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Existence and multiplicity of normalized solutions for a class of fractional Choquard equations

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Abstract In this paper, we study the existence and multiplicity of solutions with a prescribed L^2-norm for a class of nonlinear fractional Choquard equations in $\mathbb{R}^N$:

$$(-\Delta)^s u - \lambda u = (\kappa_\alpha * |u|^p)|u|^{p-2}u,$$

where $N \geqslant 3$, $s \in (0,1)$, $\alpha \in (0,N)$, $p \in (\max\{1+\frac{\alpha+2s}{N}, 2\}, \frac{N+\alpha}{N-2s})$ and $\kappa_\alpha(x) = |x|^{\alpha-N}$. To get such solutions, we look for critical points of the energy functional

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2 - \frac{1}{2p}\int_{\mathbb{R}^N} (\kappa_\alpha * |u|^p)|u|^p$$

on the constraints

$$S(c) = \{u \in H^s(\mathbb{R}^N) : \|u\|^2_{L^2(\mathbb{R}^N)} = c\}, \quad c > 0.$$

For the value $p \in (\max\{1+\frac{\alpha+2s}{N}, 2\}, \frac{N+\alpha}{N-2s})$ considered, the functional I is unbounded from below on $S(c)$. By using the constrained minimization method on a suitable submanifold of $S(c)$, we prove that for any $c > 0$, I has a critical point on $S(c)$ with the least energy among all critical points of I restricted on $S(c)$. After that, we describe a limiting behavior of the constrained critical point as c vanishes and tends to infinity. Moreover, by using a minimax procedure, we prove that for any $c > 0$, there are infinitely many radial critical points of I restricted on $S(c)$.

Keywords fractional Choquard, normalized solution, limiting behavior, constrained minimization

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1 Introduction and main results

In this paper, we study the nonlinear fractional Schrödinger equation with a Choquard nonlinearity as follows:

$$(-\Delta)^s u - \lambda u = (\kappa_\alpha * |u|^p)|u|^{p-2}u, \quad x \in \mathbb{R}^N, \tag{1.1}$$

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where $N \geqslant 3$, $s \in (0,1)$, $\alpha \in (0,N)$, $p \in (\max\{1+\frac{\alpha+2s}{N},2\},\frac{N+\alpha}{N-2s})$ and $\kappa_\alpha(x)=|x|^{\alpha-N}$. For any $u(x)$ in the Schwartz class on $\mathbb{R}^N$, the fractional Laplacian $(-\Delta)^s$ is a nonlocal operator defined as

$$(-\Delta)^s u(x) = C(N,s)\text{P.V.}\int_{\mathbb{R}^N}\frac{u(x)-u(y)}{|x-y|^{N+2s}}dy,$$

where P.V. means the Cauchy principal value on the integral and $C(N,s)$ is some positive normalization constant (see [13] for details).

Equation (1.1) arises from seeking the standing wave solutions for the following time-dependent fractional Choquard equation:

$$\mathrm{i}v_t = (-\Delta)^s v - (\kappa_\alpha * |v|^p)|v|^{p-2}v, \quad (x,t)\in\mathbb{R}^N\times(0,+\infty). \tag{1.2}$$

The wave function $v(x,t):\mathbb{R}^N\times\mathbb{R}^+\to\mathbb{C}$ is normalized according to

$$\int_{\mathbb{R}^N}|v(x,t)|^2dx = c, \tag{1.3}$$

where c is the total number of particles. We point out that Choquard nonlinearities arise in several models of mathematical physics, as in the mean field limit of weakly interacting molecules [23], in the Pekar theory of polarons [20,26–28], in the Schrödinger-Newton systems [16] and in the modeling of boson stars [15]. Specially, for the case $s=\frac{1}{2}$, (1.2) has been used to model the dynamics of pseudo-relativistic boson stars [14].

Throughout this paper, we denote the norm of $L^p(\mathbb{R}^N)$ by $\|u\|_p := (\int_{\mathbb{R}^N}|u|^p)^{\frac{1}{p}}$ for any $1\leqslant p<\infty$. The Hilbert space $H^s(\mathbb{R}^N)$ is defined as

$$H^s(\mathbb{R}^N) := \{u\in L^2(\mathbb{R}^N) : (-\Delta)^{\frac{s}{2}}u\in L^2(\mathbb{R}^N)\},$$

with the inner product and norm given respectively by

$$(u,v) := \int_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}v + \int_{\mathbb{R}^N}uv, \quad \|u\| := (\|(-\Delta)^{\frac{s}{2}}u\|_2^2 + \|u\|_2^2)^{\frac{1}{2}},$$

where

$$\|(-\Delta)^{\frac{s}{2}}u\|_2^2 := \frac{C(N,s)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}.$$

$H^{-s}(\mathbb{R}^N)$ is the dual space of $H^s(\mathbb{R}^N)$ and

$$H_r^s(\mathbb{R}^N) := \{u\in H^s(\mathbb{R}^N)\,|\,u(x)=u(|x|)\}$$

with the $H^s(\mathbb{R}^N)$ norm. We use respectively "$\to$" and "$\rightharpoonup$" to denote the strong and weak convergence in the related function spaces. C will denote a positive constant unless specified.

We say that $u\in H^s(\mathbb{R}^N)$ is a weak solution to (1.1) if

$$\int_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}v - \lambda\int_{\mathbb{R}^N}uv = \int_{\mathbb{R}^N}(\kappa_\alpha * |u|^p)|u|^{p-2}uv, \quad \text{for all} \quad v\in H^s(\mathbb{R}^N)$$

and $(u_c,\lambda_c)\in H^s(\mathbb{R}^N)\times\mathbb{R}$ is a couple of weak solution to (1.1) if u_c is a weak solution to (1.1) with $\lambda=\lambda_c$.

Motivated by the fact that physicists are interested in normalized solutions to (1.2), i.e., solutions to (1.2) satisfying (1.3), we set in (1.2),

$$v(x,t) = \mathrm{e}^{-\mathrm{i}\lambda t}u(x), \quad x\in\mathbb{R}^N, \quad t>0,$$

and then u satisfies (1.1) with

$$\int_{\mathbb{R}^N}|u(x)|^2dx = \int_{\mathbb{R}^N}|v(x,0)|^2dx = c.$$

Thus, we consider for each $c > 0$ the following problem:

(P_c) To find a couple $(u_c, \lambda_c) \in H^s(\mathbb{R}^N) \times \mathbb{R}$ of weak solution to (1.1) such that $\|u_c\|_2^2 = c$.

Define

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 - \frac{1}{2p}\int_{\mathbb{R}^N} (\kappa_\alpha * |u|^p)|u|^p \tag{1.4}$$

for $u \in H^s(\mathbb{R}^N)$. Then $I \in C^1(H^s(\mathbb{R}^N), \mathbb{R})$ and a critical point of I restricted on the constraint

$$S(c) = \{u \in H^s(\mathbb{R}^N) : \|u\|^2_{L^2(\mathbb{R}^N)} = c\}, \quad c > 0 \tag{1.5}$$

corresponds to a couple $(u_c, \lambda_c) \in H^s(\mathbb{R}^N) \times \mathbb{R}$ of weak solution to (1.1) such that $\|u_c\|_2^2 = c$.

The $\lambda \in \mathbb{R}$ in (1.1) is called a frequency. For fixed λ, d'Avenia et al. [12] obtained weak solutions to (1.1) by looking for critical points of the C^1 functional

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 - \frac{\lambda}{2}\int_{\mathbb{R}^N} |u|^2 - \frac{1}{2p}\int_{\mathbb{R}^N} (\kappa_\alpha * |u|^p)|u|^p$$

in $H^s(\mathbb{R}^N)$. By minimizing

$$\Sigma(u) := \frac{\|(-\Delta)^{\frac{s}{2}} u\|_2^2 - \lambda\|u\|_2^2}{(\int_{\mathbb{R}^N} (\kappa_\alpha * |u|^p)|u|^p)^{\frac{1}{p}}}$$

on $H^s(\mathbb{R}^N)\backslash\{0\}$, d'Avenia et al. [12] obtained a ground state $u \in H^s(\mathbb{R}^N)$ to (1.1) when $p \in (1+\frac{\alpha}{N}, \frac{N+\alpha}{N-2s})$ (see [12, Theorem 4.2]). What is more, by using the symmetric mountain pass theorem for the functional $J(u)$ in $H^s(\mathbb{R}^N)$, d'Avenia et al. [12] obtained a multiplicity result for solutions to (1.1) when $p \in (1+\frac{\alpha}{N}, \frac{N+\alpha}{N-2s})$ (see [12, Theorem 1.2]). In addition, d'Avenia et al. [12] also considered normalized solutions to (1.1), by minimizing $I(u)$ defined by (1.4) on the constraints $S(c)$ defined by (1.5), and proved that there is a couple $(u_c, \lambda_c) \in H^s(\mathbb{R}^N) \times \mathbb{R}$ of weak solution to (1.1) with $\|u_c\|_2^2 = c$ when $p \in (1+\frac{\alpha}{N}, 1+\frac{\alpha+2s}{N})$ (see [12, Theorem 4.5]). Note that, in this case, the frequency $\lambda_c \in \mathbb{R}$ appears as a Lagrange multiplier.

Recently, normalized solutions to elliptic PDEs and systems attract much attention of researchers [3–9, 18, 19, 22, 25]. In [18], Jeanjean considered the following semi-linear elliptic equation:

$$-\Delta u - \lambda u = g(u), \quad \lambda \in \mathbb{R}, \quad x \in \mathbb{R}^N, \tag{1.6}$$

where $N \geqslant 1$ and g satisfies some suitable conditions. By a minimax procedure, Jeanjean [18] proved that for each $c > 0$, (1.6) has at least one couple $(u_c, \lambda_c) \in H^1(\mathbb{R}^N) \times \mathbb{R}^-$ of weak solution with $\|u_c\|_2^2 = c$. Furthermore, an $H^1(\mathbb{R}^N)$-bifurcation result of problem (1.6), i.e., the dependence of $\|\nabla u_c\|_2$ and λ_c on c was given.

In [7], Bellazzini et al. considered the following Schrödinger-Poisson equation:

$$-\Delta u - \lambda u + (|x|^{-1} * u^2)u = |u|^{p-2}u, \quad \lambda \in \mathbb{R}, \quad x \in \mathbb{R}^3. \tag{1.7}$$

By developing a mountain pass argument on

$$S_1(c) = \{u \in H^1(\mathbb{R}^N) : \|u\|^2_{L^2(\mathbb{R}^N)} = c\}, \quad c > 0, \tag{1.8}$$

Bellazzini et al. [7] proved that if $p \in (\frac{10}{3}, 6)$, there exists $c_0 > 0$ such that for any $c \in (0, c_0)$, (1.7) possesses at least one couple $(u_c, \lambda_c) \in H^1(\mathbb{R}^N) \times \mathbb{R}^-$ of weak solution with $\|u_c\|_2^2 = c$.

