

# DG polynomial algebras and their homological properties

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**Abstract** In this paper, we introduce and study differential graded (DG for short) polynomial algebras. In brief, a DG polynomial algebra  $\mathcal{A}$  is a connected cochain DG algebra such that its underlying graded algebra  $\mathcal{A}^\#$  is a polynomial algebra  $\mathbb{K}[x_1, x_2, \dots, x_n]$  with  $|x_i| = 1$  for any  $i \in \{1, 2, \dots, n\}$ . We describe all possible differential structures on DG polynomial algebras, compute their DG automorphism groups, study their isomorphism problems, and show that they are all homologically smooth and Gorenstein DG algebras. Furthermore, it is proved that the DG polynomial algebra  $\mathcal{A}$  is a Calabi-Yau DG algebra when its differential  $\partial_{\mathcal{A}} \neq 0$  and the trivial DG polynomial algebra  $(\mathcal{A}, 0)$  is Calabi-Yau if and only if  $n$  is an odd integer.

**Keywords** DG polynomial algebra, cohomology graded algebra, homologically smooth, Gorenstein, Calabi-Yau

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## 1 Introduction

In the literature, there are many papers on the research of homological properties of connected cochain DG algebras. For example, Gorenstein properties of DG algebras are studied in [3, 5–9, 13, 16] and [20, 22]; He and Wu [12] introduced and studied Koszul connected cochain DG algebras; recently, He and Mao [11] gave a criterion for a connected cochain DG algebra to be 0-Calabi-Yau, and proved that a locally finite connected cochain DG algebra is 0-Calabi-Yau if and only if it is defined by a potential. In spite of these, it is still difficult to determine whether a given DG algebra has some good homological properties such as formality, homological smoothness, Gorensteinness and Calabi-Yau property.

Generally, the homological properties of a DG algebra are determined by the joint effects of its underlying graded algebra structure and differential structure. However, it is feasible, at least in some special cases, to determine some homological properties of a DG algebra from its underlying graded algebra. For example, it is shown in [16] that a connected cochain DG algebra  $\mathcal{B}$  is Gorenstein if its underlying graded algebra  $\mathcal{B}^\#$  is an Artin-Schelter regular algebra of dimension 2. Especially, if  $\mathcal{B}^\#$  is generated

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by degree 1 elements  $x, y$  and subject to the relation  $xy + yx = 0$ , then  $\mathcal{B}$  is a Koszul Calabi-Yau DG algebra (see [17]). Recently, DG down-up algebras are introduced and studied in [18]. It is proved that all non-trivial Noetherian DG down-up algebras are Calabi-Yau DG algebras.

This paper deals with DG polynomial algebras, which are connected cochain DG algebras whose underlying graded algebras are polynomial algebras generated by degree 1 elements. We describe all possible differential structures on such DG polynomial algebras by the following theorem (see Theorem 3.1).

**Theorem A.** *Let  $(\mathcal{A}, \partial_{\mathcal{A}})$  be a connected cochain DG algebra such that  $\mathcal{A}^{\#}$  is a polynomial graded algebra  $\mathbb{K}[x_1, x_2, \dots, x_n]$  with  $|x_i| = 1$  for any  $i \in \{1, 2, \dots, n\}$ . Then there exist some  $t_1, t_2, \dots, t_n \in \mathbb{K}$  such that  $\partial_{\mathcal{A}}$  is defined by*

$$\partial_{\mathcal{A}}(x_i) = \sum_{j=1}^n t_j x_i x_j = \sum_{j=1}^{i-1} t_j x_j x_i + t_i x_i^2 + \sum_{j=i+1}^n t_j x_i x_j,$$

for any  $i \in \{1, 2, \dots, n\}$ . Conversely, for any point  $(t_1, t_2, \dots, t_n)$  in the affine  $n$ -space  $\mathbb{A}_{\mathbb{K}}^n$ , we can define a differential  $\partial$  on  $\mathbb{K}[x_1, x_2, \dots, x_n]$  by

$$\partial(x_i) = \sum_{j=1}^{i-1} t_j x_j x_i + t_i x_i^2 + \sum_{j=i+1}^n t_j x_i x_j, \quad \forall i \in \{1, 2, \dots, n\},$$

such that  $(\mathbb{K}[x_1, x_2, \dots, x_n], \partial)$  is a DG polynomial algebra.

By Theorem A, we can define  $\mathcal{A}(t_1, t_2, \dots, t_n)$  as a cochain DG algebra such that  $\mathcal{A}(t_1, t_2, \dots, t_n)^{\#} = \mathbb{K}[x_1, x_2, \dots, x_n]$  and its differential  $\partial_{\mathcal{A}}$  is defined by

$$\partial_{\mathcal{A}}(x_i) = \sum_{j=1}^{i-1} t_j x_j x_i + t_i x_i^2 + \sum_{j=i+1}^n t_j x_i x_j, \quad \forall i \in \{1, 2, \dots, n\}.$$

Then the set

$$\Omega(x_1, x_2, \dots, x_n) = \{\mathcal{A}(t_1, t_2, \dots, t_n) \mid t_i \in \mathbb{K}, i = 1, 2, \dots, n\} \cong \mathbb{A}_{\mathbb{K}}^n.$$

To consider the homological properties of  $\mathcal{A}(t_1, t_2, \dots, t_n)$ , it is necessary to study the isomorphism problem of DG polynomial algebras. We have the following theorem (see also Theorem 4.1).

**Theorem B.** *Let  $\mathcal{A}(t_1, t_2, \dots, t_n)$  and  $\mathcal{A}(t'_1, t'_2, \dots, t'_n)$  be two points in the space  $\Omega(x_1, x_2, \dots, x_n)$ . Then the DG algebras satisfy*

$$\mathcal{A}(t_1, t_2, \dots, t_n) \cong \mathcal{A}(t'_1, t'_2, \dots, t'_n)$$

if and only if there is a matrix  $M \in \text{GL}_n(\mathbb{K})$  such that

$$(t'_1, t'_2, \dots, t'_n) = (t_1, t_2, \dots, t_n)M.$$

By Theorem B, we have only two isomorphism classes in  $\Omega(x_1, x_2, \dots, x_n)$  represented by  $\mathcal{A}(0, 0, \dots, 0)$  and  $\mathcal{A}(1, 0, \dots, 0)$  (see Corollary 4.2), and we obtain the automorphism group (see Corollary 4.3)

$$\text{Aut}_{dg}(\mathcal{A}(t_1, t_2, \dots, t_n)) \cong \{M \in \text{GL}_n(\mathbb{K}) \mid (t_1, t_2, \dots, t_n) = (t_1, t_2, \dots, t_n)M\}.$$

Generally, it is difficult to determine whether a given DG algebra has some nice homological properties. Comparatively speaking, it is much easier to compute its cohomology graded algebra. For this, we compute the cohomology graded algebra of  $\mathcal{A}(1, 0, \dots, 0)$ , which is (see Proposition 5.1)

$$\mathbb{K}[[x_2^2], [x_2 x_3], \dots, [x_2 x_n], [x_3^2], \dots, [x_3 x_n], \dots, [x_{n-1}^2], [x_{n-1} x_n], [x_n^2]].$$

It implies that any DG polynomial algebra  $\mathcal{A}(t_1, t_2, \dots, t_n)$  is a formal, homologically smooth and Gorenstein DG algebra (see Theorem 5.2).

It is natural for one to ask whether DG polynomial algebras are Calabi-Yau. In [17], it is proved that a connected cochain DG algebra  $\mathcal{A}$  is a Koszul Calabi-Yau DG algebra if  $H(\mathcal{A})$  belongs to one of the following cases:

- (a)  $H(\mathcal{A}) \cong \mathbb{K}$ ;
- (b)  $H(\mathcal{A}) = \mathbb{K}[[z]]$ ,  $z \in \ker(\partial_{\mathcal{A}}^1)$ ;
- (c)  $H(\mathcal{A}) = \frac{\mathbb{K}\langle [z_1], [z_2] \rangle}{([z_1][z_2] + [z_2][z_1])}$ ,  $z_1, z_2 \in \ker(\partial_{\mathcal{A}}^1)$ .

Recently, it is proved in [18, Proposition 6.5] that a connected cochain DG algebra  $\mathcal{A}$  is Calabi-Yau if  $H(\mathcal{A}) = \mathbb{K}[[z_1], [z_2]]$  where  $z_1 \in \ker(\partial_{\mathcal{A}}^1)$  and  $z_2 \in \ker(\partial_{\mathcal{A}}^2)$ . In addition, a connected cochain DG algebra  $\mathcal{A}$  is not Calabi-Yau if  $H(\mathcal{A}) = \mathbb{K}[[z_1], [z_2]]$ , where  $z_1 \in \ker(\partial_{\mathcal{A}}^1)$  and  $z_2 \in \ker(\partial_{\mathcal{A}}^1)$  (see [17, Theorem B]). These motivate us to consider more general cases.

We have the following two theorems (see also Theorems 6.2 and 6.4).

**Theorem C.** *Let  $\mathcal{A}$  be a connected cochain DG algebra such that*

$$H(\mathcal{A}) = \mathbb{K}[[y_1], \dots, [y_m]],$$

*for some central, cocycle and degree 2 elements  $y_1, \dots, y_m$  in  $\mathcal{A}$ . Then  $\mathcal{A}$  is a  $(-m)$ -Calabi-Yau DG algebra.*

**Theorem D.** *Let  $\mathcal{A}$  be a connected cochain DG algebra such that*

$$H(\mathcal{A}) = \mathbb{K}[[y_1], \dots, [y_m]],$$

*for some central, cocycle and degree 1 elements  $y_1, \dots, y_m$  in  $\mathcal{A}$ . Then  $\mathcal{A}$  is a Koszul, homologically smooth and Gorenstein DG algebra. Moreover,  $\mathcal{A}$  is Calabi-Yau if and only if  $n$  is an odd integer.*

With the help of Theorems C and D, we get the following conclusion (see Corollaries 6.3 and 6.5):  $\mathcal{A}(t_1, t_2, \dots, t_n)$  is a Calabi-Yau DG algebra if  $(t_1, t_2, \dots, t_n) \neq (0, 0, \dots, 0)$  and  $\mathcal{A}(0, 0, \dots, 0)$  is a Calabi-Yau DG algebra if and only if  $n$  is an odd integer.

## 2 Notation and conventions

We assume that the reader is familiar with basic definitions concerning DG homological algebras. If this is not the case, we refer to [4, 8, 19, 20] and the manuscript<sup>1)</sup> for more details on them. We begin by fixing some notation and terminologies. Some overlap with those in [18].

Throughout this paper,  $\mathbb{K}$  is an algebraically closed field of characteristic 0. For any  $\mathbb{K}$ -vector space  $V$ , we write

$$V^* = \text{Hom}_{\mathbb{K}}(V, \mathbb{K}).$$

Let  $\{e_i \mid i \in I\}$  be a basis of a finite dimensional  $\mathbb{K}$ -vector space  $V$ . We denote the dual basis of  $V$  by  $\{e_i^* \mid i \in I\}$ , i.e.,  $\{e_i^* \mid i \in I\}$  is a basis of  $V^*$  such that  $e_i^*(e_j) = \delta_{i,j}$ . For any graded vector space  $W$  and  $j \in \mathbb{Z}$ , the  $j$ -th suspension  $\Sigma^j W$  of  $W$  is a graded vector space defined by  $(\Sigma^j W)^i = W^{i+j}$ .

