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Global well-posedness of coupled parabolic systems

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Abstract The initial boundary value problem of a class of reaction-diffusion systems (coupled parabolic systems) with nonlinear coupled source terms is considered in order to classify the initial data for the global existence, finite time blowup and long time decay of the solution. The whole study is conducted by considering three cases according to initial energy: the low initial energy case, critical initial energy case and high initial energy case. For the low initial energy case and critical initial energy case the sufficient initial conditions of global existence, long time decay and finite time blowup are given to show a sharp-like condition. In addition, for the high initial energy case the possibility of both global existence and finite time blowup is proved first, and then some sufficient initial conditions of finite time blowup and global existence are obtained, respectively.

Keywords reaction-diffusion systems, coupled parabolic systems, global existence, asymptotic behavior, finite time blowup

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1 Introduction

In this paper, we consider the following nonlinear parabolic systems with power type source terms:

$$\begin{cases} u_t - \Delta u = (|u|^{2p} + |v|^{p+1}|u|^{p-1})u, & x \in \Omega, \quad t > 0, \\ v_t - \Delta v = (|v|^{2p} + |u|^{p+1}|v|^{p-1})v, & x \in \Omega, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \\ v(x,0) = v_0(x), & x \in \Omega, \\ u(x,t) = v(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T], \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $n \geqslant 2$ and p satisfies the following assumptions:

$$(\mathrm{H}): \quad p > 1 \quad \text{if} \quad n = 2, \quad 1 < p < \frac{2}{n-2} \quad \text{if} \quad n \geqslant 3. \tag{1.2}$$

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Problem (1.1) is usually referred to as a simple example of a semilinear reaction diffusion system with coupling source terms to predict the time evolution of the various density distributions. It also describes heat propagation in a two-component combustible mixture [4, 7, 12], where $u(x,t)$ and $v(x,t)$ represent the temperatures of the two interacting components, thermal conductivity is supposed constant and equal for both substances, and a volume energy release given by some powers of $u(x,t)$ and $v(x,t)$ is assumed. Moreover, it is assumed that the temperatures not only respectively depend on the components themselves (represented by the terms $|u|^{2p}u$ and $|v|^{2p}v$, respectively), but also are affected by each other (represented by the terms $|v|^{p+1}|u|^{p-1}u$ and $|u|^{p+1}|v|^{p-1}v$). Furthermore, it can be used to describe the interaction of two biological groups where the speed of diffusion is slow [10] and the model of Bose-Einstein condensation [8]. The coupled special nonlinear terms in (1.1) can be also found in the interaction of two scalar fields [35] and the motion of charged mesons in an electromagnetic field [23]. It shows that the nuclear reactors exchange heat energy with outside, u and v indicate the neutron flux and temperature of the nuclear reactors [17]. This model is also used in subjects like chemistry [25], physics [1, 9, 11], biology [41] and ecology [5, 22] systems.

In order to further motivate the studies of this paper, we recall some established results and we like to begin with the most fundamental model as follows:

$$\begin{cases} u_t - \Delta u = f(u), & x \in \Omega, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega, \quad t \geqslant 0. \end{cases} \tag{1.3}$$

The local solvability of (1.3) with $f(u) = |u|^{p-1}u$ was given in [6,18]. A powerful technique for treating the above problem is the so-called potential well method, which was established by Payne and Sattinger [27]. The technique for proving the global nonexistence of solutions of abstract problems that include (1.3) was developed in [21]. In [3], a strong result for (1.3) was established, namely pointwise blowup in finite time. By constructing a family of potential wells, Liu and Zhao [24] and Xu [36] respectively considered the initial boundary value problem (IBVP) (1.3) with the initial data $J(u_0) < d$ and $J(u_0) = d$ (here, $J(u)$ is the so-called potential energy functional in the form $J(u) = \frac{1}{2}\|\nabla u\|^2 - \int_\Omega F(u)dx$, $F(u)$ is the anti-derivative of $f(u)$, i.e., $F(u) = \int_0^u f(s)ds$, and the similar functional will be introduced in the present paper later) and proved that there exist global and non-global solutions under different conditions on the initial data. Gazzola and Weth [16] studied the initial boundary value problem (1.3), where $f(u) = |u|^{p-1}u$ and $u_0 \in H_0^1(\Omega)$. They paid much more attention to the initial data at the high energy level, and obtained the global existence (or nonexistence) solution with an arbitrarily large initial datum. Following the ideas in [16], Xu and Niu [37] and Xu and Su [38] extended the corresponding results to the IBVP of the nonlinear pseudo-parabolic equation. In addition, the Cauchy problem of the corresponding nonlinear pseudo-parabolic system was considered in [40].

For semilinear reaction diffusion systems, we first mention the following model:

$$\begin{cases} u_t - \Delta u = v^p, \\ v_t - \Delta v = u^q. \end{cases} \tag{1.4}$$

For the initial boundary value problem of (1.4), Galaktionov et al. [14, 15] proved its local and global existence of the solution. Escobedo and Herrero [12] considered the initial boundary value problem of (1.4) in a bounded open domain in $\mathbb{R}^N$ with smooth boundary. They characterized that every solution exists globally in time if $0 < pq \leqslant 1$; but if $pq > 1$, solutions may exist globally in time with small enough initial value data or blow up in finite time with large enough initial value data. The Cauchy problem of (1.4) was considered in [11] and the global well-posedness of the solutions was obtained. Sato [31] showed the relationship between the life span and initial data. Kwembe and Zhang [19] considered the system (1.4) with the general Wentzell boundary condition and obtained the results on Fujita-type global existence and finite time blowup of solutions.

Considering the combination of interacted nonlinearities, several researchers devoted to the Cauchy problem of the following system:

$$\begin{cases} u_t - \Delta u = v^p + u^r, \\ v_t - \Delta v = u^q + v^s. \end{cases} \tag{1.5}$$

Souplet and Tayachi [33] proved that one component of a blowup solution may stay bounded until the blowup time, and they also investigated the blowup rates of a class of positive radial solutions. Based on a continuity argument, Rossi and Souplet [30] studied the initial boundary value problem of (1.5). They showed that, in the range of exponents where either component may blow up alone, there also exist the initial data for which both components blow up simultaneously.

For the initial boundary value problem of the following system [13, 33]:

$$\begin{cases} u_t - \Delta u = u^{p_1} v^{q_1}, \\ v_t - \Delta v = u^{p_2} v^{q_2}, \end{cases} \tag{1.6}$$

we can view it as an intermediate problem between the uncoupled problem and coupled problem (1.4), which is a simpler case of the problem (1.1). The local solution for the IBVP of (1.6) was inferred in [20,26]. Escobedo and Levine [13] obtained Fujita-type global existence and global nonexistence results for the initial value problem of (1.6) analogous to the classical results of Fujita and others for $u_t = \Delta u + u^p$, $u(x, 0) = u_0(x) \geqslant 0$. By reviewing the above known results and also [2, 7, 13, 28, 34, 39, 42], the authors got the critical global existence exponent and the critical Fujita exponent for (1.6). The main results are as follows:

(i) If $p_1 \leqslant 1$, $q_2 \leqslant 1$, and $p_2 q_1 \leqslant (1 - p_1)(1 - q_2)$, then all the non-negative solutions of the initial boundary problem of (1.6) are global.

(ii) If $p_1 > 1$, or $q_2 > 1$, or $q_2 p_1 > (1 - p_1)(1 - q_2)$, then there are global solutions (or solutions which blow up in finite time), depending on the sufficient small (or large) initial values.

It is obvious that the source terms $|v|^{p+1}|u|^{p-1}u$ and $|u|^{p+1}|v|^{p-1}v$ in the problem (1.1) satisfy the case (ii), i.e., $p_1 = q_2 > 1$. From the above discussions, we find that the solution of the problem (1.1) may exist globally when the initial datum is sufficiently small, and oppositely, it may blow up in finite time when the initial datum is sufficiently large. Hence these efforts mainly focus on the role of the power index of the nonlinear term in the global well-posedness of the solution rather than the initial data, because these results require an extremely strict condition on the initial data, such as large enough or small enough. Inspired by these studies, we are naturally interested in more relaxed restrictions and more precise descriptions of the initial data; in other words, we are interested in how small or how large the initial data are in order to ensure the global existence or non-global existence. To answer the above questions, in this paper we will discuss these problems in the frame of variational methods. The potential well theory works well when the energy is controlled by the mountain pass level, also known as the depth of the potential well, i.e., $J(u_0) < d$, where $J(u)$ is the potential energy of the problem (1.3) and d is the mountain pass level, that will be introduced later. Then we will also extend the above corresponding results to the critical case, i.e., $J(u) = d$ in a proper way. The most interesting part of the present paper is to consider the sup-critical case, also known as the high energy case, i.e., $J(u) > d$ (later we shall show that the potential energy functional $J(u)$ will be replaced by $J(u, v)$ due to the system of parabolic equations). We are also aware of the need and the importance of considering the system of two equations rather than a single equation, especially considering the coupling effects and interactions of the nonlinear terms. But so far we have no standard method to deal with this issue, because we cannot simply solve this problem by parallelizing the method for a single equation due to the interactions in the nonlinearities. At least we need to answer some basic questions. First of all we realize that we are still unable to handle all but the most important coupling nonlinearities. So we need decide which nonlinearities can be prioritized. The nonlinear coupling terms we considered in this paper are relatively general and representative. In addition, they include some simple coupling cases even when they are relatively complex. At the same time, these nonlinear features have a very clear physical and applied background. Even so, we must be honest to say that this particular nonlinear case also brings a lot of convenience to us for constructing the variational structure and conducting corresponding analysis.

Meanwhile the difficulties associated with these nonlinear couplings are also obvious. We must design some new variational functionals and variational structures for the coupling nonlinear terms, and as far as possible make these structures convenient for us to use the well-established techniques for a single equation. Even so, we still have to face many difficulties in the selection and calculation of the auxiliary functions for blowup. In addition, the big difficulties always arise in the case of supercritical initial energy, as small as the re-estimation of the Gâteaux derivative, as large as the analysis of the manifolds and the differential inequalities, which bring us new challenges and also general interest. Although there are still a lot of nonlinear cases unsolved in the frame of the potential well method, it is not helpful to point out some open problems because the special case considered in this paper may be the only one that can be dealt with now. Therefore, our research can also be regarded as the exploration and a test of the bounds of the potential well theory.

The structure of the present paper is as follows:

(i) The low initial energy case ($J(u_0, v_0) < d$) in Section 3: by using the Galerkin method [24] and the concave function method [21, 24], we obtain the global existence and finite time blowup of the solution for the problem (1.1). Furthermore, we characterize the global solution vanishing as $t \to \infty$.

(ii) The critical initial energy case ($J(u_0, v_0) = d$) in Section 4: we prove the global solution, blowup solution and asymptotic behavior of the problem (1.1) by scaling the initial data [36, 37].

(iii) The high initial energy case ($J(u_0, v_0) > d$) in Section 5: we discuss the possibility of both global existence and finite time blowup and try to find out the corresponding initial data with arbitrarily high initial energy. First, we prove the comparison principle of the coupled parabolic systems. By using it and the ideas in [16, 38], we describe the structures of the initial data and give some sufficient conditions of the initial data which ensure the finite time blowup and global existence of the solution, respectively.

2 Notation and primary lemmas

We denote by $\|\cdot\|_q$ the $L^q(\Omega)$ norm for $1 \leqslant q \leqslant \infty$ and by $\|\cdot\|_{H_0^1}$ the Dirichlet norm in $H_0^1(\Omega)$. In the bounded domain, the Poincaré inequality holds to give two equivalent norms $\|u\|_{H_0^1}$ and $\|\nabla u\|_2$, that is to say there exist constants C_1 and C_2 such that

$$C_1\|\nabla u\|_2^2 \leqslant \|u\|_{H_0^1}^2 \leqslant C_2\|\nabla u\|_2^2,$$

which is denoted by $\|u\|_{H_0^1}^2 \simeq \|\nabla u\|_2^2$. In addition, we use $\|f\|_A \lesssim \|f\|_B$ to denote the estimate $\|f\|_A \leqslant C\|f\|_B$ if the constant $C > 0$ can be found in a proper way.

It is obvious that if $(\phi, \psi) = (\phi(x), \psi(x))$ verifies the semilinear elliptic systems

$$\begin{cases} -\Delta\phi = (|\phi|^{2p} + |\psi|^{p+1}|\phi|^{p-1})\phi, & x \in \Omega, \\ -\Delta\psi = (|\psi|^{2p} + |\phi|^{p+1}|\psi|^{p-1})\psi, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega, \\ \psi = 0, & x \in \partial\Omega, \end{cases} \tag{2.1}$$

and $(\phi, \psi) \in H_0^1(\Omega) \times H_0^1(\Omega) \setminus \{(0, 0)\}$, then

$$(u, v) = (u(x, t), v(x, t)) = (\phi(x), \psi(x)), \quad (x, t) \in \mathbb{R}^n \times (0, \infty)$$

verifies (1.1), which is the stationary solution of (1.1). Consider the energy functional J and the Nehari functional I defined by

$$\begin{aligned} J(u, v) = &\frac{1}{2}\bigg(\int_\Omega |\nabla u|^2 dx + \int_\Omega |\nabla v|^2 dx\bigg) \\ &- \frac{1}{2(p+1)}\bigg(\int_\Omega |u|^{2p+2} dx + \int_\Omega |u|^{p+1}|v|^{p+1} dx + \int_\Omega |v|^{2p+2} dx\bigg) \end{aligned}$$

$$\simeq \frac{1}{2}(\|u\|_{H_0^1}^2 + \|v\|_{H_0^1}^2) - \frac{1}{2(p+1)}(\|u\|_{2p+2}^{2p+2} + 2\|uv\|_{p+1}^{p+1} + \|v\|_{2p+2}^{2p+2}) \tag{2.2}$$

and

$$\begin{aligned} I(u,v) &= \left(\int_\Omega |\nabla u|^2 dx + \int_\Omega |\nabla v|^2 dx\right) \\ &\quad - \left(\int_\Omega |u|^{2p+2} dx + \int_\Omega |u|^{p+1}|v|^{p+1} dx + \int_\Omega |v|^{2p+2} dx\right) \\ &\simeq (\|u\|_{H_0^1}^2 + \|v\|_{H_0^1}^2) - (\|u\|_{2p+2}^{2p+2} + 2\|uv\|_{p+1}^{p+1} + \|v\|_{2p+2}^{2p+2}). \end{aligned} \tag{2.3}$$

Next, define the Nehari manifold

$$\mathcal{N} = \{(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega) \setminus \{(0,0)\} \,|\, I(u,v) = 0\}. \tag{2.4}$$

Then J is bounded from below on $\mathcal{N}$. Next, we introduce the mountain pass level (also called the depth of the potential well)

$$d = \min_{(u,v)\in H_0^1(\Omega)\times H_0^1(\Omega)\setminus\{(0,0)\}} \max_{s_1,s_2\geqslant 0} J(s_1 u, s_2 v). \tag{2.5}$$

It is obvious that the mountain pass level d defined in (2.5) may also be characterized as

$$d = \inf_{(u,v)\in\mathcal{N}} J(u,v). \tag{2.6}$$

Clearly, $\mathcal{N}$ separates the two unbounded sets

$$\mathcal{N}_+ = \{(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega) \,|\, I(u,v) > 0\} \cup \{(0,0)\} \tag{2.7}$$

and

$$\mathcal{N}_- = \{(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega) \,|\, I(u,v) < 0\}. \tag{2.8}$$

We also need to consider the (open) sublevels of J:

$$J^k = \{(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega) \,|\, J(u,v) < k\}.$$

Hence,

$$\mathcal{N}_\alpha = \mathcal{N} \cap J^\alpha = \left\{(u,v) \in \mathcal{N} \,\middle|\, \|u\|_{H_0^1}^2 + \|v\|_{H_0^1}^2 < \frac{2\alpha(p+1)}{p}\right\} \neq \emptyset \quad \text{for all} \quad \alpha > d.$$

The above alternative characterization of d also shows that

$$\operatorname{dist}(0,\mathcal{N}) = \min_{(u,v)\in\mathcal{N}} (\|u\|_{H_0^1}^2 + \|v\|_{H_0^1}^2) = \delta := \frac{2d(p+2)}{p} > 0.$$

For all $\alpha > d$, we define

$$\lambda_\alpha = \inf\{\|u\|_2^2 + \|v\|_2^2 \,|\, (u,v) \in \mathcal{N}_\alpha\} \quad \text{and} \quad \Lambda_\alpha = \sup\{\|u\|_2^2 + \|v\|_2^2 \,|\, (u,v) \in \mathcal{N}_\alpha\}.$$

Clearly, we have the following monotonicity properties:

$$\alpha \mapsto \lambda_\alpha \text{ is nonincreasing}, \quad \alpha \mapsto \Lambda_\alpha \text{ is nondecreasing}.$$

In the following, let T denote the maximal existence time of the solution with initial condition $(u_0,v_0) \in H_0^1(\Omega) \times H_0^1(\Omega)$. We denote by $S(t)$ the nonlinear semigroup associated with (1.1). Instead of

$$(u,v) = (u(t),v(t)),$$

we will also write

$$\left(S(t)u_0 + \int_0^t S(t-s)f(u,v)ds, S(t)v_0 + \int_0^t S(t-s)g(u,v)ds\right) \quad \text{for} \quad t < T,$$

where

$$f(u,v) = (|u|^{2p} + |v|^{p+1}|u|^{p-1})u \quad \text{and} \quad g(u,v) = (|v|^{2p} + |u|^{p+1}|v|^{p-1})v.$$

The smoothing properties of this semigroup suggest that we consider the space

$$C_0^1(\Omega) := \{u \in C_0^1(\bar{\Omega}) \,|\, u = 0 \text{ on } \partial\Omega\} = C^1(\bar{\Omega}) \cap H_0^1(\Omega),$$

endowed with the standard norm $\|\cdot\|_{C^1}$ of $C^1(\bar{\Omega})$. If $T = \infty$, we denote by

$$\omega(u_0, v_0) := \bigcap_{t \geqslant 0} \overline{\{(u(s), v(s)) : s \geqslant t\}}$$

the ω-limit set of

$$(u_0, v_0) \in H_0^1(\Omega) \times H_0^1(\Omega).$$

Now we introduce the stable set W and the unstable set V defined by

$$W = J^d \cap \mathcal{N}_+ \quad \text{and} \quad V = J^d \cap \mathcal{N}_-.$$

For $\delta > 0$, we further define the functional

$$I_\delta(u,v) \simeq \delta(\|u\|_{H_0^1}^2 + \|v\|_{H_0^1}^2) - (\|u\|_{2p+2}^{2p+2} + 2\|uv\|_{p+1}^{p+1} + \|v\|_{2p+2}^{2p+2}). \tag{2.9}$$

Now, we define the depth of a family of potential wells for $\delta > 0$,

$$d(\delta) = \inf_{(u,v) \in \mathcal{N}_\delta} J(u,v).$$

For the problem (1.1), it is ready for us to introduce a family of potential wells W_δ together with the outside V_δ,

$$W_\delta = \{(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega) \,|\, I_\delta(u,v) > 0, J(u,v) < d(\delta)\} \cap \{(0,0)\}, \quad \delta > 0$$

and

$$V_\delta = \{(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega) \,|\, I_\delta(u,v) < 0, J(u,v) < d(\delta)\}, \quad \delta > 0.$$

Let us introduce the sets

$$\mathcal{B} = \{(u_0, v_0) \in H_0^1(\Omega) \times H_0^1(\Omega) \,|\, \text{the solution } (u(t), v(t)) \text{ of (1.1) blows up in finite time}\},$$
$$\mathcal{G} = \{(u_0, v_0) \in H_0^1(\Omega) \times H_0^1(\Omega) \,|\, \text{the solution } (u(t), v(t)) \text{ of (1.1) exists for all } t > 0\}$$

and

$$\mathcal{G}_0 = \{(u_0, v_0) \in \mathcal{G} \,|\, u(t) \to 0,\ v(t) \to 0 \text{ in } H_0^1(\Omega) \text{ as } t \to \infty\}.$$

Next, we give four lemmas (see Lemmas 2.1–2.4) without proving them as we can derive them easily from the arguments in [38]. Lemma 2.1 discusses the monotonicity of the map $\lambda \mapsto J(\lambda)$, which can be derived by a scaling of $J(u)$ referring the proof of Lemma 1 in [38]. Lemma 2.2 states the relations between a ball in $H_0^1(\Omega) \times H_0^1(\Omega)$ and the sign of the Nehari function $I(u,v)$, which can be proved by the Sobolev embedding inequality and simple estimates like those in the proof of Lemma 2 in [38]. Lemma 2.3 ensures that the sign of $I_\delta(u,v)$ does not change for different δ, which is very important if we only have the information of some fixed δ, and can be proved by contradictory arguments like the proof of Lemma 5 in [38]. Finally, Lemma 2.4 exhibits the behaviour of $d(\delta)$ in δ (similar to the arguments of Lemma 4 in [38]).

