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The local Hölder exponent for the entropy of real unimodal maps

In Memory of Lei Tan

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Abstract We consider the topological entropy $h(\theta)$ of real unimodal maps as a function of the kneading parameter θ (equivalently, as a function of the external angle in the Mandelbrot set). We prove that this function is locally Hölder continuous where $h(\theta) > 0$, and more precisely for any θ which does not lie in a plateau the local Hölder exponent equals exactly, up to a factor $\log 2$, the value of the function at that point. This confirms a conjecture of Isola and Politi (1990), and extends a similar result for the dimension of invariant subsets of the circle.

Keywords entropy, unimodal maps, quadratic polynomials, Hölder exponent

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1 Introduction

In order to encode and classify the topological dynamics of interval maps, Milnor and Thurston [11] defined the *kneading sequence* of a unimodal map f by recording the relative position of the iterates of the critical point. This information can be packaged in a binary number, known as the *kneading parameter*, or *kneading angle* θ (see Section 2 for a precise definition). One can check that the topological entropy of f only depends on the kneading angle θ, and hence we can define

$$h(\theta) := h_{\text{top}}(f)$$

the topological entropy of any unimodal map f which has angle θ (see Figure 1).

The function $h(\theta)$ has also the following interpretation in complex dynamics. Let $\theta \in \mathbb{R}/\mathbb{Z}$, and suppose that the external ray of angle θ for the Mandelbrot set lands on some real parameter c_θ. Then $h(\theta)$ equals the topological entropy of the quadratic polynomial $f(z) = z^2 + c_\theta$.

The entropy of unimodal maps has been explored for several decades. In particular, the function $h(\theta)$

(1) is a continuous, weakly increasing function of θ [11], and its maximum value is $\log 2$;

(2) is constant on small copies of the Mandelbrot set whose root has positive entropy [5];

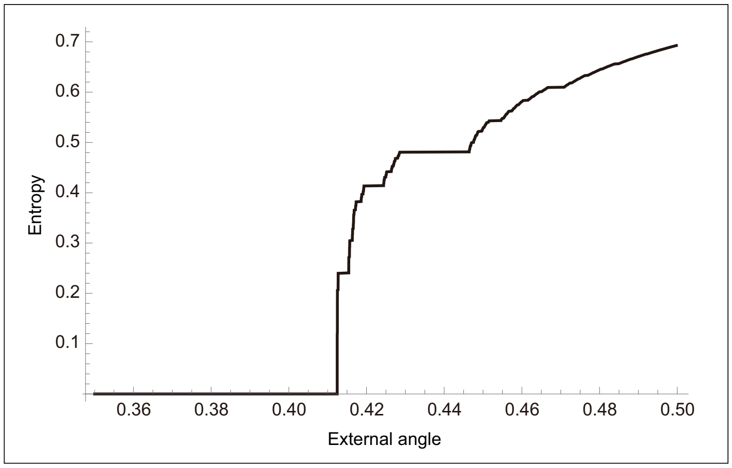

Figure 1 The entropy of real unimodal maps as a function of the external angle. Note the *Feigenbaum angle* $\theta_\star = 0.412454\cdots$, where $h(\theta_\star) = 0$ and the function is continuous but not Hölder continuous. To the right of $\theta_\star$, the function is locally Hölder continuous, and becomes more and more regular; it is actually almost Lipschitz continuous near the right endpoint $\theta = 1/2$

(3) is positive for all $\theta > \theta_\star$, where $\theta_\star$ is the kneading angle of the *Feigenbaum map*, whose binary expansion is the *Thue-Morse* sequence.

In this paper, we are interested in the regularity of h. Our main result states that the local Hölder exponent of the entropy function $h(\theta)$ equals, up to a constant factor of $\log 2$, the value of the function itself. Let us denote by $\alpha(f, z)$ the local Hölder exponent of the function f at z (see Section 2 for a precise definition). It is well known that not all binary sequences are indeed kneading sequences of unimodal maps: let us denote by $\mathcal{R}$ the set of all possible kneading angles. This coincides (up to possibly a set of Hausdorff dimension zero) with the set of angles of external rays, which land on the real slice of the Mandelbrot set [5]. Each connected component U of the complement $[0, 1/2] \setminus \mathcal{R}$ corresponds to a real hyperbolic component, and it is well known that the entropy at the two endpoints of U is the same, and hence one can extend the definition of entropy $h(\theta)$ to each value $\theta \in [0, 1/2]$ by setting it constant over each component U. Moreover, we define a plateau for the entropy function as an open interval in combinatorial parameter space on which the entropy is constant.