A cochain DG algebra is a graded  $\mathbb{K}$ -algebra  $\mathcal{A}$  together with a differential  $\partial_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{A}$  of degree 1 such that

$$\partial_{\mathcal{A}}(ab) = (\partial_{\mathcal{A}}a)b + (-1)^{|a|}a(\partial_{\mathcal{A}}b)$$

for all graded elements  $a, b \in \mathcal{A}$ . For any DG algebra  $\mathcal{A}$ , we denote  $\mathcal{A}^{\text{op}}$  as its opposite DG algebra, whose multiplication is defined as  $a \cdot b = (-1)^{|a||b|}ba$  for all graded elements  $a$  and  $b$  in  $\mathcal{A}$ . A cochain DG algebra  $\mathcal{A}$  is called non-trivial if  $\partial_{\mathcal{A}} \neq 0$ , and  $\mathcal{A}$  is said to be connected if its underlying graded algebra  $\mathcal{A}^{\#}$  is a connected graded algebra. Given a cochain DG algebra  $\mathcal{A}$ , we denote by  $\mathcal{A}^i$  its  $i$ -th

<sup>1)</sup> Avramov L L, Foxby H B, Halperin S. Differential graded homological algebra. Version from 02.07.2004, 2004

homogeneous component. The differential  $\partial_{\mathcal{A}}$  is a sequence of linear maps  $\partial_{\mathcal{A}}^i : \mathcal{A}^i \rightarrow \mathcal{A}^{i+1}$  such that  $\partial_{\mathcal{A}}^{i+1} \circ \partial_{\mathcal{A}}^i = 0$  for all  $i \in \mathbb{Z}$ . The cohomology graded algebra of  $\mathcal{A}$  is the graded algebra

$$H(\mathcal{A}) = \bigoplus_{i \in \mathbb{Z}} \frac{\ker(\partial_{\mathcal{A}}^i)}{\text{im}(\partial_{\mathcal{A}}^{i-1})}.$$

For any cocycle element  $z \in \ker(\partial_{\mathcal{A}}^i)$ , we write  $[z]$  as the cohomology class in  $H(\mathcal{A})$  represented by  $z$ . One sees that  $H(\mathcal{A})$  is a connected graded algebra if  $\mathcal{A}$  is a connected cochain DG algebra. For any connected cochain DG algebra  $\mathcal{A}$ , we denote by  $\mathfrak{m}_{\mathcal{A}}$  its maximal DG ideal

$$\dots \rightarrow 0 \rightarrow \mathcal{A}^1 \xrightarrow{\partial_{\mathcal{A}}^1} \mathcal{A}^2 \xrightarrow{\partial_{\mathcal{A}}^2} \dots \xrightarrow{\partial_{\mathcal{A}}^{n-1}} \mathcal{A}^n \xrightarrow{\partial_{\mathcal{A}}^n} \dots$$

Clearly,  $\mathbb{K}$  has a structure of DG  $\mathcal{A}$ -module via the augmentation map

$$\varepsilon : \mathcal{A} \rightarrow \mathcal{A}/\mathfrak{m}_{\mathcal{A}} = \mathbb{K}.$$

It is easy to check that the enveloping DG algebra  $\mathcal{A}^e = \mathcal{A} \otimes \mathcal{A}^{\text{op}}$  of  $\mathcal{A}$  is also a connected cochain DG algebra with  $H(\mathcal{A}^e) \cong H(\mathcal{A})^e$ , and

$$\mathfrak{m}_{\mathcal{A}^e} = \mathfrak{m}_{\mathcal{A}} \otimes \mathcal{A}^{\text{op}} + \mathcal{A} \otimes \mathfrak{m}_{\mathcal{A}^{\text{op}}}.$$

A morphism  $f : \mathcal{A} \rightarrow \mathcal{A}'$  of DG algebras is a chain map of complexes which respects multiplication and unit;  $f$  is said to be a DG algebra isomorphism (resp. quasi-isomorphism) if  $f$  (resp.  $H(f)$ ) is an isomorphism. A DG algebra isomorphism  $f$  is called a DG automorphism when  $\mathcal{A}' = \mathcal{A}$ . The set of all DG algebra automorphisms of  $\mathcal{A}$  is a group, denoted by  $\text{Aut}_{\text{dg}}(\mathcal{A})$ .

Let  $\mathcal{A}$  be a connected cochain DG algebra. The derived category of left DG modules over  $\mathcal{A}$  (DG  $\mathcal{A}$ -modules for short) is denoted by  $\text{D}(\mathcal{A})$ . A DG  $\mathcal{A}$ -module  $M$  is compact if the functor  $\text{Hom}_{\text{D}(\mathcal{A})}(M, -)$  preserves all coproducts in  $\text{D}(\mathcal{A})$ . By [19, Proposition 3.3], a DG  $\mathcal{A}$ -module is compact if and only if it admits a minimal semi-free resolution with a finite semi-basis. The full subcategory of  $\text{D}(\mathcal{A})$  consisting of compact DG  $\mathcal{A}$ -modules is denoted by  $\text{D}^c(\mathcal{A})$ .

In the rest of this section, we review some important homological properties for DG algebras. We have the following definitions.

**Definition 2.1.** Let  $\mathcal{A}$  be a connected cochain DG algebra.

- (1) If  $\dim_{\mathbb{K}} H(\text{RHom}_{\mathcal{A}}(\mathbb{K}, \mathcal{A})) = 1$ , then  $\mathcal{A}$  is called Gorenstein (see [3]).
- (2) If  $\mathcal{A}$  can be connected with the trivial DG algebra  $(H(\mathcal{A}), 0)$  by a zig-zag

$$\leftarrow \rightarrow \leftarrow \dots \rightarrow$$

of quasi-isomorphisms, then  $\mathcal{A}$  is called formal (see [14, 15]).

- (3) If  ${}_{\mathcal{A}}\mathbb{K}$ , or equivalently  ${}_{\mathcal{A}^e}\mathcal{A}$ , has a minimal semi-free resolution with a semi-basis concentrated in degree 0, then  $\mathcal{A}$  is called Koszul (see [12]).

- (4) If  ${}_{\mathcal{A}}\mathbb{K}$ , or equivalently the DG  $\mathcal{A}^e$ -module  $\mathcal{A}$  is compact, then  $\mathcal{A}$  is called homologically smooth (see [21, Corollary 2.7]).

- (5) If  $\mathcal{A}$  is homologically smooth and

$$\text{RHom}_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}^e) \cong \Sigma^{-n}\mathcal{A}$$

in the derived category  $\text{D}((\mathcal{A}^e)^{\text{op}})$  of right DG  $\mathcal{A}^e$ -modules, then  $\mathcal{A}$  is called an  $n$ -Calabi-Yau DG algebra (see [10, 23]).

The motivation of this paper is to study whether DG polynomial algebras have these homological properties mentioned in Definition 2.1.

### 3 Differential structures on polynomial algebras

In this section, we study the differential structures of DG polynomial algebras. We have the following theorem.

**Theorem 3.1.** *Let  $(\mathcal{A}, \partial_{\mathcal{A}})$  be a connected cochain DG algebra such that  $\mathcal{A}^{\#}$  is a polynomial graded algebra  $\mathbb{K}[x_1, x_2, \dots, x_n]$  with  $|x_i| = 1$  for any  $i \in \{1, 2, \dots, n\}$ . Then there exist some  $t_1, t_2, \dots, t_n \in \mathbb{K}$  such that  $\partial_{\mathcal{A}}$  is defined by*

$$\partial_{\mathcal{A}}(x_i) = \sum_{j=1}^n t_j x_i x_j = \sum_{j=1}^{i-1} t_j x_j x_i + t_i x_i^2 + \sum_{j=i+1}^n t_j x_i x_j, \quad \forall i \in \{1, 2, \dots, n\}.$$

Conversely, for any point  $(t_1, t_2, \dots, t_n) \in \mathbb{A}_{\mathbb{K}}^n$ , we can define a differential  $\partial$  on  $\mathbb{K}[x_1, x_2, \dots, x_n]$  by

$$\partial(x_i) = \sum_{j=1}^{i-1} t_j x_j x_i + t_i x_i^2 + \sum_{j=i+1}^n t_j x_i x_j, \quad \forall i \in \{1, 2, \dots, n\},$$

such that  $(\mathbb{K}[x_1, x_2, \dots, x_n], \partial)$  is a DG polynomial algebra.

*Proof.* Since the differential  $\partial_{\mathcal{A}}$  of  $\mathcal{A}$  is a  $\mathbb{K}$ -linear map of degree 1, we may let

$$\partial_{\mathcal{A}}(x_i) = \sum_{j=1}^n \sum_{k=j}^n c_{j,k}^i x_j x_k,$$

where  $c_{j,k}^i \in \mathbb{K}$  for any  $i, j \in \{1, 2, \dots, n\}$  and  $k \in \{j+1, \dots, n\}$ . By definition,  $(\mathcal{A}, \partial_{\mathcal{A}})$  is a cochain DG algebra if and only if  $\partial_{\mathcal{A}}$  satisfies the Leibniz rule and

$$\begin{cases} \partial_{\mathcal{A}} \circ \partial_{\mathcal{A}}(x_i) = 0, & \forall i \in \{1, 2, \dots, n\}, \\ \partial_{\mathcal{A}}(x_j x_k - x_k x_j) = 0, & \forall 1 \leq j < k \leq n. \end{cases} \tag{3.1}$$

