SCIENCE CHINA
Mathematics

• ARTICLES •

October 2018 Vol. 61 No. 10: 1789–1806
https://doi.org/10.1007/s11425-017-9175-2

Stability of non-monotone traveling waves for a discrete diffusion equation with monostable convolution type nonlinearity

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Received February 13, 2017; accepted October 23, 2017; published online March 16, 2018

Abstract This paper is concerned with the stability of non-monotone traveling waves for a discrete diffusion equation with monostable convolution type nonlinearity. By using the anti-weighted energy method and nonlinear Halanay's inequality, we prove that all noncritical traveling waves (waves with speeds $c > c_*$, c_* is minimal speed) are time-exponentially stable, when the initial perturbations around the waves are small. As a corollary of our stability result, we immediately obtain the uniqueness of the traveling waves.

Keywords discrete diffusion equations, stability, non-monotone traveling waves, anti-weighted energy method

MSC(2010) 35K57, 35B35, 92D25

Citation: Yang Z X, Zhang G B. Stability of non-monotone traveling waves for a discrete diffusion equation with monostable convolution type nonlinearity. Sci China Math, 2018, 61: 1789–1806, https://doi.org/10.1007/s11425-017-9175-2

1 Introduction

In population biology, lattice differential equations can be used to study the spatial spread of a species over a patchy environment. The simplest lattice differential equation model (see [2, 22, 23]) describing population growth and spread may take the form

$$\frac{\partial u_n(t)}{\partial t} = d[u_{n+1}(t) - 2u_n(t) + u_{n-1}(t)] + f(u_n(t)), \quad t > 0, \quad n \in \mathbb{Z}, \tag{1.1}$$

where $u_n(t)$ represents the population density at site n and time t, $d > 0$ denotes the diffusion rate, and $f(u_n)$ is the growth function.

Because of the influence of maturation period and the random walk of individuals in space, time delay and global interaction have to be taken into account. In 2003, Weng et al. [26] derived a delayed lattice differential equation with global interaction

$$\frac{\partial u_n(t)}{\partial t} = d[u_{n+1}(t) - 2u_n(t) + u_{n-1}(t)] - u_n(t) + \sum_{i\in\mathbb{Z}} K(i)g(u_{n-i}(t-\tau)), \tag{1.2}$$

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where $t > 0$ and $n \in \mathbb{Z}$. In this model, $u_n(t)$ represents the matured population density at site n and time t, $-u_n(t)$ denotes the death, $\sum_{i\in\mathbb{Z}} K(i)g(u_{n-i}(t-\tau))$ involves an infinite summation accounting for the non-local interaction, $g(\cdot)$ is the birth rate function of population density which interacts with neighbors by the non-negative weighted function K, and $\tau > 0$ is the maturation delay (the time required for a newborn to become matured).

Recently, a continuum version of the above lattice differential equation (1.2) has been considered by Guo and Lin [9], i.e.,

$$\frac{\partial u(t,x)}{\partial t} = \mathcal{D}_2[u](t,x) - u(t,x) + \sum_{i\in\mathbb{Z}} K(i)g(u(t-\tau, x-i)), \quad t > 0, \quad x \in \mathbb{R}, \tag{1.3}$$

where $\tau > 0$, and

$$\mathcal{D}_2[u](t,x) = d[u(t,x+1) - 2u(t,x) + u(t,x-1)].$$

Equation (1.3) can also model the matured population dynamics of a single species with nonzero maturation delay. Here, $u(t,x)$ represents the density of the matured population at the time t and location x. The other terms in (1.3) have the same meaning as those in (1.2). Guo and Lin [9] constructed three different types of entire solutions of (1.3) under bistable and monotone increasing conditions on $g(u)$.

The spatially discrete diffusion equations like (1.1)–(1.3) have been widely studied recently (see [3, 8, 10–13, 18] and the references cited therein). One of the important issues of those equations is the traveling wave solution (in short, traveling wave), since the traveling wave can describe spatial spread or invasion of the species in population dynamics. Mathematically, a traveling wave of (1.2) (or (1.3)) is a special translation invariant solution of the form $u_n(t) = \phi(\xi)$, $\xi = n + ct$ (or $u(t,x) = \phi(\xi)$, $\xi = x + ct$), and ϕ is the wave profile that propagates through the one dimensional spatial domain at a constant velocity $c > 0$. Moreover, if $\phi(\xi)$ is monotone in $\xi \in \mathbb{R}$, then it is called a traveling wavefront.

It is easy to see that the wave profile equation of (1.3) is the same as the lattice differential equation (1.2), i.e.,

$$c\phi'(\xi) - \mathcal{D}_2[\phi](\xi) + \phi(\xi) = \sum_{i\in\mathbb{Z}} K(i)g(\phi(\xi - c\tau - i)), \tag{1.4}$$

where $' = \frac{d}{d\xi}$, $\mathcal{D}_2[\phi](\xi) = d[\phi(\xi+1) - 2\phi(\xi) + \phi(\xi-1)]$. From the study of traveling waves of evolution equations, we can see that if the properties, such as existence, monotonicity and uniqueness, of traveling waves of (1.2) are obtained, then these properties also hold for (1.3). This is because the acquisition of these properties depends only on the wave profile equation (1.4). To the best of our knowledge, the traveling waves of (1.2) are well-investigated. When (1.2) is bistable, Ma and Zou [19] have proved the existence, uniqueness, global asymptotic stability and propagation failure of traveling wavefronts. When (1.2) is monostable, Weng et al. [26] established the existence of monotone traveling waves with speeds $c > c_*$, and showed that the minimal wave speed c_* is also the asymptotic speed of propagation. Later, Ma et al. [17] obtained the existence of traveling wavefront of (1.2) with speed $c = c_*$ and the uniqueness of traveling wavefront with speed $c > c_*$ under some extra assumption that the traveling wavefronts decay exponentially at $-\infty$, i.e.,

$$\limsup_{\xi\to-\infty} \phi(\xi)\mathrm{e}^{-\lambda_1(c)\xi} < +\infty, \tag{1.5}$$

where $\lambda_1(c)$ is the smallest positive solution of the characteristic equation $\mathcal{P}(c,\lambda) = 0$ (see (2.2)). Aguerrea et al. [1] proved the uniqueness of traveling waves of (1.2) by adapting the uniqueness theorem developed by Diekmann and Kaper [5]. The condition (1.5) is not assumed in their proof. In [8], Guo and Lin studied the equation (1.2) without delay ($\tau = 0$) and with short range interaction, i.e., $K(i) = 0$ for all $|i| \geqslant p$ with $p = 3$, and investigated the asymptotic behavior, monotonicity and uniqueness of traveling wavefronts without the assumption (1.5).