Theorem 1.1. *For any $\theta \in \mathcal{R}$ with $\theta > \theta_\star$, the entropy function h is locally Hölder continuous at θ with exponent $\frac{h(\theta)}{\log 2}$. Moreover, if $\theta \in \mathcal{R}$ does not lie in a plateau, then the local Hölder exponent of the entropy function at θ is related to the value of the function by the equation*

$$\alpha(h, \theta) = \frac{h(\theta)}{\log 2}.$$

This result confirms the experimental evidence by Isola and Politi [9, p. 282]. The proof turns out to be a simple computation using the fact that entropy is the zero of a power series known as the *kneading series*: basically, two nearby angles will produce power series with the same leading coefficients, and one needs to estimate how the zeros of a power series change as the coefficients change. In order to get the lower bound, however, one needs to analyze carefully the combinatorics of the set $\mathcal{R}$.

By a classical result of Guckenheimer [8], the topological entropy of any C^1 family of unimodal maps is a Hölder continuous function of the parameter. More recently, a formula for the Hölder exponent of the entropy with respect to the analytic parameter has been obtained in [4]; in this paper, we look instead at the dependence on the combinatorial parameter.

A natural generalization of this discussion would be to extend the result to the *core entropy* for the complex quadratic family, as defined by Thurston[1)] and studied in [6, 13]. In particular, it is proved in [13] that the entropy $h(\theta)$ is locally Hölder continuous in a neighbourhood of rational external angles

1) Thurston W, Baik H, Gao Y, et al. Degree-d invariant laminations. Preprint, 2015

θ with $h(\theta) > 0$, and it is not locally Hölder where $h(\theta) = 0$. The exact value for the Hölder exponent has been conjectured independently by the author (see, e.g., [3, end of introduction]) and by Bruin and Schleicher [2]. Estimates on the Hölder exponent at dyadic tips have been worked out by Jung [10], and a complete proof has recently been announced by Fels (private communication).

Relation to open dynamical systems. Theorem 1.1 can be also reformulated in terms of open dynamical systems, and in this setting it is very close to the results (and the proof) of [3].

Let $D(x) := 2x \mod 1$ be the doubling map, and for each $\theta \in [0, 1/2]$ let us consider the set $K(\theta)$ of points in the circle whose forward orbit never intersects the interval $(\theta, 1 - \theta)$, i.e.,

$$K(\theta) := \{x \in \mathbb{R}/\mathbb{Z} : D^n(x) \notin (\theta, 1-\theta), \forall\, n \geqslant 0\}.$$

Then for each $\theta \in [0, 1/2]$ one has

$$\mathrm{H.dim}\, K(\theta) = \frac{h(\theta)}{\log 2}.$$

Corollary 1.2. *Consider for each $\theta \in [0, 1/2]$ the dimension function*

$$\eta(\theta) := \mathrm{H.dim}\, K(\theta).$$

Then, for each θ not in a plateau, the local Hölder exponent of the dimension function satisfies

$$\alpha(\eta, \theta) = \eta(\theta).$$

A completely analogous statement for the set of points whose orbit does not intersect the forbidden interval $(0, t)$ is proved in [3]. The continuity of entropy (or dimension) for expanding circle maps with holes has been established in the 1980s by Urbański [14], while bounds on the Hölder exponent of the dimension for more general holes are proved in [1].

Note that the dimension function of [3] and the one considered in this paper are genuinely different functions: for example, we will check in Subsection 6.1 that the modulus of continuity of h near $\theta_\star$ is of order $\frac{1}{\log(1/x)}$, while for the dimension function considered in [3] the modulus of continuity at the point where it is not Hölder is of order $\frac{\log\log(1/x)}{\log(1/x)}$.

2 Background material

Hölder exponents. Let $I \subseteq \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ is *locally Hölder continuous* at a point x of exponent α if there exist a neighborhood U of x and a constant $C > 0$ such that

$$|f(y) - f(z)| \leqslant C|y - z|^\alpha \quad \text{for all} \quad y, z \in U.$$

The *local Hölder exponent* of f at z is

$$\alpha(f, z) := \sup\left\{\eta > 0 : \lim_{\epsilon \to 0} \sup_{\substack{|x-z|<\epsilon \\ |y-z|<\epsilon}} \frac{|f(x) - f(y)|}{|x - y|^\eta} < +\infty\right\}.$$

By definition, f is locally Hölder continuous at z if and only if $\alpha(f, z) > 0$.

Kneading theory. Let now $I = [0, 1]$. Recall that a *unimodal map* is a continuous function $f : I \to I$ with $f(0) = f(1) = 0$, and for which there exists a point $c \in (0, 1)$, which we call *critical point*, such that f is increasing on $[0, c)$ and decreasing on $(c, 1]$. In order to capture the symbolic dynamics of f, one defines the *address* of a point $x \neq c$ as

$$A(x) := \begin{cases} 0, & \text{if } x < c \\ 1, & \text{if } x > c. \end{cases}$$