Lemma 2.1 (See [38]). *Let $(u,v)\in H_0^1(\Omega)\times H_0^1(\Omega)\setminus\{(0,0)\}$. Then*

(i) $\lim_{\lambda\to 0} J(\lambda u,\lambda v)=0$, $\lim_{\lambda\to+\infty} J(\lambda u,\lambda v)=-\infty$;

(ii) *on the interval* $0<\lambda<\infty$ *there exists a unique* λ^* *such that* $\frac{d}{d\lambda}J(\lambda u,\lambda v)\,|_{\lambda=\lambda^*}=0$ *and* $J(\lambda u,\lambda v)$ *is increasing on* $0\leqslant\lambda\leqslant\lambda^*$, *decreasing on* $\lambda^*\leqslant\lambda<\infty$ *and takes the maximum at* $\lambda=\lambda^*$;

(iii) $I(\lambda u,\lambda v)>0$ *for* $0<\lambda<\lambda^*$, $I(\lambda u,\lambda v)<0$ *for* $\lambda^*<\lambda<\infty$, *and* $I(\lambda^* u,\lambda^* v)=0$.

Lemma 2.2 (See [38]). *Assume* $(u,v)\in H_0^1(\Omega)\times H_0^1(\Omega)$ *and set* $r(\delta):=(\frac{\delta}{2C_*^{2p+2}})^{\frac{1}{p}}$, *where* C_* *is the embedding constant from* H_0^1 *into* L^{2p+2}.

(i) *If* $0<\|u\|_{H_0^1}^2+\|v\|_{H_0^1}^2<r(\delta)$, *then* $I_\delta(u,v)>0$. *In particular, if* $0<\|u\|_{H_0^1}^2+\|v\|_{H_0^1}^2<r(1)$, *then* $I(u,v)>0$.

(ii) *If* $I_\delta(u,v)<0$, *then* $\|u\|_{H_0^1}^2+\|v\|_{H_0^1}^2>r(\delta)$. *In particular, if* $I(u,v)<0$, *then* $\|u\|_{H_0^1}^2+\|v\|_{H_0^1}^2>r(1)$.

(iii) *If* $I_\delta(u,v)=0$ *and* $\|u\|_{H_0^1}^2+\|v\|_{H_0^1}^2\neq 0$, *then* $\|u\|_{H_0^1}^2+\|v\|_{H_0^1}^2>r(\delta)$. *In particular, if* $I(u,v)=0$ *then* $\|u\|_{H_0^1}^2+\|v\|_{H_0^1}^2\geqslant r(1)$.

(iv) *If* $I_\delta(u,v)=0$ *and* $\|u\|_{H_0^1}^2+\|v\|_{H_0^1}^2\neq 0$, *then* $J(u,v)>0$ *for* $0<\delta<p+1$, $J(u,v)=0$ *for* $\delta=p+1$, $J(u,v)<0$ *for* $\delta>p+1$.

Lemma 2.3 (See [38]). *Assume* $0<J(u,v)<d$ *for some* $(u,v)\in H_0^1(\Omega)\times H_0^1(\Omega)$, *and* $\delta_1<\delta_2$ *are the two roots of the equation* $d(\delta)=J(u,v)$. *Then the sign of* $I_\delta(u,v)$ *does not change for* $\delta_1<\delta<\delta_2$.

Lemma 2.4 (See [38]). *For* $d(\delta)$, *the following properties hold*:

(i) $d(\delta)\geqslant a(\delta)r^2(\delta)$ *for* $a(\delta)=\frac{1}{2}-\frac{\delta}{2(p+1)}$, $0<\delta<p+1$;

(ii) $\lim_{\delta\to 0} d(\delta)=0$, $d(p+1)=0$ *and* $d(\delta)<0$ *for* $\delta>p+1$;

(iii) $d(\delta)$ *is increasing on* $0<\delta\leqslant 1$, *decreasing on* $1\leqslant\delta\leqslant p+1$ *and takes the maximum* $d=d(1)$ *at* $\delta=1$.

We define the weak solution and the maximal existence time for the problem (1.1) as follows.

Definition 2.5 (Weak solution). A function $(u,v)\in L^\infty([0,T),H_0^1(\Omega)\times H_0^1(\Omega))$ with $(u_t,v_t)\in L^2([0,T),L^2(\Omega)\times L^2(\Omega))$ is called a weak solution of the problem (1.1) on $\Omega\times[0,T)$, if the following conditions are satisfied:

(i) for all $\omega_1,\omega_2\in H_0^1(\Omega)$ and for all $t\in(0,T)$, we have

$$(u_t,\omega_1)+(\nabla u,\omega_1)=((|u|^{2p}+|v|^{p+1}|u|^{p-1})u,\omega_1) \tag{2.10}$$

and

$$(v_t,\omega_2)+(\nabla v,\omega_2)=((|v|^{2p}+|u|^{p+1}|v|^{p-1})v,\omega_2); \tag{2.11}$$

(ii) $u(x,0)=u_0(x)$ in $H_0^1(\Omega)$, $v(x,0)=v_0(x)$ in $H_0^1(\Omega)$;

(iii) for all $t\in(0,T)$, we have

$$\int_0^t(\|u_\tau\|_2^2+\|v_\tau\|_2^2)d\tau+J(u,v)\leqslant J(u_0,v_0). \tag{2.12}$$

Definition 2.6 (Maximal existence time). Let (u,v) be a weak solution of the problem (1.1). We define the maximal existence time $T=T(u,v)$ of $(u(t),v(t))$ as follows:

(i) if (u,v) exists for $0\leqslant t<\infty$, then $T=\infty$;

(ii) if there exists a $t_0\in(0,\infty)$ such that $(u(t),v(t))$ exists for $0\leqslant t<t_0$, then $T=t_0$.

Moreover, we present the following local existence and uniqueness theorems referenced in [7, 9].

Proposition 2.7 (Local existence). *Let* $(u_0,v_0)\in H_0^1(\Omega)\times H_0^1(\Omega)$. *Assume* (1.2) *holds and* u_0 *and* v_0 *are non-negative. Then the problem* (1.1) *admits a local solution* (u,v), *i.e.,*

$$u\in C([0,T);H_0^1(\Omega)),\quad u_t\in C([0,T);L^2(\Omega))$$

and

$$v\in C([0,T);\ H_0^1(\Omega)),\quad v_t\in C([0,T);L^2(\Omega)),$$

where T *is the maximal existence time of* $(u(t),v(t))$.

Proposition 2.8 (Uniqueness, see [9]). *Assume $u_0 \geqslant 0$, $v_0 \geqslant 0$, $(u_0, v_0) \in L^\infty(\Omega) \times L^\infty(\Omega)$. For f and g continuous,* (1.1) *has a non-negative solution defined in an interval $[0, T)$. Moreover, f and g are locally Lipschitz, then the solution is unique in $L^\infty((0,T)\times\Omega) \times L^\infty((0,T)\times\Omega)$ and the mapping $(u_0, v_0) \to (u(t), v(t))$ is continuous from $L^\infty(\Omega)\times L^\infty(\Omega)$ to $L^\infty((0,T)\times\Omega)\times L^\infty((0,T)\times\Omega)$.*

3 Global existence and blowup when $J(u_0, v_0) < d$

In this section, we first prove the invariant sets under the flow of (1.1), and we show the global existence (in time) and finite time blowup of the solution. Furthermore, we characterize the asymptotic behavior of the solution for the problem (1.1).

Theorem 3.1 (Invariant sets). *Let* (H) *hold, $(u_0(x), v_0(x)) \in H_0^1(\Omega) \times H_0^1(\Omega)$, $0 < e < d$, $\delta_1 < \delta_2$ be the two roots of the equation $d(\delta) = e$. Then*

(i) *the solution (u, v) of the problem* (1.1) *with $J(u_0, v_0) = e$ belongs to W_δ for $\delta_1 < \delta < \delta_2$, $0 \leqslant t < T$, provided $I(u_0, v_0) > 0$;*

(ii) *the solution (u, v) of the problem* (1.1) *with $J(u_0, v_0) = e$ belongs to V_δ for $\delta_1 < \delta < \delta_2$, $0 \leqslant t < T$, provided $I(u_0, v_0) < 0$, where T is the maximal existence time of u.*

Proof. Let (u, v) be any weak solution of the problem (1.1) with $J(u_0, v_0) = e$, $I(u_0, v_0) > 0$, and T be the maximal existence time of $(u(t), v(t))$. From $J(u_0, v_0) = e$, $I(u_0, v_0) > 0$ and Lemma 2.3, it follows that $I_\delta(u_0, v_0) > 0$ and $J(u_0, v_0) < d(\delta)$. Then $(u_0(x), v_0(x)) \in W_\delta$ for $\delta_1 < \delta < \delta_2$. We prove $(u(t), v(t)) \in W_\delta$ for $\delta_1 < \delta < \delta_2$ and $0 < t < T$. Arguing by contradiction, by continuity of $I(u, v)$ in time we suppose that there exist $\delta_0 \in (\delta_1, \delta_2)$ and $t_0 \in (0, T)$ such that $(u(t_0), v(t_0)) \in \partial W_{\delta_0}$, and $I_{\delta_0}(u(t_0), v(t_0)) = 0$, $\|u(t_0)\|_{H_0^1} \neq 0$, $\|v(t_0)\|_{H_0^1} \neq 0$ or $J(u(t_0), v(t_0)) = d(\delta_0)$. From

$$\int_0^t (\|u_\tau\|_2^2 + \|v_\tau\|_2^2) d\tau + J(u, v) \leqslant J(u_0, v_0) < d(\delta), \quad \delta_1 < \delta < \delta_2, \quad 0 \leqslant t < T, \tag{3.1}$$

we can see that $J(u(t_0), v(t_0)) \neq d(\delta_0)$. If $I_{\delta_0}(u(t_0), v(t_0)) = 0$, $\|u(t_0)\|_{H_0^1}^2 + \|v(t_0)\|_{H_0^1}^2 \neq 0$, then by the definition of $d(\delta)$ we have $J(u_0, v_0) \geqslant d(\delta_0)$, which contradicts (3.1). Similarly, we can obtain the second statement. □

Next, we give a global existence theorem for the weak solution of the problem (1.1) in the case of $J(u_0, v_0) < d$.

Theorem 3.2 (Global existence for $J(u_0, v_0) < d$). *Let $(u_0, v_0) \in H_0^1(\Omega) \times H_0^1(\Omega)$ and* (H) *hold. Assume that $0 < J(u_0, v_0) < d$, $I(u_0, v_0) > 0$. Then the solution of the problem* (1.1) *exists globally on $[0, \infty)$, i.e., $(u(t), v(t)) \in L^\infty(0, \infty; H_0^1(\Omega) \times H_0^1(\Omega))$ with $(u_t(t), v_t(t)) \in L^2(0, \infty; L^2(\Omega) \times L^2(\Omega))$ and $(u(t), v(t)) \in W$ for $0 \leqslant t < \infty$.*

Proof. Let $\{\omega_j(x)\}$ be a system of base functions in $H_0^1(\Omega)$. By the elliptic operator theory, $\{\omega_j(x)\}$ forms base functions in $H_0^1(\Omega) \cap L^p(\Omega)$ $(1 < p < \infty)$ and $\omega_j \in C^\infty(\bar{\Omega})$. Construct the Galerkin approximate solutions $(u_m(x, t), v_m(x, t))$ of the problem (1.1),

$$\begin{cases} u_m(x,t) = \displaystyle\sum_{j=1}^m g_{jm}(t)\omega_j(x), \quad m = 1, 2, \ldots, \\ v_m(x,t) = \displaystyle\sum_{j=1}^m h_{jm}(t)\omega_j(x), \quad m = 1, 2, \ldots \end{cases}$$

satisfying

$$(u_{mt}, \omega_s) + (\nabla u_m, \nabla \omega_s) = ((|u_m|^{2p} + |v_m|^{p+1}|u_m|^{p-1})u_m, \omega_s), \tag{3.2}$$

$$(v_{mt}, \omega_s) + (\nabla v_m, \nabla \omega_s) = ((|v_m|^{2p} + |u_m|^{p+1}|v_m|^{p-1})v_m, \omega_s) \tag{3.3}$$

and

$$\begin{cases} u_m(0) = u_{0m} = \sum_{j=1}^{m} g_{jm}(0)\omega_j(x) \to u_0 \quad \text{in} \quad H_0^1(\Omega), \\ v_m(0) = v_{0m} = \sum_{j=1}^{m} h_{jm}(0)\omega_j(x) \to v_0 \quad \text{in} \quad H_0^1(\Omega). \end{cases} \tag{3.4}$$

According to the standard ordinary differential equation theory, (3.2)–(3.4) admit a solution

$$(g_{jm}(t), h_{jm}(t)) \in C^1([0, t_m)) \times C^1([0, t_m)),$$

where t_m is the minimum of the existence time of $g_{jm}(t)$ and $h_{jm}(t)$ for each m.

Next, we shall extend this local approximate solution constructed by $g_{jm}(t)$ and $h_{jm}(t)$ to the global one. Multiplying (3.2) and (3.3) by $g'_{sm}(t)$ and $h'_{sm}(t)$, respectively, summing for s, integrating with respect to t from zero to t and adding these two equations, we can deduce

$$\int_0^t (\|u_{m\tau}\|_2^2 + \|v_{m\tau}\|_2^2) d\tau + J(u_m, v_m) = J(u_m(0), v_m(0)), \quad t > 0. \tag{3.5}$$

Noticing that (3.4) gives $J(u_m(0), v_m(0)) \to J(u_0, v_0)$, we have

$$\int_0^t (\|u_{m\tau}\|_2^2 + \|v_{m\tau}\|_2^2) d\tau + J(u_m, v_m) < d, \quad t > 0 \tag{3.6}$$

for sufficiently large m. By (3.6) and Theorem 3.1, we can prove that $(u_m(t), v_m(t)) \in W$ for $0 \leqslant t < \infty$ and sufficiently large m. Thus from (3.6) and

$$\begin{aligned} J(u_m, v_m) &\simeq \frac{1}{2}(\|u_m\|_{H_0^1}^2 + \|v_m\|_{H_0^1}^2) - \frac{1}{2(p+1)}(\|u_m\|_{2p+2}^{2p+2} + 2\|u_m v_m\|_{p+1}^{p+1} + \|v_m\|_{2p+2}^{2p+2}) \\ &= \left(\frac{1}{2} - \frac{1}{2(p+1)}\right)(\|u_m\|_{H_0^1}^2 + \|v_m\|_{H_0^1}^2) + \frac{1}{2(p+1)} I(u_m, v_m) \\ &= \frac{p}{2(p+1)}(\|u_m\|_{H_0^1}^2 + \|v_m\|_{H_0^1}^2) + \frac{1}{2(p+1)} I(u_m, v_m), \end{aligned} \tag{3.7}$$

we obtain

$$\int_0^t (\|u_{m\tau}\|_2^2 + \|v_{m\tau}\|_2^2) d\tau + \frac{p}{2(p+1)}(\|u_m\|_{H_0^1}^2 + \|v_{m\tau}\|_{H_0^1}^2) < d, \quad 0 \leqslant t < \infty \tag{3.8}$$

for sufficiently large m, which yields for $0 \leqslant t < \infty$,

$$\begin{aligned} &\|u_m\|_{H_0^1}^2 < \frac{2(p+1)}{p} d, \\ &\|v_m\|_{H_0^1}^2 < \frac{2(p+1)}{p} d, \\ &\|u_m\|_{2(p+1)}^2 \leqslant C_*^2 \|u_m\|_{H_0^1}^2 < C_*^2 \left(\frac{2(p+1)}{p} d\right), \\ &\|v_m\|_{2(p+1)}^2 \leqslant C_*^2 \|v_m\|_{H_0^1}^2 < C_*^2 \left(\frac{2(p+1)}{p} d\right), \\ &\int_0^t (\|u_{m\tau}\|_2^2 + \|v_{m\tau}\|_2^2) d\tau < d, \\ &\||u_m|^{2p} u_m\|_q^q = \|u_m\|_{2p+2}^{2p+2} \leqslant C_*^{2p+2} \left(\frac{2(p+1)}{p} d\right)^{p+1}, \quad q = \frac{2p+2}{2p+1}, \\ &\||v_m|^{2p} v_m\|_q^q = \|v_m\|_{2p+2}^{2p+2} \leqslant C_*^{2p+2} \left(\frac{2(p+1)}{p} d\right)^{p+1}, \quad q = \frac{2p+2}{2p+1} \end{aligned}$$

and

$$\|v_m u_m\|_{p+1}^{p+1} \leqslant \|v_m\|_{2p+2}^{p+1}\|u_m\|_{2p+2}^{p+1} \leqslant \frac{1}{2}(\|v_m\|_{2p+2}^{2p+2}+\|u_m\|_{2p+2}^{2p+2}) \leqslant C_*^{2p+2}\left(\frac{p}{2(p+1)}d\right)^{p+1}.$$

Hence, there exist u, v with their subsequences u_ν of u_m and v_ν of v_m such that

$u_\nu \to u$ in $L^\infty((0,\infty);H_0^1(\Omega))$ weak-star and a.e. in $Q=\Omega\times[0,\infty)$;

$v_\nu \to v$ in $L^\infty((0,\infty);H_0^1(\Omega))$ weak-star and a.e. in $Q=\Omega\times[0,\infty)$;

$(|u_\nu|^{2p}+|v_\nu|^{p+1}|u_\nu|^{p-1})u_\nu \to (|u|^{2p}+|u|^{p+1}|v|^{p-1})u$ in $L^\infty((0,\infty);L^q(\Omega))$ weak-star and a.e. in $Q=\Omega\times[0,\infty)$;

$(|v_\nu|^{2p}+|u_\nu|^{p+1}|v_\nu|^{p-1})v_\nu \to (|v|^{2p}+|v|^{p+1}|u|^{p-1})v$ in $L^\infty((0,\infty);L^q(\Omega))$ weak-star and a.e. in $Q=\Omega\times[0,\infty)$;

$u_{\nu t}\to u_t$ in $L^2((0,\infty);L^2(\Omega))$ weakly;

$v_{\nu t}\to v_t$ in $L^2((0,\infty);L^2(\Omega))$ weakly.