In [3], Bartsch and de Valeriola also considered the semi-linear Schrödinger equation (1.6). Under suitable assumptions on g, Bartsch and de Valeriola [3] proved there are infinitely many normalized solutions to (1.6). Inspired by [3], Luo [25] proved that if $p \in (\frac{10}{3}, 6)$, there exists a $c_0 > 0$ such that for any $c \in (0, c_0)$, there exists an unbounded sequence of couples of weak solutions $\{(\pm u_n, \lambda_n)\} \subseteq H^1_r(\mathbb{R}^N) \times \mathbb{R}^-$ to (1.7) with $\|u_n\|_2^2 = c$ for each $n \in \mathbb{N}^+$.

In [19], Jeanjean et al. considered the quasi-linear Schrödinger equation

$$-\Delta u - u\Delta(u^2) - \lambda u = |u|^{p-1}u \quad \text{in} \quad \mathbb{R}^N, \tag{1.9}$$

where $p \in (1, \frac{3N+2}{N-2})$ if $N \geqslant 3$ and $p \in (1, +\infty)$ if $N = 1, 2$. Using a perturbation method, Jeanjean et al. [19] proved that there exist at least two normalized solutions to (1.9). One is a mountain pass solution and the other is a minimizer either local or global.

Motivated by the above papers, in this paper, we discuss the existence of normalized solutions to (1.1), including the normalized ground state solution and the high energy normalized solution. Since our definition of the normalized ground state solution does not seem to be completely standard, we now give the precise definition of it. Following [6], for any fixed $c > 0$, we say that $u_c \in S(c)$ is a **normalized ground state solution** to (1.1) if $I'|_{S(c)}(u_c) = 0$ and

$$I(u_c) = \inf\{I(u) \,|\, u \in S(c),\, I'\,|_{S(c)}(u) = 0\}.$$

Furthermore, we say that $v_c \in S(c)$ is a **high energy normalized solution** to (1.1) if $I'\,|_{S(c)}(v_c) = 0$ and

$$I(v_c) > \inf\{I(u) \,|\, u \in S(c),\, I'\,|_{S(c)}(u) = 0\}.$$

For any $c > 0$, we set $\gamma(c) := \inf_{u\in S(c)} I(u)$. It is standard that the minimizers of $\gamma(c)$ are critical points of $I|_{S(c)}$ as well as normalized ground state solutions to (1.1). By scaling $u^t(x) = t^{\frac{N}{2}} u(tx)$, $t > 0$, it is easy to know that $p = 1 + \frac{\alpha+2s}{N}$ is the L^2-critical or mass-critical exponent for our minimizing problem in the sense that for any $c > 0$, $\gamma(c) > -\infty$ if $p \in (1 + \frac{\alpha}{N}, 1 + \frac{\alpha+2s}{N})$ and $\gamma(c) = -\infty$ if $p \in (1 + \frac{\alpha+2s}{N}, \frac{N+\alpha}{N-2s})$. However, we cannot deduce that $\gamma(c) > -\infty$ or $\gamma(c) = -\infty$ for $c > 0$ if $p = 1 + \frac{\alpha+2s}{N}$. In the mass-subcritical case $p \in (1 + \frac{\alpha}{N}, 1 + \frac{\alpha+2s}{N})$, $I(u)$ is bounded from below and coercive on $S(c)$ (see [12, Lemma 4.4]). As mentioned above, d'Avenia et al. [12] proved that when $p \in (1 + \frac{\alpha}{N}, 1 + \frac{\alpha+2s}{N})$, $I(u)$ has a minimum point on $S(c)$, that can be assumed non-negative, radially symmetric and decreasing (see [12, Theorem 4.5]). To the best of our knowledge, in the mass-supercritical case where $p \in (1 + \frac{\alpha+2s}{N}, \frac{N+\alpha}{N-2s})$, the existence of critical points of $I(u)$ restricted on $S(c)$ is still unknown. In this paper, we consider the mass-supercritical case where $p \in (1 + \frac{\alpha+2s}{N}, \frac{N+\alpha}{N-2s})$.

Our main results are as follows:

Theorem 1.1. *Let $p \in (\max\{1 + \frac{\alpha+2s}{N}, 2\}, \frac{N+\alpha}{N-2s})$, $N \geqslant 3$, $s \in (0, 1)$, $\alpha \in (0, N)$ and $c > 0$. Then there exists a couple of weak solution $(u_c, \lambda_c) \in H^s(\mathbb{R}^N) \times \mathbb{R}^-$ to (1.1) with $\|u_c\|_2^2 = c$ and u_c can be assumed positive and radially symmetric-decreasing in $\mathbb{R}^N$. Furthermore, $u_c \in S(c)$ is a normalized ground state of (1.1) with*

$$\begin{cases} \|(-\Delta)^{\frac{s}{2}} u_c\|_2 \to +\infty, \\ \lambda_c \to -\infty, \\ I(u_c) \to +\infty \end{cases}$$

as $c \to 0$ and

$$\begin{cases} \|(-\Delta)^{\frac{s}{2}} u_c\|_2 \to 0, \\ \lambda_c \to 0, \\ I(u_c) \to 0 \end{cases}$$

as $c \to +\infty$.

Theorem 1.2. *Let $p \in (\max\{1 + \frac{\alpha+2s}{N}, 2\}, \frac{N+\alpha}{N-2s})$, $N \geqslant 3$, $s \in (0, 1)$, $\alpha \in (0, N)$ and $c > 0$. Then there exists a sequence of couples of weak solutions $\{(v_n, \widetilde{\lambda}_n)\} \subseteq H_r^s(\mathbb{R}^N) \times \mathbb{R}^-$ to (1.1) with $\|v_n\|_2^2 = c$ and $\|v_n\|_{H^s(\mathbb{R}^N)} \to +\infty$ as $n \to +\infty$, while $\{v_n\}$ is uniformly bounded, i.e., there exists a constant $C > 0$ such that $|v_n| \leqslant C$ on $\mathbb{R}^N$.*

Remark 1.3. To the best of our knowledge, the main results in this paper are new. Theorem 1.1 indicates that a normalized ground state of (1.1) exists even if the corresponding energy $I(u)$ is unbounded from below on $S(c)$. This is a complement of the results in [12] about the existence of solutions to (1.1) with a prescribed L^2-norm for $p \in (1 + \frac{\alpha}{N}, 1 + \frac{\alpha+2s}{N})$. Roughly speaking, Theorem 1.1 generalizes the result of Theorem 4.5 in [12] to the mass-supercritical case $p \in (1 + \frac{\alpha+2s}{N}, \frac{N+\alpha}{N-2s})$.

Remark 1.4. Theorem 1.2 indicates that (1.1) has infinitely many high energy normalized solutions. Notice that, although when $p \in (1 + \frac{\alpha+2s}{N}, \frac{N+\alpha}{N-2s})$ and the frequency $\lambda \in \mathbb{R}$ is a fixed and assigned parameter, as mentioned above, d'Avenia et al. [12] obtained a multiplicity result for solutions to (1.1) (see [12, Theorem 1.2]), there is no information about the L^2-norm of the solutions. So Theorem 1.2 in this paper can also be viewed as a complement of the main results in [12].

Now, we explain the main idea of the proofs of Theorems 1.1 and 1.2. In the mass-supercritical case $p \in (1 + \frac{\alpha+2s}{N}, \frac{N+\alpha}{N-2s})$, the functional $I(u)$ is unbounded from below on $S(c)$ (see Lemma 2.1) and the minimization argument on $S(c)$ used in [12] does not work any more. For this reason, we try to construct a submanifold of $S(c)$, on which $I(u)$ is bounded from below and coercive, and we turn to look for minimizers on such a submanifold. This is motivated by the minimization method developed on the Nehari manifold and the recent works [17, 21, 29].

We give the idea of constructing such a submanifold. Notice that, if u is a critical point of $I|_{S(c)}$, then $I'(u) - \lambda u = 0$ in $H^{-s}(\mathbb{R}^N)$, where $\lambda \in \mathbb{R}$ appears as a Lagrange multiplier. Hence u satisfies $N_\lambda(u) = \langle I'(u) - \lambda u, u\rangle = 0$ and the Pohozaev identity (see Lemma 2.5):

$$P_\lambda(u) := (N-2s)\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2 - \lambda N\int_{\mathbb{R}^N}|u|^2 - \frac{\alpha+N}{p}\int_{\mathbb{R}^N}(\kappa_\alpha * |u|^p)|u|^p = 0. \tag{1.10}$$

Combining the Nehari functional $N_\lambda(u)$ with the Pohozaev functional $P_\lambda(u)$, we construct a submanifold $V(c)$ as follows:

$$V(c) := \{u \in S(c) : Q(u) = 0\}, \tag{1.11}$$

where

$$Q(u) := N \cdot N_\lambda(u) - P_\lambda(u) = 2s\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2 - \frac{pN-N-\alpha}{p}\int_{\mathbb{R}^N}(\kappa_\alpha * |u|^p)|u|^p. \tag{1.12}$$

Next, we consider the following minimization problem:

$$m(c) := \inf_{u\in V(c)} I(u). \tag{1.13}$$

We get a critical point of I restricted to $S(c)$ by proving that every minimizer of I restricted to $V(c)$ is indeed a critical point of I restricted to $S(c)$. Notice that we have two restrictions in $V(c)$, which is different from the situation in [17, 21, 29]. In order to use Lagrange's theorem, we need to prove that $Q'(u)$ and $D'(u)$ are linearly independent if u is a minimizer of I restricted to $V(c)$, where $D(u) := \|u\|_2^2$ (see Lemma 2.9 for details). The set of minimizers of $I(u)$ on $V(c)$ is defined as

$$\mathcal{M}_c := \Big\{u \in V(c) : I(u) = \inf_{v\in V(c)} I(v)\Big\}. \tag{1.14}$$

Furthermore, we obtain a result concerning some properties of the elements in $\mathcal{M}_c$ (see Proposition 2.16). Then we prove the first part of Theorem 1.1.

The idea of proving the dependence of $\|(-\Delta)^{\frac{s}{2}}u_c\|_2$ and λ_c on the value of c comes from [18, 31, 32]. The fact that u_c is a minimizer of $I(u)$ restricted on $V(c)$ and $Q(u_c) = 0$ are crucial. The case of $c \to 0^+$ is easy to prove. However, in the case of $c \to +\infty$, for a prescribed function $u \in V(1)$, we need to construct a function $\widetilde{u} \in V(c)$ whose energy $I(\widetilde{u}) \to 0$ as $c \to +\infty$. By the fact that $m(c) > 0$ and u_c is a minimizer, we end the proof with a careful analysis.