The *kneading sequence* of f is then defined as the sequence of addresses of the iterates of the critical point; namely, for any $k \geqslant 1$ set, if $f^k(c) \neq c$,

$$s_k := A(f^k(c))$$

while, if $f^k(c) = c$, then set

$$s_k := \lim_{x \to c} A(f^k(x)),$$

which is still well-defined as f "folds" a neighbourhood of c. Finally, one defines the *kneading angle* as

$$\theta := \sum_{k=1}^{\infty} \frac{\theta_k}{2^{k+1}} \quad \text{with} \quad \theta_k := s_1 + \cdots + s_k \mod 2.$$

Then the *kneading series* associated to the angle θ is the power series

$$P_\theta(t) = 1 + \sum_{k=1}^{\infty} \epsilon_k t^k,$$

where $\epsilon_k = (-1)^{\theta_k}$. Note that the coefficients of $P_\theta(t)$ are uniformly bounded, and hence the power series defines a holomorphic function in the unit disk $\{t \in \mathbb{C} : |t| < 1\}$. The main result of kneading theory is the following theorem.

Theorem 2.1 (See [11]). *Let f be a unimodal map with kneading angle θ and topological entropy $h(f)$, and let $r := \mathrm{e}^{-h(f)}$. Then the holomorphic function $P_\theta(t)$ is non-zero on the disk $\{t \in \mathbb{C} : |t| < r\}$, and if $r < 1$ one has $P_\theta(r) = 0$.*

The set of real angles. It is well known that not all binary sequences are indeed kneading sequences of unimodal maps: let us denote by $\mathcal{R}$ the set of all possible kneading angles. This coincides (up to possibly a set of Hausdorff dimension zero) with the set of angles of external rays which land on the real slice of the Mandelbrot set.

The angles corresponding to real parameters are characterized in terms of the dynamics of the angle doubling map $D(\theta) := 2\theta \mod 1$. In fact, from [5] one has the identity

$$\mathcal{R} = \{\theta \in [0, 1/2] : D^n(\theta) \notin (\theta, 1-\theta) \text{ for all } n \geqslant 0\}.$$

(Obviously, the set of angles of external rays landing on the real slice of the Mandelbrot set is symmetric about 1/2, but we will only focus on the interval $[0, 1/2]$ in this paper.) Many facts are known about the structure of the set $\mathcal{R}$. In particular, it is a closed set of Lebesgue measure zero and Hausdorff dimension one [15]. Moreover, one knows that for each angle $\theta \in \mathcal{R}$ which is purely periodic for the doubling map, one can produce its *period doubling* θ' as follows: if θ has (minimal) period p and its binary expansion is

$$\theta = .\overline{s_1 \cdots s_p},$$

then we define

$$\theta' = .\overline{s_1 \cdots s_p \check{s}_1 \cdots \check{s}_p},$$

where $\check{s}_i = 1 - s_i$. The following lemma is well-known.

Lemma 2.2. *If θ' is the period doubling of $\theta \in \mathcal{R}$, then $\theta' \in \mathcal{R}$, and moreover,*

$$h(\theta') = h(\theta).$$

Proof. It is immediate to check by the definitions that the two kneading series are related by

$$P_{\theta'}(t) = P_\theta(t) \frac{1 - t^p}{1 + t^p},$$

from which the claim holds, as $\frac{1-t^p}{1+t^p}$ does not have any root inside the unit disk. □

Moreover, the connected components of the complement of $\mathcal{R}$ are precisely (see, e.g., [15])

$$[0,1/2]\setminus\mathcal{R}=\bigcup_{\theta\in\mathcal{R}^0}(\theta,\mathrm{pd}(\theta)),$$

where $\mathrm{pd}(\theta)$ is the period doubling of θ, and $\mathcal{R}^0$ is the subset of $\mathcal{R}$ consisting of purely periodic angles. Note that $\theta=0=.\overline{0}$ belongs to $\mathcal{R}$, and the above equation is correct if one interprets its period doubling to be

$$\mathrm{pd}(0)=.\overline{01}=\frac{1}{3}.$$

For each purely periodic

$$\theta=.\overline{s_1\cdots s_p}\in\mathcal{R},$$

one defines the *small copy* of root θ as the interval

$$I(\theta):=(\theta,\overline{\theta}),$$

where $\overline{\theta}$ has binary expansion

$$\overline{\theta}:=.s_1\cdots s_p\overline{\check{s}_1\cdots\check{s}_p}.$$

The reason for the name "small copy" is that the interval $I(\theta)$ corresponds to the set of external rays landing on the real slice of the small copy of the Mandelbrot set with root of external angle θ.

Lemma 2.3 (See [5])**.** *If $h(\theta)>0$, then the entropy is constant on the small copy $I(\theta)$.*

3 Simplicity of the minimal root

We start by proving that the smallest root of the kneading series is actually simple. This fact may be of independent interest, and is probably known to experts even though we could not find it in the literature.