Since

$$\begin{aligned} \partial_{\mathcal{A}}(x_j x_k - x_k x_j) &= \partial_{\mathcal{A}}(x_j)x_k - x_j \partial_{\mathcal{A}}(x_k) - \partial_{\mathcal{A}}(x_k)x_j + x_k \partial_{\mathcal{A}}(x_j) \\ &= 2\partial_{\mathcal{A}}(x_j)x_k - 2x_j \partial_{\mathcal{A}}(x_k) \\ &= 2 \left[ \sum_{p=1}^n \sum_{q=p}^n c_{p,q}^j x_p x_q \right] x_k - 2x_j \left[ \sum_{p=1}^n \sum_{q=p}^n c_{p,q}^k x_p x_q \right] \\ &= 2 \left[ \sum_{p=1}^n \sum_{q=p}^n c_{p,q}^j x_p x_q \right] x_k - 2x_j \left[ \sum_{q=1}^n \sum_{p=1}^q c_{p,q}^k x_p x_q \right], \\ \left[ \sum_{p=1}^n \sum_{q=p}^n c_{p,q}^j x_p x_q \right] x_k &= \sum_{p=1}^{j-1} \sum_{q=p}^{j-1} c_{p,q}^j x_p x_q x_k + \sum_{p=1}^{j-1} c_{p,j}^j x_p x_j x_k + \sum_{p=1}^{j-1} \sum_{q=j+1}^n c_{p,q}^j x_p x_q x_k + c_{j,j}^j x_j^2 x_k \\ &\quad + \sum_{q=j+1}^n c_{j,q}^j x_j x_q x_k + \sum_{p=j+1}^n \sum_{q=p}^n c_{p,q}^j x_p x_q x_k \\ &=: R_1 + R_2 \end{aligned}$$

and

$$\begin{aligned} &x_j \left[ \sum_{q=1}^n \sum_{p=1}^q c_{p,q}^k x_p x_q \right] \\ &= \sum_{q=1}^{k-1} \sum_{p=1}^q c_{p,q}^k x_j x_p x_q + \sum_{p=1}^{k-1} c_{p,k}^k x_j x_p x_k + c_{k,k}^k x_j x_k^2 + \sum_{q=k+1}^n \sum_{p=1}^{k-1} c_{p,q}^k x_j x_p x_q \end{aligned}$$

$$\begin{aligned}
& + \sum_{q=k+1}^n c_{k,q}^k x_j x_k x_q + \sum_{q=k+1}^n \sum_{p=k+1}^q c_{p,q}^k x_j x_p x_q \\
& =: S_1 + S_2,
\end{aligned}$$

where

$$\begin{cases}
R_1 = \sum_{p=1}^{j-1} \sum_{q=p}^{j-1} c_{p,q}^j x_p x_q x_k + \sum_{p=1}^{j-1} \sum_{q=j+1}^n c_{p,q}^j x_p x_q x_k + \sum_{p=j+1}^n \sum_{q=p}^n c_{p,q}^j x_p x_q x_k, \\
R_2 = \sum_{p=1}^{j-1} c_{p,j}^j x_p x_j x_k + c_{j,j}^j x_j^2 x_k + \sum_{q=j+1}^n c_{j,q}^j x_j x_q x_k, \\
S_1 = \sum_{q=1}^{k-1} \sum_{p=1}^q c_{p,q}^k x_j x_p x_q + \sum_{q=k+1}^n \sum_{p=1}^{k-1} c_{p,q}^k x_j x_p x_q + \sum_{q=k+1}^n \sum_{p=k+1}^q c_{p,q}^k x_j x_p x_q, \\
S_2 = \sum_{p=1}^{k-1} c_{p,k}^k x_j x_p x_k + c_{k,k}^k x_j x_k^2 + \sum_{q=k+1}^n c_{k,q}^k x_j x_k x_q,
\end{cases}$$

we obtain that

$$\partial_{\mathcal{A}}(x_j x_k - x_k x_j) = 0$$

if and only if

$$R_1 + R_2 - S_1 - S_2 = 0,$$

which is equivalent to

$$\begin{cases}
R_1 = 0, \\
S_1 = 0, \\
R_2 - S_2 = 0.
\end{cases}$$

Since

$$R_1 = \left[ \sum_{p=1}^{j-1} \sum_{q=p}^{j-1} c_{p,q}^j x_p x_q + \sum_{p=1}^{j-1} \sum_{q=j+1}^n c_{p,q}^j x_p x_q + \sum_{p=j+1}^n \sum_{q=p}^n c_{p,q}^j x_p x_q \right] x_k$$

and

$$S_1 = x_j \left[ \sum_{q=1}^{k-1} \sum_{p=1}^q c_{p,q}^k x_p x_q + \sum_{q=k+1}^n \sum_{p=1}^{k-1} c_{p,q}^k x_p x_q + \sum_{q=k+1}^n \sum_{p=k+1}^q c_{p,q}^k x_p x_q \right],$$

we get that  $R_1 = S_1 = 0$  if and only if

$$\begin{cases}
c_{p,q}^j = 0, & \forall p \in \{1, \dots, j-1\}, \quad q \in \{p, \dots, j-1\}, \\
c_{p,q}^j = 0, & \forall p \in \{1, \dots, j-1\}, \quad q \in \{j+1, \dots, n\}, \\
c_{p,q}^j = 0, & \forall p \in \{j+1, \dots, n\}, \quad q \in \{p, \dots, n\}
\end{cases}$$

and

$$\begin{cases}
c_{p,q}^k = 0, & \forall q \in \{1, \dots, k-1\}, \quad p \in \{1, \dots, q\}, \\
c_{p,q}^k = 0, & \forall q \in \{k+1, \dots, n\}, \quad p \in \{1, \dots, k-1\}, \\
c_{p,q}^k = 0, & \forall q \in \{k+1, \dots, n\}, \quad p \in \{k+1, \dots, q\},
\end{cases}$$

which are equivalent to

$$\begin{cases}
c_{p,q}^j = 0, & \text{if } p \neq j \text{ and } q \neq j, \\
c_{p,q}^k = 0, & \text{if } p \neq k \text{ and } q \neq k.
\end{cases}$$

Since

$$\begin{aligned}
 R_2 - S_2 &= \sum_{p=1}^{j-1} c_{p,j}^j x_p x_j x_k + c_{j,j}^j x_j^2 x_k + \sum_{q=j+1}^n c_{j,q}^j x_j x_q x_k \\
 &\quad - \left[ \sum_{p=1}^{k-1} c_{p,k}^k x_j x_p x_k + c_{k,k}^k x_j x_k^2 + \sum_{q=k+1}^n c_{k,q}^k x_j x_k x_q \right] \\
 &= \sum_{p=1}^{j-1} (c_{p,j}^j - c_{p,k}^k) x_p x_j x_k + c_{j,j}^j x_j^2 x_k + \sum_{q=k+1}^n (c_{j,q}^j - c_{k,q}^k) x_j x_k x_q \\
 &\quad + \sum_{q=j+1}^{k-1} c_{j,q}^j x_j x_q x_k + c_{j,k}^j x_j x_k^2 - c_{j,k}^k x_j^2 x_k - \sum_{p=j+1}^{k-1} c_{p,k}^k x_j x_p x_k - c_{k,k}^k x_j x_k^2 \\
 &= \sum_{p=1}^{j-1} (c_{p,j}^j - c_{p,k}^k) x_p x_j x_k + (c_{j,j}^j - c_{j,k}^k) x_j^2 x_k + \sum_{q=k+1}^n (c_{j,q}^j - c_{k,q}^k) x_j x_k x_q \\
 &\quad + \sum_{r=j+1}^{k-1} (c_{j,r}^j - c_{r,k}^k) x_j x_r x_k + (c_{j,k}^j - c_{k,k}^k) x_j x_k^2,
 \end{aligned}$$

we conclude that  $R_2 - S_2 = 0$  if and only if

$$\begin{cases} c_{p,j}^j = c_{p,k}^k, & \forall p \in \{1, 2, \dots, j-1\}, \\ c_{j,j}^j = c_{j,k}^k, \\ c_{j,q}^j = c_{k,q}^k, & \forall q \in \{k+1, k+2, \dots, n\}, \\ c_{j,r}^j = c_{r,k}^k, & \forall r \in \{j+1, j+2, \dots, k-1\}, \\ c_{j,k}^j = c_{k,k}^k, \end{cases}$$

which is equivalent to

$$\begin{cases} c_{p,j}^j = c_{p,k}^k, & \forall p \in \{1, 2, \dots, j\}, \\ c_{j,r}^j = c_{r,k}^k, & \forall r \in \{j+1, j+2, \dots, k-1\}, \\ c_{j,q}^j = c_{k,q}^k, & \forall q \in \{k, k+2, \dots, n\}. \end{cases}$$

Therefore, for any  $1 \leq j < k \leq n$ ,  $\partial_{\mathcal{A}}(x_j x_k - x_k x_j) = 0$  if and only if

$$\begin{cases} c_{p,q}^j = 0, & \text{if } p \neq j \text{ and } q \neq j, \\ c_{p,q}^k = 0, & \text{if } p \neq k \text{ and } q \neq k, \\ c_{p,j}^j = c_{p,k}^k, & \forall p \in \{1, 2, \dots, j\}, \\ c_{j,r}^j = c_{r,k}^k, & \forall r \in \{j+1, j+2, \dots, k-1\}, \\ c_{j,q}^j = c_{k,q}^k, & \forall q \in \{k, k+2, \dots, n\}. \end{cases} \tag{3.2}$$

Let  $t_i = c_{1,i}^1, i = 1, 2, \dots, n$ . Then (3.2) implies

$$\partial_{\mathcal{A}}(x_i) = \sum_{j=1}^{i-1} t_j x_j x_i + t_i x_i^2 + \sum_{j=i+1}^n t_j x_i x_j, \quad \forall i \in \{1, 2, \dots, n\}. \tag{3.3}$$

For any  $(t_1, t_2, \dots, t_n) \in \mathbb{A}_{\mathbb{K}}^n$ , we claim that  $\partial_{\mathcal{A}} \circ \partial_{\mathcal{A}}(x_i) = 0$ , if (3.3) holds,  $\forall i \in \{1, 2, \dots, n\}$ . Indeed, if (3.3) holds, then

$$\begin{aligned}
 &\partial_{\mathcal{A}} \circ \partial_{\mathcal{A}}(x_i) \\
 &= \partial_{\mathcal{A}} \left[ \sum_{j=1}^{i-1} t_j x_j x_i + t_i x_i^2 + \sum_{j=i+1}^n t_j x_i x_j \right]
 \end{aligned}$$

$$\begin{aligned}
&= \sum_{j=1}^{i-1} t_j \partial_{\mathcal{A}}(x_j) x_i - \sum_{j=1}^{i-1} t_j x_j \partial_{\mathcal{A}}(x_i) + \sum_{j=i+1}^n t_j \partial_{\mathcal{A}}(x_i) x_j - \sum_{j=i+1}^n t_j x_i \partial_{\mathcal{A}}(x_j) \\
&= \sum_{j=1}^{i-1} t_j \left[ \left( \sum_{l=1}^{j-1} t_l x_l x_j + t_j x_j^2 + \sum_{l=j+1}^n t_l x_j x_l \right) x_i - x_j \left( \sum_{l=1}^{i-1} t_l x_l x_i + t_i x_i^2 + \sum_{l=i+1}^n t_l x_i x_l \right) \right] \\
&\quad + \sum_{j=i+1}^n t_j \left[ \left( \sum_{l=1}^{i-1} t_l x_l x_i + t_i x_i^2 + \sum_{l=i+1}^n t_l x_i x_l \right) x_j - x_i \left( \sum_{l=1}^{j-1} t_l x_l x_j + t_j x_j^2 + \sum_{l=j+1}^n t_l x_j x_l \right) \right] \\
&= \sum_{j=1}^{i-1} t_j x_i \left[ \sum_{l=1}^{j-1} t_l x_l x_j + t_j x_j^2 + \sum_{l=j+1}^n t_l x_j x_l - x_j \left( \sum_{l=1}^{i-1} t_l x_l + t_i x_i + \sum_{l=i+1}^n t_l x_l \right) \right] \\
&\quad + \sum_{j=i+1}^n t_j x_i \left[ \left( \sum_{l=1}^{i-1} t_l x_l + t_i x_i + \sum_{l=i+1}^n t_l x_l \right) x_j - \left( \sum_{l=1}^{j-1} t_l x_l x_j + t_j x_j^2 + \sum_{l=j+1}^n t_l x_j x_l \right) \right].
\end{aligned}$$