We should point out that in the above work (see [8,17,26]), the monotonicity of the birth rate function g is needed. When the function g is not monotone, the problem on the existence, uniqueness of traveling waves of (1.2) has also been solved (see [6, 7, 28]). In [6], Fang et al. proved the existence of traveling waves of (1.2) for $c \geqslant c_*$ by Schauder's fixed point theorem. The existing traveling waves in [6] may be non-monotone and oscillatory around positive equilibrium of (1.2). Furthermore, Fang et al. [7] established the uniqueness of traveling waves of (1.2) for $c > c_*$ without assuming that the wave profile is monotone. In [28], Yu obtained the uniqueness of traveling waves of (1.2) for $c = c_*$. But, to the best of our knowledge, when the function g is not monotone, the stability of traveling waves of (1.2) and (1.3) is still unknown. Hence, we try to make our efforts to tackle this problem. We should remark that the study of stability of traveling waves depends not only on the wave profile equation (1.4), but also on the equations (1.2) and (1.3). In this paper, we are devoted to studying the stability of traveling waves (monotone or non-monotone) of (1.3) when g is not monotone. We leave the stability of traveling waves of (1.2) for future study. To this end, we assume the kernel function K satisfies

$$K(i) = K(-i) \geqslant 0, \quad \sum_{i\in\mathbb{Z}} K(i) = 1 \quad \text{and} \quad \sum_{i\in\mathbb{Z}} K(i)\mathrm{e}^{-\lambda i} < \infty$$

for any $\lambda > 0$, and the birth rate function $g : [0,\infty) \to [0,\infty)$ satisfies the following hypotheses:

(G1) $g(0) = 0$, $g(u_+) = u_+$ for some positive constant u_+, $g(u) > u$ for $u \in (0, u_+)$, $g'(0) > 1$ and $g'(u_+) < 1$;

(G2) $g(u) > 0$ has only one positive local maximum at the point $u_* \in (0, u_+)$, and $g(u)$ is increasing on $[0, u_*]$ and decreasing on $[u_*, +\infty)$;

(G3) $g \in C^2[0,\infty)$ and $|g'(u)| \leqslant g'(0)$ for $u \in [0,\infty)$.

Remark 1.1. Hypothesis (G1) means that (1.3) has two constant equilibria $u = 0$ and $u = u_+$. In addition, 0 is unstable and u_+ is stable. Hence, (1.3) is a monostable system. Hypothesis (G2) implies that $g(u)$ is not monotone for $u \in [0, u_+]$.

The stability of traveling waves for various evolution equations has been extensively studied. We refer the readers to [14, 18, 20, 21, 24, 25, 27, 29]. In particular, Tian et al. [25] and Yang et al. [27] have considered the stability of traveling waves of (1.3) when $K(0) = 1$ and $K(i) = 0$ for all $i \neq 0$, i.e.,

$$\frac{\partial u(t,x)}{\partial t} = \mathcal{D}_2[u](t,x) - u(t,x) + g(u(t-\tau,x)). \tag{1.6}$$

Under the assumption that $g(u)$ is not monotone on the interval $[0, u_+]$, Yang et al. [27] proved the stability of traveling waves of (1.6) with noncritical speed $c > c_*$ by the technical weighted energy method. Meanwhile, Tian et al. [25] established the stability of traveling waves of (1.6) with critical speed $c = c_*$ by the same method as in [27] but with some new flavors.

It should be pointed out that the technical weighted energy method in [27] for the local equation (1.6) cannot be perfectly applied to the nonlocal equation (1.3), since the nonlocal term yields some gaps in the L^2-energy estimates, which cause us to need to take the wave speed c large enough. More recently, Huang et al. [14] adopted the so-called anti-weighted energy method (see [4]) and the nonlinear Halanay's inequality (see [15]) to prove the stability of all non-critical traveling waves for a nonlocal dispersion equation with time-delay. Inspired by [14], in this paper, we still take the anti-weighted energy method to prove the stability of traveling waves of (1.3) with noncritical speed $c > c_*$. More specifically, we first introduce a suitable transform function (an anti-weight) to switch the original equation to a new equation, and then give the a priori energy estimates for the solutions of this new equation. We leave the stability of traveling waves of (1.3) with critical speed $c = c_*$ for the future study.

The rest of this paper is organized as follows. In Section 2, we first show the existence of traveling waves of (1.3) with a general non-monotone function $g(u)$, and then state the stability theorem. In Section 3, we first reformulate the original equation to the perturbed equation around the given non-critical traveling wave. Then we give the corresponding stability theorem for the new equation. Finally, by taking the anti-weighted energy method and the nonlinear Halanay's inequality, we establish the desired a priori estimates. Based on the stability theorem, in Section 4, we prove the uniqueness of those monotone or non-monotone traveling waves.

2 Traveling waves and stability theorem

In this section, we first give the existence of traveling waves of (1.3), and then state the main result on the stability of traveling waves.

Throughout this paper, we assume that (1.3) satisfies the initial data

$$u(s,x)=u_0(s,x),\quad s\in[-\tau,0],\quad x\in\mathbb{R}. \tag{2.1}$$

The characteristic function for (1.4) with respect to the trivial equilibrium 0 can be represented by

$$\mathcal{P}(c,\lambda)=c\lambda-d(\mathrm{e}^{\lambda}+\mathrm{e}^{-\lambda}-2)+1-g'(0)G(\lambda), \tag{2.2}$$

where

$$G(\lambda)=\sum_{i\in\mathbb{Z}}K(i)\mathrm{e}^{-\lambda(i+c\tau)}<\infty.$$

The following lemma gives some properties on the characteristic equation $\mathcal{P}(c,\lambda)=0$.

Lemma 2.1. *Assume that $g'(0)>1$. Then there exist $\lambda_*>0$ and $c_*>0$ such that*

$$\mathcal{P}(c_*,\lambda_*)=0\quad\text{and}\quad \frac{\partial}{\partial\lambda}\mathcal{P}(c_*,\lambda)\bigg|_{\lambda=\lambda_*}=0.$$

Furthermore, if $c>c_*$, then $\mathcal{P}(c,\lambda)=0$ has two distinct positive real roots $\lambda_1(c)$ and $\lambda_2(c)$ with $\lambda_1(c)<\lambda_*<\lambda_2(c)$, and $\mathcal{P}(c,\lambda)>0$ for $\lambda\in(\lambda_1(c),\lambda_2(c))$.

When $g(u)$ is increasing on $[0,u_+]$, the existence of a traveling wavefront has been established in [17,26] by applying sub-super solutions, monotone iteration technique and a limiting argument.

Lemma 2.2. *Assume that (G1) holds, $g(u)$ is increasing on $[0,u_+]$ and $g'(0)u-g(u)\leqslant Nu^{1+\nu}$ for all $u\in(0,u_+)$, some $N>0$ and some $\nu\in(0,1]$. Let $c_*>0$ be defined as in Lemma 2.1. Then for each $c\geqslant c_*$, (1.3) admits a strictly increasing traveling wave $u(x,t)=\phi(x+ct)$ satisfying $\phi(-\infty)=0$ and $\phi(+\infty)=u_+$, while for any $0<c<c_*$, (1.3) has no traveling wave $\phi(x+ct)$ connecting 0 and u_+. Moreover, when $c>c_*$,*

$$\lim_{\xi\to-\infty}\phi(\xi)\mathrm{e}^{-\lambda_1(c)\xi}=1,\quad \lim_{\xi\to-\infty}\phi'(\xi)\mathrm{e}^{-\lambda_1(c)\xi}=\lambda_1(c),$$

where $\lambda_1(c)>0$ is the smallest solution to the equation $\mathcal{P}(c,\lambda)=0$.

When $g(u)$ is not monotone on $[0,u_+]$, the existence of traveling waves can also be obtained by using the idea of auxiliary equations and Schauder's fixed point theorem (see [6,16]).