Theorem 3.1. *Let $f:I\to I$ be a unimodal map with topological entropy $h(f)>0$. Denote by $P(t)$ its kneading series, and let $s=\mathrm{e}^{h(f)}$. Then $r=\frac{1}{s}$ is a simple root of $P(t)$.*

Proof. Let us assume first that f is piecewise linear with slope $\pm s$, and consider the *lap counting function*

$$L(t):=\sum_{n\geqslant 0}\ell(f^{n+1})t^n,$$

where $\ell(f)$ is the number of monotonicity intervals (also known as *laps*) of f. Recall also the kneading identity (see [11, Corollary 5.9])

$$L(t)=\frac{1}{1-t}+\frac{1}{P(t)(1-t)^2}.$$

Now, by [12, Proposition 9.6] r is a simple pole of $L(t)$, and hence by the above identity it is a simple zero of $P(t)$. In the general case, by Milnor and Thurston [11], for any unimodal map of entropy $h(f)=\log s$ there exists a semiconjugacy $\pi:I\to J$ of f to a piecewise linear unimodal map $g:J\to J$ of slope $\pm s$, i.e., there is a continuous, surjective, weakly monotone map $\pi:I\to J$ such that $\pi\circ f=g\circ\pi$. Denote by c the critical point of f, and let $\widetilde{c}=\pi(c)$ be the critical (or *turning*) point of g. Moreover, let $L:=\pi^{-1}(\widetilde{c})$, which is a closed interval containing c. There are two cases:

(1) Either $f^n(c)\notin L$ for all $n\geqslant 1$. This implies that

$$P_f(t)=P_g(t),$$

and hence the claim follows, since from the above we know that r is a simple root of $P_g(t)$.

(2) Otherwise, there exists n such that $f^n(c)\in L$. Let p be the smallest such n. This implies that

$$g^p(\widetilde{c})=g^p(\pi(c))=\pi(f^p(c))=\widetilde{c},$$

since $f^p(c) \in L$. Then we get the factorisation

$$P_f(t) = \widetilde{P}_g(t) P_h(t^p),$$

where $h = f^p|_L$ is the first return map of f to L, which is also a unimodal map, and $\widetilde{P}_g(t)$ is the polynomial (of degree $p-1$) such that

$$P_g(t) = \frac{\widetilde{P}_g(t)}{1-t^p}.$$

We know in [11] that r is a root of $P_g(t)$, and we now claim that r is *not* a root of $P_h(t^p)$; by the above factorisation, this implies that r is a simple root of $P_f(t)$. In order to prove the claim, let $k \geqslant 1$ be the integer such that

$$\frac{\log 2}{2^k} < h(g) \leqslant \frac{\log 2}{2^{k-1}}.$$

Note that the above inequality implies that g must be renormalizable of period 2 at least $k-1$ times, i.e., that there exists an interval I' which contains the critical point and such that the restriction of $g^{2^{k-1}}$ to I' is a unimodal map $\widehat{g}$. Note now that

$$\mathrm{Period}(g) = 2^{k-1}\mathrm{Period}(\widehat{g}),$$

and $\mathrm{Period}(\widehat{g}) \geqslant 2$ since g, and hence $\widehat{g}$, has positive entropy. Thus, the period p of g is at least 2^k. On the other hand, since h is unimodal, each of the roots of $P_h(t)$ has modulus $\geqslant \frac{1}{2}$, and thus each root of $P_h(t^p)$ has modulus at least

$$\frac{1}{2^{1/p}} \geqslant \frac{1}{2^{1/2^k}} > \mathrm{e}^{-h(g)} = r.$$

This means that r is not a root of $P_h(t^p)$, and by the above discussion r is a simple root of $P_g(t)$. This proves that r is a simple root of $P_f(t)$. □

Lemma 3.2. *Let $\theta \in (0,1/2]$ be a real angle with $h(\theta) > 0$, and let $r = \mathrm{e}^{-h(\theta)}$. Then there exists $\epsilon > 0$ such that for any $\theta' \in \mathcal{R} \cap (0,1/2]$ with $|\theta - \theta'| < \epsilon$ the kneading series $P_{\theta'}(t)$ has exactly one root (counted with multiplicity) inside the disk $\{t \in \mathbb{C} : |t-r| < \epsilon\}$. Moreover, there exists $c > 0$ such that $|P'_{\theta'}(t)| \geqslant c$ for all $|\theta' - \theta| < \epsilon$ and $|t - r| < \epsilon$.*

Proof. There are two cases. If θ is not purely periodic for the doubling map, then for θ' sufficiently close to θ the coefficients of $P_{\theta'}(t)$ eventually stabilize to the coefficients of $P_\theta(t)$. Hence, $P_{\theta'}(t)$ converges to $P_\theta(t)$ uniformly on compact subsets of the unit disk. Moreover, Theorem 3.1 implies that r is the only root of $P_\theta(t)$ in a neighborhood of $z = r$, counting with multiplicities, so the first claim follows by Rouché's theorem. On the other hand, if θ is purely periodic of period p, then there are two possible limits of the power series $P_{\theta'}(t)$ as $\theta' \to \theta$. Indeed, one checks that

$$\lim_{\theta' \to \theta^-} P_{\theta'}(t) = P_\theta(t)$$

while

$$\lim_{\theta' \to \theta^+} P_{\theta'}(t) = \widehat{P}_\theta(t) = P_\theta(t)\frac{1-t^p}{1+t^p}$$