Since

$$\begin{aligned}
&\sum_{j=1}^{i-1} t_j \left[ \sum_{l=1}^{j-1} t_l x_l x_j + t_j x_j^2 + \sum_{l=j+1}^n t_l x_j x_l - x_j \left( \sum_{l=1}^{i-1} t_l x_l + t_i x_i + \sum_{l=i+1}^n t_l x_l \right) \right] \\
&\quad + \sum_{j=i+1}^n t_j \left[ \left( \sum_{l=1}^{i-1} t_l x_l + t_i x_i + \sum_{l=i+1}^n t_l x_l \right) x_j - \left( \sum_{l=1}^{j-1} t_l x_l x_j + t_j x_j^2 + \sum_{l=j+1}^n t_l x_j x_l \right) \right] \\
&= \sum_{j=1}^{i-1} t_j \left[ - \sum_{l=j}^{i-1} t_l x_l x_j + t_j x_j^2 - t_i x_i x_j + \sum_{l=j+1}^i t_l x_j x_l \right] \\
&\quad + \sum_{j=i+1}^n t_j \left[ - \sum_{l=i}^{j-1} t_l x_l x_j + t_i x_i x_j + \sum_{l=i+1}^j t_l x_l x_j - t_j x_j^2 \right] \\
&= \sum_{j=1}^{i-1} t_j \left[ - \sum_{l=j+1}^{i-1} t_l x_l x_j + \sum_{l=j+1}^{i-1} t_l x_j x_l \right] + \sum_{j=i+1}^n t_j \left[ - \sum_{l=i+1}^{j-1} t_l x_l x_j + \sum_{l=i+1}^{j-1} t_l x_j x_l \right] \\
&= 0,
\end{aligned}$$

we have  $\partial_{\mathcal{A}}^2(x_i) = 0, \forall i \in \{1, 2, \dots, n\}$ .

Therefore,  $(\mathcal{A}, \partial_{\mathcal{A}})$  is a cochain DG algebra if there exist some  $t_1, t_2, \dots, t_n \in \mathbb{K}$  such that

$$\partial_{\mathcal{A}}(x_i) = \sum_{j=1}^{i-1} t_j x_j x_i + t_i x_i^2 + \sum_{j=i+1}^n t_j x_i x_j, \quad \forall i \in \{1, 2, \dots, n\}.$$

Conversely,  $\forall (t_1, t_2, \dots, t_n) \in \mathbb{A}_{\mathbb{K}}^n$ , we can define a differential  $\partial$  on  $\mathbb{K}[x_1, x_2, \dots, x_n]$  by

$$\partial(x_i) = \sum_{j=1}^{i-1} t_j x_j x_i + t_i x_i^2 + \sum_{j=i+1}^n t_j x_i x_j, \quad \forall i \in \{1, 2, \dots, n\},$$

such that  $(\mathbb{K}[x_1, x_2, \dots, x_n], \partial)$  is a DG polynomial algebra, since one can check as above that

$$\begin{cases} \partial \circ \partial(x_i) = 0, & \forall i \in \{1, 2, \dots, n\}, \\ \partial(x_j x_k - x_k x_j) = 0, & \forall 1 \leq j < k \leq n, \end{cases}$$

if  $\partial$  satisfies the Leibniz rule. □

By Theorem 3.1, the following definition is reasonable.

**Definition 3.2.** Define  $\mathcal{A}(t_1, t_2, \dots, t_n)$  as a cochain DG algebra such that

$$\mathcal{A}(t_1, t_2, \dots, t_n)^{\#} = \mathbb{K}[x_1, x_2, \dots, x_n], \quad |x_i| = 1,$$



and its differential  $\partial_{\mathcal{A}}$  is defined by

$$\partial_{\mathcal{A}}(x_i) = \sum_{j=1}^{i-1} t_j x_j x_i + t_i x_i^2 + \sum_{j=i+1}^n t_j x_i x_j, \quad \forall i \in \{1, 2, \dots, n\}.$$

Define

$$\Omega(x_1, x_2, \dots, x_n) = \{\mathcal{A}(t_1, t_2, \dots, t_n) \mid t_i \in \mathbb{K}, i = 1, 2, \dots, n\}.$$

Clearly,

$$\Omega(x_1, x_2, \dots, x_n) \cong \mathbb{A}_{\mathbb{K}}^n.$$

### 4 Isomorphism classes of DG polynomial algebras

In this section, we consider the isomorphism problem for DG polynomial algebras in  $\Omega(x_1, x_2, \dots, x_n)$ . We have the following theorem.

**Theorem 4.1.** *Let  $\mathcal{A}(t_1, t_2, \dots, t_n)$  and  $\mathcal{A}(t'_1, t'_2, \dots, t'_n)$  be two points in the space  $\Omega(x_1, x_2, \dots, x_n)$ . Then the DG algebras satisfy*

$$\mathcal{A}(t_1, t_2, \dots, t_n) \cong \mathcal{A}(t'_1, t'_2, \dots, t'_n)$$

*if and only if there is a matrix  $M \in \text{GL}_n(\mathbb{K})$  such that*

$$(t'_1, t'_2, \dots, t'_n) = (t_1, t_2, \dots, t_n)M.$$

*Proof.* We write  $\mathcal{A} = \mathcal{A}(t_1, t_2, \dots, t_n)$  and  $\mathcal{A}' = \mathcal{A}(t'_1, t'_2, \dots, t'_n)$  for simplicity. In order to distinguish, we assume that  $\mathcal{A}'^{\#} = \mathbb{K}[x'_1, x'_2, \dots, x'_n]$  with  $|x'_i| = 1$  for any  $i \in \{1, 2, \dots, n\}$ . If  $\mathcal{A} \cong \mathcal{A}'$ , then there exists an isomorphism  $f : \mathcal{A} \rightarrow \mathcal{A}'$  of DG algebras. Since  $f^1 : \mathcal{A}^1 \rightarrow \mathcal{A}'^1$  is a  $\mathbb{K}$ -linear isomorphism, we may let

$$\begin{pmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{pmatrix} = M \begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{pmatrix}$$

for some  $M = (a_{ij})_{n \times n} \in \text{GL}_n(\mathbb{K})$ . Since  $f$  is a chain map, we have  $f \circ \partial_{\mathcal{A}} = \partial_{\mathcal{A}'} \circ f$ . For any  $i \in \{1, 2, \dots, n\}$ , we have

$$\begin{aligned} \partial_{\mathcal{A}'} \circ f(x_i) &= \partial_{\mathcal{A}'} \left( \sum_{j=1}^n a_{ij} x'_j \right) \\ &= \sum_{j=1}^n a_{ij} \left( \sum_{l=1}^{j-1} t'_l x'_l x'_j + t'_j x'^2_j + \sum_{l=j+1}^n t'_l x'_j x'_l \right) \\ &= \sum_{j=1}^n a_{ij} x'_j \left( \sum_{l=1}^{j-1} t'_l x'_l + t'_j x'_j + \sum_{l=j+1}^n t'_l x'_l \right) \\ &= \left( \sum_{j=1}^n a_{ij} x'_j \right) \left( \sum_{l=1}^n t'_l x'_l \right) \\ &= \left( \sum_{l=1}^n a_{il} x'_l \right) \left( \sum_{s=1}^n t'_s x'_s \right) \end{aligned} \tag{Eq1}$$

and

$$f \circ \partial_{\mathcal{A}}(x_i) = f \left( \sum_{j=1}^{i-1} t_j x_j x_i + t_i x_i^2 + \sum_{j=i+1}^n t_j x_i x_j \right)$$

$$\begin{aligned}
&= \sum_{j=1}^{i-1} t_j \left( \sum_{l=1}^n a_{jl} x'_l \right) \left( \sum_{l=1}^n a_{il} x'_l \right) + t_i \left( \sum_{l=1}^n a_{il} x'_l \right)^2 + \sum_{j=i+1}^n t_j \left( \sum_{l=1}^n a_{il} x'_l \right) \left( \sum_{l=1}^n a_{jl} x'_l \right) \\
&= \sum_{j=1}^{i-1} t_j \left( \sum_{s=1}^n a_{js} x'_s \right) \left( \sum_{l=1}^n a_{il} x'_l \right) + t_i \left( \sum_{l=1}^n a_{il} x'_l \right)^2 + \sum_{j=i+1}^n t_j \left( \sum_{l=1}^n a_{il} x'_l \right) \left( \sum_{s=1}^n a_{js} x'_s \right) \\
&= \left( \sum_{l=1}^n a_{il} x'_l \right) \left[ \sum_{j=1}^{i-1} t_j \left( \sum_{s=1}^n a_{js} x'_s \right) + t_i \left( \sum_{l=1}^n a_{il} x'_l \right) + \sum_{j=i+1}^n t_j \left( \sum_{s=1}^n a_{js} x'_s \right) \right] \\
&= \left( \sum_{l=1}^n a_{il} x'_l \right) \left[ \sum_{j=1}^{i-1} t_j \left( \sum_{s=1}^n a_{js} x'_s \right) + t_i \left( \sum_{s=1}^n a_{is} x'_s \right) + \sum_{j=i+1}^n t_j \left( \sum_{s=1}^n a_{js} x'_s \right) \right] \\
&= \left( \sum_{l=1}^n a_{il} x'_l \right) \left[ \sum_{j=1}^n t_j \left( \sum_{s=1}^n a_{js} x'_s \right) \right] \\
&= \left( \sum_{l=1}^n a_{il} x'_l \right) \left[ \sum_{s=1}^n \left( \sum_{j=1}^n a_{js} t_j \right) x'_s \right]. \tag{Eq2}
\end{aligned}$$

Since  $(a_{ij})_{n \times n} \in \text{GL}_n(\mathbb{K})$ , we have  $(a_{i1}, a_{i2}, \dots, a_{in}) \neq (0, 0, \dots, 0)$ . So

$$\sum_{l=1}^n a_{il} x'_l \neq 0$$

and

$$\partial_{\mathcal{A}'} \circ f(x_i) = f \circ \partial_{\mathcal{A}}(x_i)$$

implies that

$$t'_s = \sum_{j=1}^n a_{js} t_j,$$

for any  $s \in \{1, 2, \dots, n\}$ . Then we get

$$(t'_1, t'_2, \dots, t'_n) = (t_1, t_2, \dots, t_n)M.$$

Conversely, if there exists a matrix  $M = (a_{ij})_{n \times n} \in \text{GL}_n(\mathbb{K})$  such that

$$(t'_1, t'_2, \dots, t'_n) = (t_1, t_2, \dots, t_n)M,$$

then

$$t'_s = \sum_{j=1}^n a_{js} t_j,$$

for any  $s \in \{1, 2, \dots, n\}$ . Hence

$$\left( \sum_{l=1}^n a_{il} x'_l \right) \left( \sum_{s=1}^n t'_s x'_s \right) = \left( \sum_{l=1}^n a_{il} x'_l \right) \left[ \sum_{s=1}^n \left( \sum_{j=1}^n a_{js} t_j \right) x'_s \right], \quad \forall i \in \{1, 2, \dots, n\}. \tag{4.1}$$

Define a linear map  $f : \mathcal{A}^1 \rightarrow \mathcal{A}'^1$  by

$$\begin{pmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{pmatrix} = M \begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{pmatrix}.$$