Theorem 2.3. *Assume that (G1)–(G3) hold. Then for every $c\geqslant c_*$, (1.3) admits a traveling wave $u(t,x)=\phi(x+ct)$ satisfying $\phi(-\infty)=0$ and $0<\liminf_{\xi\to+\infty}\phi(\xi)\leqslant\limsup_{\xi\to+\infty}\phi(\xi)\leqslant u_+^*$ for some positive constant $u_+^*>u_+$.*

Before stating our main result, let us make the following notation. Throughout the paper, $C>0$ always denotes a generic constant, while $C_i>0$ $(i=0,1,2,\ldots)$ represents a specific constant. Let I be an interval, typically $I=\mathbb{R}$. $L^2(I)$ is the space of the square integrable defined on I, and $H^k(I)$ $(k\geqslant0)$ is the Sobolev space of the L^2-functions $f(x)$ defined on the interval I whose derivatives $\frac{d^i}{dx^i}f$ $(i=1,\ldots,k)$ also belong to $L^2(I)$. Let $T>0$ be a number and $\mathcal{B}$ be a Banach space. We denote by $C([0,T];\mathcal{B})$ the space of the $\mathcal{B}$-valued continuous functions on $[0,T]$ and by $L^2([0,T];\,\mathcal{B})$ the space of the $\mathcal{B}$-valued L^2-functions on $[0,T]$.

For the technical reason, in what follows, we shall assume that

(G4) $K(i)=0$ for $|i|>m$ for some $m\in\mathbb{N}$.

Define a weight function related to such a number $\lambda>0$,

$$w(x)=\mathrm{e}^{-2\lambda x}\quad\text{for}\quad\lambda\in(\lambda_1,\lambda_2).$$

Now we state the stability theorem for (1.3) with a general non-monotone function $g(u)$.

Theorem 2.4 (Stability of traveling waves). *Assume that* (G1)–(G4) *hold. For any given traveling wave* $\phi(x+ct)$ *with* $c > c_*$ *to* (1.3)*, whether it is monotone or non-monotone, suppose that* $U_0(s,x) := u_0(s,x) - \phi(x+cs) \in C([-\tau,0];C(\mathbb{R}))$, $\sqrt{w}U_0(s,x) \in C([-\tau,0];H^1(\mathbb{R})) \cap L^2([-\tau,0];H^1(\mathbb{R}))$, *and* $\lim_{x\to+\infty} U_0(s,x) =: U_{0,\infty}(s) \in C[-\tau,0]$ *exists uniformly with respect to* $s \in [-\tau,0]$*, and*

$$\max_{s\in[-\tau,0]}\|U_0(s)\|_C^2 + \|\sqrt{w}U_0(0)\|_{H^1}^2 + \int_{-\tau}^0 \|\sqrt{w}U_0(s)\|_{H^1}^2 ds \leqslant \delta_0^2$$

for some positive number δ_0*. Then the solution* $u(t,x)$ *of* (1.3) *and* (2.1) *uniquely and globally exists in time, and satisfies*

$$\begin{aligned}&u(t,x) - \phi(x+ct) \in \mathcal{C}_{\rm unif}[-\tau,\infty),\\ &\sqrt{w}[u(t,x) - \phi(x+ct)] \in C([-\tau,\infty);H^1(\mathbb{R})) \cap L^2([-\tau,\infty);H^1(\mathbb{R}))\end{aligned}$$

and

$$\sup_{x\in\mathbb{R}}|u(t,x) - \phi(x+ct)| \leqslant C\mathrm{e}^{-\mu t}, \quad t > 0,$$

for some constant $\mu > 0$*, where* $\mathcal{C}_{\rm unif}[-\tau,T]$ *for* $0 < T \leqslant \infty$*, is defined by*

$$\mathcal{C}_{\rm unif}[-\tau,T] = \Big\{U(t,x) \in C([-\tau,T]\times\mathbb{R}) \text{ such that } \lim_{x\to+\infty} U(t,x) \text{ exists uniformly in } t \in [-\tau,T]\Big\}.$$

Corollary 2.5 (Uniqueness of traveling waves). *Assume that* (G1)–(G4) *hold. Then, for any traveling waves* $\phi(x+ct)$ *of* (1.3)*, whether they are monotone or non-monotone, with the same speed* $c > c_*$ *and the same exponential decay at* $\xi \to -\infty$:

$$\phi(\xi) = O(1)\mathrm{e}^{-\lambda|\xi|} \quad as \quad \xi \to -\infty,$$

they are unique up to translation.

3 Stability of traveling waves

This section is devoted to the proof of stability of those monotone or non-monotone non-critical traveling waves of (1.3) when g is non-monotone.

3.1 Reformulation of the problem

Let $\phi(x+ct) = \phi(\xi)$ be a given traveling wave with speed $c > c_*$, and

$$\begin{aligned}U(t,\xi) &:= u(t,x) - \phi(x+ct) = u(t,\xi-ct) - \phi(\xi),\\ U_0(s,\xi) &:= u_0(s,x) - \phi(x+cs).\end{aligned}$$

Then, from (1.3) and (1.4), we can see that $U(t,\xi)$ satisfies

$$\begin{cases}\dfrac{\partial U}{\partial t} + c\dfrac{\partial U}{\partial \xi} - \mathcal{D}_2[U] + U - \displaystyle\sum_{i=-m}^{m} K(i)g'(\phi(\xi-c\tau-i))U(t-\tau,\xi-c\tau-i)\\ \quad = \displaystyle\sum_{i=-m}^{m} K(i)Q(U(t-\tau,\xi-c\tau-i)), \quad (t,\xi) \in \mathbb{R}_+\times\mathbb{R},\\ U(s,\xi) = U_0(s,\xi), \quad s \in [-\tau,0], \quad \xi \in \mathbb{R},\end{cases} \tag{3.1}$$

where

$$Q(U) := g(\phi+U) - g(\phi) - g'(\phi)U,$$

with $\phi = \phi(\xi - c\tau - i)$ and $U = U(t-\tau, \xi - c\tau - i)$. By Taylor's expansion formula, we know

$$|Q(U)| \leqslant C|U|^2, \tag{3.2}$$

for some positive constant C.

Let $0 \leqslant T \leqslant \infty$. We define the solution space for (3.1) as follows:

$$\begin{aligned} X(-\tau, T) = \{U \,|\, & U(t,\xi) \in C([-\tau, T]; C(\mathbb{R})) \cap \mathcal{C}_{\text{unif}}[-\tau, T], \sqrt{w}U \in C([-\tau, T]; H^1(\mathbb{R})), \\ & \text{and } \sqrt{w}U \in L^2([-\tau, T]; H^1(\mathbb{R}))\}, \end{aligned}$$

equipped with the norm

$$M_U(T)^2 = \sup_{t\in[-\tau,T]} (\|U(t)\|_C^2 + \|\sqrt{w}U(t)\|_{H^1}^2) + \int_{-\tau}^{T} \|\sqrt{w}U(s)\|_{H^1}^2 ds.$$

Particularly, when $T = \infty$, we denote the solution space by $X(-\tau, \infty)$ and the norm of the solution space by $M_U(\infty)$.

Now we state the stability result for the perturbed equation (3.1), which automatically implies Theorem 2.4.