The proof of Theorem 1.2 is inspired by Bartsch and de Valeriola [3]. Since I is unbounded from below on $S(c)$ if $p \in (1 + \frac{\alpha+2s}{N}, \frac{N+\alpha}{N-2s})$, the genus of the sublevel set

$$I^d := \{u \in S(c) : I(u) \leqslant d\}$$

is always infinite. Thus the classical argument based on the Kranoselski genus [31] does not work in obtaining the existence of infinitely many normalized solutions to (1.1). First, we present a new type of linking geometry for the functional I restricted on

$$S_r(c) = \{u \in H_r^s(\mathbb{R}^N) : \|u\|_2^2 = c\}, \quad c > 0. \tag{1.15}$$

Then a min-max procedure is set up to construct an unbounded sequence $\{\gamma_n(c)\}$ of critical values for I on $S_r(c)$. At each level $\gamma_n(c)$, by using an abstract lemma developed by Jeanjean [18, Lemma 2.3], we get a Palais-Smale sequence $\{v_k^n\}_{k=1}^{+\infty}$ with an additional condition $Q(v_k^n) \to 0$ as $k \to +\infty$ (see Proposition 2.21), where $Q(u)$ is defined in (1.12). With this extra condition, we prove the boundness and non-vanishing of $\{v_k^n\}$ (see the proof of Proposition 2.23). Since the radially symmetric Sobolev space $H_r^s(\mathbb{R}^N)$ embeds compactly in $L^q(\mathbb{R}^N)$ for $2 < q < 2_s^*$, where $2_s^* = \frac{2N}{N-2s}$, we get the compactness of the Palais-Smale sequence. Thus, we get a critical point v_n at each level $\gamma_n(c)$. Finally, we prove that the critical point sequence $\{v_n\}$ is unbounded in $H^s(\mathbb{R}^N)$. In addition, by using a decay and regularity property of solutions to (1.1) (see Lemma 2.5), we prove that $\{v_n\}$ is in $C^2(\mathbb{R}^N)$ and uniformly bounded on $\mathbb{R}^N$.

Remark 1.5. The hypothesis $p > 2$ is used to get a better regularity of solutions to (1.1) and to guarantee the Pohozaev identity (see Lemma 2.5), which is useful for our purpose.

2 Preliminary results

In this section, we give some preliminary results. First, we define, for short, the following quantities:

$$A(u) := \|(-\Delta)^{\frac{s}{2}} u\|_2^2 = \frac{C(N,s)}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}},$$
$$B(u) := \int_{\mathbb{R}^N} (\kappa_\alpha * |u|^p)|u|^p = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^p |u(y)|^p}{|x-y|^{N-\alpha}},$$
$$D(u) := \|u\|_2^2 = \int_{\mathbb{R}^N} |u|^2.$$

Next, we introduce a scaling. For $u \in S(c)$, set $u^t(x) = t^{\frac{N}{2}} u(tx)$, $t > 0$. Then

$$A(u^t) = t^{2s} A(u), \quad D(u^t) = D(u), \quad B(u^t) = t^{Np-N-\alpha} B(u)$$

and

$$I(u^t) = \frac{1}{2} t^{2s} A(u) - \frac{1}{2p} t^{Np-N-\alpha} B(u). \tag{2.1}$$

Lemma 2.1. *Let $p \in (1 + \frac{\alpha+2s}{N}, \frac{N+\alpha}{N-2s})$, $N \geqslant 3$, $s \in (0,1)$ and $\alpha \in (0,N)$. Then for any $u \in S(c)$, $u^t \in S(c)$, $A(u^t) \to +\infty$ and $I(u^t) \to -\infty$ as $t \to \infty$.*

Proof. For any $u \in S(c)$, since $D(u^t) = D(u)$, $u^t(x) \in S(c)$. By (2.1), $A(u^t) \to +\infty$ and $I(u^t) \to -\infty$ as $t \to \infty$ follow from the fact that $p \in (1 + \frac{\alpha+2s}{N}, \frac{N+2\alpha}{N-2s})$. □

Lemma 2.2 (See [12, Lemma 2.1])**.** *Let $p \in (1 + \frac{\alpha}{N}, \frac{N+\alpha}{N-2s})$, $N \geqslant 3$, $s \in (0,1)$, $\alpha \in (0,N)$. Then for any $u \in H^s(\mathbb{R}^N)$,*

$$\int_{\mathbb{R}^N} (\kappa_\alpha * |u|^p)|u|^p \leqslant C \|u\|_{\frac{2Np}{N+\alpha}}^{2p}, \tag{2.2}$$
$$\int_{\mathbb{R}^N} (\kappa_\alpha * |u|^p)|u|^p \leqslant C \|(-\Delta)^{\frac{s}{2}} u\|_2^{2\delta p} \|u\|_2^{2(1-\delta)p}, \quad \delta = \frac{Np-N-\alpha}{2sp}. \tag{2.3}$$

Now we show that the set $V(c)$ constructed in (1.11) is nonempty in $H^s(\mathbb{R}^N)$.

Lemma 2.3. *Let $p \in (1 + \frac{2s+\alpha}{N}, \frac{N+\alpha}{N-2s})$, $N \geqslant 3$, $s \in (0,1)$ and $\alpha \in (0,N)$. Then for any $u \in S(c)$, $c > 0$, there exists a unique $t_0 > 0$ such that $I(u^{t_0}) = \max_{t>0} I(u^t)$ and $u^{t_0} \in V(c)$. In particular,*

(i) $t_0 < 1 \Leftrightarrow Q(u) < 0$;

(ii) $t_0 = 1 \Leftrightarrow Q(u) = 0$.

Proof. Define

$$\tau(t) := I(u^t) = \frac{1}{2} t^{2s} A(u) - \frac{1}{2p} t^{Np-N-\alpha} B(u).$$

By Lemma 2.1 and an elementary analysis, we know that $\tau(t)$ has a unique critical point $t_0 > 0$ corresponding to its maximum on $(0, +\infty)$. Hence

$$I(u^{t_0}) = \max_{t>0} I(u^t) \quad \text{and} \quad \tau'(t_0) = st_0^{2s-1}A(u) - \frac{Np-N-\alpha}{2p} t_0^{Np-N-\alpha-1}B(u) = 0.$$

Thus $Q(u^{t_0}) = 2st_0^{2s}A(u) - \frac{Np-N-\alpha}{p} t_0^{Np-N-\alpha}B(u) = 0$, i.e., $u^{t_0} \in V(c)$. Moreover,

$$Q(u) = 2sA(u) - \frac{Np-N-\alpha}{p}B(u) = 2sA(u)(1 - t_0^{2s+N+\alpha-Np}),$$

which concludes (i) and (ii). □

Let X_0 be a subset of a Banach space X. Recall that a functional $E : X \to \mathbb{R}$ is called coercive on X_0 if, for every sequence $\{u_k\} \subset X_0$, $\|u_k\| \to +\infty$ implies $E(u_k) \to +\infty$ (see [1, Definition 1.5.5]). Having this in mind, we give the following result.

Lemma 2.4. *Let $p \in (1 + \frac{2s+\alpha}{N}, \frac{N+\alpha}{N-2s})$, $N \geqslant 3$, $s \in (0,1)$ and $\alpha \in (0,N)$. Then $I(u)$ is bounded from below and coercive on $V(c)$. Moreover, there exists a constant $C_0 > 0$ such that $I(u) \geqslant C_0$ for all $u \in V(c)$.*

Proof. For any $u \in V(c)$, $Q(u) = 2sA(u) - \frac{Np-N-\alpha}{p}B(u) = 0$, and then $B(u) = \frac{2sp}{Np-N-\alpha}A(u)$. We have

$$I(u) = \frac{1}{2}A(u) - \frac{1}{2p}B(u) = \left(\frac{1}{2} - \frac{s}{Np-N-\alpha}\right)A(u) \geqslant 0,$$

and I is coercive on $V(c)$. Furthermore, by Lemma 2.2,

$$\frac{2sp}{Np-N-\alpha}A(u) = B(u) \leqslant C \cdot A(u)^{\frac{Np-N-\alpha}{2s}} D(u)^{\frac{2sp-Np+N+\alpha}{2s}}.$$

Since $p \in (1 + \frac{2s+\alpha}{N}, \frac{N+\alpha}{N-2s})$, there exists a constant $\widetilde{C_0} > 0$ such that $A(u) \geqslant \widetilde{C_0} > 0$. Then there exists

$$C_0 = \left(\frac{1}{2} - \frac{s}{Np-N-\alpha}\right)\widetilde{C_0}$$

such that

$$I(u) = \left(\frac{1}{2} - \frac{s}{Np-N-\alpha}\right)A(u) \geqslant C_0.$$

This completes the proof. □

Next, we show some results related to weak solutions to (1.1).

Lemma 2.5 (See [12, Theorem 3.3] and [30, Proposition 2])**.** *Let $p \in (1 + \frac{\alpha}{N}, \frac{N+\alpha}{N-2s})$, $N \geqslant 3$, $s \in (0,1)$, $\alpha \in (0,N)$ and u is a weak solution to (1.1). Then*

(i) *there exists $C > 0$ such that*

$$|u(x)| \leqslant C \frac{1}{(1+|x|^2)^{\frac{N+2s}{2}}}, \quad \text{if} \quad p \geqslant 2;$$

(ii) *$u \in C^2(\mathbb{R}^N)$ and satisfies the following Pohozaev identity:*

$$(N-2s)\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2 - \lambda N \int_{\mathbb{R}^N} |u|^2 - \frac{\alpha+N}{p}\int_{\mathbb{R}^N} (\kappa_\alpha * |u|^p)|u|^p = 0, \quad \text{if} \quad p > 2.$$

Lemma 2.6. *Let $p \in (\max\{1 + \frac{\alpha+2s}{N}, 2\}, \frac{N+\alpha}{N-2s})$, $N \geqslant 3$, $s \in (0,1)$, $\alpha \in (0,N)$ and $\lambda \in \mathbb{R}$. If $v \in H^s(\mathbb{R}^N)$ is a weak solution to (1.1), then $Q(v) = 0$. Moreover, $v = 0$ if $\lambda \geqslant 0$.*

Proof. By Lemma 2.5, the following Pohozaev identity holds for $v \in H^s(\mathbb{R}^N)$:

$$(N-2s)\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}v|^2 - \lambda N\int_{\mathbb{R}^N} |v|^2 - \frac{\alpha+N}{p}\int_{\mathbb{R}^N} (\kappa_\alpha * |v|^p)|v|^p = 0.$$

Multiplying (1.1) by v and integrating we derive a second identity,

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}v|^2 - \lambda \int_{\mathbb{R}^N} |v|^2 - \int_{\mathbb{R}^N} (\kappa_\alpha * |v|^p)|v|^p = 0.$$

Thus we immediately have

$$Q(v) = 2s\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}v|^2 - \frac{pN-N-\alpha}{p}\int_{\mathbb{R}^N} (\kappa_\alpha * |v|^p)|v|^p = 0.$$

Also with the simple calculations, we obtain

$$\lambda D(v) = \frac{(N-2s)p-(N+\alpha)}{Np-(N+\alpha)}A(v).$$

(1) If $\lambda > 0$, we get $v \equiv 0$ immediately.
(2) If $\lambda = 0$, $A(v) = 0$ then $v \equiv 0$. □

Lemma 2.7. *Let $p \in (\max\{1+\frac{\alpha}{N}, 2\}, \frac{N+\alpha}{N-2s})$, $N \geqslant 3$, $s \in (0,1)$ and $\alpha \in (0,N)$. If u is a critical point of $I|_{S(c)}$, then $I'(u) - \lambda_c u = 0$ in $H^{-s}(\mathbb{R}^N)$ for some $\lambda_c < 0$.*

Proof. Since u is a critical point of $I|_{S(c)}$, there exists $\lambda_c \in \mathbb{R}$ such that $I'(u) - \lambda_c u = 0$ in $H^{-s}(\mathbb{R}^N)$. Thus

$$\langle I'(u) - \lambda_c u, u\rangle = A(u) - B(u) - \lambda_c D(u) = 0. \tag{2.4}$$

By the Pohozaev identity (see Lemma 2.5), u satisfies

$$(N-2s)A(u) - \frac{\alpha+N}{p}B(u) - \lambda_c N\cdot D(u) = 0. \tag{2.5}$$

Combining (2.4) with (2.5), we have

$$\lambda_c = \frac{(N-2s)p-(N+\alpha)}{[Np-(N+\alpha)]c}A(u) < 0$$

for $p \in (1+\frac{\alpha}{N}, \frac{N+\alpha}{N-2s})$. □

Recall a useful result for constrained minimization problems in the following lemma.