(in both cases, the limit is taken over $\theta' \in \mathcal{R}$). Since the function $\varphi(t) = \frac{1-t^p}{1+t^p}$ is never vanishing inside the unit disk, then the claim follows by Rouché's theorem as before. To prove the second claim, note that we just proved that $P'_\theta(r) \neq 0$, and $\widehat{P}'_\theta(r) \neq 0$, so the claim follows by noting that the derivative $P'_{\theta'}(t)$ also converges uniformly on compact sets to either $P'_\theta(t)$ or $\widehat{P}'_\theta(t)$. □

4 The local Hölder exponent: The upper bound

Let us start with an elementary lemma.

Lemma 4.1. *Let θ and θ' be two real angles, and let us assume $\theta' < \theta \leqslant \frac{1}{2}$. Let their binary expansions be*

$$\theta = \sum_{k=1} \theta_k 2^{-k} \quad \textit{and} \quad \theta' = \sum_{k=1} \theta'_k 2^{-k}$$

and let $n := \min\{k : \theta_k \neq \theta'_k\}$. Then

$$c2^{-n} \leqslant |\theta - \theta'| \leqslant 2^{-n+1},$$

where $c = 2(1-2\theta)$ if $\theta < 1/2$, and $c = 1$ if $\theta = 1/2$.

Proof. Let us first consider the case $\theta < 1/2$. Recall that the set $\mathcal{R}$ is characterized as

$$\mathcal{R} = \{\theta \in [0, 1/2] : D^n(\theta) \notin (\theta, 1-\theta), \forall\, n \geqslant 0\}.$$

Now, by definition of n one has

$$D^{n-1}(\theta') < \frac{1}{2} < D^{n-1}(\theta),$$

and hence by the definition of $\mathcal{R}$,

$$D^{n-1}(\theta') \leqslant \theta' < \theta < \frac{1}{2} < 1 - \theta \leqslant D^{n-1}(\theta).$$

This implies

$$2^{n-1}(\theta - \theta') = D^{n-1}(\theta) - D^{n-1}(\theta') \geqslant 1 - 2\theta,$$

which yields the lower bound.

If $\theta = 1/2$, then $D^{n-1}(\theta) = 1$, and hence $D^{n-1}(\theta) - D^{n-1}(\theta') \geqslant 1/2$, and the proof proceeds as before.

The upper bound follows simply because

$$2^{n-1}(\theta - \theta') = D^{n-1}(\theta) - D^{n-1}(\theta') \leqslant 1.$$

This completes the proof. □

Proposition 4.2. *For each $\theta \in \mathcal{R}$ with $h(\theta) > 0$, there exists $C = C(\theta) > 0$ such that the modulus of continuity of the entropy is bounded by*

$$|h(\theta) - h(\theta')| \leqslant C|\theta - \theta'|^{\frac{h(\theta)}{\log 2}}$$

for each $\theta, \theta' \in \mathcal{R}$ with $\theta' < \theta \leqslant 1/2$.

Proof. Let θ and θ' be two real angles, with $\theta' < \theta \leqslant \frac{1}{2}$. Let their binary expansions be

$$\theta = \sum_{k=1} \theta_k 2^{-k} \quad \text{and} \quad \theta' = \sum_{k=1} \theta'_k 2^{-k}$$

and let $n := \min\{k : \theta_k \neq \theta'_k\}$. Then by Lemma 4.1 one gets

$$c2^{-n} \leqslant |\theta - \theta'| \leqslant 2^{-n+1}.$$

Let us now compare the two kneading series $P_\theta(t)$ and $P_{\theta'}(t)$. As the first $n-1$ coefficients of the two series coincide, one gets

$$P_\theta(t) - P_{\theta'}(t) = 2t^n + \sum_{k=n+1}^{\infty} (\epsilon_k - \epsilon'_k)t^k = t^n g(t), \tag{4.1}$$

where

$$g(t) = 2 + \sum_{k=1}^{\infty} (\epsilon_{n+k} - \epsilon'_{n+k})t^k.$$

On the other hand, as $P_\theta(r) = P_{\theta'}(r') = 0$, one has

$$P_\theta(r) - P_{\theta'}(r) = P_{\theta'}(r') - P_{\theta'}(r) = P'_{\theta'}(\xi)(r' - r) \tag{4.2}$$

with $\xi \in [r, r']$. Thus, combining the two previous equations we get

$$r' - r = r^n \frac{g(r)}{P'_{\theta'}(\xi)}. \tag{4.3}$$

In order to get the upper bound, let us note that $|g(r)| \leqslant \frac{2}{1-r}$ as the coefficients of $g(t)$ are bounded in the absolute value by 2. Moreover, by Lemma 3.2 we have

$$\inf_{\substack{|\theta-\theta'|<\epsilon \\ |\xi-\theta|<\epsilon}} |P'_{\theta'}(\xi)| = c_1 > 0.$$