Obviously,  $f$  is invertible since  $M \in \text{GL}_n(\mathbb{K})$ . We have

$$f(x_i)f(x_j) = f(x_j)f(x_i), \quad \forall 1 \leq i < j \leq n,$$

since  $\mathcal{A}'$  is a commutative algebra. Hence  $f : \mathcal{A}^1 \rightarrow \mathcal{A}'^1$  can be extended to a morphism of graded algebras between  $\mathbb{K}[x_1, x_2, \dots, x_n]$  and  $\mathbb{K}[x'_1, x'_2, \dots, x'_n]$ . We still denote it by  $f$ . For any  $i \in \{1, 2, \dots, n\}$ , we still have (Eq1) and (Eq2). Then (4.1) indicates that  $f \circ \partial_{\mathcal{A}}(x_i) = \partial_{\mathcal{A}'} \circ f(x_i)$ , which implies that  $f$  is a chain map. Hence,  $f : \mathcal{A} \rightarrow \mathcal{A}'$  is an isomorphism of DG algebras.  $\square$

**Corollary 4.2.** *In the space  $\Omega(x_1, x_2, \dots, x_n)$ , there are only two isomorphism classes  $\mathcal{A}(0, 0, \dots, 0)$  and  $\mathcal{A}(1, 0, \dots, 0)$ .*

*Proof.* By Theorem 4.1,  $\mathcal{A}(t_1, t_2, \dots, t_n) \cong \mathcal{A}(t'_1, t'_2, \dots, t'_n)$  if and only if there is an  $M \in \text{GL}_n(\mathbb{K})$  such that  $(t'_1, t'_2, \dots, t'_n) = (t_1, t_2, \dots, t_n)M$ . For any  $(t_1, t_2, \dots, t_n) \neq (0, 0, \dots, 0)$ , there exists an  $i \in \{1, 2, \dots, n\}$  such that  $t_i \neq 0$ . We have

$$(t_1, t_2, \dots, t_n) = (1, 0, \dots, 0) \begin{pmatrix} t_1 & t_2 & \cdots & t_{i-1} & t_i & t_{i+1} & \cdots & t_n \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Since

$$\begin{vmatrix} t_1 & t_2 & \cdots & t_{i-1} & t_i & t_{i+1} & \cdots & t_n \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{vmatrix} = (-1)^{i+1} t_i \neq 0,$$

we have  $\mathcal{A}(t_1, t_2, \dots, t_n) \cong \mathcal{A}(1, 0, \dots, 0)$ . Clearly,  $\mathcal{A}(1, 0, \dots, 0) \neq \mathcal{A}(0, 0, \dots, 0)$  since  $(1, 0, \dots, 0) \neq (0, 0, \dots, 0)M$  for any  $M \in \text{GL}_n(\mathbb{K})$ . Therefore, there are only two isomorphism classes  $\mathcal{A}(0, 0, \dots, 0)$  and  $\mathcal{A}(1, 0, \dots, 0)$  in  $\Omega(x_1, x_2, \dots, x_n)$ .  $\square$

**Corollary 4.3.** *For any  $\mathcal{A}(t_1, t_2, \dots, t_n) \in \Omega(x_1, x_2, \dots, x_n)$ , we have*

$$\text{Aut}_{dg}(\mathcal{A}(t_1, t_2, \dots, t_n)) \cong \{M \in \text{GL}_n(\mathbb{K}) \mid (t_1, t_2, \dots, t_n) = (t_1, t_2, \dots, t_n)M\}.$$

**Remark 4.4.** We want to consider the homological properties of DG polynomial algebras. This can be reduced to studying those of  $\mathcal{A}(0, 0, \dots, 0)$  and  $\mathcal{A}(1, 0, \dots, 0)$  by Corollary 4.2. Corollary 4.3 characterizes the automorphism group of any DG polynomial algebra.

### 5 Cohomology of DG polynomial algebras

Generally, the cohomology algebra of a DG algebra usually contains much useful information on its properties (see [1, 2]). In this section, we will compute the cohomology algebra of the non-trivial DG polynomial algebra. By Corollary 4.2, we only need to compute  $H(\mathcal{A}(1, 0, \dots, 0))$ . We have the following proposition.

**Proposition 5.1.** *The cohomology algebra of  $\mathcal{A}(1, 0, \dots, 0)$  is the polynomial algebra*

$$\mathbb{K}[[x_2^2], [x_2x_3], \dots, [x_2x_n], [x_3^2], \dots, [x_3x_n], \dots, [x_{n-1}^2], [x_{n-1}x_n], [x_n^2]].$$

*Proof.* For simplicity, let  $\mathcal{A} = \mathcal{A}(1, 0, \dots, 0)$ . We have  $\partial_{\mathcal{A}}(x_i) = x_1x_i$  for any  $i \in \{1, 2, \dots, n\}$ . Hence,

$$\text{im}(\partial_{\mathcal{A}}^1) = \bigoplus_{i=1}^n \mathbb{K}x_1x_i.$$

For any  $i, j \in \{1, 2, \dots, n\}$ , we have

$$\partial_{\mathcal{A}}(x_ix_j) = \partial_{\mathcal{A}}(x_i)x_j - x_i\partial_{\mathcal{A}}(x_j) = x_1x_ix_j - x_ix_1x_j = 0.$$

Then  $\ker(\partial_{\mathcal{A}}^2) = \mathcal{A}^2$  and hence  $\ker(\partial_{\mathcal{A}}^{2k}) = \mathcal{A}^{2k}$  by the Leibniz rule for any  $k \geq 2$ . Since  $\mathcal{A}^2 = \bigoplus_{i=1}^n \bigoplus_{j=i}^n \mathbb{K}x_ix_j$ , we have

$$H^2(\mathcal{A}) = \bigoplus_{i=2}^n \bigoplus_{j=i}^n \mathbb{K}x_ix_j.$$

Since  $\partial_{\mathcal{A}}(x_i) = x_1x_i$  and  $\ker(\partial_{\mathcal{A}}^{2k-2}) = \mathcal{A}^{2k-2}$ , it is easy to check that

$$\text{im}(\partial_{\mathcal{A}}^{2k-1}) = \bigoplus_{\omega_1=1}^{2k} \bigoplus_{\substack{\sum_{j=2}^n \omega_j=2k-\omega_1 \\ x_j \geq 0, j=2, \dots, n}} \mathbb{K}x_1^{\omega_1}x_2^{\omega_2} \cdots x_n^{\omega_n}.$$

Since

$$\mathcal{A}^{2k} = \bigoplus_{\substack{\sum_{j=1}^n \omega_j=2k \\ x_j \geq 0, j=1, \dots, n}} \mathbb{K}x_1^{\omega_1}x_2^{\omega_2} \cdots x_n^{\omega_n},$$

we have

$$H^{2k}(\mathcal{A}) = \bigoplus_{\substack{\sum_{j=2}^n \omega_j=2k \\ x_j \geq 0, j=2, \dots, n}} \mathbb{K}x_2^{\omega_2} \cdots x_n^{\omega_n}.$$

For any  $k \geq 2$ , any cocycle element in  $\mathcal{A}^{2k+1}$  can be written as  $\sum_{i=1}^n x_i f_i$  for some  $f_i \in \mathcal{A}^{2k}, i = 1, 2, \dots, n$ . We have

$$\partial_{\mathcal{A}}\left(\sum_{i=1}^n x_i f_i\right) = \sum_{i=1}^n x_1x_i f_i = x_1 \sum_{i=1}^n x_i f_i = 0.$$

So  $\sum_{i=1}^n x_i f_i = 0$ . Hence,  $\ker(\partial_{\mathcal{A}}^{2k+1}) = 0$  and then  $H^{2k+1}(\mathcal{A}) = 0$ . Therefore,

$$H(\mathcal{A}) = \mathbb{K}[[x_2^2], [x_2x_3], \dots, [x_2x_n], [x_3^2], \dots, [x_3x_n], \dots, [x_{n-1}^2], [x_{n-1}x_n], [x_n^2]].$$

This completes the proof. □

By Proposition 5.1, one can deduce the following interesting results for DG polynomial algebras.

**Theorem 5.2.** *Any DG polynomial algebra  $\mathcal{A}(t_1, t_2, \dots, t_n)$  is a formal, homologically smooth and Gorenstein DG algebra.*

*Proof.* For briefness, let  $\mathcal{A} = \mathcal{A}(t_1, t_2, \dots, t_n)$ .

If  $(t_1, t_2, \dots, t_n) = (0, 0, \dots, 0)$ , then  $\partial_{\mathcal{A}} = 0$  and  $H(\mathcal{A}) = \mathcal{A}^{\#} = k[x_1, x_2, \dots, x_n]$ . Hence  $H(\mathcal{A})$  is a Noetherian, Gorenstein graded algebra with  $\text{gl.dim}H(\mathcal{A}) = n$ . Therefore,  $\mathcal{A}$  is Gorenstein and  $\mathbb{K} \in D^c(\mathcal{A})$  by [9, Proposition 1] and [20, Corollary 3.6], respectively. Hence  $\mathcal{A}$  is homologically smooth by [21, Corollary 2.7].

If  $(t_1, t_2, \dots, t_n) \neq (0, 0, \dots, 0)$ , then  $\mathcal{A} \cong \mathcal{A}(1, 0, \dots, 0)$  by Corollary 4.2. We have

$$H(\mathcal{A}) \cong \mathbb{K}[[x_2^2], [x_2x_3], \dots, [x_2x_n], [x_3^2], \dots, [x_3x_n], \dots, [x_{n-1}^2], [x_{n-1}x_n], [x_n^2]].$$

by Proposition 5.1. So  $H(\mathcal{A})$  is a Noetherian, Gorenstein graded algebra with

$$\text{gl.dim}H(\mathcal{A}) = \frac{n(n-1)}{2} < \infty.$$

By [9, Proposition 1],  $\mathcal{A}$  is Gorenstein. In addition,  $\mathcal{A}$  is homologically smooth by [20, Corollary 3.6] and [21, Corollary 2.7]. We can define a morphism of DG algebras  $\iota : (H(\mathcal{A}), 0) \rightarrow (\mathcal{A}, \partial_{\mathcal{A}})$  by  $\iota([x_i x_j]) = x_i x_j$  for all  $2 \leq i \leq j \leq n$ . One sees easily that  $\iota$  is a quasi-isomorphism. So  $\mathcal{A}$  is formal.  $\square$

### 6 Calabi-Yau properties of DG polynomial algebras

By Theorem 5.2, we know that any DG polynomial algebra  $\mathcal{A}(t_1, t_2, \dots, t_n)$  is a formal, homologically smooth and Gorenstein DG algebra. It is natural for one to ask whether  $\mathcal{A}(t_1, t_2, \dots, t_n)$  is Calabi-Yau. We completely solve this problem in this section.