In (3.2) and (3.3) for fixed s, letting $m=\nu\to\infty$, we get

$$(u_t,\omega_s)+(\nabla u,\nabla\omega_s)=((|u|^{2p}+|v|^{p+1}|u|^{p-1})u,\omega_s)$$

and

$$(v_t,\omega_s)+(\nabla v,\nabla\omega_s)=((|v|^{2p}+|u|^{p+1}|v|^{p-1})v,\omega_s).$$

On the other hand, (3.4) gives $u(x,0)=u_0(x)$ and $v(x,0)=v_0(x)$ in $H_0^1(\Omega)$. Finally, from Theorem 3.1, we have $(u(t),v(t))\in W$ for $0\leqslant t<\infty$. □

Corollary 3.3. *Let $(u_0,v_0)\in H_0^1(\Omega)\times H_0^1(\Omega)$ and* (H) *hold. Assume that $0<J(u_0,v_0)<d$, $I_{\delta_2}(u_0,v_0)>0$, where $\delta_1<\delta_2$ are the two roots of the equation $d(\delta)=J(u_0,v_0)$. Then the problem* (1.1) *admits a global weak solution $(u(t),v(t))\in L^\infty(0,\infty;H_0^1(\Omega)\times H_0^1(\Omega))$ with $(u_t(t),v_t(t))\in L^2(0,\infty;L^2(\Omega)\times L^2(\Omega))$ and $(u(t),v(t))\in W_\delta$ for $0\leqslant t<\infty$.*

Theorem 3.4 (Global nonexistence for $J(u_0,v_0)<d$)**.** *Let p satisfy* (H) *and $(u_0(x),v_0(x))\in H_0^1(\Omega)\times H_0^1(\Omega)$. Assume that $J(u_0,v_0)<d$ and $I(u_0,v_0)<0$. Then the weak solution $(u(t),v(t))$ of the problem* (1.1) *blows up in finite time, i.e., there exists a $T>0$ such that*

$$\lim_{t\to T}\int_0^t(\|u\|_2^2+\|v\|_2^2)d\tau=+\infty. \tag{3.9}$$

Proof. Let $u(t)$ be any weak solution of the problem (1.1) with $J(u_0,v_0)<d$, $I(u_0,v_0)<0$. We define $F(t)=\int_0^t(\|u\|_2^2+\|v\|_2^2)d\tau$. Then $F'(t)=\|u\|_2^2+\|v\|_2^2$ and

$$\begin{aligned}F''(t)&=2((u_t,u)+(v_t,v))\\&=2(-(\|\nabla u\|_2^2+\|\nabla v\|_2^2)+(\|u\|_{2p+2}^{2p+2}+2\|uv\|_{p+1}^{p+1}+\|v\|_{2p+2}^{2p+2}))\\&=-2I(u,v).\end{aligned} \tag{3.10}$$

By Theorem 3.1, we see that $F''(t)>0$. From (3.10), (2.12) and

$$J(u,v)\simeq\frac{p}{2(p+1)}(\|u\|_{H_0^1}^2+\|v\|_{H_0^1}^2)+\frac{1}{2(p+1)}I(u,v), \tag{3.11}$$

we deduce

$$\begin{aligned}F''(t)&\simeq 2p(\|u\|_{H_0^1}^2+\|v\|_{H_0^1}^2)-4(p+1)J(u,v)\\&\geqslant 2p(\|u\|_{H_0^1}^2+\|v\|_{H_0^1}^2)-4(p+1)J(u_0,v_0)+4(p+1)\int_0^t(\|u_\tau\|_2^2+\|v_\tau\|_2^2)d\tau\\&\geqslant 4(p+1)\int_0^t(\|u_\tau\|_2^2+\|v_\tau\|_2^2)d\tau+2pCF'(t)-4(p+1)J(u_0,v_0),\end{aligned} \tag{3.12}$$

where the constant C is from the Poincaré inequality $\|u\|_2^2 \leqslant C\|\nabla u\|_2^2$. Note that

$$\begin{aligned}
&\bigg(\int_0^t ((u_\tau, u) + (v_\tau, v))d\tau\bigg)^2 \\
&= \bigg(\frac{1}{2}\int_0^t \frac{d}{d\tau}(\|u\|_2^2 + \|v\|_2^2)d\tau\bigg)^2 \\
&= \frac{1}{4}((\|u\|_2^2 + \|v\|_2^2)^2 - 2(\|u_0\|_2^2\|v_0\|_2^2)(\|u\|_2^2 + \|v\|_2^2) + (\|u\|_2^2 + \|v\|_2^2)^2) \\
&= \frac{1}{4}(F'^2(t) - 2F'(t)(\|u_0\|_2^2 + \|v_0\|_2^2) + (\|u_0\|_2^2 + \|v_0\|_2^2)^2).
\end{aligned} \tag{3.13}$$

Thus we get

$$\begin{aligned}
F''(t)F(t) - (p+1)(F'(t))^2 \geqslant\ & 4(p+1)\int_0^t (\|u_\tau\|_2^2 + \|v_\tau\|_2^2)d\tau \int_0^t (\|u\|_2^2 + \|v\|_2^2)d\tau \\
& + 2pCF'(t)F(t) - 4(p+1)J(u_0, v_0)F(t) \\
& - (p+1)\bigg(4\bigg(\int_0^t ((u, u_t) + (v, v_t))d\tau\bigg)^2 \\
& + 2F'(t)(\|u_0\|_2^2 + \|v_0\|_2^2) - (\|u_0\|_2^2 + \|v_0\|_2^2)^2\bigg) \\
=\ & 4(p+1)\xi + 2pCF'(t)F(t) - 4(p+1)J(u_0, v_0)F(t) \\
& - 2(p+1)F'(t)(\|u_0\|_2^2 + \|v_0\|_2^2) \\
& + (p+1)(\|u_0\|_2^2 + \|v_0\|_2^2)^2,
\end{aligned} \tag{3.14}$$

where

$$\begin{aligned}
\xi :=\ & \int_0^t (\|u_\tau\|_2^2 + \|v_\tau\|_2^2)d\tau \int_0^t (\|u\|_2^2 + \|v\|_2^2)d\tau - \bigg(\int_0^t ((u, u_\tau) + (v, v_\tau))d\tau\bigg)^2 \\
\geqslant\ & \int_0^t (\|u_\tau\|_2^2 + \|v_\tau\|_2^2)d\tau \int_0^t (\|u\|_2^2 + \|v\|_2^2)d\tau - \bigg(\int_0^t (\|u\|_2\|u_\tau\|_2 + \|v\|_2\|v_\tau\|_2)d\tau\bigg)^2 \\
\geqslant\ & \int_0^t (\|u_\tau\|_2^2 + \|v_\tau\|_2^2)d\tau \int_0^t (\|u\|_2^2 + \|v\|_2^2)d\tau - \bigg(\int_0^t \sqrt{\|u_\tau\|_2^2 + \|v_\tau\|_2^2}\sqrt{\|u\|_2^2 + \|v\|_2^2}d\tau\bigg)^2 \\
\geqslant\ & 0.
\end{aligned}$$

Hence we can obtain

$$\begin{aligned}
F''(t)F(t) - (p-1)(F'(t))^2 \geqslant\ & 2pCF'(t)F(t) - 4(p+1)J(u_0, v_0)F(t) \\
& - 2(p+1)F'(t)(\|u_0\|_2^2 + \|v_0\|_2^2) \\
& + (p+1)(\|u_0\|_2^2 + \|v_0\|_2^2)^2.
\end{aligned} \tag{3.15}$$

Then we need to determine the sign of $F''(t)F(t) - (p-1)(F'(t))^2$. In order to simplify the discussion, we respectively consider the following two cases, i.e., $J(u_0, v_0) \leqslant 0$ and $0 < J(u_0, v_0) < d$.

(i) If $J(u_0, v_0) \leqslant 0$, (3.15) becomes

$$\begin{aligned}
& F''(t)F(t) - (p-1)(F'(t))^2 \\
& \geqslant 2pCF'(t)F(t) - 2(p+1)F'(t)(\|u_0\|_2^2 + \|v_0\|_2^2) \\
& = 2F'(t)(pCF(t) - (p+1)(\|u_0\|_2^2 + \|v_0\|_2^2)).
\end{aligned} \tag{3.16}$$

So we only need to consider the sign of $pCF(t) - (p+1)(\|u_0\|_2^2 + \|v_0\|_2^2)$. From $F'(t) = \|u\|_2^2 + \|v\|_2^2 \geqslant 0$ and $F'(0) = \|u_0\|_2^2 + \|v_0\|_2^2 > 0$, we have $pCF(t) > (p+1)(\|u_0\|_2^2 + \|v_0\|_2^2)$. Then

$$F(t)F''(t) - (p+1)(F'(t))^2 > 0 \quad \text{for sufficiently large } t. \tag{3.17}$$

(ii) If $0 < J(u_0, v_0) < d$, then from Theorem 3.1 it follows that $(u(t), v(t)) \in V_\delta$ for $1 < \delta < \delta_2$ and $t > 0$, where δ_2 is the larger root of the equation $d(\delta) = e$. Hence $I_\delta(u, v) \leqslant 0$ and $\|u\|^2_{H^1_0} + \|v\|^2_{H^1_0} > r(\delta)$ for $0 < \delta < \delta_2$ and $t > 0$. So we get $I_{\delta_2}(u, v) \leqslant 0$ and $\|u\|^2_{H^1_0} + \|v\|^2_{H^1_0} \geqslant r(\delta_2)$ for $t > 0$, and by (3.12) we obtain

$$\begin{aligned} F''(t) &= -2I(u,v) \simeq 2(\delta_2 - 1)(\|u\|^2_{H^1_0} + \|v\|^2_{H^1_0}) - 2I_{\delta_2}(u,v) \\ &\geqslant 2(\delta_2 - 1)r(\delta_2) > 0, \quad t \geqslant 0, \\ F'(t) &= 2(\delta_2 - 1)r(\delta_2)t + F'(0) \geqslant 2(\delta_2 - 1)r(\delta_2)t, \quad t \geqslant 0 \end{aligned}$$

and

$$F(t) = (\delta_2 - 1)r(\delta_2)t^2 + F(0) = (\delta_2 - 1)r(\delta_2)t^2, \quad t \geqslant 0.$$

Hence for sufficiently large t we have

$$pCF(t) > 2(p+1)(\|u_0\|^2_2 + \|v_0\|^2_2) \quad \text{and} \quad pCF'(t) > 4(p+1)J(u_0, v_0).$$

Therefore by (3.15) we again obtain (3.17) for sufficiently large t.

For the both two cases above, (3.17) shows

$$(F^{-\alpha}(t))'' = \frac{-\alpha}{F^{\alpha+2}(t)}(F(t)F''(t) - (\alpha+1)(F'(t))^2) < 0, \quad \alpha = p, \tag{3.18}$$

which gives that there exists a $T > 0$ such that $\lim_{t\to T} F^{-\alpha}(t) = 0$, i.e., $\lim_{t\to T} F(t) = +\infty$, which contradicts $T = +\infty$. □

In the following, we prove the asymptotic behavior of the solution for the problem (1.1) in the frame of a family of potential wells with $J(u_0, v_0) < d$.

Theorem 3.5 (Asymptotic behavior of the solution for $J(u_0, v_0) < d$)**.** *Let p satisfy* (H) *and* $(u_0, v_0) \in H^1_0(\Omega) \times H^1_0(\Omega)$. *Assume that $J(u_0, v_0) < d$ and $I(u_0, v_0) > 0$. Then for the global solution of the problem* (1.1), *there exists a constant $\lambda > 0$ such that*

$$\|u\|^2_2 + \|v\|^2_2 \leqslant (\|u_0\|^2_2 + \|v_0\|^2_2)e^{-\lambda t}. \tag{3.19}$$

Proof. First, Theorem 3.2 implies the existence of the global weak solution for the problem (1.1). Now we only need to prove (3.19). Let (u, v) be a global solution of the problem (1.1) with $J(u_0, v_0) < d$ and $I(u_0, v_0) > 0$. Then for $\omega_1, \omega_2 \in L^\infty([0,T), H^1_0(\Omega)) \cap L^2([0,T), L^2(\Omega))$, (2.10) and (2.11) imply that

$$(u_t, \omega_1) + (\nabla u, \nabla \omega_1) = ((|u|^{2p} + |v|^{p+1}|u|^{p-1})u, \omega_1), \tag{3.20}$$

$$(v_t, \omega_2) + (\nabla v, \nabla \omega_2) = ((|v|^{2p} + |u|^{p+1}|v|^{p-1})v, \omega_2). \tag{3.21}$$

Setting $\omega_1 = u$, $\omega_2 = v$, and adding (3.20) and (3.21) together, we obtain

$$\frac{1}{2}\frac{d}{dt}(\|u\|^2_2 + \|v\|^2_2) + I(u,v) = 0, \quad 0 \leqslant t < \infty. \tag{3.22}$$

From $0 < J(u_0, v_0) < d$, $I(u_0, v_0) > 0$ and Theorem 3.1, we have $(u(t), v(t)) \in W_\delta$ for $\delta_1 < \delta < \delta_2$ and $0 \leqslant t < \infty$, where $\delta_1 < \delta_2$ are two roots of the equation $d(\delta) = J(u_0, v_0)$. Hence, we get $I_\delta(u, v) \geqslant 0$ for $\delta_1 < \delta < \delta_2$ and $I_{\delta_1}(u, v) \geqslant 0$ for $0 \leqslant t < 0$. Thus, (3.22) gives

$$\frac{1}{2}\frac{d}{dt}(\|u\|^2_2 + \|v\|^2_2) + (1 - \delta_1)(\|\nabla u\|^2_2 + \|\nabla v\|^2_2) + I_{\delta_1}(u,v) = 0, \quad 0 \leqslant t < \infty. \tag{3.23}$$

From (3.23) and the Poincaré inequality $\|\nabla u\|^2_2 \geqslant C\|u\|^2_2$, we also have

$$\frac{1}{2}\frac{d}{dt}(\|u\|^2_2 + \|v\|^2_2) + C(1 - \delta_1)(\|u\|^2_2 + \|v\|^2_2) \leqslant 0, \quad 0 \leqslant t < \infty.$$

Integrating the above inequality we have

$$\|u\|_2^2+\|v\|_2^2 \leqslant (\|u_0\|_2^2+\|v_0\|_2^2)-2C(1-\delta_1)\int_0^t(\|u(\tau)\|_2^2+\|v(\tau)\|_2^2)d\tau, \quad 0\leqslant t<\infty.$$

Then by Grönwall's inequality, we arrive at

$$\|u\|_2^2+\|v\|_2^2 \leqslant (\|u_0\|_2^2+\|v_0\|_2^2)\mathrm{e}^{-2C(1-\delta_1)t}, \quad 0\leqslant t<\infty.$$

Therefore, there exists a constant $\lambda>0$ such that

$$\|u\|_2^2+\|v\|_2^2 \leqslant (\|u_0\|_2^2+\|v_0\|_2^2)\mathrm{e}^{-\lambda t}, \quad 0\leqslant t<\infty.$$

This completes the proof. □

4 Global existence and blowup when $J(u_0,v_0)=d$

In this section, we consider the global existence, nonexistence and the asymptotic behavior of the solution for the problem (1.1) with the critical initial conditions.

Theorem 4.1 (Global existence for $J(u_0,v_0)=d$). *Assume that p satisfies* (H) *and* $(u_0,v_0)\in H_0^1(\Omega)\times H_0^1(\Omega)$. *If $J(u_0,v_0)=d$ and $I(u_0,v_0)\geqslant 0$, then the solution $(u(t),v(t))$ of the problem* (1.1) *exists globally, i.e., $(u(t),v(t))\in L^\infty(0,\infty;H_0^1(\Omega)\times H_0^1(\Omega))$ with $(u_t(t),v_t(t))\in L^2(0,\infty;L^2(\Omega)\times L^2(\Omega))$ and $(u(t),v(t))\in \overline{W}=W\cup\partial W$ for $0\leqslant t<\infty$.*

Proof. First, $J(u_0,v_0)=d$ implies that $\|u_0\|_2^2+\|v_0\|_2^2\neq 0$. Let $(u_{0m},v_{0m})=\lambda_m(u_0,v_0)$, for $m=1,2,\ldots$, where $0<\lambda_m<1$ and $\lambda_m=1-\frac{1}{m}\to 1$ as $m\to\infty$. Consider the following initial boundary value problem:

$$\begin{cases} u_{mt}-\Delta u_m=(|u_m|^{2p}+|v_m|^{p+1}|u_m|^{p-1})u_m, & x\in\Omega, \quad t>0,\\ v_{mt}-\Delta v_m=(|v_m|^{2p}+|u_m|^{p+1}|v_m|^{p-1})v_m, & x\in\Omega, \quad t>0,\\ u_m(x,0)=u_{0m}(x), & x\in\Omega,\\ v_m(x,0)=v_{0m}(x), & x\in\Omega,\\ u_m(x,t)=v_m(x,t)=0, & (x,t)\in\partial\Omega\times(0,T]. \end{cases} \tag{4.1}$$

From $I(u_0,v_0)\geqslant 0$ and $0<\lambda<1$, we obtain

$$\begin{aligned} I(u_{0m},v_{0m}) &= I(\lambda_m u_0,\lambda_m v_0)\\ &\simeq \lambda_m^2(\|u_0\|_{H_0^1}^2+\|v_0\|_{H_0^1}^2)-\lambda_m^{2p+2}(\|u_0\|_{2p+2}^{2p+2}+\|u_0v_0\|_{p+1}^{p+1}+\|v_0\|_{2p+2}^{2p+2})\\ &= (\lambda_m^2-\lambda_m^{2p+2})(\|u_0\|_{H_0^1}^2+\|v_0\|_{H_0^1}^2)+\lambda_m^{2p+2}I(u_0,v_0)>0. \end{aligned} \tag{4.2}$$

Furthermore, by Lemma 2.1(iii) and $I(u_0,v_0)\geqslant 0$, we obtain

$$\lambda^*=\left(\frac{\|u\|_{H_0^1}^2+\|v\|_{H_0^1}^2}{\|u\|_{2p+2}^{2p+2}+2\|uv\|_{p+1}^{p+1}+\|v\|_{2p+2}^{2p+2}}\right)^{\frac{1}{2p}}\geqslant 1,$$

which means $\lambda_m<1\leqslant\lambda^*$. Then combining Lemma 2.1(ii) we have

$$J(u_{0m},v_{0m})=J(\lambda_m u_0,\lambda_m v_0)<J(u_0,v_0)=d. \tag{4.3}$$

By recalling Theorem 3.2, we obtain that for each m the problem (4.1) admits a global weak solution

$$(u_m(t),v_m(t))\in L^\infty(0,\infty;H_0^1(\Omega)\times H_0^1(\Omega))$$

with

$$(u_{mt}(t),v_{mt}(t))\in L^2(0,\infty;L^2(\Omega)\times L^2(\Omega))$$

and $(u_m(t), v_m(t)) \in W$ for $0 \leqslant t < \infty$ satisfying

$$(u_{mt}, \omega_1) + (\nabla u_m, \omega_1) = ((|u_m|^{2p} + |v_m|^{p+1}|u_m|^{p-1})u_m, \omega_1), \quad \omega_1 \in H_0^1(\Omega), \quad t > 0,$$
$$(v_{mt}, \omega_2) + (\nabla v_m, \omega_2) = ((|v_m|^{2p} + |u_m|^{p+1}|v_m|^{p-1})v_m, \omega_2), \quad \omega_2 \in H_0^1(\Omega), \quad t > 0$$

and

$$\int_0^t (\|u_{m\tau}\|_2^2 + \|v_{m\tau}\|_2^2)d\tau + J(u_m, v_m) \leqslant J(u_{0m}, v_{0m}) < d, \quad 0 \leqslant t < \infty. \tag{4.4}$$