Finally, by Lemma 4.1 one gets

$$n \geqslant \frac{\log c - \log|\theta - \theta'|}{\log 2},$$

and hence

$$r^n = \mathrm{e}^{n \log r} \leqslant c_2 |\theta - \theta'|^{\frac{-\log r}{\log 2}}, \tag{4.4}$$

where $c_2 = \mathrm{e}^{\frac{\log r \log c}{\log 2}}$. Thus, putting together the previous estimates

$$r' - r \leqslant \frac{c_2}{c_1(1-r)} |\theta - \theta'|^{\frac{-\log r}{\log 2}}, \tag{4.5}$$

which using the definition $h(\theta) = -\log r$ yields the upper bound

$$r' - r \leqslant C|\theta - \theta'|^{\frac{h(\theta)}{\log 2}},$$

where we set $C = \frac{c_2}{c_1(1-r)}$. The claim then follows as $h(\theta) = -\log r$ and the function $x \mapsto \log x$ is differentiable with the bounded derivative (hence Lipschitz) on the interval $[1, 2]$. □

5 Primitive angles

In order to prove the lower bound for the local Hölder exponent we need the following definition.

Definition 5.1. An angle $\theta \in \mathcal{R}$ is called *primitive* if it is purely periodic for the doubling map, and moreover such that $D^k(\theta) \neq 1 - \theta$ for all $k \geqslant 0$.

A purely periodic, real angle which is not primitive will be called *satellite*. The external rays corresponding to these parameters land at roots of satellite components of the Mandelbrot set.

Lemma 5.2. *If $\theta \in \mathcal{R}$ is satellite, then the entropy is locally constant at θ.*

Proof. Indeed, if $D^k(\theta) = 1 - \theta$ then the binary expansion of θ is of the form

$$\theta = .\overline{s_1 \cdots s_k \check{s}_1 \cdots \check{s}_k},$$

where $\check{s}_i := 1 - s_i$. This means that θ is the period doubling of the angle $\theta' = .\overline{s_1 \cdots s_k}$, and thus $h(\theta) = h(\theta')$ by Lemma 2.2. □

The reason we introduce this definition is because we need it to prove that primitive angles can be approximated by real angles with controlled combinatorics.

Lemma 5.3. *Let $\theta \in \mathcal{R} \cap (0, \frac{1}{2})$ be a primitive angle with $D^p(\theta) = \theta$. Pick $\delta > 0$ such that $D^k(\theta) \notin [\theta, 1 - \theta + 2\delta]$ for $0 < k < p$, and let $\theta' \in \mathcal{R}$ be a purely periodic angle with $\theta - \delta < \theta' < \theta$ and $D^q(\theta') = \theta'$. Let*

$$\theta = \sum_{k=1}^{\infty} s_k 2^{-k} \qquad \textit{and} \qquad \theta' = \sum_{k=1}^{\infty} t_k 2^{-k}$$

be the binary expansions of θ *and* θ'*, respectively. Then the point of binary expansion*

$$\xi := .\overline{s_1 \cdots s_p t_1 \cdots t_q}$$

belongs to $\mathcal{R}$*.*

Proof. In order to simplify the notation, let us introduce the binary sequences $s = s_1 \cdots s_p$ and $t = t_1 \cdots t_q$. Since the map D^p is uniformly expanding, if we let $x = .s\bar{t} \in [\theta', \theta]$, then D^p is a homeomorphism between $[x, \theta]$ and $[\theta', \theta]$. Similarly, the point $y = .t\bar{s} \in [\theta', \theta]$ is so that D^q is a homeomorphism between $[\theta', y]$ and $[\theta', \theta]$. We have that $\xi \in [x, \theta]$ and $D^p(\xi) \in [\theta', y]$, with $D^{p+q}(\xi) = \xi$. We now check that ξ belongs to $\mathcal{R}$. In order to do so, we will check that for all iterates $0 < k < p+q$ the point $D^k(\xi)$ does not lie in the "forbidden" interval $(\xi, 1-\xi)$. Let us first consider the earlier iterates, $D^k(\xi)$ with $0 < k \leqslant p$. Then as D^k is expanding and orientation-preserving on $[x, \theta]$ one gets the estimates

$$\theta - \xi \leqslant D^k(\theta) - D^k(\xi) \leqslant D^p(\theta) - D^p(\xi) \leqslant \theta - \theta'.$$

There are two cases:

- If $D^k(\theta) < \frac{1}{2}$, then $D^k(\theta) \leqslant \theta$, and hence

$$D^k(\xi) \leqslant (D^k(\theta) - \theta) + \xi \leqslant \xi.$$

- If $D^k(\theta) > \frac{1}{2}$, then by hypothesis

$$D^k(\theta) \geqslant 1 - \theta + 2\delta \geqslant 1 - \theta + 2(\theta - \theta'),$$

which implies

$$D^k(\xi) \geqslant D^k(\theta) - \theta + \theta' \geqslant 1 - \theta' \geqslant 1 - \xi$$

as required.