Lemma 2.8 (See [11, Corollary 4.1.2])**.** *Let X be a real Banach space, $U \subset X$ be an open set. Suppose that $f, g_1, \dots, g_m : U \to \mathbb{R}^1$ are C^1 functions and $x_0 \in M$ is such that $f(x_0) = \inf_{x\in M} f(x)$ with*

$$M = \{x \in U \mid g_i(x) = 0, i = 1, 2, \dots, m\}.$$

If $\{g_i'(x_0)\}_{i=1}^m$ is linearly independent, then there exist $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ such that

$$f'(x_0) + \sum_{i=1}^m \lambda_i g_i'(x_0) = 0.$$

Lemma 2.9. *Let $p \in (\max\{1+\frac{\alpha+2s}{N}, 2\}, \frac{N+\alpha}{N-2s})$, $N \geqslant 3$, $s \in (0,1)$ and $\alpha \in (0,N)$. Then each minimizer of $I|_{V(c)}$ is a critical point of $I|_{S(c)}$.*

Proof. Suppose that u is a minimizer of $I|_{V(c)}$. Then by Lemma 2.8, either (i) $Q'(u)$ and $D'(u)$ are linearly dependent, or (ii) there exist $\lambda_1, \lambda_2 \in \mathbb{R}$ such that

$$I'(u) - \lambda_1 Q'(u) - \lambda_2 u = 0 \quad \text{in} \quad H^{-s}(\mathbb{R}^N). \tag{2.6}$$

If (i) holds, then u satisfies

$$2s(-\Delta)^s u - \lambda^* u = (Np - N - \alpha)(\kappa_\alpha * |u|^p)|u|^{p-2}u$$

for some $\lambda^* \in \mathbb{R}$. Multiplying the above equation by u and integrating, we get

$$2sA(u) - (Np - N - \alpha)B(u) - \lambda^* D(u) = 0.$$

By the Pohozaev identity, we derive

$$2s(N-2s)A(u) - \frac{\alpha+N}{p}(Np-N-\alpha)B(u) - \lambda^* N D(u) = 0.$$

Thus we have

$$4s^2 A(u) - \frac{(Np-N-\alpha)^2}{p} B(u) = 0.$$

Notice that $Q(u) = 0$ and $p > 1 + \frac{2s+\alpha}{N}$. Then we have immediately $B(u) = 0$, which leads to a contradiction. This implies that (i) does not occur and (ii) is true. It is enough to show that $\lambda_1 = 0$. By (2.6) we have

$$\begin{aligned} &\langle I'(u) - \lambda_1 Q'(u) - \lambda_2 u, u\rangle \\ &\quad = (1 - 4\lambda_1 s)A(u) - [1 - 2\lambda_1(Np - N - \alpha)]B(u) - \lambda_2 D(u) = 0. \end{aligned} \tag{2.7}$$

By the Pohozaev identity,

$$(1 - 4\lambda_1 s)(N - 2s)A(u) - [1 - 2\lambda_1(Np - N - \alpha)]\frac{N+\alpha}{p}B(u) - \lambda_2 N \cdot D(u) = 0. \tag{2.8}$$

Combining (2.7) with (2.8), we have

$$(1 - 4\lambda_1 s) \cdot 2sA(u) = [1 - 2\lambda_1(Np - N - \alpha)]\left(N - \frac{N+\alpha}{p}\right)B(u). \tag{2.9}$$

Since $u \in V(c)$, $B(u) = \frac{2sp}{Np-N-\alpha}A(u)$, and then by (2.9) we have $4\lambda_1 s = 2\lambda_1(Np - N - \alpha)$. Hence $\lambda_1 = 0$, for $p > 1 + \frac{2s+\alpha}{N}$. Finally, by Lemma 2.7, we have $\lambda_2 < 0$. □

Lemma 2.9 indicates that the restriction $Q(u) = 0$ in $V(c)$ is a natural constraint. In order to prove the minimizing problem (1.13) is attained, we need the following monotonic condition of $m(c)$.

Lemma 2.10. *Let $p \in (1 + \frac{\alpha+2s}{N}, \frac{N+\alpha}{N-2s})$, $N \geqslant 3$, $s \in (0,1)$, $\alpha \in (0,N)$ and $c > 0$. Define $m(c) := \inf_{u\in V(c)} I(u)$. Then the function $c \to m(c)$ is strictly decreasing on $(0,+\infty)$, where $V(c)$ is given in* (1.11).

Proof. By Lemma 2.4, $m(c) \geqslant C_0 > 0$ is well-defined. For any $0 < c_1 < c_2 < +\infty$, by Lemma 2.3, there exists $\{u_n\} \subseteq V(c_1)$ such that

$$I(u_n) = \max_{t>0} I(u_n^t) \leqslant m(c_1) + \frac{1}{n}.$$

Similar to the proof of Lemma 2.4, we get that there exist constants $k_i > 0$ $(i = 1,2,3,4)$ which are independent of n, such that $k_1 \leqslant A(u_n) \leqslant k_2$ and $k_3 \leqslant B(u_n) \leqslant k_4$. Set

$$v_n(x) = \left(\frac{c_1}{c_2}\right)^{\frac{1}{2s}\cdot\frac{N-2s}{2}} u_n\left(\left(\frac{c_1}{c_2}\right)^{\frac{1}{2s}} x\right).$$

Then

$$A(v_n) = A(u_n), \quad B(v_n) = \left(\frac{c_1}{c_2}\right)^{\frac{(N-2s)p-N-\alpha}{2s}} B(u_n), \quad D(v_n) = \frac{c_2}{c_1}D(u_n) = c_2.$$

Moreover, by Lemma 2.3, there exists $t_n > 0$ such that $v_n^{t_n} \in V(c_2)$ and $I(v_n^{t_n}) = \max_{t>0} I(v_n^t)$. By $Q(v_n^{t_n}) = 0$ and the boundness of $A(u_n)$ and $B(u_n)$, there exists a positive constant $C > 0$ such that $t_n \geqslant C > 0$. Therefore,

$$\begin{aligned}
m(c_2) &\leqslant I(v_n^{t_n}) \\
&= \frac{1}{2} t_n^{2s} A(u_n) - \frac{1}{2p} t_n^{Np-N-\alpha} \left(\frac{c_1}{c_2}\right)^{\frac{(N-2s)p-N-\alpha}{2s}} B(u_n) \\
&= I(u_n^{t_n}) + \frac{1}{2p} t_n^{Np-N-\alpha} B(u_n) - \frac{1}{2p} t_n^{Np-N-\alpha} \left(\frac{c_1}{c_2}\right)^{\frac{(N-2s)p-N-\alpha}{2s}} B(u_n) \\
&= I(u_n^{t_n}) - \frac{1}{2p} t_n^{Np-N-\alpha} B(u_n) \left[\left(\frac{c_1}{c_2}\right)^{\frac{(N-2s)p-N-\alpha}{2s}} - 1\right] \\
&\leqslant m(c_1) + \frac{1}{n} - \left[\left(\frac{c_1}{c_2}\right)^{\frac{(N-2s)p-N-\alpha}{2s}} - 1\right] C^{Np-N-\alpha} k_3,
\end{aligned} \tag{2.10}$$

which implies that $m(c_2) < m(c_1)$, by letting $n \to +\infty$. □

Lemma 2.11 (See [12, Lemma 2.3]). *If $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$ and for some $\sigma > 0$ and $2 \leqslant q < 2_s^*$ we have*

$$\sup_{x\in\mathbb{R}^N} \int_{B_\sigma(x)} |u_n|^q \to 0, \quad \text{as} \quad n \to +\infty,$$

then $u_n \to 0$ in $L^r(\mathbb{R}^N)$ for $2 < r < 2_s^*$.

Lemma 2.12 (Brezis-Lieb type theorem, see [26, Lemma 2.4]). *Let $N \in \mathbb{N}$, $\alpha \in (0, N)$, $p \in [1, \frac{2N}{N+\alpha})$ and $\{u_n\}$ be a bounded sequence in $L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$. If $u_n \to u$ a.e. in $\mathbb{R}^N$, as $n \to +\infty$, then*

$$\lim_{n\to+\infty} (B(u_n) - B(u_n - u)) = B(u).$$

Lemma 2.13 (See [24, Theorem 3.7]). *Let f, g and h be three Lebesgue measurable non-negative functions on $\mathbb{R}^N$. Then, with*

$$\Phi(f, g, h) := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f(x) g(x-y) h(y) dx dy,$$

we have

$$\Phi(f, g, h) \leqslant \Phi(f^\star, g^\star, h^\star),$$

where $f^\star$, $g^\star$ and $h^\star$ denote the symmetric-decreasing rearrangement of f, g and h.

Then we prove that $m(c)$ can be attained.