By (3.7) and (4.4), we can obtain

$$\int_0^t (\|u_{m\tau}\|_2^2 + \|v_{m\tau}\|_2^2)d\tau + \frac{p}{2(p+1)}(\|u_m\|_{H_0^1}^2 + \|v_m\|_{H_0^1}^2) < d, \quad 0 \leqslant t < \infty.$$

The remainder of the proof is similar to that in the proof of Theorem 3.2. □

Theorem 4.2 (Global nonexistence for $J(u_0, v_0) = d$). *Assume that p satisfies* (H) *and* $(u_0, v_0) \in H_0^1(\Omega) \times H_0^1(\Omega)$. *If $J(u_0, v_0) = d$ and $I(u_0, v_0) < 0$, then the solution $(u(t), v(t))$ of the problem* (1.1) *blows up in finite time.*

Proof. Let $(u(t), v(t))$ be any solution of the problem (1.1) with $J(u_0, v_0) = d$, $I(u_0, v_0) < 0$, T being the existence time of $(u(t), v(t))$. Let us prove $T < \infty$. Arguing by contradiction, we suppose $T = +\infty$. Let

$$F(t) = \int_0^t (\|u\|_2^2 + \|v\|_2^2)d\tau.$$

Taking into account that (3.15) still holds, combining the fact $J(u_0, v_0) = d$ we arrive at

$$\begin{aligned} F''(t)F(t) - (p+1)(F'(t))^2 \geqslant{} & (pCF(t) - 2(p+1))(\|u_0\|_2^2 + \|v_0\|_2^2)F'(t) \\ & + (pCF'(t) - 4(p+1)d)F(t). \end{aligned} \tag{4.5}$$

On the other hand, from $J(u_0, v_0) = d > 0$, $I(u_0, v_0) < 0$ and the continuity of $J(u, v)$ and $I(u, v)$ with respect to t, it follows that there exists a sufficiently small $t_1 > 0$ such that $J(u(t_1), v(t_1)) > 0$ and $I(u, v) < 0$ for $0 < t < t_1$. Hence $(u_t, u) + (v_t, v) = -I(u, v) > 0$ and $\|u_t\|_2^2 + \|v_t\|_2^2 > 0$ for $0 \leqslant t \leqslant t_1$. From this and the continuity of $\int_0^t (\|u_\tau\|_2^2 + \|v_\tau\|_2^2)d\tau$ it follows that we can choose a t_1 such that

$$0 < d_1 := d - \int_0^{t_1} (\|u_t\|_2^2 + \|v_t\|_2^2)dt < d. \tag{4.6}$$

In addition, by (2.12) we have

$$0 < J(u(t_1), v(t_1)) \leqslant d - \int_0^{t_1} (\|u_t\|_2^2 + \|v_t\|_2^2)dt = d_1 < d.$$

Thus we take $t = t_1$ as the initial time, and then we have $(u(t), v(t)) \in V_\delta$ for $\delta \in (\delta_1, \delta_2)$, $t_1 \leqslant t < \infty$, where (δ_1, δ_2) is the maximal interval including $\delta = 1$ such that $d(\delta) > d_1$ for $\delta \in (\delta_1, \delta_2)$. Hence we have $I_\delta(u, v) < 0$ and $\|u\|_{H_0^1}^2 + \|v\|_{H_0^1}^2 > r(\delta)$ for $\delta \in (1, \delta_2)$, $t_1 \leqslant t < \infty$, and $I_{\delta_2}(u, v) \leqslant 0$, $\|u\|_{H_0^1}^2 + \|v\|_{H_0^1}^2 \geqslant r(\delta_2)$ for $t_1 \leqslant t < \infty$. Thus from (3.10) we obtain

$$\begin{aligned} F''(t) = -2I(u, v) &\simeq 2(\delta_2 - 1)(\|u\|_{H_0^1}^2 + \|v\|_{H_0^1}^2) - 2I_{\delta_2}(u, v) \\ &\geqslant 2(\delta_2 - 1)r(\delta_2), \quad t_1 \leqslant t < \infty, \end{aligned} \tag{4.7}$$

$$F'(t) \geqslant 2(\delta_2 - 1)r(\delta_2)(t - t_1) + F'(t_1) \geqslant 2(\delta_2 - 1)r(\delta_2)(t - t_1), \quad t_1 \leqslant t < \infty \tag{4.8}$$

and

$$F(t) \geqslant (\delta_2 - 1)r(\delta_2)(t - t_1)^2 + F(t_1) > (\delta_2 - 1)r(\delta_2)(t - t_1)^2, \quad t_1 \leqslant t < \infty. \tag{4.9}$$

From (4.8) and (4.9) it follows that for sufficiently large t we have

$$pCF(t) > 2(p+1)(\|u_0\|_2^2 + \|v_0\|_2^2)$$

and

$$pCF'(t) > 4(p+1)d.$$

Thus (3.15) yields $F(t)F''(t) - (p+1)(F'(t))^2 > 0$. The remainder of the proof is similar to that in the proof of Theorem 3.4. □

Recalling Theorems 4.1 and 4.2, we can conclude a sharp condition for the global existence of the solution for the problem (1.1) with $J(u_0, v_0) = d$ as follows.

Corollary 4.3 (Sharp condition for $J(u_0, v_0) = d$). *Assume that p satisfies* (H), $(u_0, v_0) \in H_0^1(\Omega) \times H_0^1(\Omega)$ *and* $J(u_0, v_0) = d$. *Then when* $I(u_0, v_0) \geqslant 0$, *the problem* (1.1) *admits a global weak solution* $(u(t), v(t)) \in L^\infty(0, \infty; H_0^1(\Omega) \times H_0^1(\Omega))$ *with* $(u_t(t), v_t(t)) \in L^2(0, \infty; L^2(\Omega) \times L^2(\Omega))$ *and* $(u(t), v(t)) \in \overline{W} = W \cup \partial W$ *for* $0 \leqslant t < \infty$; *when* $I(u_0, v_0) < 0$, *the problem does not admit any global weak solution.*

Then we show the long time behavior of the solution for the problem (1.1) with the critical initial condition $J(u_0, v_0) = d$.

Theorem 4.4 (Asymptotic behavior for $J(u_0, v_0) = d$). *Assume that p satisfies* (H), $(u_0, v_0) \in H_0^1(\Omega) \times H_0^1(\Omega)$, $J(u_0, v_0) = d$ *and* $I(u_0, v_0) > 0$. *Then for the global weak solution* (u, v) *of the problem* (1.1), *there exist constants* $C > 0$, $t_0 > 0$ *and* $\lambda > 0$ *such that*

$$\|u\|_2^2 + \|v\|_2^2 \leqslant C\mathrm{e}^{-\lambda t}, \quad t_0 \leqslant t < \infty. \tag{4.10}$$

Proof. First, Theorem 4.1 gives the global weak solution for the problem (1.1). Next, we shall show that if $(u(t), v(t))$ is a global weak solution of the problem (1.1) with $J(u_0, v_0) = d$, $I(u_0, v_0) > 0$, one sees $I(u, v) > 0$ for any $t > 0$. Let us suppose by contradiction that $t_1 > 0$ is the first time such that $I(u(t_1), v(t_1)) = 0$. By the definition of the mountain pass level d in (2.6), we see $J(u(t_1), v(t_1)) \geqslant d$. Meanwhile, (2.12) gives

$$J(u(t_1), v(t_1)) \leqslant d - \int_0^{t_1} (\|u_\tau\|_2^2 + \|v_\tau\|_2^2) d\tau \leqslant d. \tag{4.11}$$

Then from (4.11) we get $\int_0^{t_1} (\|u_\tau\|_2^2 + \|v_\tau\|_2^2) = 0$, i.e., $u_t \equiv 0$ and $v_t \equiv 0$ for $0 \leqslant t \leqslant t_1$, which contradicts $I(u_0, v_0) > 0$. Hence we have $I(u, v) > 0$ for $0 \leqslant t < \infty$.

From the continuity of $J(u, v)$ and $I(u, v)$ with respect to t, we reset the initial time to a sufficiently small t_0 such that $0 < J(u(t_0), v(t_0)) < d$ and $I(u(t_0), v(t_0)) > 0$. Hence, by Theorem 3.5, we obtain the conclusion. □

5 High initial energy $J(u_0, v_0) > d$

First, we should recall some simple results. Let

$$u(t) = S(t)u_0 + \int_0^t S(t-s) f(u(s), v(s)) ds,$$
$$v(t) = S(t)v_0 + \int_0^t S(t-s) g(u(s), v(s)) ds$$

and

$$\mathbb{K} = \{(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega) \mid u \geqslant 0, v \geqslant 0 \ \text{ a.e. in } \Omega\},$$

where $S(t)$ is the heat semigroup generator [11].

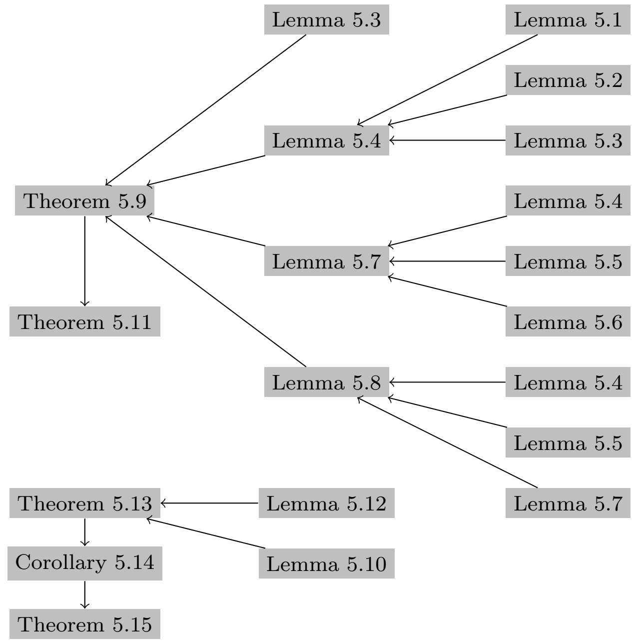

Figure 1 The relationship among conclusions in Section 5

For any $u \in H_0^1(\Omega)$, its positive part and negative part are defined as follows:

$$u^+(x) := \max\{u(x), 0\}, \quad u^-(x) := \min\{u(x), 0\}.$$

In this section, we will give four theorems, ten lemmas and one corollary. In order to well organize these conclusions to clear show their connections, this section is divided into three subsections according to their different aspects of the contents, and also the relationship among these conclusions is shown in Figure 1.

Lemma 5.1 (Grönwall's inequality). *Let $y(t) \in L^1[0,T]$ and $y(0) = a$. If there exists a constant b such that $\frac{d}{dt}y(t) \leqslant by(t)$, then $y(t) \leqslant a\mathrm{e}^{bt}$.*

Lemma 5.2 (See [13, 29]). *The function $T^* : H_0^1(\Omega) \to (0,\infty]$ is continuous. Moreover, for all $u_0 \in H_0^1(\Omega)$ and for all $t \in (0, T^*(u_0))$, the semigroup $S(t)$ maps an $H_0^1(\Omega)$ neighborhood of u_0 continuously into $C_0^1(\Omega)$.*

From [29, 32], we can conclude the following result.

Lemma 5.3. *Assume that $(u_0, v_0) \in \mathcal{G}$. Then there exists a solution*

$$\left(S(t)u_0 + \int_0^t S(t-s)f(u(s),v(s))ds, S(t)v_0 + \int_0^t S(t-s)g(u(s),v(s))ds\right)$$

of (1.1) *which converges to the solution of* (2.1).

5.1 Comparison principle

In this subsection, we prove the comparison principle of the problem (1.1) in order to facilitate the description of the structure of the manifolds for the initial data.

Lemma 5.4 (Comparison principle). *Let $(\widetilde{u}_0, \widetilde{v}_0)$, $(u_0, v_0) \in H_0^1(\Omega) \times H_0^1(\Omega)$ be such that*

$$(\widetilde{u}_0 - u_0, \widetilde{v}_0 - v_0) \in \mathbb{K}.$$

Then $(S(t)\widetilde{u}_0 - S(t)u_0, S(t)\widetilde{v}_0 - S(t)v_0) \in \mathbb{K}$ for all $t \geqslant 0$.

Moreover, if $u_0 \not\equiv \widetilde{u}_0$, $v_0 \not\equiv \widetilde{v}_0$, then we have that for $t > 0$,

$$\widetilde{u}(t) - u(t) = S(t)\widetilde{u}_0 - S(t)u_0 > 0$$

and

$$\widetilde{v}(t)-v(t)=S(t)\widetilde{v}_0-S(t)v_0>0.$$

Proof. It is well known that $C_c^{\infty}(\Omega)$ is the dense subspace of $H_0^1(\Omega)$. In order to get the comparison principle for general initial data in $H_0^1(\Omega)$, we first prove the statement for $u_0,v_0,\widetilde{u}_0,\widetilde{v}_0\in C_c^{\infty}(\Omega)$ so that $u,v,\widetilde{u},\widetilde{v}\in C(\overline{\Omega}\times[0,\overline{T}])$. We recall

$$\widetilde{u}(t):=S(t)\widetilde{u}_0+\int_0^t S(t-s)(((|u(s)|^{2p}+|v(s)|^{p+1}|u(s)|^{p-1})u(s))ds,$$
$$u(t):=S(t)u_0+\int_0^t S(t-s)(((|u(s)|^{2p}+|v(s)|^{p+1}|u(s)|^{p-1})u(s))ds,$$
$$\widetilde{v}(t):=S(t)\widetilde{v}_0+\int_0^t S(t-s)(((|v(s)|^{2p}+|u(s)|^{p+1}|v(s)|^{p-1})v(s))ds$$

and

$$v(t):=S(t)v_0+\int_0^t S(t-s)(((|v(s)|^{2p}+|u(s)|^{p+1}|v(s)|^{p-1})v(s))ds.$$

For maximum existence time $\overline{T}:=\min\{T(u_0,v_0),\ T(\widetilde{u}_0,\widetilde{v}_0)\}$, let $\omega:=\widetilde{u}-u$ and $\varphi:=\widetilde{v}-v$. From (1.1), we have

$$\begin{cases}\omega_t-\Delta\omega=((|\widetilde{u}|^{2p}+|\widetilde{v}|^{p+1}|\widetilde{u}|^{p-1})\widetilde{u}-(|u|^{2p}+|v|^{p+1}|u|^{p-1})u), & x\in\Omega\times(0,\overline{T}),\\ \varphi_t-\Delta\varphi=((|\widetilde{v}|^{2p}+|\widetilde{u}|^{p+1}|\widetilde{v}|^{p-1})\widetilde{v}-(|v|^{2p}+|u|^{p+1}|v|^{p-1})v), & x\in\Omega\times(0,\overline{T}),\\ \omega(0)=\widetilde{u}(0)-u(0)=\widetilde{u}_0-u_0\geqslant 0, & x\in\Omega,\\ \varphi(0)=\widetilde{v}(0)-v(0)=\widetilde{v}_0-v_0\geqslant 0, & x\in\Omega,\\ \omega=\varphi=0, & x\in\partial\Omega,\quad t\in(0,\overline{T}).\end{cases}$$

In order to prove this lemma, it is enough to prove $\omega>0$ and $\phi>0$. Firstly, we set

$$f(s,l):=(|s|^{2p}+|l|^{p+1}|s|^{p-1})s\quad\text{and}\quad g(s,l):=(|l|^{2p}+|s|^{p+1}|l|^{p-1})l.$$

Then for $\theta\in(0,1)$ we have

$$\begin{aligned}f(\widetilde{u},\widetilde{v})-f(u,v)=\omega&\int_0^1((2p+1)|u+\theta(\widetilde{u}-u)|^{2p}+p|v|^{p+1}|u+\theta(\widetilde{u}-u)|^{p-1})d\theta\\ &+(p+1)\varphi\int_0^1|v+\theta(\widetilde{v}-v)|^p|\widetilde{u}|^{p-1}\widetilde{u}d\theta\end{aligned}\tag{5.1}$$

and

$$\begin{aligned}g(\widetilde{u},\widetilde{v})-g(u,v)=\varphi&\int_0^1((2p+1)|v+\theta(\widetilde{v}-v)|^{2p}+p|u|^{p+1}|v+\theta(\widetilde{v}-v)|^{p-1})d\theta\\ &+(p+1)\omega\int_0^1|u+\theta(\widetilde{u}-u)|^p|\widetilde{v}|^{p-1}\widetilde{v}d\theta.\end{aligned}\tag{5.2}$$

For $x\in\Omega$, $t\geqslant 0$, we define

$$h_{11}(x,t):=\int_0^1((2p+1)|u+\theta(\widetilde{u}-u)|^{2p}+p|v|^{p+1}|u+\theta(\widetilde{u}-u)|^{p-1})d\theta\geqslant 0,$$
$$h_{12}(x,t):=\int_0^1(p+1)|v+\theta(\widetilde{v}-v)|^p|\widetilde{u}|^{p-1}\widetilde{u}d\theta\geqslant 0,$$
$$h_{21}(x,t):=\int_0^1((2p+1)|v+\theta(\widetilde{v}-v)|^{2p}+p|u|^{p+1}|v+\theta(\widetilde{v}-v)|^{p-1})d\theta\geqslant 0,$$
$$h_{22}(x,t):=\omega\int_0^1(p+1)|u+\theta(\widetilde{u}-u)|^p|\widetilde{v}|^{p-1}\widetilde{v}d\theta\geqslant 0.$$

Then we have

$$\omega_t - \Delta\omega = h_{11}(t)\omega + h_{12}(t)\varphi, \quad (x,t) \in \Omega \times (0, \overline{T}), \tag{5.3}$$
$$\varphi_t - \Delta\varphi = h_{21}(t)\varphi + h_{22}(t)\omega, \quad (x,t) \in \Omega \times (0, \overline{T}), \tag{5.4}$$
$$\omega(0) = \widetilde{u}(0) - u(0) = \widetilde{u}_0 - u_0 \geqslant 0, \quad x \in \Omega, \tag{5.5}$$
$$\varphi(0) = \widetilde{v}(0) - v(0) = \widetilde{v}_0 - v_0 \geqslant 0, \quad x \in \Omega, \tag{5.6}$$
$$\omega = \varphi = 0, \quad (x,t) \in \partial\Omega \times (0, \overline{T}). \tag{5.7}$$