Theorem 3.1 (Stability). *Assume that* (G1)–(G4) *hold. For any given traveling wave $\phi(x+ct) = \phi(\xi)$ with $c > c_*$, suppose that $U_0(s,\xi) \in X(-\tau, 0)$ is small enough, namely, there exists a constant $\delta_0 > 0$ such that $M_U(0) \leqslant \delta_0$. Then the solution $U(t,\xi)$ of* (3.1) *uniquely and globally exists in $X(-\tau, \infty)$ and satisfies*

$$\sup_{\xi\in\mathbb{R}} |U(t)| \leqslant C\mathrm{e}^{-\mu t}, \quad t > 0$$

for some constant $\mu > 0$.

The global existence of $U(t,\xi)$ can be obtained by the continuity extension method (see [20,21]), if we get the following local existence result and the a priori estimate.

Proposition 3.2 (Local existence). *Assume that* (G1)–(G4) *hold. For any given traveling wave $\phi(x + ct) = \phi(\xi)$ with $c > c_*$, suppose $U_0(s,\xi) \in X(-\tau, 0)$, and $M_U(0) \leqslant \delta_1$ for a given positive constant $\delta_1 > 0$. Then there exists a small $t_0 = t_0(\delta_1) > 0$ such that the local solution $U(t,\xi)$ of* (3.1) *uniquely exists for $t \in [-\tau, t_0]$, and satisfies $U(t,\xi) \in X(-\tau, t_0)$ and $M_U(t_0) \leqslant C_1 M_U(0)$ for some constant $C_1 > 1$.*

Proof. The proof for the local existence of the solution is standard, since it can be proved by the well-known iteration technique. We just sketch the proof as follows.

Let $U^{(0)}(t,\xi) := U_0(t,\xi) \in X(-\tau, 0) \subseteq X(-\tau, t_0)$. Then we define the iteration $U^{(n+1)} = \mathcal{F}(U^{(n)})$ for $n \geqslant 0$ by

$$\begin{cases} \dfrac{\partial U^{(n+1)}}{\partial t} + c\dfrac{\partial U^{(n+1)}}{\partial \xi} + (2d+1)U^{(n+1)}(t,\xi) \\ \quad = dU^{(n+1)}(t,\xi+1) + dU^{(n+1)}(t,\xi-1) + P(U^{(n)}(t-\tau,\xi-c\tau)), \quad (t,\xi) \in \mathbb{R}_+ \times \mathbb{R}, \\ U^{(n+1)}(s,\xi) = U_0(s,\xi), \quad s \in [-\tau, 0], \quad \xi \in \mathbb{R}, \end{cases} \tag{3.3}$$

where

$$P(U^{(n)}(t-\tau,\xi-c\tau)) := \sum_{i=-m}^{m} K(i)(g(\phi + U^{(n)}) - g(\phi)),$$

with $\phi = \phi(\xi - c\tau)$ and $U^{(n)} = U^{(n)}(t-\tau, \xi - c\tau)$. By Taylor's expansion formula, we obtain

$$|P(U^{(n)}(t-\tau,\xi-c\tau))| \leqslant g'(0) \sum_{i=-m}^{m} K(i)|U^{(n)}(t-\tau,\xi-c\tau-i)|. \tag{3.4}$$

The solution of (3.3) can be written in the form

$$U^{(n+1)}(t,\xi) = \mathrm{e}^{-(2d+1)t}U_0(0,\xi-ct) + \mathrm{e}^{-(2d+1)t}\int_0^t \mathrm{e}^{(2d+1)s}[dU^{(n+1)}(s,\xi+1+c(s-t)) \\ + dU^{(n+1)}(s,\xi-1+c(s-t)) + P(U^{(n)}(s-\tau,\xi+c(s-t-\tau)))]ds. \tag{3.5}$$

Combining (3.4) and (3.5), one has

$$\begin{aligned}\|U^{(n+1)}(t)\|_C &\leqslant \mathrm{e}^{-(2d+1)t}\|U_0(0)\|_C + 2d\int_0^t \mathrm{e}^{-(2d+1)(t-s)}\|U^{(n+1)}(s)\|_C ds \\ &\quad + C\int_0^t \mathrm{e}^{-(2d+1)(t-s)}\|U^{(n)}(s-\tau)\|_C ds \\ &\leqslant \|U_0(0)\|_C + Ct_0 \sup_{t\in[-\tau,t_0]}\|U^{(n)}(t)\|_C + 2d\int_0^t \|U^{(n+1)}(s)\|_C ds.\end{aligned}$$

Applying Gronwall's inequality, we get

$$\|U^{(n+1)}(t)\|_C \leqslant \Big(\|U_0(0)\|_C + Ct_0\sup_{t\in[-\tau,t_0]}\|U^{(n)}(t)\|_C\Big)\mathrm{e}^{2dt_0}, \quad t\in[0,t_0]. \tag{3.6}$$

Notice that $U^{(n)}(t,\xi)\in\mathcal{C}_{\mathrm{unif}}[-\tau,t_0]$, namely, $\lim_{\xi\to\infty}U^{(n)}(t,\xi) =: U^{(n)}_\infty(t)\in C[-\tau,t_0]$. We are going to prove $U^{(n+1)}(t,\xi)\in\mathcal{C}_{\mathrm{unif}}[-\tau,t_0]$. We rewrite the solution of (3.3) as

$$\begin{aligned}U^{(n+1)}(t,\xi) = \mathrm{e}^{-t}U_0(0,\xi-ct) + \mathrm{e}^{-t}\int_0^t \mathrm{e}^{s}[&dU^{(n+1)}(s,\xi+1+c(s-t)) \\ &- 2dU^{(n+1)}(s,\xi+c(s-t)) + dU^{(n+1)}(s,\xi-1+c(s-t)) \\ &+ P(U^{(n)}(s-\tau,\xi+c(s-t-\tau)))]ds.\end{aligned}$$

It is clear that

$$\begin{aligned}\lim_{\xi\to\infty}U^{(n+1)}(t,\xi) &= \mathrm{e}^{-t}\lim_{\xi\to\infty}U_0(0,\xi-ct) + \mathrm{e}^{-t}\int_0^t \mathrm{e}^{s}\lim_{\xi\to\infty}P(U^{(n)}(s-\tau,\xi+c(s-t-\tau)))ds \\ &= U_{0,\infty}(0)\mathrm{e}^{-t} + \int_0^t \mathrm{e}^{-(t-s)}P(U^{(n)}_\infty(s-\tau))ds \\ &=: U^{(n+1)}_\infty(t), \quad \text{uniformly with respect to } t\in[-\tau,t_0].\end{aligned}$$

We further prove that $U^{(n+1)}(t,\xi)$ is uniformly convergent as $\xi\to\infty$. In fact,

$$\begin{aligned}&\lim_{\xi\to\infty}\sup_{0\leqslant t\leqslant t_0}|U^{(n+1)}(t,\xi) - U^{(n+1)}_\infty(t)| \\ &\quad= \lim_{\xi\to\infty}\sup_{0\leqslant t\leqslant t_0}\bigg|\int_0^t \mathrm{e}^{-(t-s)}[P(U^{(n)}(s-\tau,\xi+c(s-t-\tau))) - P(U^{(n)}_\infty(s-\tau))]ds\bigg| \\ &\quad\leqslant \lim_{\xi\to\infty}\sup_{0\leqslant t\leqslant t_0}\int_0^t \sup_{s\in[0,t_0]}(\mathrm{e}^{-(t-s)}|P(U^{(n)}(s-\tau,\xi+c(s-t-\tau))) - P(U^{(n)}_\infty(s-\tau))|)ds \\ &\quad\leqslant C\lim_{\xi\to\infty}\sup_{0\leqslant t\leqslant t_0}\int_0^t \sup_{s\in[0,t_0]}(\mathrm{e}^{-(t-s)}|U^{(n)}(s-\tau,\xi+c(s-t-\tau)) - U^{(n)}_\infty(s-\tau)|)ds \\ &\quad= C\sup_{0\leqslant t\leqslant t_0}\int_0^t \lim_{\xi\to\infty}\sup_{s\in[0,t_0]}(\mathrm{e}^{-(t-s)}|U^{(n)}(s-\tau,\xi+c(s-t-\tau)) - U^{(n)}_\infty(s-\tau)|)ds \\ &\quad= 0.\end{aligned}$$