Proposition 2.14. *Let $p \in (1 + \frac{\alpha+2s}{N}, \frac{N+\alpha}{N-2s})$, $N \geqslant 3$, $s \in (0, 1)$, $\alpha \in (0, N)$ and $c > 0$. Then $m(c) := \inf_{u\in V(c)} I(u)$ is attained, where $V(c)$ is given in (1.11).*

Proof. Let $\{u_n\}$ be a minimizing sequence for $m(c)$. By Lemma 2.4, $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$. Then up to a subsequence and up to a translation, there exists $u \neq 0$ in $H^s(\mathbb{R}^N)$ such that

$$\begin{cases} u_n \rightharpoonup u & \text{in} \quad H^s(\mathbb{R}^N), \\ u_n \to u & \text{in} \quad L^q_{\text{loc}}(\mathbb{R}^N), \\ u_n \to u & \text{a.e. in} \quad \mathbb{R}^N, \end{cases}$$

for $2 < q < 2_s^*$. Otherwise, by Lemma 2.11, $u_n \to 0$ in $L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$. Then by Lemma 2.2, $B(u_n) \to 0$. Since $Q(u_n) = 2sA(u_n) - \frac{Np-N-\alpha}{p} B(u_n) = 0$, $A(u_n) \to 0$. Therefore, $I(u_n) \to 0$ and $m(c) = 0$, which contradicts the fact that $m(c) > 0$. Let $u_n^\star$ be the symmetric-decreasing rearrangement of u_n. By Lemma 2.13 with $f(x) = |u_n(x)|^p$, $h(y) = |u_n(y)|^p$ and $g(x) = |x|^{\alpha-N}$. We have $B(u_n^\star) \geqslant B(u_n)$. In addition, from the fact that $A(u_n^\star) \leqslant A(u_n)$ (see [2, Theorem 3]), we have $I(u_n^\star) \leqslant I(u_n)$, $Q(u_n^\star) \leqslant$

$Q(u_n)=0$. By Lemma 2.3, there exists $t_n\in(0,1]$ such that $Q((u_n^\star)^{t_n})=0$. For $u_n\in V(c)$, $(u_n^\star)^{t_n}\in V(c)$, by the relationship between $I(u)$ and $Q(u)$,

$$I(u)-\frac{1}{2(Np-N-\alpha)}Q(u)=\left(\frac{1}{2}-\frac{s}{Np-N-\alpha}\right)A(u),$$

we have $I(u_n),I((u_n^\star)^{t_n})\geqslant 0$. Thus,

$$\begin{aligned}
I((u_n^\star)^{t_n}) &= I((u_n^\star)^{t_n})-\frac{1}{2(Np-N-\alpha)}Q((u_n^\star)^{t_n})\\
&=\frac{Np-N-\alpha-2s}{2(Np-N-\alpha)}A((u_n^\star)^{t_n})\\
&=\frac{Np-N-\alpha-2s}{2(Np-N-\alpha)}{t_n}^{2s}A((u_n^\star))\\
&\leqslant\frac{Np-N-\alpha-2s}{2(Np-N-\alpha)}{t_n}^{2s}A(u_n)\\
&=t_n^{2s}\left(I(u_n)-\frac{1}{2(Np-N-\alpha)}Q(u_n)\right)\\
&=t_n^{2s}I(u_n)\\
&\leqslant I(u_n).
\end{aligned}\tag{2.11}$$

Denote $\widehat{u}_n=(u_n^\star)^{t_n}$. By (2.11), $\{\widehat{u}_n\}$ is a minimizing sequence for $m(c)$. Then by Lemma 2.4, $\{\widehat{u}_n\}$ is bounded in $H^s(\mathbb{R}^N)$ and up to a subsequence, there exists $\widehat{u}\neq 0$ in $H^s(\mathbb{R}^N)$ such that

$$\begin{cases}\widehat{u}_n\rightharpoonup\widehat{u} & \text{in} \quad H_r^s(\mathbb{R}^N),\\ \widehat{u}_n\to\widehat{u} & \text{in} \quad L^q(\mathbb{R}^N),\\ \widehat{u}_n\to\widehat{u} & \text{a.e. in} \quad \mathbb{R}^N,\end{cases}$$

for $2<q<2_s^*$. Next, we shall prove that $\|\widehat{u}\|_2^2=c$. Just suppose that $\|\widehat{u}\|_2^2=\overline{c}\in(0,c)$, and then by Lemma 2.10, $m(\overline{c})>m(c)$. Since $\widehat{u}_n\rightharpoonup\widehat{u}$ in $H_r^s(\mathbb{R}^N)$, $Q(\widehat{u})\leqslant\lim_{n\to\infty}Q(\widehat{u}_n)=0$. By Lemma 2.3, there exists $t_0\in(0,1]$ such that $\widehat{u}^{t_0}\in V(\overline{c})$. Then

$$\begin{aligned}
m(\overline{c})\leqslant I(\widehat{u}^{t_0}) &= I(\widehat{u}^{t_0})-\frac{1}{2(Np-N-\alpha)}Q(\widehat{u}^{t_0})\\
&=\frac{Np-N-\alpha-2s}{2(Np-N-\alpha)}A(\widehat{u}^{t_0})\\
&=\frac{Np-N-\alpha-2s}{2(Np-N-\alpha)}t_0^{2s}A(\widehat{u})\\
&\leqslant\frac{Np-N-\alpha-2s}{2(Np-N-\alpha)}A(\widehat{u})\\
&\leqslant\lim_{n\to\infty}\frac{Np-N-\alpha-2s}{2(Np-N-\alpha)}A(\widehat{u}_n)\\
&=\lim_{n\to\infty}\left[I(\widehat{u}_n)-\frac{1}{2(Np-N-\alpha)}Q(\widehat{u}_n)\right]\\
&=m(c),
\end{aligned}\tag{2.12}$$

which leads to a contradiction. So $t_0=1$, $c=\overline{c}$, i.e., $\|\widehat{u}\|_2^2=c$ and $I(\widehat{u})=m(c)$. Then by (2.12) we have $A(\widehat{u}_n-\widehat{u})=o(1)$, $\widehat{u}_n\to\widehat{u}$ in $H^s(\mathbb{R}^N)$ and $\widehat{u}$ is a minimizer for $m(c)$. □

In order to get a positive normalized ground state solution to (1.1), we need the following property of the set $\mathcal{M}_c$ defined in (1.14).

Proposition 2.15. *Let* $p\in(\max\{1+\frac{\alpha+2s}{N},2\},\frac{N+\alpha}{N-2s})$, $N\geqslant 3$, $s\in(0,1)$, $\alpha\in(0,N)$ *and* $c>0$. *Then*
(i) $|u_c|\in\mathcal{M}_c$ *if* $u_c\in\mathcal{M}_c$;
(ii) $|u_c|>0$ *if* $u_c\in\mathcal{M}_c$.

Proof. Let $u_c \in H^s(\mathbb{R}^N)$ with $u_c \in V(c)$. Since $B(|u_c|) = B(u_c)$, $A(|u_c|) \leqslant A(u_c)$, we have that $I(|u_c|) \leqslant I(u_c)$ and $Q(|u_c|) \leqslant Q(u_c) = 0$. In addition, by Lemma 2.3, there exists $t_0 \in (0,1]$ such that $Q(|u_c|^{t_0}) = 0$. We claim that $I(|u_c|^{t_0}) \leqslant t_0^{2s} \cdot I(u_c)$. Indeed, for $u_c \in V(c)$, $|u_c|^{t_0} \in V(c)$, by the relationship between $I(u)$ and $Q(u)$, we have $I(u_c), I(|u_c|^{t_0}) > 0$. Thus,

$$
\begin{aligned}
I(|u_c|^{t_0}) &= I(|u_c|^{t_0}) - \frac{1}{2(Np-N-\alpha)} Q(|u_c|^{t_0}) \\
&= \frac{Np-N-\alpha-2s}{2(Np-N-\alpha)} A(|u_c|^{t_0}) \\
&= \frac{Np-N-\alpha-2s}{2(Np-N-\alpha)} {t_0}^{2s} A(|u_c|) \\
&\leqslant \frac{Np-N-\alpha-2s}{2(Np-N-\alpha)} {t_0}^{2s} A(u_c) \\
&= t_0^{2s}\left(I(u_c) - \frac{1}{2(Np-N-\alpha)} Q(u_c)\right) \\
&= t_0^{2s} I(u_c). \qquad (2.13)
\end{aligned}
$$

Therefore, if $u_c \in H^s(\mathbb{R}^N)$ is a minimizer of $I(u)$ on $V(c)$ we have

$$
I(u_c) = \inf_{u \in V(c)} I(u) \leqslant I(|u_c|^{t_0}) \leqslant t_0^{2s} I(u_c),
$$

which implies $t_0 = 1$. Then $Q(|u_c|) = 0$ and we conclude that $A(|u_c|) = A(u_c)$ and $I(|u_c|) = I(u_c)$. (i) is proved.

Now, since $|u_c|$ is a minimizer of $I(u)$ on $V(c)$, by Lemmas 2.7 and 2.9, $(|u_c|, \lambda_c) \in H^s(\mathbb{R}^N) \times \mathbb{R}^-$ satisfies (1.1) for some $\lambda_c < 0$. Next, we claim that $|u_c| > 0$ for all $x \in \mathbb{R}^N$. We argue by contradiction. Just suppose that there exists $x_0 \in \mathbb{R}^N$ such that $|u_c|(x_0) = 0$. Then, it follows from (1.1) that $(-\Delta)^s|u|(x_0) = 0$. So

$$
(-\Delta)^s |u_c|(x_0) = C(N,s)\mathrm{P.V.}\left(\lim_{\varepsilon \to 0} \int_{\varepsilon \leqslant |x_0-y| \leqslant r} \frac{-|u_c|(y)}{|x_0-y|^{N+2s}} dy + \int_{\mathbb{R}^N \setminus B(x_0,r)} \frac{-|u_c|(y)}{|x_0-y|^{N+2s}} dy\right) = 0,
$$

which implies that

$$
\int_{\mathbb{R}^N \setminus B(x_0,r)} \frac{-|u_c|(y)}{|x_0-y|^{N+2s}} dy = 0 \quad \text{for all} \quad r > 0.
$$

Therefore, $|u_c| \equiv 0$ in $\mathbb{R}^N$, which leads to a contradiction. Thus the proof is completed. □

Afterwards, we give some preliminaries for the proof of Theorem 1.2. Let $\{V_n\} \subset H_r^s(\mathbb{R}^N)$ be a strictly increasing sequence of finite-dimensional linear subspaces in $H_r^s(\mathbb{R}^N)$ such that $\bigcup_n V_n$ is dense in $H_r^s(\mathbb{R}^N)$. We denote by $V_n^\perp$ the orthogonal space of V_n in $H_r^s(\mathbb{R}^N)$.

Lemma 2.16. *Let $p \in (1 + \frac{\alpha}{N}, \frac{N+\alpha}{N-2s})$, $N \geqslant 3$, $s \in (0,1)$ and $\alpha \in (0,N)$. Then it holds that*

$$
\mu_n := \inf_{u \in V_{n-1}^\perp} \frac{\int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u|^2 + |u|^2)}{(\int_{\mathbb{R}^N} (\kappa_\alpha * |u|^p)|u|^p)^{\frac{1}{p}}} = \inf_{u \in V_{n-1}^\perp} \frac{\|u\|^2}{B(u)^{\frac{1}{p}}} \to +\infty, \quad n \to \infty.
$$

Proof. Just suppose that there exists a sequence $\{u_n\} \subseteq H_r^s(\mathbb{R}^N)$ such that $u_n \in V_{n-1}^\perp$, $B(u_n) = 1$ and $\|u_n\| \to c^* < +\infty$. Then there exists $u \in H_r^s(\mathbb{R}^N)$ such that, up to a subsequence

$$
\begin{cases}
u_n \rightharpoonup u & \text{in} \quad H_r^s(\mathbb{R}^N), \\
u_n \to u & \text{in} \quad L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N), \\
u_n \to u & \text{a.e. in} \quad \mathbb{R}^N.
\end{cases}
$$

Let $v \in H_r^s(\mathbb{R}^N)$ and $\{v_n\} \subseteq H_r^s(\mathbb{R}^N)$ such that $v_n \in V_{n-1}$ and $v_n \to v$ in $H^s(\mathbb{R}^N)$. Then

$$
|(u_n, v)| \leqslant |(u_n, v - v_n)| + |(u_n, v_n)| \leqslant \|u_n\| \|v - v_n\| \to 0.
$$

Thus $u_n \rightharpoonup 0 = u$ in $H^s_r(\mathbb{R}^N)$. While by (2.2), $1 = B(u_n) \leqslant C\|u_n\|^{2p}_{\frac{2Np}{N+\alpha}} \to 0$, which leads to a contradiction. □

Now for $c > 0$ fixed and for each $n \in \mathbb{N}^+$ and $n \geqslant 2$, we define $S_r(c)$ by (1.15),

$$\rho_n := \left(\frac{N+\alpha+2s}{NL}\mu_n^p\right)^{\frac{1}{p-1}} \quad \text{with} \quad L = \max_{x>0}\frac{(x+c)^p}{x^p+c^p},$$
$$B_n := \{u \in V_{n-1}^{\perp} \cap S_r(c) : \|(-\Delta)^{\frac{s}{2}}u\|_2^2 = \rho_n\} \tag{2.14}$$

and

$$b_n := \inf_{u\in B_n} I(u). \tag{2.15}$$

Then we have the following lemma.