Since u, v, $\widetilde{u}$ and $\widetilde{v}$ are all continuous functions, we know that

$$M_{1T} := \sup_{\Omega\times(0,T)} h_{11}(x,t) < \infty,$$
$$M_{2T} := \sup_{\Omega\times(0,T)} h_{12}(x,t) < \infty,$$
$$M_{3T} := \sup_{\Omega\times(0,T)} h_{21}(x,t) < \infty$$

and

$$M_{4T} := \sup_{\Omega\times(0,T)} h_{22}(x,t) < \infty.$$

We multiply (5.3) by ω^- and integrate on Ω, and then

$$\int_\Omega \omega_t\omega^- dx = \int_\Omega \Delta\omega\omega^- dx + \int_\Omega h_{11}(t)\omega\omega^- dx + \int_\Omega h_{12}(t)\varphi\omega^- dx,$$

which gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\omega^-\|_2^2 &= -\|\nabla\omega^-\|_2^2 + \int_\Omega h_{11}(t)|\omega^-|^2\, dx + \int_\Omega h_{12}(t)\varphi\omega^- dx \\ &\leqslant M_{1T}\|\omega^-\|_2^2 + \int_\Omega h_{12}(t)(\varphi^+\omega^- + \varphi^-\omega^-)dx \\ &\leqslant M_{1T}\|\omega^-\|_2^2 + M_{2T}\int_\Omega \varphi^-\omega^- dx \\ &\leqslant M_{1T}\|\omega^-\|_2^2 + M_{2T}\|\varphi^-\|_2\|\omega^-\|_2 \\ &\leqslant M_{1T}\|\omega^-\|_2^2 + \frac{1}{2}M_{2T}(\|\varphi^-\|_2^2 + \|\omega^-\|_2^2).\end{aligned} \tag{5.8}$$

Similar to (5.8), we get from (5.4) that

$$\frac{1}{2}\frac{d}{dt}\|\varphi^-\|_2^2 \leqslant M_{3T}\|\varphi^-\|_2^2 + \frac{1}{2}M_{4T}(\|\omega^-\|_2^2 + \|\varphi^-\|_2^2). \tag{5.9}$$

Adding (5.8) and (5.9), we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\omega^-\|_2^2 + \|\varphi^-\|_2^2) &\leqslant \left(M_{1T} + \frac{1}{2}M_{2T} + \frac{1}{2}M_{4T}\right)\|\omega^-\|_2^2 \\ &\quad + \left(M_{3T} + \frac{1}{2}M_{2T} + \frac{1}{2}M_{4T}\right)\|\varphi^-\|_2^2 \\ &\leqslant M_T(\|\omega^-\|_2^2 + \|\varphi^-\|_2^2),\end{aligned}$$

where

$$M_T = \max\left\{\left(M_{1T} + \frac{1}{2}M_{2T} + \frac{1}{2}M_{4T}\right), \left(M_{3T} + \frac{1}{2}M_{2T} + \frac{1}{2}M_{4T}\right)\right\}.$$

By Grönwall's inequality and the arbitrariness of T, this proves that $\omega^-(t) \equiv 0$ and $\varphi^-(t) \equiv 0$. So for the initial data $u_0, \widetilde{u}_0, v_0, \widetilde{v}_0 \in C_c^\infty(\Omega)$, we have $\widetilde{u} - u \geqslant 0$ and $\widetilde{v} - v \geqslant 0$, i.e.,

$$(S(t)\widetilde{u}_0 - S(t)u_0, S(t)\widetilde{v}_0 - S(t)v_0) \in \mathbb{K},$$

where $u, v, \widetilde{u}, \widetilde{v} \in C(\overline{\Omega} \times [0, \overline{T}])$.

Next, we consider the initial data $u_0, \widetilde{u}_0, v_0, \widetilde{v}_0 \in H_0^1(\Omega)$, and pick four sequences $\{u_0^m\}, \{\widetilde{u}_0^m\}, \{v_0^m\}, \{\widetilde{v}_0^m\} \subset C_0^\infty(\Omega)$. From the denseness, we have $u_0^m \to u_0$, $\widetilde{u}_0^m \to \widetilde{u}_0$, $v_0^m \to v_0$, $\widetilde{v}_0^m \to \widetilde{v}_0$ in $H_0^1(\Omega)$ as $m \to \infty$, $u_0^m \leqslant u_0 \leqslant \widetilde{u}_0 \leqslant \widetilde{u}_0^m$ and $v_0^m \leqslant v_0 \leqslant \widetilde{v}_0 \leqslant \widetilde{v}_0^m$ in Ω for all m. Then we just need to prove $u^m \leqslant u \leqslant \widetilde{u} \leqslant \widetilde{u}^m$ and $v^m \leqslant v \leqslant \widetilde{v} \leqslant \widetilde{v}^m$, where

$$u^m(t) = S(t)u_0^m + \int_0^t S(t-s)((|u(s)|^{2p} + |v(s)|^{p+1}|u(s)|^{p-1})u(s))ds, \tag{5.10}$$

$$\widetilde{u}^m(t) = S(t)\widetilde{u}_0^m + \int_0^t S(t-s)((|u(s)|^{2p} + |v(s)|^{p+1}|u(s)|^{p-1})u(s))ds, \tag{5.11}$$

$$v^m(t) = S(t)v_0^m + \int_0^t S(t-s)((|v(s)|^{2p} + |u(s)|^{p+1}|v(s)|^{p-1})v(s))ds \tag{5.12}$$

and

$$\widetilde{v}^m(t) = S(t)\widetilde{v}_0^m + \int_0^t S(t-s)((|v(s)|^{2p} + |u(s)|^{p+1}|v(s)|^{p-1})v(s))ds. \tag{5.13}$$

Arguing by contradiction, we suppose that there exists a point $(X, T) \in \Omega \times (0, \overline{T})$ such that $u(X, T) > \widetilde{u}(X, T)$. Then by Lemma 5.2 and (5.10)–(5.11) we also have $u^m(X, T) > \widetilde{u}^m(X, T)$ for sufficiently large m. This contradicts the just proved comparison principle for smooth initial data. Then we have

$$\omega(t) = S(t)\widetilde{u}_0 - S(t)u_0 \geqslant 0.$$

Similarly we also have

$$\varphi(t) = S(t)\widetilde{v}_0 - S(t)v_0 \geqslant 0,$$

i.e.,

$$(S(t)\widetilde{u}_0 - S(t)u_0, S(t)\widetilde{v}_0 - S(t)v_0) \in \mathbb{K}.$$

Since $h_{11}(t) \geqslant 0$, $h_{12}(t) \geqslant 0$, $h_{21}(t) \geqslant 0$ and $h_{22}(t) \geqslant 0$, we get

$$\omega_t - \Delta\omega = h_{11}\omega + h_{12}\varphi \geqslant 0$$

and

$$\varphi_t - \Delta\varphi = h_{21}\varphi + h_{22}\omega \geqslant 0.$$

Therefore, from the maximum principle, if $\omega(0) > 0$, $\varphi(0) > 0$ with $\omega = 0$, $\varphi = 0$ on $\partial\Omega \times (0, \overline{T})$, together with

$$\omega_t - \Delta\omega = h_{11}\omega + h_{12}\varphi \geqslant 0$$

and

$$\varphi_t - \Delta\varphi = h_{21}\varphi + h_{22}\omega \geqslant 0,$$

we have $\omega > 0$ and $\varphi > 0$ in Ω. The comparison principle is proved. □

5.2 Stationary problem

In order to show the comparison between the nontrivial solution of (2.1) and the initial data $u_0, v_0 \in H_0^1(\Omega)$ (see Theorem 5.9), we shall first consider the characteristic of the nontrivial solution of (2.1) in Lemmas 5.5–5.8. Before this we define the first order derivatives and the second order derivatives of energy functional J as follows:

$$J_u(u, v)u := \lim_{\varepsilon \to 0} \frac{J((1+\varepsilon)u, v) - J(u, v)}{\varepsilon},$$

$$J_v(u,v)v := \lim_{\varepsilon\to0}\frac{J(u,(1+\varepsilon)v)-J(u,v)}{\varepsilon},$$
$$J_{uu}(u,v)u^2 := 2\lim_{\varepsilon\to0}\frac{J((1+\varepsilon)u,v)-J(u,v)}{\varepsilon^2}$$

and

$$J_{vv}(u,v)v^2 := 2\lim_{\varepsilon\to0}\frac{J(u,(1+\varepsilon)v)-J(u,v)}{\varepsilon^2},$$

where $J(u,v)$ is the same as (2.2).

Lemma 5.5. *If (u,v) is a nontrivial solution of* (2.1)*, then we have $J_u(u,v)u=0$, $J_v(u,v)v=0$, $J_{uu}(u,v)u^2<0$, $J_{vv}(u,v)v^2<0$ and the first eigenvalue (λ,ρ) of the eigenvalue problem*

$$-\Delta\phi-((2p+1)|u|^{2p}+p|v|^{p+1}|u|^{p-1})\phi=\lambda\phi \quad in \quad \Omega, \tag{5.14}$$
$$-\Delta\psi-((2p+1)|v|^{2p}+p|u|^{p+1}|v|^{p-1})\psi=\rho\psi \quad in \quad \Omega, \tag{5.15}$$
$$\psi=\phi=0 \quad on \quad \partial\Omega$$

is negative.

Proof. A nontrivial solution (u,v) of (2.1) satisfies

$$\|\nabla u\|_2^2=(\|u\|_{2p+2}^{2p+2}+\|uv\|_{p+1}^{p+1}),$$
$$\|\nabla v\|_2^2=(\|v\|_{2p+2}^{2p+2}+\|uv\|_{p+1}^{p+1}),$$

which implies that $I(u,v)=0$. Next, we have

$$\begin{aligned}
J_u(u,v)u &= \lim_{\varepsilon\to0}\frac{J((1+\varepsilon)u,v)-J(u,v)}{\varepsilon}\\
&\simeq \lim_{\varepsilon\to0}\Bigg(\frac{\frac12(\|(1+\varepsilon)u\|_{H_0^1}^2+\|v\|_{H_0^1}^2)}{\varepsilon}\\
&\quad-\frac{\frac{1}{2(p+1)}(\|(1+\varepsilon)u\|_{2p+2}^{2p+2}+2\|(1+\varepsilon)uv\|_{p+1}^{p+1}+\|v\|_{2p+2}^{2p+2})}{\varepsilon}\\
&\quad-\frac{\frac12(\|u\|_{H_0^1}^2+\|v\|_{H_0^1}^2)}{\varepsilon}+\frac{\frac{1}{2(p+1)}(\|u\|_{2p+2}^{2p+2}+2\|uv\|_{p+1}^{p+1}+\|v\|_{2p+2}^{2p+2})}{\varepsilon}\Bigg)\\
&= \lim_{\varepsilon\to0}\Bigg(\frac{\frac12(\|u\|_{H_0^1}^2+2\varepsilon\|u\|_{H_0^1}^2+o(\varepsilon)+\|v\|_{H_0^1}^2)}{\varepsilon}\\
&\quad-\frac{\frac{1}{2(p+1)}(\|u\|_{2p+2}^{2p+2}+(2p+2)\varepsilon\|u\|_{2p+2}^{2p+2}+o(\varepsilon))}{\varepsilon}\\
&\quad-\frac{\frac{1}{p+1}(\|uv\|_{p+1}^{p+1}+(p+1)\varepsilon\|uv\|_{p+1}^{p+1}+o(\varepsilon))}{\varepsilon}-\frac{\frac{1}{2p+2}\|v\|_{2p+2}^{2p+2}}{\varepsilon}\\
&\quad-\frac{\frac12(\|u\|_{H_0^1}^2+\|v\|_{H_0^1}^2)}{\varepsilon}+\frac{\frac{1}{2(p+1)}(\|u\|_{2p+2}^{2p+2}+2\|uv\|_{p+1}^{p+1}+\|v\|_{2p+2}^{2p+2})}{\varepsilon}\Bigg)\\
&\simeq \|\nabla u\|_2^2-(\|u\|_{2p+2}^{2p+2}+\|uv\|_{p+1}^{p+1})=0.
\end{aligned}$$

In addition, similarly we have

$$\begin{aligned}
J_v(u,v)v &= \lim_{\varepsilon\to0}\frac{J(u,(1+\varepsilon)v)-J(u,v)}{\varepsilon}\\
&= \|\nabla v\|_2^2-(\|v\|_{2p+2}^{2p+2}+\|uv\|_{p+1}^{p+1})=0.
\end{aligned}$$

It is interesting to see that (even we shall not use it in the following discussion)

$$J_uu+J_vv=I(u,v)=0.$$

Moreover, we also have the following observations:

$$
\begin{aligned}
J_{uu}(u,v)u^2 &= 2\lim_{\varepsilon\to 0}\frac{J((1+\varepsilon)u,v)-J(u,v)}{\varepsilon^2}\\
&\simeq 2\lim_{\varepsilon\to 0}\Bigg(\frac{\frac{1}{2}(\|(1+\varepsilon)u\|_{H_0^1}^2+\|v\|_{H_0^1}^2)}{\varepsilon^2}\\
&\quad-\frac{\frac{1}{2(p+1)}(\|(1+\varepsilon)u\|_{2p+2}^{2p+2}+2\|(1+\varepsilon)uv\|_{p+1}^{p+1}+\|v\|_{2p+2}^{2p+2})}{\varepsilon^2}\\
&\quad-\frac{\frac{1}{2}(\|u\|_{H_0^1}^2+\|v\|_{H_0^1}^2)}{\varepsilon^2}+\frac{\frac{1}{2(p+1)}(\|u\|_{2p+2}^{2p+2}+2\|uv\|_{p+1}^{p+1}+\|v\|_{2p+2}^{2p+2})}{\varepsilon^2}\Bigg)\\
&= 2\lim_{\varepsilon\to 0}\Bigg(\frac{\frac{1}{2}(\|u\|_{H_0^1}^2+2\varepsilon\|u\|_{H_0^1}^2+\varepsilon^2\|u\|_{H_0^1}^2+\|v\|_{H_0^1}^2)}{\varepsilon^2}\\
&\quad-\frac{\frac{1}{2(p+1)}(\|u\|_{2p+2}^{2p+2}+(2p+2)\varepsilon\|u\|_{2p+2}^{2p+2})}{\varepsilon^2}\\
&\quad-\frac{\frac{1}{2(p+1)}((p+1)(2p+1)\varepsilon^2\|u\|_{2p+2}^{2p+2}+o(\varepsilon^2))}{\varepsilon^2}\\
&\quad-\frac{\frac{1}{2(p+1)}(2\|uv\|_{p+1}^{p+1}+4(p+1)\varepsilon\|uv\|_{p+1}^{p+1})}{\varepsilon^2}\\
&\quad-\frac{\frac{1}{2(p+1)}(p(p+1)\varepsilon^2\|uv\|_{p+1}^{p+1}+o(\varepsilon^2)+\|v\|_{2p+2}^{2p+2})}{\varepsilon^2}\\
&\quad-\frac{\frac{1}{2}(\|u\|_{H_0^1}^2+\|v\|_{H_0^1}^2)}{\varepsilon^2}+\frac{\frac{1}{2(p+1)}(\|u\|_{2p+2}^{2p+2}+2\|uv\|_{p+1}^{p+1}+\|v\|_{2p+2}^{2p+2})}{\varepsilon^2}\Bigg)\\
&\simeq \|\nabla u\|_2^2-((2p+1)\|u\|_{2p+2}^{2p+2}+p\|uv\|_{p+1}^{p+1})\\
&= \|\nabla u\|_2^2-\|u\|_{2p+2}^{2p+2}-\|uv\|_{p+1}^{p+1}-(2p\|u\|_{2p+2}^{2p+2}+(p-1)\|uv\|_{p+1}^{p+1})\\
&= -2p\|u\|_{2p+2}^{2p+2}-(p-1)\|uv\|_{p+1}^{p+1}<0.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
J_{vv}(u,v)v^2 &= 2\lim_{\varepsilon\to 0}\frac{J(u,(1+\varepsilon)v)-J(u,v)}{\varepsilon^2}\\
&= \|\nabla v\|_2^2-((2p+1)\|v\|_{2p+2}^{2p+2}+p\|uv\|_{p+1}^{p+1})<0
\end{aligned}
$$

and

$$
J_{uu}u^2+J_{vv}v^2<0.
$$

As (u,v) is the nontrivial solution of the problem (2.1), together with $J_{uu}(u,v)u^2<0$ and $J_{vv}(u,v)v^2<0$, we can show that the corresponding eigenvalue of (u,v) is negative as follows:

$$
\begin{aligned}
&\|\nabla u\|_2^2-((2p+1)\|u\|_{2p+2}^{2p+2}+p\|uv\|_{p+1}^{p+1})=\lambda\|u\|_2^2<0,\\
&\|\nabla v\|_2^2-((2p+1)\|v\|_{2p+2}^{2p+2}+p\|uv\|_{p+1}^{p+1})=\rho\|v\|_2^2<0.
\end{aligned}
$$

Hence the first eigenvalue of (5.14) is negative. □

Lemma 5.6. *Let $(u_0,v_0)\in\mathcal{G}$ and put*

$$
u(t)=S(t)u_0+\int_0^t S(t-s)f(u(s),v(s))ds
$$

and

$$
v(t)=S(t)v_0+\int_0^t S(t-s)g(u(s),v(s))ds
$$

for $t \in [0, T^*(u_0, v_0))$. *Then*

$$\frac{dJ(u(t), v(t))}{dt} = -\int_\Omega u_t^2 dx - \int_\Omega v_t^2 dx \tag{5.16}$$

for all $t \in (0, T^*(u_0, v_0))$.

Proof. Multiplying the two equations in (1.1) by u_t and v_t respectively and integrating by parts, we can obtain

$$\int_\Omega u_t u_t dx + \int_\Omega \nabla u \nabla u_t dx = \int_\Omega (|u|^{2p} + |v|^{p+1}|u|^p) u u_t dx \tag{5.17}$$

and

$$\int_\Omega v_t v_t dx + \int_\Omega \nabla v \nabla v_t dx = \int_\Omega (|v|^{2p} + |u|^{p+1}|v|^p) v v_t dx. \tag{5.18}$$

Combining (2.2), (5.17) and (5.18), we can directly calculate

$$\begin{aligned} \frac{dJ(u(t), v(t))}{dt} &= \int_\Omega \nabla u \nabla u_t dx + \int_\Omega \nabla v \nabla v_t dx - \int_\Omega |u|^{2p} u u_t dx \\ &\quad - \int_\Omega |v|^{p+1}|u|^p u u_t dx - \int_\Omega |u|^{p+1}|v|^p v v_t dx - \int_\Omega |v|^{2p} v v_t dx \\ &= -\int_\Omega u_t^2 dx - \int_\Omega v_t^2 dx. \end{aligned}$$

This completes the proof. □

Lemma 5.7. *Assume that* $(u_1, v_1), (u_2, v_2) \in H_0^1(\Omega)\backslash\{0\} \times H_0^1(\Omega)\backslash\{0\}$ *solve* (2.1) *with* $u_1 \leqslant u_2$ *and* $v_1 \leqslant v_2$. *Then either* $u_1 < 0 < u_2$, $v_1 < 0 < v_2$ *or* $u_1 \equiv u_2$, $v_1 \equiv v_2$.