**Remark 6.1.** For any connected cochain DG algebra  $\mathcal{A}$ , one sees that  $H(\mathcal{A}^e) = H(\mathcal{A})^e$  is a connected graded algebra. We should emphasize that the multiplication of  $H(\mathcal{A})^e$  is defined by

$$([c] \otimes [d]) \cdot ([e] \otimes [f]) = (-1)^{|d| \cdot |e|} [c][e] \otimes [d] \diamond [f] = (-1)^{|d| \cdot |e| + |d| \cdot |f|} [ce] \otimes [fd],$$

for any cocycle elements  $c, e \in \mathcal{A}$  and  $d, f \in \mathcal{A}^{\text{op}}$ . In the proof of the following two theorems, we need to construct minimal free resolutions of  ${}_{H(\mathcal{A})^e}H(\mathcal{A})$  and  $H(\mathcal{A})_{H(\mathcal{A})^e}$ . We remind the readers to remember our emphasis above.

**Theorem 6.2.** Let  $\mathcal{A}$  be a connected cochain DG algebra such that

$$H(\mathcal{A}) = \mathbb{K}[[y_1], \dots, [y_m]],$$

for some central, cocycle and degree 2 elements  $y_1, \dots, y_m$  in  $\mathcal{A}$ . Then  $\mathcal{A}$  is a  $(-m)$ -Calabi-Yau DG algebra.

*Proof.* The graded left  $H(\mathcal{A})^e$ -module  $H(\mathcal{A})$  admits a minimal free resolution:

$$0 \leftarrow H(\mathcal{A}) \xleftarrow{\mu} H(\mathcal{A})^e \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_2 \xleftarrow{d_3} \dots \xleftarrow{d_{m-1}} F_{m-1} \xleftarrow{d_m} F_m \xleftarrow{0} 0,$$

where  $\mu : H(\mathcal{A})^e \rightarrow H(\mathcal{A})$  is defined by  $\mu([a] \otimes [b]) = [ab]$ , each  $F_k$  is a free  $H(\mathcal{A})^e$ -module

$$\bigoplus_{e_{i_1 \dots i_k} \in E_k} H(\mathcal{A})^e e_{i_1 \dots i_k}$$

of rank  $C_m^k$  with basis

$$E_k = \{e_{i_1 \dots i_k} \mid 1 \leq i_1 < i_2 < \dots < i_k \leq m\}, \quad |e_{i_1 \dots i_k}| = 2k,$$

and  $d_k$  is defined by

$$d_k(e_{i_1 \dots i_k}) = \sum_{j=1}^k (-1)^{j-1} ([y_{i_j}] \otimes 1 - 1 \otimes [y_{i_j}]) e_{i_1 \dots \hat{i}_j \dots i_k},$$

for any  $k = 1, 2, \dots, m$ . Applying the constructing procedure of Eilenberg-Moore resolution, we can construct a minimal semi-free resolution  $F$  of the DG  $\mathcal{A}^e$ -module  $\mathcal{A}$ . Since  $y_1, y_2, \dots, y_m$  are central elements in  $\mathcal{A}$ , it is not difficult to check that

$$F^\# = \mathcal{A}^{\#} \oplus \left[ \bigoplus_{k=1}^m \bigoplus_{e_{i_1 \dots i_k} \in E_k} \mathcal{A}^{\# \Sigma^k} e_{i_1 \dots i_k} \right]$$

and  $\partial_F$  is defined by

$$\partial_F(\Sigma^k e_{i_1 \dots i_k}) = \sum_{j=1}^k (-1)^{j-1} (y_{i_j} \otimes 1 - 1 \otimes y_{i_j}) \Sigma^{k-1} e_{i_1 \dots \hat{i}_j \dots i_k},$$

for any  $e_{i_1 \dots i_k} \in E_k$  and  $k = 1, 2, \dots, m$ . Clearly,  $\mathcal{A}$  is homologically smooth since  $F$  has a finite semi-basis.

Similarly, the right graded  $H(\mathcal{A})^e$ -module  $H(\mathcal{A})$  has a minimal free resolution

$$0 \leftarrow H(\mathcal{A}) \xleftarrow{\tau} H(\mathcal{A})^e \xleftarrow{d_1} G_1 \xleftarrow{d_2} G_2 \xleftarrow{d_3} \dots \xleftarrow{d_{m-1}} G_{m-1} \xleftarrow{d_m} G_m \xleftarrow{0} 0, \tag{6.1}$$

where  $\tau : H(\mathcal{A})^e \rightarrow H(\mathcal{A})$  is defined by  $\tau([a] \otimes [b]) = (-1)^{|a| \cdot |b|} [ba]$ , each  $G_k$  is a free right  $H(\mathcal{A})^e$ -module

$$\bigoplus_{\lambda_{i_1 \dots i_k} \in \Lambda_k} \lambda_{i_1 \dots i_k} H(\mathcal{A})^e$$

of rank  $C_m^k$  with basis

$$\Lambda_k = \{ \lambda_{i_1 \dots i_k} \mid 1 \leq i_1 < i_2 < \dots < i_k \leq m \}, \quad |\lambda_{i_1 \dots i_k}| = 2k,$$

and the differentials are defined by

$$\begin{aligned} d_1(\lambda_{i_1}) &= (1 \otimes [y_{i_1}] - [y_{i_1}] \otimes 1), \quad \forall \lambda_{i_1} \in \Lambda_1, \\ d_k(\lambda_{i_1 \dots i_k}) &= \sum_{j=1}^k (-1)^{k-j} \lambda_{i_1 \dots \hat{i}_j \dots i_k} (1 \otimes [y_{i_j}] - [y_{i_j}] \otimes 1), \quad \forall \lambda_{i_1 \dots i_k} \in \Lambda_k, \end{aligned}$$

where  $k = 2, \dots, m$ . From the free resolution (6.2), we can construct the Eilenberg-Moore resolution  $G$  of  $\mathcal{A}_{\mathcal{A}^e}$ . We have

$$G^\# = \mathcal{A}^{e\#} \oplus \left[ \bigoplus_{k=1}^m \bigoplus_{\lambda_{i_1 \dots i_k} \in \Lambda_k} \Sigma^k \lambda_{i_1 \dots i_k} \mathcal{A}^{e\#} \right]$$

and  $\partial_G$  is defined by

$$\begin{aligned} \partial_G(1) &= 0, \quad \partial_G(\Sigma \lambda_{i_1}) = (1 \otimes y_{i_1} - y_{i_1} \otimes 1), \quad \forall \lambda_{i_1} \in \Lambda_1, \\ \partial_G(\Sigma^k \lambda_{i_1 \dots i_k}) &= \sum_{j=1}^k (-1)^{k-j} \Sigma^{k-1} \lambda_{i_1 \dots \hat{i}_j \dots i_k} (1 \otimes y_{i_j} - y_{i_j} \otimes 1), \end{aligned}$$

for any  $\lambda_{i_1 \dots i_k} \in \Lambda_k$  and  $k = 2, \dots, m$ . Hence

$$\Sigma^m G^\# = \Sigma^m 1 \mathcal{A}^{e\#} \oplus \left[ \bigoplus_{k=1}^m \bigoplus_{\lambda_{i_1 \dots i_k} \in \Lambda_k} \Sigma^{m+k} \lambda_{i_1 \dots i_k} \mathcal{A}^{e\#} \right]$$

with

$$\begin{aligned} \partial_{\Sigma^m G}(\Sigma^m 1) &= 0, \quad \partial_{\Sigma^m G}(\Sigma^{m+1} \lambda_{i_1}) = (-1)^m \Sigma^m 1 (1 \otimes y_{i_1} - y_{i_1} \otimes 1), \quad \forall \lambda_{i_1} \in \Lambda_1, \\ \partial_{\Sigma^m G}(\Sigma^{m+k} \lambda_{i_1 \dots i_k}) &= \sum_{j=1}^k (-1)^{m+k-j} \Sigma^{m+k-1} \lambda_{i_1 \dots \hat{i}_j \dots i_k} (1 \otimes y_{i_j} - y_{i_j} \otimes 1). \end{aligned}$$

The DG right  $\mathcal{A}^e$ -module  $\text{Hom}_{\mathcal{A}^e}(F, \mathcal{A}^e)$  is a minimal semi-free DG module whose underlying graded module is

$$\left\{ \mathbb{K}1^* \oplus \left[ \bigoplus_{k=1}^m \bigoplus_{e_{i_1 \dots i_k} \in E_k} \mathbb{K}(\Sigma^k e_{i_1 \dots i_k})^* \right] \right\} \otimes (\mathcal{A}^e)^\#.$$

The differential  $\partial_{\text{Hom}}$  of  $\text{Hom}_{\mathcal{A}^e}(F, \mathcal{A}^e)$  is defined by

$$\partial_{\text{Hom}}(f) = \partial_{\mathcal{A}^e} \circ f - (-1)^{|f|} f \circ \partial_F,$$

for any graded element  $f \in \text{Hom}_{\mathcal{A}^e}(F, \mathcal{A}^e)$ . So we have  $\partial_{\text{Hom}}[(\Sigma^m e_{1\dots m})^*] = 0$ ,

$$\partial_{\text{Hom}}[(\Sigma^k e_{i_1\dots i_k})^*] = \sum_{e_{i_1\dots i_j l i_{j+1}\dots i_k} \in E_{k+1}} (-1)^{j-k} (\Sigma^{k+1} e_{i_1\dots i_j l i_{j+1}\dots i_k})^* (1 \otimes y_l - y_l \otimes 1),$$

for any  $k \in \{1, 2, \dots, m-1\}$ , and

$$\partial_{\text{Hom}}[1^*] = \sum_{i=1}^m (\Sigma e_i)^* (1 \otimes y_i - y_i \otimes 1).$$

Define an  $(\mathcal{A}^e)^{\text{op}}$ -linear map  $\theta : \text{Hom}_{\mathcal{A}^e}(F, \mathcal{A}^e) \rightarrow \Sigma^m G$  by

$$\theta : \begin{cases} (\Sigma^m e_{1\dots m})^* \mapsto (-1)^{\frac{m(m+1)}{2}} \Sigma^m 1, \\ (\Sigma^k e_{i_1\dots i_k})^* \mapsto (-1)^{\sum_{j=1}^k i_j} \Sigma^{2m-k} \lambda_{\overline{i_1\dots i_k}}, \quad \forall k \in \{1, \dots, m-1\}, \\ 1^* \mapsto \Sigma^{2m} \lambda_{1\dots m}, \end{cases}$$

where  $\overline{i_1 \dots i_k}$  is the  $m-k$  integers arranged from small to large obtained by deleting  $i_1, i_2, \dots, i_k$  from  $\{1, 2, \dots, m\}$ . We claim that  $\theta$  is a chain map. Indeed, we have

$$\theta \circ \partial_{\text{Hom}}[(\Sigma^m e_{1\dots m})^*] = 0 = (-1)^{\frac{m(m+1)}{2}} \partial_{\Sigma^m G}[\Sigma^m 1] = \partial_{\Sigma^m G} \circ \theta[(\Sigma^m e_{1\dots m})^*]$$

and

$$\theta \circ \partial_{\text{Hom}}[1^*] = \partial_{\Sigma^m G} \circ \theta[1^*],$$

since

$$\begin{aligned} \theta \circ \partial_{\text{Hom}}[1^*] &= \theta \left[ \sum_{i=1}^m (\Sigma e_i)^* (1 \otimes y_i - y_i \otimes 1) \right] \\ &= \sum_{i=1}^m (-1)^i \Sigma^{2m-1} \lambda_{1\dots(i-1)(i+1)\dots m} (1 \otimes y_i - y_i \otimes 1) \end{aligned}$$

and

$$\begin{aligned} \partial_{\Sigma^m G} \circ \theta[1^*] &= \partial_{\Sigma^m G}[\Sigma^{2m} \lambda_{1\dots m}] \\ &= \sum_{i=1}^m (-1)^{m+m-i} \Sigma^{2m-1} \lambda_{1\dots(i-1)(i+1)\dots m} (1 \otimes y_i - y_i \otimes 1) \\ &= \sum_{i=1}^m (-1)^i \Sigma^{2m-1} \lambda_{1\dots(i-1)(i+1)\dots m} (1 \otimes y_i - y_i \otimes 1). \end{aligned}$$