Lemma 2.17. *Let* $p \in (1+\frac{\alpha+2s}{N}, \frac{N+\alpha}{N-2s})$, $N \geqslant 3$, $s \in (0,1)$ *and* $\alpha \in (0,N)$. *Then* $b_n \to \infty$ *as* $n \to \infty$.

Proof. For any $u \in B_n$, we have that

$$\begin{aligned} I(u) &= \frac{1}{2}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2 - \frac{1}{2p}\int_{\mathbb{R}^N}(\kappa_\alpha * |u|^p)|u|^p \\ &\geqslant \frac{1}{2}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2 - \frac{1}{2p}\cdot\frac{1}{\mu_n^p}\cdot\left(\int_{\mathbb{R}^N}(|(-\Delta)^{\frac{s}{2}}u|^2+|u|^2)\right)^p \\ &\geqslant \frac{1}{2}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2 - \frac{1}{2p}\cdot\frac{1}{\mu_n^p}\cdot L\left[\left(\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2\right)^p + c^p\right] \\ &= \left(\frac{Np-N-\alpha-2s}{2Np}\right)\rho_n - \frac{Lc^p}{2p\mu_n^p}. \end{aligned} \tag{2.16}$$

From this estimate and Lemma 2.16, it follows since $p > 1+\frac{\alpha+2s}{N}$, that $b_n \to \infty$ as $n \to \infty$. □

Next, we begin to set up our min-max procedure. First we introduce the map

$$\begin{aligned} &\kappa : H^s_r(\mathbb{R}^N)\times\mathbb{R} \to H^s_r(\mathbb{R}^N) \\ &(u,\theta) \to \kappa(u,\theta) := \mathrm{e}^{\frac{N}{2}\theta}u(\mathrm{e}^\theta x). \end{aligned} \tag{2.17}$$

Observe that for any given $u \in S_r(c)$, we have $\kappa(u,\theta) \in S_r(c)$ for all $\theta \in \mathbb{R}$. Also we know from Lemma 2.1 that

$$\begin{cases} A(\kappa(u,\theta)) \to 0, \quad I(\kappa(u,\theta)) \to 0, \quad \theta \to -\infty, \\ A(\kappa(u,\theta)) \to +\infty, \quad I(\kappa(u,\theta)) \to -\infty, \quad \theta \to +\infty. \end{cases} \tag{2.18}$$

Thus, we deduce that for each $n \in \mathbb{N}$, there exists $\theta_n > 0$, such that

$$\overline{g}_n : [0,1]\times(S_r(c)\cap V_n) \to S_r(c), \quad \overline{g}_n(t,u) \to \kappa(u,(2t-1)\theta_n) \tag{2.19}$$

satisfying

$$\begin{cases} A(\overline{g}_n(0,u)) < \rho_n, \quad A(\overline{g}_n(1,u)) > \rho_n, \\ I(\overline{g}_n(0,u)) < b_n, \quad I(\overline{g}_n(1,u)) < b_n. \end{cases} \tag{2.20}$$

Now we define

$$\begin{aligned} \Gamma_n := \{g : [0,1]\times(S_r(c)\cap V_n) \to S_r(c)\,|\,& g \text{ is continuous, odd in } u \\ & \text{and such that } \forall\, u : g(0,u) = \overline{g}_n(0,u),\ g(1,u) = \overline{g}_n(1,u)\}. \end{aligned} \tag{2.21}$$

Clearly $\overline{g}_n \in \Gamma_n$. Before proving the key intersection result, we need the following linking property.

Lemma 2.18. *For each* $g \in \Gamma_n$, *there exists* $(t,u) \in [0,1]\times(S_r(c)\cap V_n)$ *such that* $g(t,u) \in B_n$ *with* B_n *defined in* (2.14).

Proof. The idea of the proof of this lemma comes from [3]. First, we recall some properties of the cohomological index for spaces with an action of the group $G = \{-1, 1\}$. It associates to a G-space X an element $i(X) \in \mathbb{N} \cup \{\infty\}$. We need the following three properties (see [3, Lemma 2.3]):

(I_1) If G acts on $\mathbb{S}^{n-1}$ via multiplication, then $i(\mathbb{S}^{n-1}) = n$.

(I_2) If there exists an equivariant map $X \to Y$, then $i(X) \leqslant i(Y)$.

(I_3) Let $X = X_0 \cup X_1$ be metrisable and $X_0, X_1 \subset X$ be closed G-invariant subspaces. Let Y be a G-space and consider a continuous map $\phi : [0,1] \times Y \to X$ such that each $\phi_t = \phi(t, \cdot) : Y \to X$ is equivariant. If $\phi_0(Y) \subset X_0$ and $\phi_1(Y) \subset X_1$, then

$$i(\mathrm{Im}(\phi) \cap X_0 \cap X_1) \geqslant i(Y),$$

where $\mathrm{Im}(\phi) := \phi([0,1] \times Y)$ and a map $h : X \to X$ is equivariant if $g \circ h = h \circ g$ for every $g \in G$. Let $P_{n-1} : H_r^s(\mathbb{R}^N) \to V_{n-1}$ be the orthogonal projection and set

$$h_n : S_r(c) \to V_{n-1} \times \mathbb{R}^+, \quad u \to (P_{n-1}u, \|(-\Delta)^{\frac{s}{2}} u\|_2^2).$$

Then clearly, $B_n = h_n^{-1}(0, \rho_n)$. We fix $g \in \Gamma_n$ and consider the map

$$\phi = h_n \circ g : [0,1] \times (S_r(c) \cap V_n) \to V_{n-1} \times \mathbb{R}^+ =: X.$$

Then

$$\phi_0(S_r(c) \cap V_n) \subset V_{n-1} \times (0, \rho_n] =: X_0$$

and

$$\phi_1(S_r(c) \cap V_n) \subset V_{n-1} \times [\rho_n, +\infty) =: X_1.$$

By (I_1)–(I_3),

$$i(\mathrm{Im}(\phi) \cap X_0 \cap X_1) \geqslant i(S_r(c) \cap V_n) = \mathrm{dim} V_n.$$

Just suppose that there would not exist $(t, u) \in [0,1] \times (S_r(c) \cap V_n)$ such that $g(t, u) \in B_n$. Then

$$\mathrm{Im}(\phi) \cap X_0 \cap X_1 \subset (V_{n-1} \setminus \{0\}) \times \{\rho_n\}.$$

By (I_1) and (I_2), we have

$$i(\mathrm{Im}(\phi) \cap X_0 \cap X_1) \leqslant i((V_{n-1} \setminus \{0\}) \times \{\rho_n\}) = \mathrm{dim} V_{n-1}.$$

Then $\mathrm{dim} V_n \leqslant \mathrm{dim} V_{n-1}$, which leads to a contradiction. Thus the proof is completed. □

Lemma 2.19. *For each* $n \in \mathbb{N}^+$,

$$\gamma_n(c) := \inf_{g \in \Gamma_n} \max_{0 \leqslant t \leqslant 1, u \in S_r(c) \cap V_n} I(g(t, u)) \geqslant b_n.$$

Proof. It follows from Lemma 2.18 immediately. □

Next, we shall prove that the sequence $\{\gamma_n(c)\}$ is indeed a sequence of critical values for I restricted to $S_r(c)$. To this end, we first show that there exists a bounded Palais-Smale sequence at each level $\gamma_n(c)$. From now on we fix an arbitrary $n \in \mathbb{N}^+$. To find such a Palais-Smale sequence, we apply the approach developed by Jeanjean [18], already applied in [3]. Notice that, in [3, 18], the function space is $H_r^1(\mathbb{R}^N)$, where we only need to change it to $H_r^s(\mathbb{R}^N)$. First, we introduce the auxiliary functional

$$\widetilde{I} : S_r(c) \times \mathbb{R} \to \mathbb{R}, \quad (u, \theta) \to I(\kappa(u, \theta)),$$

where $\kappa(u, \theta)$ is given in (2.17), and the set

$$\begin{aligned} \widetilde{\Gamma}_n := \{&\widetilde{g} : [0,1] \times (S_r(c) \cap V_n) \to S_r(c) \times \mathbb{R} \,|\, \widetilde{g} \text{ is continuous, odd in } u, \\ &\text{and such that } \kappa \circ \widetilde{g} \in \Gamma_n\}. \end{aligned} \tag{2.22}$$

Clearly, for any $g \in \Gamma_n$, $\widetilde{g} := (g, 0) \in \widetilde{\Gamma}_n$.

Observe the definition

$$\widetilde{\gamma}_n(c) := \inf_{\widetilde{g} \in \widetilde{\Gamma}_n} \max_{0 \leqslant t \leqslant 1, u \in S_r(c) \cap V_n} \widetilde{I}(\widetilde{g}(t, u)),$$

we have that $\widetilde{\gamma}_n(c) = \gamma_n(c)$. Indeed, by the definitions of $\widetilde{\gamma}_n(c)$ and $\gamma_n(c)$, this identity follows immediately from the fact that the maps

$$\varphi : \Gamma_n \to \widetilde{\Gamma}_n, \quad g \to \varphi(g) := (g, 0)$$

and

$$\psi : \widetilde{\Gamma}_n \to \Gamma_n, \quad \widetilde{g} \to \psi(\widetilde{g}) := \kappa \circ \widetilde{g}$$

satisfy

$$\widetilde{I}(\varphi(g)) = I(g) \quad \text{and} \quad I(\psi(\widetilde{g})) = \widetilde{I}(\widetilde{g}).$$

We denote by E the space $H_r^s(\mathbb{R}^N) \times \mathbb{R}$ endowed with the norm $\|\cdot\|_E^2 = \|\cdot\|^2 + |\cdot|_{\mathbb{R}}^2$, and by E^* its dual space and give a useful result, which was proved by using Ekeland's variational principle.