Proof. Suppose that $u_1 \not\equiv u_2$ and $v_1 \not\equiv v_2$. By the comparison principle, we have $u_1 < u_2$ and $v_1 < v_2$ in Ω. By Lemma 5.5, the first eigenvalues $(\lambda_{u_1}, \rho_{v_1})$ and $(\lambda_{u_2}, \rho_{v_2})$ of the eigenvalue problems

$$\begin{aligned} &-\Delta\phi - ((2p+1)|u_i|^{2p} + p|v_i|^{p+1}|u_i|^{p-1})\phi = \lambda_{u_i}\phi \quad \text{in} \quad \Omega, \\ &-\Delta\psi - ((2p+1)|v_i|^{2p} + p|u_i|^{p+1}|v_i|^{p-1})\psi = \rho_{v_i}\psi \quad \text{in} \quad \Omega, \\ &\phi = \psi = 0 \quad \text{on} \quad \partial\Omega, \quad i = 1, 2 \end{aligned}$$

are negative. Due to the proof of Lemma 5.5, the corresponding positive first eigenfunctions (e_{u_1}, e_{v_1}) and (e_{u_2}, e_{v_2}) satisfy (for $\delta > 0$)

$$\begin{aligned} &J_u(u_1, v_1)e_{u_1} = 0, \\ &J_v(u_1 + \delta e_{u_1}, v_1)e_{v_1} = 0, \\ &J_u(u_2, v_2)e_{v_2} = 0, \\ &J_v(u_2 - \delta e_{u_2}, v_2)e_{v_2} = 0, \\ &J_{uu}(u_1, v_1)e_{u_1}^2 < 0, \\ &J_{vv}(u_1 + \delta e_{u_1}, v_1)e_{v_1}^2 < 0, \\ &J_{uu}(u_2, v_2)e_{v_2}^2 < 0 \end{aligned}$$

and

$$J_{vv}(u_2 - \delta e_{u_2}, v_2)e_{v_2}^2 < 0.$$

First, we treat $J(u_1 + \delta e_{u_1}, v_1)$ as the function of $u_1 + \delta e_{u_1}$, and by Taylor's theorem for $J(\delta)$ with the remainder for the Gâteaux derivative we have

$$J(u_1 + \delta e_{u_1}, v_1) = J(u_1, v_1) + \delta J_u(u_1, v_1)e_{u_1} + \frac{\delta^2}{2} J_{uu}(u_1, v_1)e_{u_1}^2 + o(\delta^2)$$

$$= J(u_1, v_1) + \frac{\delta^2}{2} J_{uu}(u_1, v_1) e_{u_1}^2 + o(\delta^2) < J(u_1, v_1). \tag{5.19}$$

Similarly, we obtain that

$$\begin{aligned} J(u_1 + \delta e_{u_1}, v_1 + \delta e_{v_1}) &= J(u_1 + \delta e_{u_1}, v_1) + \frac{\delta^2}{2} J_{vv}(u_1 + \delta e_{u_1}, v_1) e_{v_1}^2 + o(\delta^2) \\ &< J(u_1 + \delta e_{u_1}, v_1), \end{aligned} \tag{5.20}$$

$$J(u_2 - \delta e_{u_2}, v_2) = J(u_2, v_2) + \frac{\delta^2}{2} J_{uu}(u_2, v_2) e_{u_2}^2 + o(\delta^2) < J(u_2, v_2) \tag{5.21}$$

and

$$\begin{aligned} J(u_2 - \delta e_{u_2}, v_2 - \delta e_{v_2}) &= J(u_2 - \delta e_{u_2}, v_2) + \frac{\delta^2}{2} J_{vv}(u_2 - \delta e_{u_2}, v_2) e_{v_2}^2 + o(\delta^2) \\ &< J(u_2 - \delta e_{u_2}, v_2) \end{aligned} \tag{5.22}$$

for sufficiently small $\delta > 0$. We consider the closed set

$$Q := \{(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega) \,|\, u_1 < u < u_2,\ v_1 < v < v_2 \text{ a.e. in } \Omega\}$$

and

$$m := \inf_{(u,v)\in Q} J(u, v).$$

Since for some properly small $\delta > 0$,

$$u_1 < u_1 + \delta e_{u_1} < u_2 - \delta e_{u_2} < u_2 \quad \text{and} \quad v_1 < v_1 + \delta e_{v_1} < v_2 - \delta e_{v_2} < v_2,$$

that is to say

$$(u_1 + \delta e_{u_1}, v_1 + \delta e_{v_1}) \in Q \quad \text{and} \quad (u_2 - \delta e_{u_2}, v_2 - \delta e_{v_2}) \in Q,$$

and then (5.19)–(5.22) imply that

$$m \leqslant J(u_1 + \delta e_{u_1}, v_1 + \delta e_{v_1}) < J(u_1 + \delta e_{u_1}, v_1) < J(u_1, v_1)$$

and

$$m \leqslant J(u_2 - \delta e_{u_2}, v_2 - \delta e_{v_2}) < J(u_2 - \delta e_{u_2}, v_2) < J(u_2, v_2),$$

i.e.,

$$m < \min\{J(u_1, v_1), J(u_2, v_2)\}. \tag{5.23}$$

Next, we will verify that m can be achieved by some $(\omega, \mu) \in Q$. Indeed, let $\{(\omega_n, \mu_n)\} \subset Q$ be a minimizing sequence for $J\,|_Q = J(u, v)\,|_{(u,v)\in Q}$. Then (2.2) and the characteristic of the minimizing sequence give

$$\begin{aligned} \|\nabla\omega_n\|_2^2 + \|\nabla\mu_n\|_2^2 &= 2J(\omega_n, \mu_n) + \frac{1}{p+1}(\|\omega_n\|_{2p+2}^{2p+2} + 2\|\omega_n\mu_n\|_{p+1}^{p+1} + \|\mu_n\|_{2p+2}^{2p+2}) \\ &\leqslant 2(J(u_1, v_1) + J(u_2, v_2)) + \frac{1}{p+1}(\|u_1\|_{2p+2}^{2p+2} + 2\|u_1v_1\|_{p+1}^{p+1} + \|v_1\|_{2p+2}^{2p+2} \\ &\quad + \|u_2\|_{2p+2}^{2p+2} + 2\|u_2v_2\|_{p+1}^{p+1} + \|v_2\|_{2p+2}^{2p+2}) \leqslant C, \end{aligned}$$

where $C > 0$ is a constant independent of n. Passing to a subsequence, that is to select subsequences to make $\omega_n \rightharpoonup \omega$ in $H_0^1(\Omega)$ and $\mu_n \rightharpoonup \mu$ in $H_0^1(\Omega)$ (weak convergence), we have

$$\begin{aligned} &(\omega_n, \mu_n) \to (\omega, \mu), \\ &\|\omega_n\|_{2p+2}^{2p+2} \to \|\omega\|_{2p+2}^{2p+2}, \end{aligned}$$

$$\|\omega_n\mu_n\|_{p+1}^{p+1} \to \|\omega\mu\|_{p+1}^{p+1}$$

and

$$\|\mu_n\|_{2p+2}^{2p+2} \to \|\mu\|_{2p+2}^{2p+2}.$$

We conclude that $(\omega, \mu) \in Q$ and from Fatou's lemma we also have

$$\begin{aligned} J(\omega,\mu) &\simeq \frac{1}{2}(\|\omega\|_{H_0^1}^2 + \|\mu\|_{H_0^1}^2) - \frac{1}{2p+2}(\|\omega\|_{2p+2}^{2p+2} + 2\|\omega\mu\|_{p+1}^{p+1} + \|\mu\|_{2p+2}^{2p+2}) \\ &\leqslant \frac{1}{2}\liminf_{n\to\infty}(\|\omega_n\|_{H_0^1}^2 + \|\mu_n\|_{H_0^1}^2) \\ &\quad - \frac{1}{2p+2}\liminf_{n\to\infty}(\|\omega_n\|_{2p+2}^{2p+2} + 2\|\omega_n\mu_n\|_{p+1}^{p+1} + \|\mu_n\|_{2p+2}^{2p+2}) \\ &= \liminf_{n\to\infty} J(\omega_n,\mu_n) = m. \end{aligned}$$

This forces $J(\omega,\mu) = m$ so that (ω,μ) is a minimizer for $J|_Q$.

By (5.23) we have $\omega \not\equiv u_1$, $\omega \not\equiv u_2$, $\mu \not\equiv v_1$ and $\mu \not\equiv v_2$. Moreover, from $(\omega,\mu) \in Q$, we have $u_1 < \omega < u_2$ and $v_1 < \mu < v_2$. Combining the comparison principle, for all $t \geqslant 0$ or any fixed $t = t_0$, we conclude

$$\omega(x,t) = S(t)\omega + \int_0^t S(t-s)f(\omega(s),\mu(s))ds, \tag{5.24}$$

$$\mu(x,t) = S(t)\mu + \int_0^t S(t-s)g(\omega(s),\mu(s))ds, \tag{5.25}$$

$$S(t)u_1 + \int_0^t S(t-s)f(u_1,v_1)ds < \omega(x,t) < S(t)u_2 + \int_0^t S(t-s)f(u_2,v_2)ds$$

and

$$S(t)v_1 + \int_0^t S(t-s)g(u_1,v_1)ds < \mu(x,t) < S(t)v_2 + \int_0^t S(t-s)g(u_2,v_2)ds.$$

As (u_1,v_1) and (u_2,v_2) solve (2.1) (the stationary solutions to the problem (1.1)), that is to say

$$\begin{aligned} S(t)u_1 + \int_0^t S(t-s)f(u_1,v_1)ds &= u_1, \\ S(t)v_1 + \int_0^t S(t-s)g(u_1,v_1)ds &= v_1, \\ S(t)u_2 + \int_0^t S(t-s)f(u_2,v_2)ds &= u_2 \end{aligned}$$

and

$$S(t)v_2 + \int_0^t S(t-s)g(u_2,v_2)ds = v_2,$$

i.e.,

$$u_1 < \omega(x,t) < u_2, \quad v_1 < \mu(x,t) < v_2 \quad \text{and} \quad (\omega(x,t),\mu(x,t)) \in Q.$$

By the definition of m, we obtain that

$$J(\omega(x,t),\mu(x,t)) \geqslant m \quad \text{for all} \quad t \geqslant 0. \tag{5.26}$$

On the other hand, by Lemma 5.6, we know that $t \mapsto J(\omega(x,t),\mu(x,t))$ is decreasing along nonconstant trajectories. As (5.24) and (5.25) show that $\omega(x,t)$ and $\mu(x,t)$ come from the initial data ω and μ, respectively, $J(\omega,\mu) < m$ and (5.16) imply that

$$J(\omega(x,t),\mu(x,t)) \leqslant m \quad \text{for all} \quad t \geqslant 0. \tag{5.27}$$

These two facts (5.26) and (5.27) enable us to conclude that

$$J(\omega(x,t),\mu(x,t)) = J(\omega,\mu) = m \quad \text{for all} \quad t \geqslant 0,$$

which means

$$\omega(x,t) = \omega \quad \text{and} \quad \mu(x,t) = \mu.$$

Consequently, (ω,μ) is a solution of (2.1) and by the comparison principle we have $u_1 < \omega < u_2$ and $v_1 < \mu < v_2$ in Ω. For $|\varepsilon|$ sufficiently small, we have

$$((1+\varepsilon)\omega,(1+\varepsilon)\mu) \in Q$$

such that the minimization property of ω and μ yields

$$J_{\omega\omega}\omega^2 = 2\lim_{\varepsilon\to 0}\frac{J((1+\varepsilon)\omega,\mu) - J(\omega,\mu)}{\varepsilon^2} \geqslant 0$$

and

$$J_{\mu\mu}\mu^2 = 2\lim_{\varepsilon\to 0}\frac{J(\omega,(1+\varepsilon)\mu) - J(\omega,\mu)}{\varepsilon^2} \geqslant 0.$$

By Lemma 5.5, this implies $\omega \equiv 0$, $\mu \equiv 0$ and the proof is completed. □

Before starting the next lemma, we define some sets

$$S_\pm := \left\{(u,v) \in C_0^1(\Omega)\times C_0^1(\Omega) \,\middle|\, \pm u > 0, \pm v > 0 \text{ in } \Omega;\ \pm\frac{\partial u}{\partial \nu} < 0,\ \pm\frac{\partial v}{\partial \nu} < 0 \text{ on } \partial\Omega\right\}$$

and

$$S_n := \{(u,v) \in C_0^1(\Omega)\times C_0^1(\Omega) \,|\, u(x) < 0 < u(y),\ v(x) < 0 < v(y) \text{ for some points } x,y \in \Omega\},$$

which are open and disjoint in $C_0^1(\Omega)\times C_0^1(\Omega)$.

Lemma 5.8. *Let $(u_1,v_1) \in \mathcal{G}\setminus\mathcal{G}_0$. Then*

(i) *if $\omega(u_1,v_1) \subset S_+ \cup S_n$, then $(u_2,v_2) \in \mathcal{B}$ for every $u_2 \geqslant u_1$, $v_2 \geqslant v_1$, $u_2 \not\equiv u_1$ and $v_2 \not\equiv v_1$;*

(ii) *if $\omega(u_1,v_1) \subset S_- \cup S_n$, then $(u_2,v_2) \in \mathcal{B}$ for every $u_2 \leqslant u_1$, $v_2 \leqslant v_1$, $u_2 \not\equiv u_1$ and $v_2 \not\equiv v_1$.*

Proof. From the Hopf boundary lemma, every nontrivial solution of (2.1) lies either in S_+, in S_- or S_n. We just prove (i), and (ii) is similar. Let $(u_1,v_1) \in \mathcal{G}\setminus\mathcal{G}_0$, $u_2 \geqslant u_1$, $v_2 \geqslant v_1$, $u_2 \not\equiv u_1$ and $v_2 \not\equiv v_1$. Denote

$$u(t) := S(t)u_1 + \int_0^t S(t-s)f(u_1,v_1)ds,$$
$$v(t) := S(t)v_1 + \int_0^t S(t-s)g(u_1,v_1)ds,$$
$$\hat{u}(t) := S(t)u_2 + \int_0^t S(t-s)f(u_2,v_2)ds$$

and

$$\hat{v}(t) := S(t)v_2 + \int_0^t S(t-s)g(u_2,v_2)ds.$$

From the comparison principle and the definition of $\omega(u_0,v_0)$, we have

$$\hat{u}(t) > u(t) \quad \text{and} \quad \hat{v}(t) > u(t),$$

i.e., $(u_2,v_2) \notin \mathcal{G}_0$. Arguing by contradiction to prove $(u_2,v_2) \in \mathcal{B}$, considering $(u_2,v_2) \notin \mathcal{G}_0$, we suppose that $(u_2,v_2) \notin \mathcal{G}\setminus\mathcal{G}_0$ and distinguish the following two cases:

Case (1) There are an $\varepsilon > 0$ and a sequence $t_n \to \infty$ such that

$$\|\hat{u}(x,t_n) - u(x,t_n)\|_{C^1} \geqslant \varepsilon$$

and

$$\|\hat{v}(x,t_n) - v(x,t_n)\|_{C^1} \geqslant \varepsilon$$

for all n.

Case (2) $\|\hat{u}(x,t) - u(x,t)\|_{C^1} \to 0$ and $\|\hat{v}(x,t) - v(x,t)\|_{C^1} \to 0$ as $t \to \infty$.

If Case (1) occurs, by compactness of $\omega(u_1,v_1)$ and $\omega(u_2,v_2)$, we may pass to a subsequence such that as $t_n \to \infty$: $u(t_n) \to u'$, $v(t_n) \to v'$, $\hat{u}(t_n) \to \hat{u}'$, $\hat{v}(t_n) \to \hat{v}'$ in $C_0^1(\Omega)$, where (u',v') and $(\hat{u}',\hat{v}')$ are nontrivial solutions of the problem (2.1). By the comparison principle we have $\hat{u}' \geqslant u'$ and $\hat{v}' \geqslant v'$. Here from Lemma 5.3 we remind that for $(u_1,v_1) \in \mathcal{G}$ the solution

$$\left(S(t)u_1 + \int_0^t S(t-s)f(u(s),v(s))ds, S(t)v_1 + \int_0^t S(t-s)g(u(s),v(s))ds\right)$$

of the problem (1.1) converges to the solution (u',v') of (2.1) having the sequence $\omega(u_1,v_1)$ with its subsequence $(u(t_n),v(t_n)) \to (u',v')$ in $C_0^1(\Omega)$ as $t_n \to \infty$. Due to the assumption (i) of this lemma, i.e., $\omega(u_1,v_1) \subset S_+ \cup S_n$, we have that (u',v') is not negative. Hence, Lemma 5.7 implies $\hat{u}' \equiv u'$ and $\hat{v}' \equiv v'$. But this is impossible, since

$$\|\hat{u}' - u'\|_{C^1} = \lim_{n\to\infty} \|\hat{u}(t_n) - u(t_n)\|_{C^1} \geqslant \varepsilon$$

and

$$\|\hat{v}' - v'\|_{C^1} = \lim_{n\to\infty} \|\hat{v}(t_n) - v(t_n)\|_{C^1} \geqslant \varepsilon.$$

Hence Case (1) does not hold.