Here, we used the uniform convergence of

$$\lim_{\xi\to\infty}\sup_{t\in[0,t_0]}|U^{(n)}(t,\xi) - U^{(n)}_\infty(t)| = 0.$$

Next, we shall show the regular energy estimates for $U^{(n+1)}(t,\xi)$. First of all, we introduce the transformation

$$V^{(n+1)}(t,\xi)=\sqrt{w(\xi)}U^{(n+1)}(t,\xi)=\mathrm{e}^{-\lambda\xi}U^{(n+1)}(t,\xi).$$

Substituting $U^{(n+1)}(t,\xi)=\mathrm{e}^{\lambda\xi}V^{(n+1)}(t,\xi)$ to (3.3), we derive the following equation for $V^{(n+1)}(t,\xi)$:

$$\begin{cases}\dfrac{\partial V^{(n+1)}}{\partial t}+c\dfrac{\partial V^{(n+1)}}{\partial\xi}+(c\lambda+2d+1)V^{(n+1)}(t,\xi)\\ \quad=d\mathrm{e}^{\lambda}V^{(n+1)}(t,\xi+1)+d\mathrm{e}^{-\lambda}V^{(n+1)}(t,\xi-1)\\ \qquad+\mathrm{e}^{-\lambda\xi}P(U^{(n)}(t-\tau,\xi-c\tau)),\quad (t,\xi)\in\mathbb{R}_+\times\mathbb{R},\\ V^{(n+1)}(s,\xi)=\sqrt{w(\xi)}U_0(s,\xi)=:V_0(s,\xi),\quad s\in[-\tau,0],\quad \xi\in\mathbb{R}.\end{cases}\tag{3.7}$$

Multiplying (3.7) by $V^{(n+1)}(t,\xi)$, we have

$$\begin{aligned}&\left\{\frac{1}{2}(V^{(n+1)})^2(t,\xi)\right\}_t+\left\{\frac{c}{2}(V^{(n+1)})^2(t,\xi)\right\}_\xi+[c\lambda+2d+1](v^{(n+1)})^2(t,\xi)\\&\quad=d\mathrm{e}^{\lambda}V^{(n+1)}(t,\xi)V^{(n+1)}(t,\xi+1)+d\mathrm{e}^{-\lambda}V^{(n+1)}(t,\xi)V^{(n+1)}(t,\xi-1)\\&\qquad+\mathrm{e}^{-\lambda\xi}V^{(n+1)}(t,\xi)P(U^{(n)}(t-\tau,\xi-c\tau)).\end{aligned}\tag{3.8}$$

Integrating (3.8) over $\mathbb{R}\times[0,t]$ with respect to ξ and t, and noting the vanishing term at far fields, we have

$$\left\{\frac{c}{2}(V^{(n+1)})^2(t,\xi)\right\}\Big|_{\xi=-\infty}^{\infty}=0,$$

because $V^{(n+1)}(t,\xi)=\sqrt{w(\xi)}U^{(n+1)}(t,\xi)\in H^1(\mathbb{R})$. Thus, we obtain

$$\begin{aligned}&\|V^{(n+1)}(t)\|_{L^2}^2+\int_0^t\int_{\mathbb{R}}[2c\lambda+4d+2](V^{(n+1)})^2(s,\xi)d\xi ds\\&\quad=\|V_0^{(n+1)}(0)\|_{L^2}^2+2d\mathrm{e}^{\lambda}\int_0^t\int_{\mathbb{R}}V^{(n+1)}(s,\xi)V^{(n+1)}(s,\xi+1)d\xi ds\\&\qquad+2d\mathrm{e}^{-\lambda}\int_0^t\int_{\mathbb{R}}V^{(n+1)}(s,\xi)V^{(n+1)}(s,\xi-1)d\xi ds\\&\qquad+2\int_0^t\int_{\mathbb{R}}\mathrm{e}^{-\lambda\xi}V^{(n+1)}(s,\xi)P(U^{(n)}(s-\tau,\xi-c\tau))d\xi ds.\end{aligned}\tag{3.9}$$

By the Cauchy-Schwarz inequality, one has

$$\begin{aligned}&\left|\int_0^t\int_{\mathbb{R}}V^{(n+1)}(s,\xi)V^{(n+1)}(s,\xi\pm1)d\xi ds\right|\\&\quad\leqslant\frac{1}{2}\int_0^t\int_{\mathbb{R}}(V^{(n+1)})^2(s,\xi)d\xi ds+\frac{1}{2}\int_0^t\int_{\mathbb{R}}(V^{(n+1)})^2(s,\xi\pm1)d\xi ds\\&\quad=\int_0^t\int_{\mathbb{R}}(V^{(n+1)})^2(s,\xi)d\xi ds.\end{aligned}\tag{3.10}$$

Applying (3.10) to (3.9), we get

$$\begin{aligned}&\|V^{(n+1)}(t)\|_{L^2}^2+\int_0^t\int_{\mathbb{R}}[2c\lambda-2d(\mathrm{e}^{\lambda}+\mathrm{e}^{-\lambda}-2)+2](V^{(n+1)})^2(s,\xi)d\xi ds\\&\quad\leqslant\|V_0^{(n+1)}(0)\|_{L^2}^2+2\int_0^t\int_{\mathbb{R}}\mathrm{e}^{-\lambda\xi}V^{(n+1)}(s,\xi)P(U^{(n)}(s-\tau,\xi-c\tau))d\xi ds.\end{aligned}\tag{3.11}$$

From (3.4) and by using the Cauchy-Schwarz inequality, the right-hand side of (3.11) can be estimated by

$$\left|2\int_0^t\int_{\mathbb{R}}\mathrm{e}^{-\lambda\xi}V^{(n+1)}(s,\xi)P(U^{(n)}(s-\tau,\xi-c\tau))d\xi ds\right|$$