Lemma 2.20. *Let $\varepsilon > 0$. Suppose that $\widetilde{g}_0 \in \widetilde{\Gamma}_n$ satisfies*

$$\max_{0 \leqslant t \leqslant 1, u \in S_r(c) \cap V_n} \widetilde{I}(\widetilde{g}_0(t, u)) \leqslant \widetilde{\gamma}_n(c) + \varepsilon.$$

Then there exists a pair of $(u_0, \theta_0) \in S_r(c) \times \mathbb{R}$ such that

(1) $\widetilde{I}(u_0, \theta_0) \in [\widetilde{\gamma}_n(c) - \varepsilon, \widetilde{\gamma}_n(c) + \varepsilon]$;

(2) $\min_{0 \leqslant t \leqslant 1, u \in S_r(c) \cap V_n} \|(u_0, \theta_0) - \widetilde{g}_0(t, u)\|_E \leqslant \sqrt{\varepsilon}$;

(3) $\|\widetilde{I}'|_{S_r(c) \times \mathbb{R}}(u_0, \theta_0)\|_{E^*} \leqslant 2\sqrt{\varepsilon}$, *i.e.*, $|\langle \widetilde{I}'(u_0, \theta_0), z\rangle_{E^* \times E}| \leqslant 2\sqrt{\varepsilon}\|z\|_E$ *holds, for all*

$$z \in \widetilde{T}_{(u_0, \theta_0)} := \{(z_1, z_2) \in E, \langle u_0, z_1\rangle_{L^2} = 0\}.$$

Proof. The proof is the same as the proof of Lemma 2.3 in [18] and we only need to change the function space from $H_r^1(\mathbb{R}^N)$ to $H_r^s(\mathbb{R}^N)$, so we omit it here. □

Proposition 2.21. *Let $p \in (1 + \frac{\alpha + 2s}{N}, \frac{N+\alpha}{N-2s})$, $N \geqslant 3$, $s \in (0, 1)$ and $\alpha \in (0, N)$. Then for any fixed $c > 0$ and $n \in \mathbb{N}^+$, there exists a sequence $\{v_k^n\} \subset S_r(c)$ satisfying as $k \to \infty$,*

$$\begin{cases} I(v_k^n) \to \gamma_n(c), \\ I'|_{S_r(c)}(v_k^n) \to 0, \\ Q(v_k^n) \to 0. \end{cases} \tag{2.23}$$

In particular $\{v_k^n\} \subset S_r(c)$ is bounded in $H_r^s(\mathbb{R}^N)$.

Proof. From the definition of $\gamma_n(c)$, we know that for each $k \in \mathbb{N}^+$, there exists a $g_k \in \Gamma_n$ such that

$$\max_{0 \leqslant t \leqslant 1, u \in S_r(c) \cap V_n} I(g_k(t, u)) \leqslant \gamma_n(c) + \frac{1}{k}.$$

Since $\widetilde{\gamma}_n(c) = \gamma_n(c)$, $\widetilde{g}_k = (g_k, 0) \in \widetilde{\Gamma}_n$ satisfies

$$\max_{0 \leqslant t \leqslant 1, u \in S_r(c) \cap V_n} \widetilde{I}(\widetilde{g}_k(t, u)) \leqslant \widetilde{\gamma}_n(c) + \frac{1}{k}.$$

Thus applying Lemma 2.20, we obtain a sequence $\{(u_k^n, \theta_k^n)\} \subset S_r(c) \times \mathbb{R}$ such that

(i) $\widetilde{I}(u_k^n, \theta_k^n) \in [\gamma_n(c) - \frac{1}{k}, \gamma_n(c) + \frac{1}{k}]$;

(ii) $\min_{0 \leqslant t \leqslant 1, u \in S_r(c) \cap V_n} \|(u_k^n, \theta_k^n) - (g_k(t, u), 0)\|_E \leqslant \sqrt{\frac{1}{k}}$;

(iii) $\|\widetilde{I}'|_{S_r(c) \times \mathbb{R}}(u_k^n, \theta_k^n)\|_{E^*} \leqslant 2\sqrt{\frac{1}{k}}$, i.e., $|\langle \widetilde{I}'(u_k^n, \theta_k^n), z\rangle_{E^* \times E}| \leqslant 2\sqrt{\frac{1}{k}}\|z\|_E$ holds for all

$$z \in \widetilde{T}_{(u_k^n, \theta_k^n)} := \{(z_1, z_2) \in E, \langle u_k^n, z_1\rangle_{L^2} = 0\}.$$

For each $k \in \mathbb{N}^+$, let $v_k^n = \kappa(u_k^n, \theta_k^n)$. We shall prove that $\{v_k^n\} \subset S_r(c)$ satisfies (2.23). First from (i) we have that $I(v_k^n) \to \gamma_n(c)$ as $k \to \infty$, since $I(v_k^n) = I(\kappa(u_k^n, \theta_k^n)) = \widetilde{I}(u_k^n, \theta_k^n)$. Secondly, note that

$$\begin{aligned}
&\langle \widetilde{I}'(u,\theta),(\phi,r)\rangle \\
&= e^{2\theta s}\int_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}\phi + rse^{2\theta s}\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2 \\
&\quad - \frac{(Np-N-\alpha)r}{2p}e^{\theta(Np-N-\alpha)}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^p|u(y)|^p}{|x-y|^{N-\alpha}} \\
&\quad - e^{\theta(Np-N-\alpha)}\int_{\mathbb{R}^N}(\kappa_\alpha * |u|^p)|u|^{p-2}u\phi.
\end{aligned} \tag{2.24}$$

Then we obtain

$$\begin{aligned}
Q(v_k^n) &= 2sA(v_k^n) - \frac{NP-N-\alpha}{p}B(v_k^n) \\
&= 2se^{2\theta_k^n s}A(u_k^n) - \frac{NP-N-\alpha}{p}e^{\theta_k^n(Np-N-\alpha)}B(u_k^n) \\
&= 2\langle \widetilde{I}'(u_k^n,\theta_k^n),(0,1)\rangle.
\end{aligned} \tag{2.25}$$

Thus (iii) yields $Q(v_k^n) \to 0$ as $k \to \infty$, for $(0,1) \in \widetilde{T}_{(u_k^n,\theta_k^n)}$. Finally, we prove that

$$I'|_{S_r(c)}(v_k^n) \to 0 \quad \text{as} \quad k \to \infty.$$

We claim that for $k \in \mathbb{N}$ sufficiently large,

$$|\langle I'(v_k^n),\omega\rangle| \leqslant \frac{2\sqrt{2}}{\sqrt{k}}\|\omega\| \quad \text{holds for all} \quad \omega \in T_{v_k^n},$$

where $T_{v_k^n} = \{\omega \in H_r^s(\mathbb{R}^N), \langle v_k^n, \omega\rangle_{L^2} = 0\}$. Indeed, for $\omega \in T_{v_k^n}$, setting $\widetilde{\omega} = \kappa(\omega, -\theta_k^n)$, one has

$$\begin{aligned}
\langle I'(v_k^n),\omega\rangle &= \int_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}}v_k^n(-\Delta)^{\frac{s}{2}}\omega - \int_{\mathbb{R}^N}(\kappa_\alpha * |v_k^n|^p)|v_k^n|^{p-2}v_k^n\omega \\
&= e^{2\theta_k^n s}\int_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}}u_k^n(-\Delta)^{\frac{s}{2}}\widetilde{\omega} \\
&\quad - e^{\theta_k^n(Np-N-\alpha)}\int_{\mathbb{R}^N}(\kappa_\alpha * |u_k^n|^p)|u_k^n|^{p-2}u_k^n\widetilde{\omega} \\
&= \langle \widetilde{I}'(u_k^n,\theta_k^n),(\widetilde{\omega},0)\rangle.
\end{aligned} \tag{2.26}$$

Since $\int_{\mathbb{R}^N} u_k^n\widetilde{\omega} = \int_{\mathbb{R}^N} v_k^n\omega$, we obtain $(\widetilde{\omega},0) \in \widetilde{T}_{(u_k^n,\theta_k^n)} \Leftrightarrow \omega \in T_{v_k^n}$. From (ii) it follows that

$$|\theta_k^n| = |\theta_k^n - 0| \leqslant \min_{0\leqslant t\leqslant 1, u\in S_r(c)\cap V_n}\|(u_k^n,\theta_k^n) - (g_k(t,u),0)\|_E \leqslant \frac{1}{\sqrt{k}},$$

by which we deduce that, for k large enough,

$$\|(\widetilde{\omega},0)\|_E^2 = \|\widetilde{\omega}\|^2 = \int_{\mathbb{R}^N}|\omega|^2 + e^{-2\theta_k^n s}A(\omega) \leqslant 2\|\omega\|^2.$$

Thus, by (iii) we have

$$|\langle I'(v_k^n),\omega\rangle| = \langle \widetilde{I}'(u_k^n,\theta_k^n),(\widetilde{\omega},0)\rangle \leqslant \frac{2}{\sqrt{k}}\|(\widetilde{\omega},0)\|_E \leqslant \frac{2\sqrt{2}}{\sqrt{k}}\|\omega\|.$$

As a consequence,

$$\|I'|_{S_r(c)}(v_k^n)\| = \sup_{\omega\in T_{v_k^n}, \|\omega\|\leqslant 1}|\langle I'(v_k^n),\omega\rangle| \leqslant \frac{2\sqrt{2}}{\sqrt{k}} \to 0, \quad k \to \infty.$$

To end the proof of the proposition, it remains to show that $\{v_k^n\} \subset S_r(c)$ is bounded in $H_r^s(\mathbb{R}^N)$. But since $p \in (1+\frac{\alpha+2s}{N}, \frac{N+\alpha}{N-2s})$, this follows from the relationship between $I(u)$ and $Q(u)$,

$$I(u) - \frac{1}{2(Np-N-\alpha)}Q(u) = \frac{Np-N-\alpha-2s}{2(Np-N-\alpha)}A(u). \tag{2.27}$$

This completes the proof. □

Next, we show the compactness of our Palais-Smale sequence $\{v_k^n\}$ obtained in Proposition 2.21. First, we give a useful lemma.

Lemma 2.22. *Let F be a C^1 functional on $H^s(\mathbb{R}^N)$. If $\{x_k\} \subset S(c)$ is bounded in $H^s(\mathbb{R}^N)$, then*

$$F'|_{S(c)}(x_k) \to 0 \quad in \quad H^{-s}(\mathbb{R}^N) \Leftrightarrow F'(x_k) - \langle F'(x_k), x_k\rangle x_k \to 0 \quad in \quad H^{-s}(\mathbb{R}^N) \quad as \quad k\to\infty.$$

Proof. The proof is the same as the proof of Lemma 3 in [10], so we omit it here. □

Proposition 2.23. *Let $p \in (1+\frac{\alpha+2s}{N}, \frac{N+\alpha}{N-2s})$, $N \geqslant 3$, $s \in (0,1)$, $\alpha \in (0,N)$, $c>0$ and $\{v_k\} \subset S_r(c)$ be a sequence satisfying as $k \to \infty$,*

$$\begin{cases} I(v_k) \to \rho(c) \in \mathbb{R}\setminus\{0\}, \\ I'|_{S_r(c)}(v_k) \to 0, \\ Q(v_k) \to 0. \end{cases} \tag{2.28}$$

Then there exist $v \in H_r^s(\mathbb{R}^N)$ and $\{\lambda_k\} \subset \mathbb{R}$ such that up to a subsequence, as $k \to +\infty$,

(i) $v_k \rightharpoonup v \neq 0$ *in* $H_r^s(\mathbb{R}^N)$;
(ii) $\lambda_k \to \widetilde{\lambda} \leqslant 0$ *in* $\mathbb{R}$;
(iii) $(-\Delta)^s v_k - \lambda_k v_k - (\kappa_\alpha * |v_k|^p)|v_k|^{p-2}v_k \to 0$ *in* $H_r^{-s}(\mathbb{R}^N)$;
(iv) $(-\Delta)^s v - \widetilde{\lambda} v - (\kappa_\alpha * |v|^p)|v|^{p-2}v = 0$ *in* $H_r^{-s}(\mathbb{R}^N)$.