We now suppose that Case (2) occurs. For every $(u_e,v_e) \in \omega(u_1,v_1)$, let $(\lambda_{u_e},\rho_{v_e})$ be the first eigenvalue of the Dirichlet eigenvalue problem

$$-\Delta\phi - ((2p+1)|u_e|^{2p} + p|v_e|^{p+1}|u_e|^{p-1})\phi = \lambda_{u_e}\phi \quad \text{in} \quad \Omega, \tag{5.28}$$
$$-\Delta\psi - ((2p+1)|v_e|^{2p} + p|u_e|^{p+1}|v_e|^{p-1})\psi = \rho_{v_e}\psi \quad \text{in} \quad \Omega, \tag{5.29}$$
$$\phi = \psi = 0 \quad \text{on} \quad \partial\Omega,$$

and let (e_{u_e},e_{v_e}) denote the unique positive L^∞ normalized eigenfunction corresponding to $(\lambda_{u_e},\rho_{v_e})$. By Lemma 5.5 and the compactness of $\omega(u_1,v_1)$ in $C_0^1(\Omega)$, we have

$$\lambda_0 := \sup_{(u,v)\in\omega(u_0,v_0)} \lambda_u < 0$$

and

$$\rho_0 := \sup_{(u,v)\in\omega(u_0,v_0)} \rho_v < 0.$$

Moreover, let $\theta \in C(\overline{\Omega})$ denote the distance function to the boundary $\partial\Omega$, i.e., $\theta(x) = \mathrm{dist}(x,\partial\Omega)$ for $x \in \Omega$. Then, again by compactness, there are $C_1 > 0$ and $C_2 > 0$ such that

$$C_1\theta(x) \leqslant e_{u_e}(x) \leqslant C_2\theta(x) \tag{5.30}$$
$$C_1\theta(x) \leqslant e_{v_e}(x) \leqslant C_2\theta(x) \quad \text{for all} \quad (u_e,v_e) \in \omega(u_1,v_1), \quad x \in \Omega. \tag{5.31}$$

Let $\omega(t) = \hat{u}(t) - u(t)$ and $\xi(t) = \hat{v}(t) - v(t)$. Then in aid of the comparison principle and the spirits of Lemma 5.4, we see that $\omega(x,t) > 0$ and $\xi(x,t) > 0$ for $x \in \Omega$, $t > 0$, and ω and ξ solve the problem

$$\omega_t = \Delta\omega + \bar{h}_{11}\omega + \bar{h}_{12}\xi, \tag{5.32}$$
$$\xi_t = \Delta\xi + \bar{h}_{21}\xi + \bar{h}_{22}\omega, \tag{5.33}$$

where

$$\bar{h}_{11}(x,t) := \int_0^1 ((2p+1)|u+s\omega|^{2p} + p|v|^{p+1}|u+s\omega|^{p-1})ds,$$
$$\bar{h}_{12}(x,t) := \int_0^1 (p+1)|v+s\xi|^p|\hat{u}|^{p-1}\hat{u}ds,$$
$$\bar{h}_{21}(x,t) := \int_0^1 ((2p+1)|v+s\xi|^{2p} + p|u|^{p+1}|v+s\xi|^{p-1})ds$$

and

$$\bar{h}_{22}(x,t) := \int_0^1 (p+1)|u+s\omega|^p|\hat{v}|^{p-1}\hat{v}ds.$$

Now fix $\tau > 0$ such that

$$C_2 \leqslant C_1 \mathrm{e}^{\frac{|\lambda_{u_e}|}{2}\tau} \tag{5.34}$$

and

$$C_2 \leqslant C_1 \mathrm{e}^{\frac{|\rho_{v_e}|}{2}\tau}, \tag{5.35}$$

which will be used in the estimate of (5.46) later. We claim that

$$\inf_{(u_e,v_e)\in\omega(u_1,v_1)} \sup_{t\leqslant s\leqslant t+\tau} \|\bar{h}_{11}(x,s) - ((2p+1)|u_e|^{2p} + p|v_e|^{p+1}|u_e|^{p-1})\|_\infty \to 0 \quad \text{as} \quad t\to\infty \tag{5.36}$$

and

$$\inf_{(u_e,v_e)\in\omega(u_1,v_1)} \sup_{t\leqslant s\leqslant t+\tau} \|\bar{h}_{21}(x,s) - ((2p+1)|u_e|^{2p} + p|v_e|^{p+1}|u_e|^{p-1})\|_\infty \to 0 \quad \text{as} \quad t\to\infty. \tag{5.37}$$

Indeed, suppose by contradiction that for a sequence $t_n \to \infty$ and some $\varepsilon > 0$ we have

$$\inf_{(u_e,v_e)\in\omega(u_1,v_1)} \sup_{t_n\leqslant s\leqslant t_n+\tau} \|\bar{h}_{11}(x,s) - ((2p+1)|u_e|^{2p} + p|v_e|^{p+1}|u_e|^{p-1})\|_\infty > \varepsilon \quad \text{for all} \quad n \tag{5.38}$$

and

$$\inf_{(u_e,v_e)\in\omega(u_1,v_1)} \sup_{t_n\leqslant s\leqslant t_n+\tau} \|\bar{h}_{21}(x,s) - ((2p+1)|u_e|^{2p} + p|v_e|^{p+1}|u_e|^{p-1})\|_\infty > \varepsilon \quad \text{for all} \quad n. \tag{5.39}$$

There exist $(u_e,v_e)\in\omega(u_1,v_1)$ and a subsequence (still denoted by t_n) such that

$$\sup_{t_n\leqslant s\leqslant t_n+\tau} \|u(s)-u_e\|_\infty \to 0 \quad \text{as} \quad n\to\infty$$

and

$$\sup_{t_n\leqslant s\leqslant t_n+\tau} \|v(s)-v_e\|_\infty \to 0 \quad \text{as} \quad n\to\infty.$$

Moreover, when $\|\omega(t)\|_{C^1} \to 0$ and $\|\xi(t)\|_{C^1} \to 0$ as $t\to\infty$ occur, we obtain that as $t_n \to \infty$,

$$\sup_{t_n\leqslant s\leqslant t_n+\tau} \|\bar{h}_{11}(x,s) - ((2p+1)|u_e|^{2p} + p|v_e|^{p+1}|u_e|^{p-1})\|_\infty \to 0$$

and

$$\sup_{t_n\leqslant s\leqslant t_n+\tau} \|\bar{h}_{21}(x,s) - ((2p+1)|v_e|^{2p} + p|u_e|^{p+1}|v_e|^{p-1})\|_\infty \to 0.$$

These contradict (5.38)–(5.39) and prove (5.36)–(5.37). We may therefore take $T_0 > 0$ such that

$$\inf_{(u_e,v_e)\in\omega(u_1,v_1)} \sup_{t\leqslant s\leqslant t+\tau} \|\bar{h}_{11}(x,s) - ((2p+1)|u_e|^{2p} + p|v_e|^{p+1}|u_e|^{p-1})\|_\infty \leqslant \frac{|\lambda_{u_e}|}{2} \tag{5.40}$$

and

$$\inf_{(u_e,v_e)\in\omega(u_1,v_1)} \sup_{t\leqslant s\leqslant t+\tau} \|\bar{h}_{21}(x,s) - ((2p+1)|v_e|^{2p} + p|u_e|^{p+1}|v_e|^{p-1})\|_\infty \leqslant \frac{|\rho_{v_e}|}{2} \tag{5.41}$$

for $t \geqslant T_0$.

Next, we claim that

$$\int_\Omega \omega(t+\tau)\theta dx \geqslant \int_\Omega \omega(t)\theta dx \quad \text{for} \quad t \geqslant T_0 \tag{5.42}$$

and

$$\int_\Omega \xi(t+\tau)\theta dx \geqslant \int_\Omega \xi(t)\theta dx \quad \text{for} \quad t \geqslant T_0. \tag{5.43}$$

Indeed, by (5.40)–(5.41) and compactness, for any $t \geqslant T_0$ we may find $(u_e, v_e) \in \omega(u_1, v_1)$ such that

$$\|\bar{h}_{11}(s,t) - ((2p+1)|u_e|^{2p} + p|v_e|^{p+1}|u_e|^{p-1})\|_\infty \leqslant \frac{|\lambda_{u_e}|}{2} \tag{5.44}$$

and

$$\|\bar{h}_{21}(s,t) - ((2p+1)|v_e|^{2p} + p|u_e|^{p+1}|v_e|^{p-1})\|_\infty \leqslant \frac{|\rho_{v_e}|}{2} \tag{5.45}$$

for all $s \in [t, t+\tau]$. Using Green's formula, (5.28) and (5.44), for $\omega(x,t)$ and $\xi(x,t)$, we have

$$\begin{aligned}
\frac{d}{ds}\int_\Omega \omega(x,s)e_{u_e}dx &= \int_\Omega (\Delta\omega(x,s)dx + \bar{h}_{11}(s)\omega(x,s) + \bar{h}_{12}\xi(x,s))e_{u_e}dx \\
&= \int_\Omega ((\Delta e_{u_e} + \bar{h}_{11}(s)e_{u_e})\omega(x,s) + \bar{h}_{12}\xi(x,s)e_{u_e})dx \\
&= \int_\Omega (\omega(x,s)e_{u_e}(\bar{h}_{11}(s) - (2p+1)|u_e|^{2p} - p|v_e|^{p+1}|u_e|^{p-1}) \\
&\quad + \bar{h}_{12}\xi(x,s)e_{u_e})dx \\
&\geqslant \frac{|\lambda_{u_e}|}{2}\int_\Omega \omega(x,s)e_{u_e}dx + \int_\Omega \bar{h}_{12}\xi(x,s)e_{u_e}dx \\
&\geqslant \frac{|\lambda_{u_e}|}{2}\int_\Omega \omega(x,s)e_{u_e}dx,
\end{aligned} \tag{5.46}$$

where

$$\int_\Omega \bar{h}_{12}\xi(x,s)e_{u_e}dx > 0, \quad s \in [t, t+\tau].$$

Similarly, for $s \in [t, t+\tau]$ we obtain

$$\frac{d}{ds}\int_\Omega \xi(x,s)e_{v_e}dx \geqslant \frac{|\rho_{v_e}|}{2}\int_\Omega \xi(x,s)e_{v_e}dx. \tag{5.47}$$

Since $\int_\Omega \xi(x,s)e_{v_e} > 0$ for $s \in [t, t+\tau]$, integrating (5.46) and (5.47) with respect to s from t to $t+\tau$, we obtain

$$\int_\Omega \omega(t+\tau)e_{u_e}dx \geqslant \mathrm{e}^{\frac{|\lambda_{u_e}|}{2}\tau}\int_\Omega \omega(t)e_{u_e}dx \tag{5.48}$$

and

$$\int_\Omega \xi(t+\tau)e_{v_e}dx \geqslant \mathrm{e}^{\frac{|\rho_{v_e}|}{2}\tau}\int_\Omega \xi(t)e_{v_e}dx. \tag{5.49}$$

Combining (5.48)–(5.49) with (5.30)–(5.31), we get

$$C_2\int_\Omega \omega(t+\tau)\theta dx \geqslant \int_\Omega \omega(t+\tau)e_{u_e}dx \geqslant \mathrm{e}^{\frac{|\lambda_{u_e}|}{2}\tau}\int_\Omega \omega(t)e_{u_e}dx \geqslant C_1\mathrm{e}^{\frac{\lambda_{u_e}}{2}\tau}\int_\Omega \omega(t)\theta dx$$

and

$$C_2\int_\Omega \xi(t+\tau)\theta dx \geqslant \int_\Omega \xi(t+\tau)e_{v_e}dx \geqslant \mathrm{e}^{\frac{|\rho_{v_e}|}{2}\tau}\int_\Omega \xi(t)e_{v_e}dx \geqslant C_1\mathrm{e}^{\frac{\rho_{v_e}}{2}\tau}\int_\Omega \xi(t)\theta dx.$$

From the relationship between C_1 and C_2 we required in (5.34)–(5.35) and the above estimates, now we obtain (5.42)–(5.43) as we claimed, which easily indicate that

$$\int_\Omega \omega(T_0+l\tau)dx \geqslant \int_\Omega \omega(T_0)\theta dx > 0 \tag{5.50}$$

and

$$\int_\Omega \xi(T_0+l\tau)dx \geqslant \int_\Omega \xi(T_0)\theta dx > 0 \tag{5.51}$$

for every $l \in \mathbb{N}$. It is obvious that (5.50)–(5.51) contradict the assumption that $\|\omega(t)\|_{C^1} \to 0$ and $\|\xi(t)\|_{C^1} \to 0$ as $t \to \infty$. The proof is finished. □

From the essence of Lemma 5.8, we obtain the following theorem.

Theorem 5.9. *Let (u,v) be a nontrivial solution of* (2.1)*, and let $u_0, v_0 \in H_0^1(\Omega)$, $u_0 \not\equiv \pm u$, $v_0 \not\equiv \pm v$.*

(i) *If $u^+ \not\equiv 0$, $v^+ \not\equiv 0$, $u_0 \geqslant u$ and $v_0 \geqslant v$, then $(u_0,v_0) \in \mathcal{B}$.*

(ii) *If $u^- \not\equiv 0$, $v^- \not\equiv 0$, $u_0 \leqslant u$ and $v_0 \leqslant v$, then $(u_0,v_0) \in \mathcal{B}$.*

(iii) *If $0 \leqslant u_0 < u$ and $0 \leqslant v_0 < v$, then $(u_0,v_0) \in \mathcal{G}_0$.*

Proof. (i) Let (u,v) be a nontrivial solution of (2.1), so that (u,v) is the stationary solution of (1.1), i.e., $(u,v) \in \mathcal{G}\backslash\mathcal{G}_0$. If $u^+ \not\equiv 0$ and $v^+ \not\equiv 0$, considering Lemma 5.8(i), we obtain that $(u_0,v_0) \in \mathcal{B}$.

(ii) Analogously, if $u^- \not\equiv 0$ and $v^- \not\equiv 0$, considering Lemma 5.8(ii), we obtain that $(u_0,v_0) \in \mathcal{B}$.

(iii) Since $0 \leqslant u_0 < u$, $0 \leqslant v_0 < v$ and by the comparison principle, we may conclude $(u_0,v_0) \in \mathcal{G}$. Therefore, from Lemma 5.3, we have

$$\left(S(t)u_0 + \int_0^t S(t-s)f(u(s),v(s))ds, S(t)v_0 + \int_0^t S(t-s)g(u(s),v(s))ds\right) \to (\hat{u},\hat{v})$$

in $H_0^1(\Omega)\times H_0^1(\Omega)$ as $t \to \infty$, where $(\hat{u},\hat{v})$ is a nontrivial solution of (2.1). By the comparison principle and combining $0 \leqslant u_0 < u$, $0 \leqslant v_0 < v$, we may conclude that $0 \leqslant \hat{u} < u$ and $0 \leqslant \hat{v} < v$. Arguing by contradiction, we suppose that $\hat{u} \not\equiv 0$, $\hat{v} \not\equiv 0$ (a nontrivial solution). Combining $0 \leqslant u_0 < u$, $0 \leqslant v_0 < v$, $u_0 \not\equiv \pm u$, $v_0 \not\equiv \pm v$ with Lemma 5.7, we infer the following two cases:

(a) $\hat{u} < 0 < u, \hat{v} < 0 < v$ or

(b) $\hat{u} \equiv u, \hat{v} \equiv v$.

On one hand, as $u_0, v_0 \geqslant 0$ and $u_0(x) \not\equiv 0$, $v_0(x) \not\equiv 0$, the comparison principle tells that Case (a) is not possible. On the other hand, $u_0(x) < u$ and $v_0(x) < v$ with the comparison principle give that $\hat{u} \not\equiv u$ and $\hat{v} \not\equiv v$, which means that (b) is not possible either. Hence $(\hat{u},\hat{v})$ is a trivial solution of (2.1), i.e., $\omega(u_0,v_0) = \{(0,0)\}$, $(u_0,v_0) \in \mathcal{G}_0$. □

5.3 Global existence and blowup with high energy

Lemma 5.10. *Let $(u_0,v_0)\in H_0^1(\Omega)\times H_0^1(\Omega)$, and for $t\in[0,T^*(u_0,v_0))$ put*

$$u(t)=S(t)u_0+\int_0^t S(t-s)f(u(s),v(s))ds$$

and

$$v(t)=S(t)v_0+\int_0^t S(t-s)g(u(s),v(s))ds.$$

Then

$$\frac{d}{dt}(\|u\|_2^2+\|v\|_2^2)=-2I(u,v)\quad \text{for all}\quad t\in(0,T^*(u_0,v_0)). \tag{5.52}$$

Proof. Multiplying (1.1) by $u(t)$ and $v(t)$ and integrating by parts, we can obtain

$$\int_\Omega u_t u dx-\int_\Omega \Delta u u dx=\int_\Omega (|u|^{2p}+|v|^{p+1}|u|^p)uudx$$

and

$$\int_\Omega v_t v dx-\int_\Omega \Delta v v dx=\int_\Omega (|v|^{2p}+|u|^{p+1}|v|^p)vvdx.$$

Then

$$\frac{1}{2}\frac{d}{dt}(\|u(t)\|_2^2+\|v(t)\|_2^2)=-\int_\Omega(|\nabla u|^2+|\nabla v|^2)dx+(\|u\|_{2p+2}^{2p+2}+2\|uv\|_{p+1}^{p+1}+\|v\|_{2p+2}^{2p+2}).$$

Hence, for any $t\in(0,T^*(u_0,v_0))$, we have

$$\frac{d}{dt}(\|u\|_2^2+\|v\|_2^2)=-2I(u,v).$$

This completes the proof. □

Theorem 5.11 (Global existence and blowup at the high energy level)**.** *For any $M>0$, there exist*

$$(u_{M_-},v_{M_-}),(u_{M_+},v_{M_+})\in\mathcal{N}_+\cap\mathbb{K}\cap(C_0^1(\Omega)\times C_0^1(\Omega))$$

with $J(u_{M_-},v_{M_-})$, $J(u_{M_+},v_{M_+})\geqslant M$ and $(u_{M_-},v_{M_-})\in\mathcal{G}_0$, $(u_{M_+},v_{M_+})\in\mathcal{B}$.

Proof. Let $M>0$ and (u,v) denote a positive solution. Assume that

$$\Omega'=\{x\in\Omega\,|\,(u,v)\in H_0^1(\Omega)\times H_0^1(\Omega), u>\varepsilon \text{ and } v>\varepsilon\}$$

is an open subset of Ω for a positive constant ε and denote $\Omega''=\Omega\backslash\Omega'$. For fixed $k>0$, we pick functions $\phi_k\in C_0^1(\Omega')$ such that

$$\|\nabla\phi_k\|_{L^2(\Omega')}\geqslant k,\quad \|\phi_k\|_{L^\infty(\Omega')}\leqslant\varepsilon,\quad \phi_k>0\quad\text{in }\Omega'\quad\text{and}\quad \phi_k\equiv 0\quad\text{in }\Omega''$$

and we define $\omega_+:=u+\phi_k$, $\omega_-:=u-\phi_k$, $\varphi_+:=v+\phi_k$ and $\varphi_-:=v-\phi_k$. It follows that $(\omega_\pm,\varphi_\pm)\in\mathbb{K}$,

$$\begin{aligned}\|\nabla\omega_\pm\|_{L^2(\Omega')}&\geqslant\|\nabla\phi_k\|_{L^2(\Omega')}-\|\nabla u\|_{L^2(\Omega')}\geqslant k-\|\nabla u\|_{L^2(\Omega')},\\ \|\nabla\varphi_\pm\|_{L^2(\Omega')}&\geqslant\|\nabla\phi_k\|_{L^2(\Omega')}-\|\nabla v\|_{L^2(\Omega')}\geqslant k-\|\nabla v\|_{L^2(\Omega')}.\end{aligned}$$