$$
\begin{aligned}
&\leqslant 2g'(0)\int_0^t\int_{\mathbb{R}}\sum_{i=-m}^{m}K(i)\mathrm{e}^{-\lambda\xi}|U^{(n)}(s-\tau,\xi-c\tau-i)||V^{(n+1)}(s,\xi)|d\xi ds\\
&=2g'(0)\int_0^t\int_{\mathbb{R}}\sum_{i=-m}^{m}K(i)\mathrm{e}^{-\lambda(i+c\tau)}\mathrm{e}^{-\lambda(\xi-c\tau-i)}|U^{(n)}(s-\tau,\xi-c\tau-i)||V^{(n+1)}(s,\xi)|d\xi ds\\
&=2g'(0)\int_0^t\int_{\mathbb{R}}\sum_{i=-m}^{m}K(i)\mathrm{e}^{-\lambda(i+c\tau)}|V^{(n)}(s-\tau,\xi-c\tau-i)||V^{(n+1)}(s,\xi)|d\xi ds\\
&\leqslant g'(0)\int_0^t\int_{\mathbb{R}}G(\lambda)(V^{(n+1)})^2(s,\xi)d\xi ds\\
&\quad+g'(0)\int_0^t\int_{\mathbb{R}}\sum_{i=-m}^{m}K(i)\mathrm{e}^{-\lambda(i+c\tau)}(V^{(n)})^2(s-\tau,\xi-c\tau-i)d\xi ds\\
&=g'(0)\int_0^t\int_{\mathbb{R}}G(\lambda)(V^{(n+1)})^2(s,\xi)d\xi ds+g'(0)\int_{-\tau}^{t-\tau}\int_{\mathbb{R}}G(\lambda)(V^{(n)})^2(s,\xi)d\xi ds\\
&\leqslant g'(0)\int_0^t\int_{\mathbb{R}}G(\lambda)(V^{(n+1)})^2(s,\xi)d\xi ds+g'(0)\int_0^t\int_{\mathbb{R}}G(\lambda)(V^{(n)})^2(s,\xi)d\xi ds\\
&\quad+g'(0)\int_{-\tau}^{0}\int_{\mathbb{R}}G(\lambda)(V_0^{(n)})^2(s,\xi)d\xi ds.
\end{aligned}\tag{3.12}
$$

Substituting (3.12) into (3.11), we have

$$
\begin{aligned}
&\|V^{(n+1)}(t)\|_{L^2}^2+\mu_0\int_0^t\|V^{(n+1)}(s)\|_{L^2}^2ds\\
&\quad\leqslant\|V_0^{(n+1)}(0)\|_{L^2}^2+g'(0)G(\lambda)\int_0^t\|V^{(n)}(s)\|_{L^2}^2ds+g'(0)G(\lambda)\int_{-\tau}^{0}\|V_0^{(n)}(s)\|_{L^2}^2ds,
\end{aligned}
$$

where

$$
\mu_0:=2[c\lambda-d(\mathrm{e}^{\lambda}+\mathrm{e}^{-\lambda}-2)+1]-g'(0)G(\lambda)>g'(0)G(\lambda)>0.
$$

Namely,

$$
\begin{aligned}
&\|V^{(n+1)}(t)\|_{L^2}^2+\int_0^t\|V^{(n+1)}(s)\|_{L^2}^2ds\\
&\quad\leqslant C\bigg(\|V_0(0)\|_{L^2}^2+\int_0^t\|V^{(n)}(s)\|_{L^2}^2ds+\int_{-\tau}^{0}\|V_0(s)\|_{L^2}^2ds\bigg).
\end{aligned}\tag{3.13}
$$

Similarly, differentiating (3.7) with respect to ξ and multiplying it by $V_\xi^{(n+1)}(t,\xi)$, and integrating the resultant equation over $\mathbb{R}\times[0,t]$ with respect to ξ and t, we can prove

$$
\begin{aligned}
&\|V_\xi^{(n+1)}(t)\|_{L^2}^2+\int_0^t\|V_\xi^{(n+1)}(s)\|_{L^2}^2ds\\
&\quad\leqslant C\bigg(\|V_{0,\xi}(0)\|_{L^2}^2+\int_0^t\|V_\xi^{(n)}(s)\|_{L^2}^2ds+\int_{-\tau}^{0}\|V_{0,\xi}(s)\|_{L^2}^2ds\bigg).
\end{aligned}\tag{3.14}
$$

Combining (3.13) and (3.14), we further have

$$
\begin{aligned}
&\|V^{(n+1)}(t)\|_{H^1}^2+\int_0^t\|V^{(n+1)}(s)\|_{H^1}^2ds\\
&\quad\leqslant C\bigg(\|V_0(0)\|_{H^1}^2+\int_0^t\|V^{(n)}(s)\|_{H^1}^2ds+\int_{-\tau}^{0}\|V_0(s)\|_{H^1}^2ds\bigg),
\end{aligned}
$$

namely,

$$
\|\sqrt{w}U^{(n+1)}(t)\|_{H^1}^2+\int_0^t\|\sqrt{w}U^{(n+1)}(s)\|_{H^1}^2ds
$$

$$\leqslant C\bigg(\|\sqrt{w}U_0(0)\|_{H^1}^2+\int_0^t\|\sqrt{w}U^{(n)}(s)\|_{H^1}^2ds+\int_{-\tau}^0\|\sqrt{w}U_0(s)\|_{H^1}^2ds\bigg). \tag{3.15}$$

Combining (3.6) and (3.15), we get

$$\begin{aligned}M_{U^{(n+1)}}(t_0)\leqslant C\bigg(&\max_{s\in[-\tau,0]}\|U_0(s)\|_C^2+\|\sqrt{w}U_0(0)\|_{H^1}^2+\int_{-\tau}^0\|\sqrt{w}U_0(s)\|_{H^1}^2ds\bigg)\\&+Ct_0M_{U^{(n)}}(t_0).\end{aligned}$$

Thus, we can prove that $U^{(n+1)}=\mathcal{F}(U^{(n)})$ defined in (3.3) maps from $X(-\tau,t_0)$ to $X(-\tau,t_0)$ and leads to a contraction mapping in $X(-\tau,t_0)$ by providing $0<t_0\ll 1$ and

$$\max_{s\in[-\tau,0]}\|U_0(s)\|_C^2+\|\sqrt{w}U_0(0)\|_{H^1}^2+\int_{-\tau}^0\|\sqrt{w}U_0(s)\|_{H^1}^2ds\ll 1.$$

Hence, by applying the Banach fixed point theorem, we can prove local existence of the solution in $X(-\tau,t_0)$. Since the convergence $\lim_{n\to\infty}U^{(n)}(t,\xi)=U(t,\xi)$ is uniform for $(t,\xi)\in[0,t_0]\times\mathbb{R}$, and $U^{(n)}(t,\xi)\in\mathcal{C}_{\text{unif}}[0,t_0]$, we can also guarantee $U(t,\xi)\in\mathcal{C}_{\text{unif}}[0,t_0]$. □

Proposition 3.3 (A priori estimates). *Assume that* (G1)–(G4) *hold. For any given traveling wave* $\phi(x+ct)=\phi(\xi)$ *with* $c>c_*$*, let* $U(t,\xi)\in X(-\tau,T)$ *be a local solution of* (3.1) *for a given constant* $T>0$*. Then there exist positive constants* $\delta_2>0$*,* $C_2>1$ *and* $\mu>0$ *independent of* T *and* $U(t,\xi)$ *such that, when* $M_U(T)\leqslant\delta_2$*,*

$$\|U(t)\|_C^2+\|\sqrt{w}U(t)\|_{H^1}^2+\int_0^t e^{-2\mu(t-s)}\|\sqrt{w}U(s)\|_{H^1}^2ds\leqslant C_2e^{-2\mu t}M_U(0)^2, \tag{3.16}$$

for $t\in[0,T]$.

The proof for the a priori estimates of the solution in the designed solution space $X(-\tau,T)$ is technical and plays a crucial role in this paper. We leave this for the next section.