Furthermore, for any  $k \in \{1, 2, \dots, m-1\}$ , we have

$$\partial_{\Sigma^m G} \circ \theta[(\Sigma^k e_{i_1\dots i_k})^*] = \theta \circ \partial_{\text{Hom}}[(\Sigma^k e_{i_1\dots i_k})^*],$$

since

$$\begin{aligned} &\theta \circ \partial_{\text{Hom}}[(\Sigma^k e_{i_1\dots i_k})^*] \\ &= \theta \left[ \sum_{e_{i_1\dots i_j l i_{j+1}\dots i_k} \in E_{k+1}} (-1)^{j-k} (\Sigma^{k+1} e_{i_1\dots i_j l i_{j+1}\dots i_k})^* (1 \otimes y_l - y_l \otimes 1) \right] \\ &= \sum_{e_{i_1\dots i_j l i_{j+1}\dots i_k} \in E_{k+1}} (-1)^{j-k+\sum_{j=1}^k i_j+l} \Sigma^{2m-k-1} \lambda_{\overline{i_1\dots i_j l i_{j+1}\dots i_k}} (1 \otimes y_l - y_l \otimes 1) \end{aligned}$$

and

$$\begin{aligned} & \partial_{\Sigma^m G} \circ \theta[(\Sigma^k e_{i_1 \dots i_k})^*] \\ &= \partial_{\Sigma^m G} [(-1)^{\sum_{j=1}^k i_j} \Sigma^{2m-k} \lambda_{i_1 \dots i_k}] \\ &= \sum_{e_{i_1 \dots i_j i_{j+1} \dots i_k} \in E_{k+1}} (-1)^{\sum_{j=1}^k i_j + 2m - k - l + j} \Sigma^{2m-k-1} \lambda_{i_1 \dots i_j l i_{j+1} \dots i_k} (1 \otimes y_l - y_l \otimes 1). \end{aligned}$$

So  $\theta$  is an isomorphism of DG  $\mathcal{A}^{\text{op}}$ -modules and  $\text{Hom}_{\mathcal{A}^e}(F, \mathcal{A}^e) \cong \Sigma^m \mathcal{A}$  in  $\text{D}((\mathcal{A}^e)^{\text{op}})$ . Therefore,  $\mathcal{A}$  is a  $(-m)$ -Calabi-Yau DG algebra. □

**Corollary 6.3.** *The DG polynomial algebra  $\mathcal{A}(t_1, t_2, \dots, t_n)$  is a  $\frac{-n(n-1)}{2}$ -Calabi-Yau DG algebra if  $(t_1, t_2, \dots, t_n) \neq (0, 0, \dots, 0)$ .*

*Proof.* By Corollary 4.2, we have  $\mathcal{A}(t_1, t_2, \dots, t_n) \cong \mathcal{A}(1, 0, \dots, 0)$ . So we only need to show  $\mathcal{A}(1, 0, \dots, 0)$  is a  $\frac{-n(n-1)}{2}$ -Calabi-Yau DG algebra. By Proposition 5.1, the cohomology graded algebra of  $\mathcal{A}(1, 0, \dots, 0)$  is

$$\mathbb{K}[[x_2^2], [x_2 x_3], \dots, [x_2 x_n], [x_3^2], \dots, [x_3 x_n], \dots, [x_{n-1}^2], [x_{n-1} x_n], [x_n^2]].$$

Theorem 6.2 indicates that  $\mathcal{A}(1, 0, \dots, 0)$  is a  $\frac{-n(n-1)}{2}$ -Calabi-Yau DG algebra. □

**Theorem 6.4.** *Let  $\mathcal{A}$  be a connected cochain DG algebra such that*

$$H(\mathcal{A}) = \mathbb{K}[[y_1], \dots, [y_m]],$$

*for some central, cocycle and degree 1 elements  $y_1, \dots, y_m$  in  $\mathcal{A}$ . Then  $\mathcal{A}$  is a Koszul, homologically smooth and Gorenstein DG algebra. Moreover,  $\mathcal{A}$  is 0-Calabi-Yau if and only if  $m$  is an odd integer.*

*Proof.* By the assumption  $H(\mathcal{A}) = \mathbb{K}[[y_1], \dots, [y_m]]$  is a Koszul, Gorenstein and Noetherian graded algebra with  $\text{gl.dim} H(\mathcal{A}) = m < \infty$ . Hence  $\mathcal{A}$  is Koszul, Gorenstein and homologically smooth by [12, Proposition 2.3], [9, Proposition 1] and [20, Corollary 3.7], respectively.

Now, let us consider the Calabi-Yau properties of  $\mathcal{A}$ . The graded left  $H(\mathcal{A})^e$ -module  $H(\mathcal{A})$  admits a minimal free resolution

$$0 \leftarrow H(\mathcal{A}) \xleftarrow{\mu} H(\mathcal{A})^e \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_2 \xleftarrow{d_3} \dots \xleftarrow{d_{m-1}} F_{m-1} \xleftarrow{d_m} F_m \xleftarrow{0} 0,$$

where  $\mu : H(\mathcal{A})^e \rightarrow H(\mathcal{A})$  is defined by  $\mu([a] \otimes [b]) = [ab]$ , each  $F_k$  is a free  $H(\mathcal{A})^e$ -module

$$\bigoplus_{e_{i_1 \dots i_k} \in E_k} H(\mathcal{A})^e e_{i_1 \dots i_k}$$

of rank  $C_m^k$  with basis

$$E_k = \{e_{i_1 \dots i_k} \mid 1 \leq i_1 < i_2 < \dots < i_k \leq m\}, \quad |e_{i_1 \dots i_k}| = k,$$

and  $d_k$  is defined by

$$d_k(e_{i_1 \dots i_k}) = \sum_{j=1}^k (-1)^{j-1} ([y_{i_j}] \otimes 1 + (-1)^k 1 \otimes [y_{i_j}]) e_{i_1 \dots \hat{i}_j \dots i_k},$$

for any  $k = 1, 2, \dots, m$ . Applying the constructing procedure of Eilenberg-Moore resolution, we can construct a minimal semi-free resolution  $F$  of the DG  $\mathcal{A}^e$ -module  $\mathcal{A}$ . Since  $y_1, \dots, y_m$  are central elements in  $\mathcal{A}$ , it is not difficult to check that

$$F^\# = \mathcal{A}^{\#} \oplus \left[ \bigoplus_{k=1}^m \bigoplus_{e_{i_1 \dots i_k} \in E_k} \mathcal{A}^{\# \Sigma^k} e_{i_1 \dots i_k} \right]$$



and  $\partial_F$  is defined by

$$\partial_F(\Sigma^k e_{i_1 \dots i_k}) = \sum_{j=1}^k (-1)^{j-1} (y_{i_j} \otimes 1 + (-1)^k 1 \otimes y_{i_j}) \Sigma^{k-1} e_{i_1 \dots \hat{i}_j \dots i_k},$$

for any  $e_{i_1 \dots i_k} \in E_k$  and  $k = 1, 2, \dots, m$ .

The DG right  $\mathcal{A}^e$ -module  $\text{Hom}_{\mathcal{A}^e}(F, \mathcal{A}^e)$  is a minimal semi-free DG module whose underlying graded module is

$$\left\{ \mathbb{K}1^* \oplus \left[ \bigoplus_{k=1}^m \bigoplus_{e_{i_1 \dots i_k} \in E_k} \mathbb{K}(\Sigma^k e_{i_1 \dots i_k})^* \right] \right\} \otimes (\mathcal{A}^e)^\#.$$

The differential  $\partial_{\text{Hom}}$  of  $\text{Hom}_{\mathcal{A}^e}(F, \mathcal{A}^e)$  is defined by

$$\partial_{\text{Hom}}(f) = \partial_{\mathcal{A}^e} \circ f - (-1)^{|f|} f \circ \partial_F,$$

for any graded element  $f \in \text{Hom}_{\mathcal{A}^e}(F, \mathcal{A}^e)$ . So we have

$$\begin{aligned} \partial_{\text{Hom}}[(\Sigma^m e_{1 \dots m})^*] &= 0, \\ \partial_{\text{Hom}}[(\Sigma^k e_{i_1 \dots i_k})^*] &= \sum_{e_{i_1 \dots i_j l_{j+1} \dots i_k} \in E_{k+1}} (-1)^{j+1} (\Sigma^{k+1} e_{i_1 \dots i_j l_{j+1} \dots i_k})^* (y_l \otimes 1 - (-1)^k 1 \otimes y_l), \end{aligned}$$

for any  $k \in \{1, 2, \dots, m-1\}$ , and

$$\partial_{\text{Hom}}[1^*] = \sum_{i=1}^m (\Sigma e_i)^* (1 \otimes y_i - y_i \otimes 1).$$