Moreover, if $\widetilde{\lambda} < 0$, then we have $v_k \to v$ in $H_r^s(\mathbb{R}^N)$ as $k \to \infty$.

Proof. Since by (2.27) and (2.28), $\{v_k\} \subset S_r(c)$ is bounded, up to a subsequence, there exists $v \in H_r^s(\mathbb{R}^N)$ such that

$$\begin{cases} v_k \rightharpoonup v & \text{in} \quad H_r^s(\mathbb{R}^N), \\ v_k \to v & \text{in} \quad L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N), \\ v_k \to v & \text{a.e. in} \quad \mathbb{R}^N. \end{cases}$$

If $v=0$, by (2.2), we have $B(v_k) = o(1)$. Thus we obtain $A(v_k) = o(1)$ for $Q(v_k) = o(1)$. As a consequence, $I(v_k) = o(1)$, which contradicts $\rho(c) \neq 0$. Thus (i) is obtained. By Lemma 2.22,

$$I'|_{S_r(c)}(v_k) \to 0 \quad \text{in} \quad H_r^{-s}(\mathbb{R}^N) \Leftrightarrow I'(v_k) - \langle I'(v_k), v_k\rangle v_k \to 0 \quad \text{in} \quad H_r^{-s}(\mathbb{R}^N) \quad \text{as} \quad k \to \infty.$$

Since for any $\omega \in H_r^s(\mathbb{R}^N)$,

$$\begin{aligned} \langle I'(v_k) - \langle I'(v_k), v_k\rangle v_k, \omega\rangle = &\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} v_k (-\Delta)^{\frac{s}{2}}\omega \\ &- \int_{\mathbb{R}^N} (\kappa_\alpha * |v_k|^p)|v_k|^{p-2}v_k\omega - \lambda_k \int_{\mathbb{R}^N} v_k\omega, \end{aligned} \tag{2.29}$$

where

$$\lambda_k = \langle I'(v_k), v_k\rangle = A(v_k) - B(v_k). \tag{2.30}$$

Thus (iii) is proved. Since each term on the right-hand side of (2.30) is bounded, there exists $\widetilde{\lambda} \in \mathbb{R}$ such that $\lambda_k \to \widetilde{\lambda}$ as $k \to +\infty$ up to a subsequence. Furthermore, for $p \in (1+\frac{\alpha+2s}{N}, \frac{N+\alpha}{N-2s})$, $Q(v_k) = o(1)$,

$$\widetilde{\lambda} = \lim_{k\to\infty} \lambda_k = \lim_{k\to\infty} [A(v_k) - B(v_k)]$$

$$
\begin{aligned}
&= \lim_{k\to\infty}\left[\frac{(N-2s)p-(N+\alpha)}{Np-N-\alpha}A(v_k)\right]\\
&\leqslant 0. \qquad (2.31)
\end{aligned}
$$

Thus (ii) is proved and (iv) follows from (iii). By (ii) and (iii) we have

$$
\int_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}}v_k(-\Delta)^{\frac{s}{2}}(v_k-v)-\widetilde{\lambda}\int_{\mathbb{R}^N}v_k(v_k-v)-\int_{\mathbb{R}^N}(\kappa_\alpha*|v_k|^p)|v_k|^{p-2}v_k(v_k-v)=o(1). \qquad (2.32)
$$

From (iv) we have

$$
\int_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}}v(-\Delta)^{\frac{s}{2}}(v_k-v)-\widetilde{\lambda}\int_{\mathbb{R}^N}v(v_k-v)-\int_{\mathbb{R}^N}(\kappa_\alpha*|v|^p)|v|^{p-2}v(v_k-v)=0. \qquad (2.33)
$$

Combining (2.32) with (2.33), we obtain

$$
A(v_k-v)-\widetilde{\lambda}D(v_k-v)=\int_{\mathbb{R}^N}[(\kappa_\alpha*|v_k|^p)|v_k|^{p-2}v_k-(\kappa_\alpha*|v|^p)|v|^{p-2}v](v_k-v).
$$

By the Hardy-Littlewood-Sobolev inequality,

$$
\int_{\mathbb{R}^N}(\kappa_\alpha*|v|^p)|v|^{p-2}v(v_k-v)\leqslant\|v\|^{2p-1}_{\frac{2Np}{N+\alpha}}\|v_k-v\|_{\frac{2Np}{N+\alpha}}\to 0
$$

and

$$
\int_{\mathbb{R}^N}(\kappa_\alpha*|v_k|^p)|v_k|^{p-2}v_k(v_k-v)\leqslant\|v_k\|^{2p-1}_{\frac{2Np}{N+\alpha}}\|v_k-v\|_{\frac{2Np}{N+\alpha}}\to 0.
$$

Then, $A(v_k-v)-\widetilde{\lambda}D(v_k-v)=o(1)$. If $\widetilde{\lambda}<0$, $A(v_k-v)=o(1)$ and $D(v_k-v)=o(1)$, then we have $v_k\to v$ in $H^s_r(\mathbb{R}^N)$ as $k\to\infty$. □

3 Proof of main results

At this point we can prove our main results.

Proof of Theorem 1.1. The first part follows from Lemmas 2.7, 2.9 and Propositions 2.14 and 2.15. Furthermore, we shall prove that $u_c\in S(c)$ is a ground state. Indeed, if $u\in S(c)$ and $I'|_{S(c)}(u)=0$, by Lemmas 2.6 and 2.7, we get that $Q(u)=0$, i.e., $u\in V(c)$. This concludes that $I(u)\geqslant m(c)=I(u_c)$. By Lemma 2.6, $Q(u_c)=2sA(u_c)-\frac{Np-N-\alpha}{p}B(u_c)=0$, and then $B(u_c)=\frac{2sp}{Np-N-\alpha}A(u_c)$. By (2.3),

$$
\frac{2sp}{Np-N-\alpha}A(u_c)=B(u_c)\leqslant C(N,\alpha,s)A(u_c)^{\frac{Np-N-\alpha}{2s}}D(u_c)^{\frac{(N+\alpha)-(N-2s)p}{2s}},
$$

and then

$$
A(u_c)^{\frac{Np-N-\alpha-2s}{2s}}\geqslant C(N,\alpha,s)c^{\frac{(N-2s)p-(N+\alpha)}{2s}}\to+\infty
$$

as $c\to 0^+$, i.e., $A(u_c)\to+\infty$ as $c\to 0^+$. Moreover,

$$
m(c)=I(u_c)=\frac{Np-N-\alpha-2s}{2(Np-N-\alpha)}A(u_c)\to+\infty
$$

as $c\to 0^+$. From (1.1), we have $A(u_c)-\lambda_cD(u_c)=B(u_c)$. Then

$$
\begin{aligned}
\lambda_c&=\frac{1}{c}[A(u_c)-B(u_c)]\\
&=\frac{1}{c}\left[A(u_c)-\frac{2sp}{Np-N-\alpha}A(u_c)\right]\\
&=\frac{1}{c}\cdot\frac{(N-2s)p-(N+\alpha)}{Np-N-\alpha}A(u_c)
\end{aligned}
$$

$$\to -\infty \tag{3.1}$$

as $c \to 0^+$, for $p \in (1+\frac{\alpha+2s}{N}, \frac{N+\alpha}{N-2s})$. Next, we consider the case when $c \to +\infty$. Let $u \in V(1)$, $\widetilde{u}(x) = \sqrt{c}t_c^{\frac{N}{2}}u_1(t_c x)$ with $t_c = c^{-\frac{p-1}{Np-N-\alpha-2s}}$. Then $\widetilde{u} \in V(c)$, $A(\widetilde{u}) = ct_c^{2s}A(u_1)$ and $B(\widetilde{u}) = c^p t_c^{Np-N-\alpha}B(u_1)$. By calculation, we have

$$\begin{aligned} I(\widetilde{u}) &= \frac{1}{2}A(\widetilde{u}) - \frac{1}{2p}B(\widetilde{u}) \\ &= \frac{1}{2}A(\widetilde{u}) - \frac{s}{Np-N-\alpha}A(\widetilde{u}) \\ &= \frac{Np-N-\alpha-2s}{2(Np-N-\alpha)}c \cdot t_c^{2s}A(u_1) \\ &= \frac{Np-N-\alpha-2s}{2(Np-N-\alpha)}c \cdot c^{-\frac{2(p-1)s}{Np-N-\alpha-2s}}A(u_1) \\ &= \frac{Np-N-\alpha-2s}{2(Np-N-\alpha)}c^{\frac{(N-2s)p-(N+\alpha)}{Np-N-\alpha-2s}}A(u_1) \\ &\to 0 \end{aligned} \tag{3.2}$$

as $c \to +\infty$, for $p \in (1+\frac{\alpha+2s}{N}, \frac{N+\alpha}{N-2s})$. Therefore, $0 < m(c) = I(u_c) \leqslant I(\widetilde{u}) \to 0$ as $c \to +\infty$. So

$$A(u_c) = \frac{2(Np-N-\alpha)}{Np-N-\alpha-2s}m(c) \to 0$$

and

$$\lambda_c = \frac{1}{c}\frac{(N-2s)p-(N+\alpha)}{[Np-(N+\alpha)]}A(u_c) \to 0$$

as $c \to +\infty$. Thus the proof is completed. □

Proof of Theorem 1.2. By Propositions 2.21 and 2.23, it is enough to prove that if $(v_n, \widetilde{\lambda}_n) \in H^s(\mathbb{R}^N)\backslash\{0\} \times \mathbb{R}$ satisfies (1.1), then $\widetilde{\lambda}_n < 0$, for each $n \in \mathbb{N}^+$. However, this point has been proved in Lemma 2.6. Since

$$I(v_n) - \frac{1}{2(Np-N-\alpha)}Q(v_n) = \frac{Np-N-\alpha-2s}{2(Np-N-\alpha)}A(v_n) = \gamma_n(c),$$

for $Q(v_n) = 0$, then we get that $\{v_n\}$ is unbounded in $H_r^s(\mathbb{R}^N)$ from the fact in Lemmas 2.17 and 2.19 that $\gamma_n(c) \geqslant b_n \to \infty$ as $n \to \infty$. Finally, by Lemma 2.5(i), there exists a constant $C > 0$ such that

$$|v_n(x)| \leqslant \frac{C}{(1+|x|^2)^{\frac{N+2s}{2}}} \leqslant C.$$

Thus the proof is completed. □

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