Proof of Theorem 3.1. Choose

$$\delta_1=\max\{\sqrt{C_2}C_1M_U(0),\delta_2\},\quad \delta_0=\max\bigg\{\frac{\delta_2}{C_1},\frac{\delta_2}{\sqrt{C_2}C_1}\bigg\},$$

where δ_2 and C_2 are two positive constants given in Proposition 3.3, and C_1 is a positive constant given in Proposition 3.2. It follows from Proposition 3.2 that there exists $t_0=t_0(\delta_1)>0$ such that $U(t,\xi)\in X(-\tau,t_0)$. By the selection of δ_0 and δ_1, we can see that $M_U(t_0)\leqslant\delta_2$. Then by Proposition 3.3, we can obtain the exponential decay estimate (3.16) for $t\in[0,t_0]$. Next, we consider (3.1) with the new initial data $U(s,\xi)$ for $s\in[t_0-\tau,t_0]$. Again, by Proposition 3.2, we can prove that the solution to the new Cauchy problem (3.1) exists for time t in $[t_0,2t_0]$, which means the time interval of the solution has been extended to $[-\tau,2t_0]$, namely, $U(t,\xi)\in X(-\tau,2t_0)$. Furthermore, by using Proposition 3.3, we can establish the exponential decay estimate (3.16) for $t\in[0,2t_0]$. Repeating this procedure, we can prove global existence of the solution $U(t,\xi)\in X(-\tau,\infty)$ with the exponential decay estimate (3.16) for $t\in[0,\infty]$. For details, we refer the reader to [15, 20]. The proof is completed. □

3.2 A priori estimates

In this subsection, we first prove the time-exponential decay of $U(t,\xi)$ at $\xi=+\infty$.

Lemma 3.4. *There exist a large number* $x_0\gg 1$ (*independent of* t) *and a number* $\mu_1>0$*, such that*

$$\|U(t)\|_{L^\infty[x_0,+\infty)}\leqslant Ce^{-\mu_1 t}\|U_0\|_{L^\infty([-\tau,0]\times\mathbb{R})},\quad t\geqslant 0. \tag{3.17}$$

Proof. Since $U(t,\xi)\in X(-\tau,T)$, by the definition of $\mathcal{C}_{\rm unif}[0,T]$, we can see that $\lim_{\xi\to+\infty}U(t,\xi)$ exists uniformly with respect to $t\in[0,T]$. Let us go back to the original equations (1.3), (2.1) and (1.4), and denote

$$\mathcal{U}(t,x)=u(t,x)-\phi(x+ct).$$

Namely, $\mathcal{U}(t,x)=U(t,\xi)$ and satisfies

$$\begin{cases}\dfrac{\partial\mathcal{U}}{\partial t}-\mathcal{D}_2[\mathcal{U}]+\mathcal{U}-\displaystyle\sum_{i=-m}^{m}K(i)g'(\phi(x+c(t-\tau)-i))\mathcal{U}(t-\tau,x-i)\\ \quad=\displaystyle\sum_{i=-m}^{m}K(i)Q(\mathcal{U}(t-\tau,x-i)),\quad (t,x)\in\mathbb{R}_+\times\mathbb{R},\\ \mathcal{U}(s,x)=\mathcal{U}_0(s,x),\quad s\in[-\tau,0],\quad x\in\mathbb{R}.\end{cases}\tag{3.18}$$

Denote $z(t):=U(t,+\infty)=\mathcal{U}(t,+\infty)$ and $z_0(s):=\mathcal{U}_0(s,+\infty)$ for $s\in[-\tau,0]$. Since $U(t,\xi)\in\mathcal{C}_{\rm unif}[0,T]$, $\lim_{\xi\to+\infty}U(t,\xi)=\lim_{x\to+\infty}\mathcal{U}(t,x)=z(t)$ is uniformly with respect to $t\in[0,T]$. Thus, by taking $x\to+\infty$ to (3.18), we have

$$\begin{cases}z'(t)+z(t)-g'(u_+)z(t-\tau)=Q(z(t-\tau)),\\ z(s)=z_0(s),\quad s\in[-\tau,0].\end{cases}$$

Applying the nonlinear Halanay's inequality given in [15], we get

$$|z(t)|\leqslant C\|z_0\|_{L^\infty(-\tau,0)}\mathrm{e}^{-\mu_1 t},\quad t>0\tag{3.19}$$

for some $0<\mu_1<1$.

It is easy to see that (3.18) is equivalent to

$$\begin{aligned}\left\{\mathrm{e}^t\mathcal{U}\right\}_t=\mathrm{e}^t\mathcal{D}_2[\mathcal{U}]&+\mathrm{e}^t\sum_{i=-m}^{m}K(i)g'(\phi(x+c(t-\tau)-i))\mathcal{U}(t-\tau,x-i)\\ &+\mathrm{e}^t\sum_{i=-m}^{m}K(i)Q(\mathcal{U}(t-\tau,x-i)).\end{aligned}\tag{3.20}$$

Integrating (3.20) with respect to t over $[0,t]$, we obtain

$$\begin{aligned}\mathcal{U}(t,x)=\mathrm{e}^{-t}\mathcal{U}_0(0,x)&+\int_0^t\mathrm{e}^{-(t-s)}\mathcal{D}_2[\mathcal{U}](s,x)ds\\ &+\int_0^t\mathrm{e}^{-(t-s)}\sum_{i=-m}^{m}K(i)g'(\phi(x+c(s-\tau)-i))\mathcal{U}(s-\tau,x-i)ds\\ &+\int_0^t\mathrm{e}^{-(t-s)}\sum_{i=-m}^{m}K(i)Q(\mathcal{U}(s-\tau,x-i))ds.\end{aligned}$$

Thus, for $0<\mu_1<1$, we further have

$$\begin{aligned}\mathrm{e}^{\mu_1 t}\mathcal{U}(t,x)=\mathrm{e}^{-(1-\mu_1)t}\mathcal{U}_0(0,x)&+\mathrm{e}^{-(1-\mu_1)t}\int_0^t\mathrm{e}^s\mathcal{D}_2[\mathcal{U}](s,x)ds\\ &+\mathrm{e}^{-(1-\mu_1)t}\int_0^t\mathrm{e}^s\sum_{i=-m}^{m}K(i)g'(\phi(x+c(s-\tau)-i))\mathcal{U}(s-\tau,x-i)ds\\ &+\mathrm{e}^{-(1-\mu_1)t}\int_0^t\mathrm{e}^s\sum_{i=-m}^{m}K(i)Q(\mathcal{U}(s-\tau,x-i))ds.\end{aligned}\tag{3.21}$$

Taking the limits to (3.21) as $x \to +\infty$, and noting all these limits are uniformly in t, and applying the fact $|Q(z)| \leqslant Cz^2$ and the decay estimate (3.19) for $z(t)$, we get

$$\begin{aligned}\lim_{x\to+\infty} \mathrm{e}^{\mu_1 t}\mathcal{U}(t,x) &= \mathrm{e}^{-(1-\mu_1)t}\left[z_0(0) + \int_0^t \mathrm{e}^s g'(u_+)z(s-\tau)ds + \int_0^t \mathrm{e}^s Q(z(s-\tau))ds\right] \\ &\leqslant C\mathrm{e}^{-(1-\mu_1)t}\left[|z_0(0)| + \int_0^t \mathrm{e}^s|z(s-\tau)|ds + \int_0^t \mathrm{e}^s|z(s-\tau)|^2 ds\right] \\ &\leqslant C\mathrm{e}^{-(1-\mu_1)t}\left[|z_0(0)| + \int_0^t \mathrm{e}^s\mathrm{e}^{-\mu_2(s-\tau)}ds + \int_0^t \mathrm{e}^s\mathrm{e}^{-2\mu_2(s-\tau)}ds\right] \\ &\leqslant C, \quad \text{uniformly in } t.\end{aligned}$$