Let us consider now $D^{p+k}(\xi)$, with $0 < k < q$. Recall that by construction $D^{p+q}(\xi) = \xi$. Moreover, the map D^k is expanding and orientation-preserving on $[\theta', y]$, and thus

$$0 \leqslant D^{p+k}(\xi) - D^k(\theta') \leqslant D^{p+q}(\xi) - D^q(\theta') = \xi - \theta'.$$

Now, there are two cases:

- Suppose $D^k(\theta') < \frac{1}{2}$. Then $D^k(\theta') \leqslant \theta'$ as θ' belongs to $\mathcal{R}$, and hence one gets

$$D^{p+k}(\xi) \leqslant \xi + (D^k(\theta') - \theta') \leqslant \xi$$

as required.

- Suppose instead $D^k(\theta') > \frac{1}{2}$. Then $D^k(\theta') \geqslant 1 - \theta'$, and hence we can write

$$D^{p+k}(\xi) \geqslant D^k(\theta') \geqslant 1 - \theta' \geqslant 1 - \xi.$$

In both cases, $D^{p+k}(\xi)$ belongs to $[\xi, 1-\xi]$, and hence ξ belongs to $\mathcal{R}$. □

Corollary 5.4. *In the hypotheses of the previous lemma, if we denote* $s = s_1 \cdots s_p$ *and* $t = t_1 \cdots t_q$*, then for each* $m \geqslant 1$ *the sequence of points* $\theta_m = .\overline{s^m t}$ *belongs to* $\mathcal{R}$*.*

Corollary 5.5. *If* $\theta \in \mathcal{R} \cap (0, 1/2)$ *is a purely periodic, primitive angle, then for any* $\delta > 0$ *there exists a purely periodic* $\theta' \in \mathcal{R}$ *with* $\theta - \delta < \theta' < \theta$*.*

6 The lower bound

We now turn to the proof of the lower bound on the modulus of continuity, i.e., we show that the entropy function is not more regular than expected, and in particular it is not Hölder continuous of any exponent higher than $h(\theta)/\log 2$. We start by proving it for primitive angles.

Proposition 6.1. *Let $\theta \in \mathcal{R}$ be a primitive angle which does not lie in a plateau. Then one has the bound*

$$\limsup_{\theta' \to \theta} \frac{|h(\theta) - h(\theta')|}{|\theta - \theta'|^{\frac{h(\theta)}{\log 2}}} = c > 0.$$

Proof. Let $\theta \in \mathcal{R} \cap (0, \frac{1}{2})$ be a purely periodic, primitive angle of period p, and let us define

$$\delta := \frac{1}{2} \min\{D^k(\theta) - 1 + \theta : k \geqslant 0,\, D^k(\theta) > 1 - \theta\} > 0.$$

Then, by Corollary 5.5, there exists a purely periodic $\theta' \in \mathcal{R} \cap (\theta - \delta, \theta)$: let q be the period of θ', and denote $P = pq$. Now by Corollary 5.4, for any m, the angle

$$\theta_m = .\overline{(\epsilon_1 \cdots \epsilon_P)^m (\epsilon'_1 \cdots \epsilon'_P)}$$

belongs to $\mathcal{R}$, where (ϵ_k) and (ϵ'_k) are, respectively, the digits in the binary expansions of θ and θ'. Now, if we let

$$g(t) = \sum_{k=1}^{P} (\epsilon_k - \epsilon'_k) t^{k-1},$$

then for each m the difference between the two kneading series $P_{\theta_m}(t)$ and $P_\theta(t)$ can be written as

$$P_\theta(t) - P_{\theta_m}(t) = \frac{\sum_{k=1}^{P} (\epsilon_k - \epsilon'_k) t^{mP+k-1}}{1 - t^{P(m+1)}} = \frac{g(t) t^{mP}}{1 - t^{P(m+1)}}.$$

Thus, by denoting by r_m the smallest real root of P_{θ_m} and using $P_\theta(r) = P_{\theta_m}(r_m) = 0$, one has

$$P_\theta(r) - P_{\theta_m}(r) = P_{\theta_m}(r_m) - P_{\theta_m}(r) = P'_{\theta_m}(\xi)(r_m - r)$$

for some $\xi \in [r, r_m]$. By combining the previous equations we get

$$r_m - r = \frac{1}{P'_{\theta_m}(\xi)} \frac{g(r) r^{mP}}{1 - r^{P(m+1)}}. \tag{6.1}$$