If  $m = 2t$  is an even integer, we claim that  $\mathcal{A}$  is not a Calabi-Yau DG algebra. It suffices to show that  $\text{Hom}_{\mathcal{A}^e}(F, \mathcal{A}^e) \not\cong \mathcal{A}$  in  $D((\mathcal{A}^e)^{\text{op}})$ . We prove this with a proof by contradiction. If  $\text{Hom}_{\mathcal{A}^e}(F, \mathcal{A}^e) \cong \mathcal{A}$  in  $D((\mathcal{A}^e)^{\text{op}})$ , then there exist a DG right  $\mathcal{A}^e$ -module  $P$  and two quasi-isomorphisms of right DG  $\mathcal{A}^e$ -modules  $\phi : P \rightarrow \mathcal{A}$  and  $\psi : P \rightarrow \text{Hom}_{\mathcal{A}^e}(F, \mathcal{A}^e)$ . The DG right  $\mathcal{A}^e$ -module  $P$  admits a minimal semi-free resolution  $\varepsilon : G \rightarrow P$  since  $H(P) \cong H(\mathcal{A})$  is bounded below. Then

$$\psi \circ \varepsilon : G \rightarrow \text{Hom}_{\mathcal{A}^e}(F, \mathcal{A}^e)$$

is a quasi-isomorphism of right DG  $\mathcal{A}^e$ -modules. Since both  $G$  and  $\text{Hom}_{\mathcal{A}^e}(F, \mathcal{A}^e)$  are minimal,  $\psi \circ \varepsilon$  is an isomorphism by Corollary 1.3 of Section 12 in the manuscript<sup>1)</sup>. Then the composition

$$\text{Hom}_{\mathcal{A}^e}(F, \mathcal{A}^e) \xrightarrow{(\psi \circ \varepsilon)^{-1}} G \xrightarrow{\varepsilon} P \xrightarrow{\phi} \mathcal{A}$$

is a quasi-isomorphism of DG right  $\mathcal{A}^e$ -modules. Set  $g = \phi \circ \varepsilon \circ (\psi \circ \varepsilon)^{-1}$ . Since  $H(g)$  is an isomorphism and  $[(\Sigma^m e_{1 \dots m})^*] \neq 0$ , we have  $g[(\Sigma^m e_{1 \dots m})^*] = a_0 \neq 0$ , for some  $a_0 \in \mathcal{A}^0 = k$ . For any  $j \in \{1, 2, \dots, m\}$ , we have  $[(\Sigma^{m-1} e_{1 \dots j-1 j+1 \dots m})^*] = 0$ . So there exists an  $a \in \mathcal{A}^0$  such that  $g[(\Sigma^{m-1} e_{1 \dots j-1 j+1 \dots m})^*] = a$ . We have

$$\begin{aligned} \partial_{\mathcal{A}} \circ g[(\Sigma^{m-1} e_{1 \dots (j-1)(j+1) \dots m})^*] &= \partial_{\mathcal{A}}(a) = 0, \\ g \circ \partial_{\text{Hom}}[(\Sigma^{m-1} e_{1 \dots (j-1)(j+1) \dots m})^*] &= g[(-1)^j (\Sigma^m e_{1 \dots m})^* (y_j \otimes 1 - (-1)^{m-1} 1 \otimes y_j)] \\ &= (-1)^j a_0 \cdot (y_j \otimes 1 + 1 \otimes y_j) = (-1)^j 2a_0 y_j. \end{aligned}$$

Therefore,  $\partial_{\mathcal{A}} \circ g \neq g \circ \partial_{\text{Hom}}$  since  $\text{char } \mathbb{K} = 0$  and  $a_0 \neq 0$  in  $\mathcal{A}^0$ . This implies that  $g$  is not a chain map. Then we get a contradiction since  $g$  is a morphism of DG right  $\mathcal{A}^e$ -modules.

Now, it remains to consider the case that  $m = 2t - 1$  is an odd integer. The right graded  $H(\mathcal{A})^e$ -module  $H(\mathcal{A})$  has a minimal free resolution

$$0 \leftarrow H(\mathcal{A}) \xleftarrow{\tau} H(\mathcal{A})^e \xleftarrow{d_1} G_1 \xleftarrow{d_2} G_2 \xleftarrow{d_3} \dots \xleftarrow{d_{m-1}} G_{m-1} \xleftarrow{d_m} G_m \xleftarrow{0} 0, \tag{6.2}$$

where  $\tau : H(\mathcal{A})^e \rightarrow H(\mathcal{A})$  is defined by  $\tau([a] \otimes [b]) = (-1)^{|a||b|} [ba]$ , each  $G_k$  is a free right  $H(\mathcal{A})^e$ -module

$$\bigoplus_{\lambda_{i_1 \dots i_k} \in \Lambda_k} \lambda_{i_1 \dots i_k} H(\mathcal{A})^e$$

of rank  $C_m^k$  with basis

$$\Lambda_k = \{ \lambda_{i_1 \dots i_k} \mid 1 \leq i_1 < i_2 < \dots < i_k \leq m \}, \quad |\lambda_{i_1 \dots i_k}| = k,$$

and the differentials are defined by

$$d_1(\lambda_{i_1}) = (1 \otimes [y_{i_1}] - [y_{i_1}] \otimes 1), \quad \forall \lambda_{i_1} \in \Lambda_1,$$

$$d_k(\lambda_{i_1 \dots i_k}) = \sum_{j=1}^k (-1)^{k-j} \lambda_{i_1 \dots \hat{i}_j \dots i_k} (1 \otimes [y_{i_j}] + (-1)^k [y_{i_j}] \otimes 1), \quad \forall \lambda_{i_1 \dots i_k} \in \Lambda_k,$$

$k = 2, \dots, m$ . From the free resolution (6.2), we can construct the Eilenberg-Moore resolution  $G$  of  $\mathcal{A}_{\mathcal{A}^e}$ . We have

$$G^\# = \mathcal{A}^{e\#} \oplus \left[ \bigoplus_{k=1}^m \bigoplus_{\lambda_{i_1 \dots i_k} \in \Lambda_k} \Sigma^k \lambda_{i_1 \dots i_k} \mathcal{A}^{e\#} \right]$$

and  $\partial_G$  is defined by

$$\partial_G(1) = 0, \quad \partial_G(\Sigma \lambda_{i_1}) = (1 \otimes y_{i_1} - y_{i_1} \otimes 1), \quad \forall \lambda_{i_1} \in \Lambda_1,$$

$$\partial_G(\Sigma^k \lambda_{i_1 \dots i_k}) = \sum_{j=1}^k (-1)^{k-j} \Sigma^{k-1} \lambda_{i_1 \dots \hat{i}_j \dots i_k} (1 \otimes y_{i_j} + (-1)^k y_{i_j} \otimes 1),$$

for any  $\lambda_{i_1 \dots i_k} \in \Lambda_k$  and  $k = 2, \dots, m$ .

Define an  $(\mathcal{A}^e)^{\text{op}}$ -linear map  $\theta : \text{Hom}_{\mathcal{A}^e}(F, \mathcal{A}^e) \rightarrow G$  by

$$\theta : \begin{cases} (\Sigma^m e_{1 \dots m})^* \mapsto (-1)^{\frac{m(m+1)}{2} + m} 1, \\ (\Sigma^k e_{i_1 \dots i_k})^* \mapsto (-1)^{\sum_{j=1}^k i_j + k} \Sigma^{m-k} \lambda_{\overline{i_1 \dots i_k}}, \quad \forall k \in \{1, \dots, m-1\}, \\ 1^* \mapsto \Sigma^m \lambda_{1 \dots m}, \end{cases}$$

where  $\overline{i_1 \dots i_k}$  is the  $m - k$  integers arranged from small to large obtained by deleting  $i_1, i_2, \dots, i_k$  from  $\{1, 2, \dots, m\}$ . We claim that  $\theta$  is a chain map. Indeed, we have

$$\theta \circ \partial_{\text{Hom}}[(\Sigma^m e_{1 \dots m})^*] = 0 = \partial_G[(-1)^{\frac{m(m+1)}{2} + m} 1] = \partial_G \circ \theta[(\Sigma^m e_{1 \dots m})^*]$$

and  $\theta \circ \partial_{\text{Hom}}[1^*] = \partial_{\Sigma^m G} \circ \theta[1^*]$  since

$$\begin{aligned} \theta \circ \partial_{\text{Hom}}[1^*] &= \theta \left[ \sum_{i=1}^m (\Sigma e_i)^* (1 \otimes y_i - y_i \otimes 1) \right] \\ &= \sum_{i=1}^m (-1)^{i+1} \Sigma^{m-1} \lambda_{1 \dots (i-1)(i+1) \dots m} (1 \otimes y_i - y_i \otimes 1) \end{aligned}$$

and

$$\partial_G \circ \theta[1^*] = \partial_G[\Sigma^m \lambda_{1 \dots m}]$$

$$\begin{aligned} &= \sum_{i=1}^m (-1)^{m-i} \Sigma^{m-1} \lambda_{1 \dots (i-1)(i+1) \dots m} (1 \otimes y_i + (-1)^m y_i \otimes 1) \\ &= \sum_{i=1}^m (-1)^{2i-1-i} \Sigma^{m-1} \lambda_{1 \dots (i-1)(i+1) \dots m} (1 \otimes y_i - y_i \otimes 1) \\ &= \sum_{i=1}^m (-1)^{1+i} \Sigma^{m-1} \lambda_{1 \dots (i-1)(i+1) \dots m} (1 \otimes y_i - y_i \otimes 1). \end{aligned}$$

Furthermore, for any  $k \in \{1, 2, \dots, m - 1\}$ , we have

$$\partial_G \circ \theta[(\Sigma^k e_{i_1 \dots i_k})^*] = \theta \circ \partial_{\text{Hom}}[(\Sigma^k e_{i_1 \dots i_k})^*]$$

since

$$\begin{aligned} &\theta \circ \partial_{\text{Hom}}[(\Sigma^k e_{i_1 \dots i_k})^*] \\ &= \theta \left[ \sum_{e_{i_1 \dots i_j l_{j+1} \dots i_k} \in E_{k+1}} (-1)^{j+1} (\Sigma^{k+1} e_{i_1 \dots i_j l_{j+1} \dots i_k})^* (y_l \otimes 1 - (-1)^k 1 \otimes y_l) \right] \\ &= \sum_{e_{i_1 \dots i_j l_{j+1} \dots i_k} \in E_{k+1}} (-1)^{j + \sum_{s=1}^k i_s + l + k} \Sigma^{m-k-1} \lambda_{i_1 \dots i_j l_{j+1} \dots i_k} (y_l \otimes 1 - (-1)^k 1 \otimes y_l) \end{aligned}$$

and

$$\begin{aligned} &\partial_G \circ \theta[(\Sigma^k e_{i_1 \dots i_k})^*] \\ &= \partial_G [(-1)^{\sum_{s=1}^k i_s + k} \Sigma^{m-k} \lambda_{i_1 \dots i_k}] \\ &= \sum_{e_{i_1 \dots i_j l_{j+1} \dots i_k} \in E_{k+1}} (-1)^{\sum_{s=1}^k i_s + m - l + j} \Sigma^{m-k-1} \lambda_{i_1 \dots i_j l_{j+1} \dots i_k} (1 \otimes y_l - (-1)^{-k} y_l \otimes 1) \\ &= \sum_{e_{i_1 \dots i_j l_{j+1} \dots i_k} \in E_{k+1}} (-1)^{\sum_{s=1}^k i_s - l + j - k} \Sigma^{m-k-1} \lambda_{i_1 \dots i_j l_{j+1} \dots i_k} (y_l \otimes 1 - (-1)^k 1 \otimes y_l). \end{aligned}$$

So  $\theta$  is an isomorphism of DG  $\mathcal{A}^{\text{op}}$ -modules and  $\text{Hom}_{\mathcal{A}^e}(F, \mathcal{A}^e) \cong \mathcal{A}$  in  $D((\mathcal{A}^e)^{\text{op}})$ . Therefore,  $\mathcal{A}$  is a 0-Calabi-Yau DG algebra. □

**Corollary 6.5.** *The trivial DG polynomial algebra  $\mathcal{A}(0, 0, \dots, 0)$  is a Koszul, homologically smooth and Gorenstein DG algebra. Moreover, it is 0-Calabi-Yau if and only if  $n$  is an odd integer.*

**Remark 6.6.** In Theorems 6.2 and 6.4, we need that the cocycle elements  $y_1, y_2, \dots, y_m$  are central in  $\mathcal{A}$ . This simplifies the construction of Eilenberg-Moore resolution. Without this subtle condition, we are unable to get the corresponding result. Note that Theorem 6.4 is not a generalization of [17, Proposition 2.4 and Theorem 2.7] for this reason.

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