } \alpha > 4d/5)$$

$$> \left(\frac{(4 - 3 \cdot \frac{14}{17} - 2 \cdot \frac{14}{17} \cdot \frac{13}{17})d - 2/5}{4(d-\alpha+1)(2d-1)} \right) dn' - O(n^{1/4}) \quad (\alpha < \tfrac{14d}{17} \text{ by (3.17)}, \beta < \tfrac{13}{17} \text{ by (3.19)})$$

$$> \left(\frac{d/4 - 2/5}{4(d-\alpha+1)(2d-1)} \right) dn' - O(n^{1/4})$$

$$> 0 \quad \text{(since } d \geqslant 4 \text{ and } n' = n - O(n^{1/4}) \text{ is large).}$$

Again, (3.3) holds, and $W_1 \cup W_2$ gives the desired partition.

4 Concluding remarks

Using copies of K_{2d-1} and one copy of K_{2d+1}, Lee et al. [18] constructed digraphs with minimum outdegree d and minimum indegree $d-1$ to show that the main term $\frac{d-1}{2(2d-1)}$ in Conjecture 1.1 is best possible. One could even ask whether or not Conjecture 1.1 holds without the $o(1)$ term, and, in particular, for $d = 2, 3$ (as for these cases Conjecture 1.1 is known to be true). More precisely, is it true that if D is a digraph with m arcs and the minimum outdegree is at least 2 (respectively, 3) then D admits a bipartition $V(D) = V_1 \cup V_2$ such that $\min\{e(V_1, V_2), e(V_2, V_1)\} \geqslant m/6$ (respectively, $\min\{e(V_1, V_2), e(V_2, V_1)\} \geqslant m/5$)?

It is not clear to us whether or not the minimum outdegree condition alone is sufficient for Conjecture 1.1 with large d. We have demonstrated that the assertion of Conjecture 1.1 holds if we impose the additional (natural) condition that the minimum indegree of D is also at least d. (Though we do not know whether the main term in Theorem 1.2 is tight.) Below, we comment on the five places in our proof where we use this indegree assumption.

The first place is where we prove (3.6) for a lower bound on m_2 (the number of arcs in $D[Y]$). The minimum outdegree condition alone is not sufficient for this purpose. However, we checked several extreme situations (e.g., when $e(Y) = 0$), and we can always find a partition satisfying the conjectured bound. But we do not know how to deal with the general case.

The second place is in the proof of (3.7) and (3.8). We need to bound the number of arcs of D between X and A_i in both directions for $i = 1, 3$, in order to bound τ (the number of tight components in $G[Y]$) in (3.11) and m (the number of arcs in D) in (3.12).

The third place is where we obtain $\min\{e(X, Y), e(Y, X)\} \geqslant dn' - m_2$ to bound m_1 in (3.13), and the fourth place is where we prove the lower bound on m in (3.16). The final place we use this minimum indegree condition is where we give an upper bound on τ in (3.18).

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