Then we have

$$\begin{aligned}J(\omega_\pm,\varphi_\pm)=&\frac{1}{2}(\|\nabla\omega_\pm\|_{L^2(\Omega)}^2+\|\nabla\varphi_\pm\|_{L^2(\Omega)}^2)-\frac{1}{2p+2}(\|\omega_\pm\|_{L^{2p+2}(\Omega)}^{2p+2}\\ &+2\|\omega_\pm\varphi_\pm\|_{L^{p+1}(\Omega)}^{p+1}+\|\varphi_\pm\|_{L^{2p+2}(\Omega)}^{2p+2})\end{aligned}$$

$$
\begin{aligned}
&\geqslant \frac{1}{2}(\|\nabla\omega_\pm\|^2_{L^2(\Omega')}+\|\nabla\varphi_\pm\|^2_{L^2(\Omega')})-\frac{1}{p+1}(\|\omega_\pm\|^{2p+2}_{L^{2p+2}(\Omega')}+\|\omega_\pm\|^{2p+2}_{L^{2p+2}(\Omega'')}\\
&\quad+\|\varphi_\pm\|^{2p+2}_{L^{2p+2}(\Omega')}+\|\varphi_\pm\|^{2p+2}_{L^{2p+2}(\Omega'')})\\
&\geqslant \frac{1}{2}(\|\nabla\omega_\pm\|^2_{L^2(\Omega')}+\|\nabla\varphi_\pm\|^2_{L^2(\Omega')})-\frac{1}{p+1}(\|\omega_\pm\|^{2p+2}_{L^{2p+2}(\Omega')}+\|u\|^{2p+2}_{L^{2p+2}(\Omega'')}\\
&\quad+\|\varphi_\pm\|^{2p+2}_{L^{2p+2}(\Omega')}+\|v\|^{2p+2}_{L^{2p+2}(\Omega'')})\\
&\geqslant \frac{1}{2}((k-\|\nabla u\|_{L^2(\Omega')})^2+(k-\|\nabla v\|_{L^2(\Omega')})^2)-\frac{1}{p+1}(\|\omega_\pm\|^{2p+2}_{L^{2p+2}(\Omega')}\\
&\quad+\varepsilon^{2p+2}|\Omega''|+\|\varphi_\pm\|^{2p+2}_{L^{2p+2}(\Omega')}+\varepsilon^{2p+2}|\Omega''|)\\
&\geqslant \frac{1}{2}((k-\|\nabla u\|_{L^2(\Omega')})^2+(k-\|\nabla v\|_{L^2(\Omega')})^2)-\frac{1}{p+1}(2\varepsilon^{2p+2}|\Omega''|\\
&\quad+(\|u\|_{L^{2p+2}(\Omega')}+\|\phi_k\|_{L^{2p+2}(\Omega')})^{2p+2}+(\|v\|_{L^{2p+2}(\Omega')}+\|\phi_k\|_{L^{2p+2}(\Omega')})^{2p+2})\\
&\geqslant \frac{1}{2}((k-\|\nabla u\|_{L^2(\Omega')})^2+(k-\|\nabla v\|_{L^2(\Omega')})^2)-\frac{1}{p+1}(2\varepsilon^{2p+2}|\Omega''|\\
&\quad+(\|u\|_{L^{2p+2}(\Omega')}+\varepsilon|\Omega'|^{\frac{1}{2p+2}})^{2p+2}+(\|v\|_{L^{2p+2}(\Omega')}+\varepsilon|\Omega'|^{\frac{1}{2p+2}})^{2p+2}).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
I(\omega_\pm,\phi_\pm)\geqslant{}&(k-\|\nabla u\|_{L^2(\Omega')})^2+(k-\|\nabla v\|_{L^2(\Omega')})^2-2((\|u\|_{L^{2p+2}(\Omega')}+\varepsilon|\Omega'|^{\frac{1}{2p+2}})^{2p+2}\\
&+(\|v\|_{L^{2p+2}(\Omega')}+\varepsilon|\Omega'|^{\frac{1}{2p+2}})^{2p+2}+2\varepsilon^{2p+2}|\Omega''|).
\end{aligned}
$$

Therefore, for a sufficiently large k we have both $J(\omega_\pm,\varphi_\pm)\geqslant M$ and $I(\omega_\pm,\varphi_\pm)>0$, and hence $(\omega_\pm,\varphi_\pm)\in\mathcal{N}_+$. For such a number k, take $(u_{M_-},v_{M_-})=(\omega_-,\varphi_-)$ and $(u_{M_+},v_{M_+})=(\omega_+,\varphi_+)$. Since $0\leqslant u_{M_-}\leqslant u$ and $0\leqslant v_{M_-}\leqslant v$, we have $(u_{M_-},v_{M_-})\in\mathcal{G}_0$ by Theorem 5.9(iii); while $u_{M_+}\geqslant u$ and $v_{M_+}\geqslant v$ we have $(u_{M_+},v_{M_+})\in\mathcal{B}$ by Theorem 5.9(i). □

Lemma 5.12. *We have $J(u,v)>0$ for any $(u,v)\in\mathcal{N}_+$. Moreover, for all $(u,v)\in\mathcal{N}$, we have $J(u,v)=\max_{\lambda\geqslant0}J(\lambda u,\lambda v)$. Finally, for any $k>0$, the set $J^k\cap\mathcal{N}_+$ is bounded in $H^1_0(\Omega)\times H^1_0(\Omega)$.*

Proof. As in Lemma 2.1, for $s>0$, we have

$$
\begin{aligned}
J(\lambda u,\lambda v)&\simeq\frac{\lambda^2}{2}(\|u\|^2_{H^1_0}+\|v\|^2_{H^1_0})-\frac{\lambda^{2p+2}}{2(p+1)}(\|u\|^{2p+2}_{2p+2}+2\|uv\|^{p+1}_{p+1}+\|v\|^{2p+2}_{2p+2}),\\
\frac{d}{d\lambda}J(\lambda u,\lambda v)&\simeq\lambda(\|u\|^2_{H^1_0}+\|v\|^2_{H^1_0})-\lambda^{2p+1}(\|u\|^{2p+2}_{2p+2}+2\|uv\|^{p+1}_{p+1}+\|v\|^{2p+2}_{2p+2})
\end{aligned}
$$

and there exists a λ^* such that $\frac{d}{ds}J(su,sv)\,|_{s=s^*}=0$. For $(u,v)\in\mathcal{N}_+$, we have

$$
I(u,v)=\|\nabla u\|^2_2+\|\nabla v\|^2_2-(\|u\|^{2p+2}_{2p+2}+2\|uv\|^{p+1}_{p+1}+\|v\|^{2p+2}_{2p+2})>0.
$$

Then

$$
\begin{aligned}
J(u,v)&=\frac{1}{2}(\|\nabla u\|^2_2+\|\nabla v\|^2_2)-\frac{1}{2p+2}(\|u\|^{2p+2}_{2p+2}+2\|uv\|^{p+1}_{p+1}+\|v\|^{2p+2}_{2p+2})\\
&>\frac{1}{2p+2}I(u,v)+\frac{p}{2p+2}(\|\nabla u\|^2_2+\|\nabla v\|^2_2)>0.
\end{aligned}
$$

For $(u,v)\in\mathcal{N}$, we get

$$
I(u,v)=\|\nabla u\|^2_2+\|\nabla v\|^2_2-(\|u\|^{2p+2}_{2p+2}+2\|uv\|^{p+1}_{p+1}+\|v\|^{2p+2}_{2p+2})=0.
$$

Hence

$$
\frac{d}{d\lambda}J(\lambda u,\lambda v)\simeq\lambda(\|u\|^2_{H^1_0}+\|v\|^2_{H^1_0})-\lambda^{2p+1}(\|u\|^{2p+2}_{2p+2}+2\|uv\|^{p+1}_{p+1}+\|v\|^{2p+2}_{2p+2})=0,
$$

which implies that $\lambda = 1$ and $J(u,v) = \max_{\lambda\geqslant 0} J(\lambda u, \lambda v)$ for any $(u,v) \in \mathcal{N}$. Since $J(u,v) < k$ and $I(u,v) > 0$, we get

$$\begin{aligned} k > J(u,v) &= \frac{1}{2}(\|\nabla u\|_2^2 + \|\nabla v\|_2^2) - \frac{1}{2p+2}(\|u\|_{2p+2}^{2p+2} + 2\|uv\|_{p+1}^{p+1} + \|v\|_{2p+2}^{2p+2}) \\ &> \frac{1}{2p+2} I(u,v) + \frac{p}{2p+2}(\|\nabla u\|_2^2 + \|\nabla v\|_2^2) \\ &> \frac{p}{2p+2}(\|\nabla u\|_2^2 + \|\nabla v\|_2^2), \end{aligned}$$

which yields

$$\|\nabla u\|_2^2 + \|\nabla v\|_2^2 < \frac{2p+2}{p} k.$$

Then for any $k > 0$, the set $J^k \cap \mathcal{N}_+$ is bounded in $H_0^1(\Omega) \times H_0^1(\Omega)$. □

Theorem 5.13. *If $(u_0, v_0) \in \mathcal{N}_+$ and $\|u_0\|_2^2 + \|v_0\|_2^2 \leqslant \lambda_{J(u_0,v_0)}$, then $(u_0,v_0) \in \mathcal{G}_0$. If $(u_0,v_0) \in \mathcal{N}_-$ and $\|u_0\|_2^2 + \|v_0\|_2^2 \geqslant \Lambda_{J(u_0,v_0)}$, then $(u_0,v_0) \in \mathcal{B}$.*

Proof. Let

$$u(t) := S(t)u_0 + \int_0^t S(t-s) f(u,v) ds$$

and

$$v(t) := S(t)v_0 + \int_0^t S(t-s) g(u,v) ds$$

for $t \in [0, T(u_0,v_0))$. As (5.16), $u_t \not\equiv 0$ and $v_t \not\equiv 0$ give

$$\frac{dJ(u,v)}{dt} = -\int_\Omega u_t^2 dx - \int_\Omega v_t^2 dx < 0,$$

and then

$$J(u(t),v(t)) < J(u_0,v_0) \quad \text{for all} \quad t \in (0,T). \tag{5.53}$$

Assume first that $(u_0,v_0) \in \mathcal{N}_+$ satisfies

$$\|u_0\|_2^2 + \|v_0\|_2^2 \leqslant \lambda_{J(u_0,v_0)}.$$

We claim that $(u(t),v(t)) \in \mathcal{N}_+$ for all $t \in [0,T)$. By contradiction, if there is an $s > 0$ such that $(u(t),v(t)) \in \mathcal{N}_+$ for $0 \leqslant t < s$ and $(u(s),v(s)) \in \mathcal{N}$, then (5.52) and (5.53) imply

$$\|u(s)\|_2^2 + \|v(s)\|_2^2 < \|u_0\|_2^2 + \|v_0\|_2^2 \leqslant \lambda_{J(u_0,v_0)}$$

and

$$J(u(s),v(s)) < J(u_0,v_0),$$

which contradict the definition of $\lambda_{J(u_0,v_0)}$ and prove the claim. Hence, Lemma 5.12 shows that the orbit $\{(u(t),v(t))\}$ remains bounded in $H_0^1(\Omega) \times H_0^1(\Omega)$ for $t \in [0,T)$ so that $T = \infty$. Now for every $(\omega,\varphi) \in \omega(u_0,v_0)$, by (5.16) and (5.52) we get

$$\|\omega\|_2^2 + \|\varphi\|_2^2 < \lambda_{J(u_0,v_0)} \quad \text{and} \quad J(\omega,\varphi) \leqslant J(u_0,v_0).$$

By the definition of $\lambda_{J(u_0,v_0)}$, $(u_0,v_0) \in \mathcal{N}_+$ and the definition of $\omega(u_0,v_0)$, we get $\omega(u_0,v_0) \subset \mathcal{N}_+$. Hence, we conclude $\omega(u_0,v_0) \cap \mathcal{N} = \emptyset$. As $\mathcal{N}$ includes the nontrivial solutions of the problem (2.1), we know $\omega(u_0,v_0) = \{(0,0)\}$. In other words, $(u_0,v_0) \in \mathcal{G}_0$, as claimed.

Next, we consider the case that $(u_0, v_0) \in \mathcal{N}_-$ and $\|u_0\|_2^2 + \|v_0\|_2^2 \geqslant \Lambda_{J(u_0,v_0)}$. We claim that $(u(t), v(t)) \in \mathcal{N}_-$ for all $t \in [0, T)$. By contradiction, if there is an $s > 0$ such that $(u(t), v(t)) \in \mathcal{N}_-$ for $0 \leqslant t < s$ and $(u(s), v(s)) \in \mathcal{N}$, then by (5.52), we have

$$\frac{d}{dt}(\|u\|_2^2 + \|v\|_2^2) = -2I(u,v) > 0, \quad 0 \leqslant t < s. \tag{5.54}$$

Furthermore, from the above combined with (5.53), we have

$$\|u(s)\|_2^2 + \|v(s)\|_2^2 > \|u_0\|_2 + \|v_0\|_2^2 \geqslant \Lambda_{J(u_0,v_0)}$$

and

$$J(u(s), v(s)) < J(u_0, v_0),$$

which contradict the definition of $\Lambda_{J(u_0,v_0)}$. Hence for every $(\omega, \varphi) \in \omega(u_0, v_0)$, $T = \infty$, we then infer that $\omega(u_0, v_0) \cap \mathcal{N} = \emptyset$. However, since $\operatorname{dist}(0, \mathcal{N}_-) > 0$, we also have $(0, 0) \notin \omega(u_0, v_0)$. This gives $\omega(u_0, v_0) = \emptyset$, contrary to the assumption that $(u(t), v(t))$ is a global solution. We conclude that $T < \infty$. □

Corollary 5.14. *If* $(u_0, v_0) \in \mathcal{N}_-$ *and*

$$\|u_0\|_2^{2p+2} + \|v_0\|_2^{2p+2} \geqslant \Upsilon_{J(u_0,v_0)} := \sup\{\|u\|_2^{2p+2} + \|v\|_2^{2p+2} \mid (u,v) \in \mathcal{N}_{J(u_0,v_0)}\},$$

then $(u_0, v_0) \in \mathcal{B}$.

Proof. We claim that $(u(t), v(t)) \in \mathcal{N}_-$ for all $t \in [0, T)$. By contradiction, suppose that there is an $s > 0$ such that $(u(t), v(t)) \in \mathcal{N}_-$ for $0 \leqslant t < s$ and $(u(s), v(s)) \in \mathcal{N}$. Hence (5.54) tells that $\|u\|_2^2 + \|v\|_2^2$ is monotonically increasing on $0 \leqslant t < s$, which also means that $\|u\|_2^{2p+2} + \|v\|_2^{2p+2}$ is monotonically increasing on $0 \leqslant t < s$, i.e.,

$$\|u(s)\|_2^{2p+2} + \|v(s)\|_2^{2p+2} > \|u_0\|_2^{2p+2} + \|v_0\|_2^{2p+2} \geqslant \Upsilon_{J(u_0,v_0)}. \tag{5.55}$$

From another perspective, (5.53) gives

$$J(u(s), v(s)) < J(u_0, v_0). \tag{5.56}$$

We can show the contradiction between (5.55) and (5.56). Due to $(u(s), v(s)) \in \mathcal{N}$ and (5.56), we see that

$$(u(s), v(s)) \in \mathcal{N}_{J(u_0,v_0)} = \mathcal{N} \cap J^{J(u_0,v_0)},$$

and then (5.55) contradicts the definition of $\Upsilon_{J(u_0,v_0)}$ immediately. This contradiction means the solution $(u(t), v(t))$ cannot go through the boundary of $\mathcal{N}_-$, i.e., $(u(t), v(t)) \in \mathcal{N}_-$ for all $t \in [0, T)$.

Next, we go to prove $T < \infty$. By the contradiction, we assume that $T = \infty$, which means $\omega(u_0, v_0) \neq \emptyset$. Hence $\omega(u_0, v_0) \subset \mathcal{N}_-$, i.e., $\omega(u_0, v_0) \cap \mathcal{N} = \emptyset$ for $T = \infty$. Since $\operatorname{dist}(0, \mathcal{N}_-) > 0$, we also have $(0, 0) \notin \omega(u_0, v_0)$. For $T = \infty$, combining $\omega(u_0, v_0) \cap \mathcal{N} = \emptyset$, $(0, 0) \notin \omega(u_0, v_0)$ and Lemma 5.3 (which tells that the solution converges into $\mathcal{N}$ as time tends to infinity), we obtain $\omega(u_0, v_0) = \emptyset$. This contradiction proves $T < \infty$. □

Theorem 5.15. *Assume that* $(u_0, v_0) \in H_0^1(\Omega) \times H_0^1(\Omega)$ *satisfies*

$$\|u_0\|_2^{2p+2} + \|v_0\|_2^{2p+2} \geqslant \frac{2(p+1)}{p}|\Omega|^p J(u_0, v_0). \tag{5.57}$$

Then $(u_0, v_0) \in \mathcal{N}_- \cap \mathcal{B}$.

Proof. By using Hölder's inequality we obtain (the same for v_0)

$$\|u_0\|_2^2 = \int_\Omega |u_0|^2 dx \leqslant \left(\int_\Omega |u_0|^{2 \cdot \frac{2p+2}{2}} dx\right)^{\frac{2}{2p+2}} \left(\int_\Omega 1 dx\right)^{\frac{2p}{2p+2}} = \|u_0\|_{2p+2}^2 |\Omega|^{\frac{2p}{2p+2}}. \tag{5.58}$$

From the above combined with (5.57), we have

$$|\Omega|^p(\|u_0\|_{2p+2}^{2p+2}+\|v_0\|_{2p+2}^{2p+2}) \geqslant \|u_0\|_2^{2p+2}+\|v_0\|_2^{2p+2} \geqslant \frac{2(p+1)}{p}|\Omega|^p J(u_0,v_0). \tag{5.59}$$

Furthermore, from (5.59) and (noting $u_0 \not\equiv 0$ and $v_0 \not\equiv 0$)

$$\begin{aligned} J(u_0,v_0) &= \frac{1}{2}(\|\nabla u_0\|_2^2+\|\nabla v_0\|_2^2) - \frac{1}{2(p+1)}(\|u_0\|_{2p+2}^{2p+2}+2\|u_0v_0\|_{p+1}^{p+1}+\|v_0\|_{2p+2}^{2p+2}) \\ &= \frac{p}{2(p+1)}(\|u_0\|_{2p+2}^{2p+2}+2\|u_0v_0\|_{p+1}^{p+1}+\|v_0\|_{2p+2}^{2p+2}) + \frac{1}{2}I(u_0,v_0) \\ &> \frac{p}{2(p+1)}(\|u_0\|_{2p+2}^{2p+2}+\|v_0\|_{2p+2}^{2p+2}) + \frac{1}{2}I(u_0,v_0), \end{aligned} \tag{5.60}$$

we have $I(u_0,v_0)<0$, i.e., $(u_0,v_0)\in\mathcal{N}_-$.

Next, we will detect the upper bound of $\|u_0\|_2^{2p+2}+\|v_0\|_2^{2p+2}$ in order to apply Corollary 5.14. Similar to (5.58), Hölder's inequality also gives (the same for v)

$$\|u\|_2^2 \leqslant \|u\|_{2p+2}^2|\Omega|^{\frac{2p}{2p+2}}.$$

Then we notice

$$\begin{aligned} J(u,v) &= \frac{1}{2}(\|\nabla u\|_2^2+\|\nabla v\|_2^2) - \frac{1}{2(p+1)}(\|u\|_{2p+2}^{2p+2}+2\|uv\|_{p+1}^{p+1}+\|v\|_{2p+2}^{2p+2}) \\ &= \frac{p}{2(p+1)}(\|u\|_{2p+2}^{2p+2}+2\|uv\|_{p+1}^{p+1}+\|v\|_{2p+2}^{2p+2}) + \frac{1}{2}I(u,v) \\ &\geqslant \frac{p}{2(p+1)}(\|u\|_{2p+2}^{2p+2}+\|v\|_{2p+2}^{2p+2}) + \frac{1}{2}I(u,v), \end{aligned}$$

and (5.59) for any $(u,v)\in\mathcal{N}_{J(u_0,v_0)}$, which yields

$$|\Omega|^{\frac{1}{p}}(\|u\|_2^{2p+2}+\|v\|_2^{2p+2}) \leqslant \|u\|_{2p+2}^{2p+2}+\|v\|_{2p+2}^{2p+2} \leqslant \frac{2(p+1)}{p}J(u_0,v_0). \tag{5.61}$$

Therefore, taking the maximum over $\mathcal{N}_{J(u_0,v_0)}$, we immediately get

$$\Upsilon_{J(u_0,v_0)} \leqslant \frac{2(p+1)}{p}|\Omega|^p J(u_0,v_0). \tag{5.62}$$

Hence (5.57) means

$$\|u_0\|_2^{2p+2}+\|v_0\|_2^{2p+2} \geqslant \Upsilon_{J(u_0,v_0)},$$

which ensures $(u_0,v_0)\in\mathcal{B}$ by Corollary 5.14. □

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