Note that, since θ does not lie in a plateau, $r_m \neq r$ for m sufficiently large, and hence $g(r) \neq 0$. Now, observe that the binary expansions of θ_m and θ have at least mP common initial digits, and hence

$$|\theta - \theta_m| \leqslant 2^{-mP}.$$

Thus

$$r^{mP} = \mathrm{e}^{mP \log r} \geqslant |\theta - \theta_m|^{-\frac{\log r}{\log 2}} = |\theta - \theta_m|^{\frac{h(\theta)}{\log 2}}.$$

On the other hand, the coefficient of t^k in $P'_{\theta_m}(t)$ has modulus less than or equal to $k + 1$, and hence

$$|P'_{\theta_m}(t)| \leqslant \sum_{k=0} (k+1) t^k = \frac{1}{(1-t)^2},$$

so using $\xi < r_1 < 1$ one gets

$$\frac{1}{|P'_{\theta_m}(\xi)|} \geqslant (1 - r_1)^2$$

and hence setting $c_3 = g(r)(1 - r_1)^2$ yields the final estimate

$$r_m - r \geqslant c_3 |\theta - \theta_m|^{\frac{h(\theta)}{\log 2}}, \tag{6.2}$$

which as $\theta_m \to \theta$ for $m \to \infty$ establishes the required lower bound. □

Proof of Theorem 1.1. The first claim follows directly from Proposition 4.2. The second claim is proved by noticing that every $\theta \in \mathcal{R}$ which does not lie in a plateau can be approximated by primitive angles which do not lie in a plateau, and hence the lower bound on the modulus of continuity follows directly from Proposition 6.1. □

6.1 The Feigenbaum point

Let us recall in the end that the entropy function is not Hölder continuous at $\theta = \theta_\star$, and in fact one can compute its modulus of continuity using the combinatorics of period doubling.

Let us consider the binary string (S_n) defined recursively as $S_0 := 0$ and $S_{n+1} := S_n \check{S}_n$. The limit $S_\infty := \lim_{n\to\infty} S_n$ is the well-known *Thue-Morse* sequence, which is the binary expansion of the Feigenbaum angle $\theta_\star$. For each n, the angle $\eta_n := .\overline{S_n}$ lands at the root of a hyperbolic component of period 2^n which is given by n-times period doubling of the main cardioid, and there is an associated small copy M_n of the Mandelbrot set. We will consider the angle $\theta_n := .S_n\overline{\check{S}_n}$, whose ray lands at the tip of M_n, and so that $\theta_\star = \lim_{n\to\infty} \theta_n$. Since θ_n is given by tuning of the tip of the Mandelbrot set with a zero entropy map of period 2^n, one has

$$h(\theta_n) = \frac{\log 2}{2^n}$$

while by looking at the binary expansions one gets $\theta_n - \theta_\star \asymp 2^{-2^n}$, which yields the estimate

$$|h(\theta_n) - h(\theta_\star)| \asymp \frac{1}{-\log|\theta_n - \theta_\star|}$$

(up to a multiplicative constant), and thus the modulus of continuity is of order $\frac{1}{\log(\frac{1}{x})}$.

6.2 Relation to open dynamical systems

Proof of Corollary 1.2. Let f be a real quadratic polynomial of kneading angle θ, and let J be the Julia set of f, which we know to be locally connected. Thus, there exists a continuous Caratheodory map $\gamma : \mathbb{R}/\mathbb{Z} \to J$ which semiconjugates the doubling map D to f. Let us consider the set $\widetilde{K} := \gamma^{-1}(J \cap \mathbb{R})$ of external angles of rays which land on the real section of the Julia set. By Douady [5], this set can be characterized as

$$\widetilde{K} = \{x \in \mathbb{R}/\mathbb{Z} : D^n(x) \notin (\theta, 1-\theta), \forall\, n \geqslant 1\}.$$

Since the map γ is finite-to-one, one gets the equality

$$h(f) = h(f\,|_{J\cap\mathbb{R}}) = h(D\,|_{\widetilde{K}}).$$

Moreover, the map D is uniformly expanding with derivative 2, and hence (see, e.g., [7, Proposition III.1])

$$\text{H.dim}\, \widetilde{K} = \frac{h(D\,|_{\widetilde{K}})}{\log 2}.$$

Finally, if we denote

$$K(\theta) := \{x \in \mathbb{R}/\mathbb{Z} : D^n(x) \notin (\theta, 1-\theta), \forall\, n \geqslant 0\}$$

then clearly

$$D(\widetilde{K}) \subseteq K(\theta) \subseteq \widetilde{K},$$

and hence

$$\text{H.dim}\, K(\theta) = \text{H.dim}\, \widetilde{K} = \frac{h(f)}{\log 2} = \frac{h(\theta)}{\log 2},$$

and thus the Corollary follows directly from Theorem 1.1. □

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