Therefore, there exists a number $x_0 \gg 1$ independent of t, such that when $x \geqslant x_0$, one has

$$\sup_{x\in[x_0,+\infty)} |\mathcal{U}(t,x)| \leqslant C\mathrm{e}^{-\mu_1 t}\|\mathcal{U}_0\|_{L^\infty([-\tau,0]\times\mathbb{R})}, \quad t \geqslant 0. \tag{3.22}$$

Again, notice that $U(t,\xi) = \mathcal{U}(t,x)$ and $\xi = x+ct \geqslant x \geqslant x_0$ for $x \geqslant x_0$ and $t \geqslant 0$, and then (3.22) immediately implies

$$\sup_{\xi\in[x_0,+\infty)} |U(t,\xi)| \leqslant C\mathrm{e}^{-\mu_1 t}\|U_0\|_{L^\infty([-\tau,0]\times\mathbb{R})}, \quad t \geqslant 0.$$

The proof is completed. □

In order to establish the a priori estimate (3.16), we adopt a new transformed energy method, which is different from the standard weighted energy method by multiplying (3.1) by $w(\xi)U(t,\xi)$. We first shift $U(t,\xi)$ to $U(t,\xi+x_0)$ by the constant x_0 given in Lemma 3.4, and then introduce the following transformation:

$$V(t,\xi) = \sqrt{w(\xi)}U(t,\xi+x_0) = \mathrm{e}^{-\lambda\xi}U(t,\xi+x_0),$$

where $\mathrm{e}^{-\lambda\xi} \to +\infty$ as $\xi \to -\infty$, and $\mathrm{e}^{-\lambda\xi} \to 0$ as $\xi \to +\infty$. Substituting $U = w^{-1/2}V$ to (3.1) yields

$$\begin{cases} \dfrac{\partial V}{\partial t} + c\dfrac{\partial V}{\partial \xi} + (c\lambda+2d+1)V(t,\xi) - d\mathrm{e}^{\lambda}V(t,\xi+1) - d\mathrm{e}^{-\lambda}V(t,\xi-1) \\ \quad - \displaystyle\sum_{i=-m}^{m} K(i)g'(\phi(\xi-c\tau-i+x_0))\mathrm{e}^{-\lambda(i+c\tau)}V(t-\tau,\xi-c\tau-i) \\ \quad = \displaystyle\sum_{i=-m}^{m} K(i)\sqrt{w(\xi)}Q(U(t-\tau,\xi-c\tau-i+x_0)), \quad (t,\xi)\in\mathbb{R}_+\times\mathbb{R}, \\ V(s,\xi) = \sqrt{w(\xi)}U_0(s,\xi+x_0) =: V_0(s,\xi), \quad s\in[-\tau,0], \quad \xi\in\mathbb{R}. \end{cases} \tag{3.23}$$

Now we are going to prove the a priori estimate (3.16) by the following several lemmas.

Lemma 3.5. *It holds that*

$$\frac{1}{2}\frac{d}{dt}\|V(t)\|_{L^2}^2 + \mu_2\|V(t)\|_{L^2}^2 + C_3[\|V(t)\|_{L^2}^2 - \|V(t-\tau)\|_{L^2}^2] \leqslant \mathcal{R}(t), \tag{3.24}$$

where

$$\begin{aligned} &C_3 := \frac{1}{2}g'(0)G(\lambda) > 0, \\ &\mu_2 := c\lambda - d(\mathrm{e}^{\lambda}+\mathrm{e}^{-\lambda}-2) + 1 - g'(0)G(\lambda) > 0, \quad (\textit{see Lemma } 2.1), \\ &\mathcal{R}(t) := \int_{\mathbb{R}} V(t,\xi)\sum_{i=-m}^{m} K(i)\sqrt{w(\xi)}Q(U(t-\tau,\xi-c\tau-i+x_0))d\xi. \end{aligned}$$

Proof. Multiplying (3.23) by $V(t,\xi)$ and integrating the resultant equation over $\mathbb{R}$ with respect to ξ, we have

$$\frac{1}{2}\frac{d}{dt}\|V(t)\|_{L^2}^2 + \int_{\mathbb{R}}[c\lambda+2d+1]V^2(t,\xi)d\xi$$

$$
\begin{aligned}
&- de^{\lambda}\int_{\mathbb{R}} V(t,\xi)V(t,\xi+1)d\xi - de^{-\lambda}\int_{\mathbb{R}} V(t,\xi)V(t,\xi-1)d\xi \\
&- \int_{\mathbb{R}} V(t,\xi)\bigg(\sum_{i=-m}^{m} K(i)g'(\phi(\xi-c\tau-i+x_0))\mathrm{e}^{-\lambda(i+c\tau)}V(t-\tau,\xi-c\tau-i)\bigg)d\xi \\
&\quad = \mathcal{R}(t). \qquad (3.25)
\end{aligned}
$$

By using the Cauchy-Schwarz inequality, one has

$$
\begin{aligned}
\bigg|\int_{\mathbb{R}} V(t,\xi)V(t,\xi\pm 1)d\xi\bigg| &\leqslant \frac{1}{2}\bigg(\int_{\mathbb{R}} V^2(t,\xi)d\xi + \int_{\mathbb{R}} V^2(t,\xi\pm 1)d\xi\bigg) \\
&= \int_{\mathbb{R}} V^2(t,\xi)d\xi. \qquad (3.26)
\end{aligned}
$$

It can be seen from (G3) that $|g'(u)| \leqslant g'(0)$ for $u \in [0,\infty)$. Then we can obtain the following estimate:

$$
\begin{aligned}
&\bigg|\int_{\mathbb{R}} V(t,\xi)\bigg(\sum_{i=-m}^{m} K(i)g'(\phi(\xi-c\tau-i+x_0))\mathrm{e}^{-\lambda(i+c\tau)}V(t-\tau,\xi-c\tau-i)\bigg)d\xi\bigg| \\
&\leqslant \int_{\mathbb{R}} \bigg|\sum_{i=-m}^{m} K(i)g'(\phi(\xi-c\tau-i+x_0))\mathrm{e}^{-\lambda(i+c\tau)}V(t,\xi)V(t-\tau,\xi-c\tau-i)\bigg|d\xi \\
&\leqslant g'(0)\int_{\mathbb{R}} \sum_{i=-m}^{m} K(i)\mathrm{e}^{-\lambda(i+c\tau)}|V(t,\xi)||V(t-\tau,\xi-c\tau-i)|d\xi \\
&\leqslant \frac{1}{2}g'(0)\int_{\mathbb{R}} G(\lambda)V^2(t,\xi)d\xi + \frac{1}{2}g'(0)\int_{\mathbb{R}} \sum_{i=-m}^{m} K(i)\mathrm{e}^{-\lambda(i+c\tau)}V^2(t-\tau,\xi-c\tau-i)d\xi \\
&= \frac{1}{2}g'(0)G(\lambda)\|V(t)\|_{L^2}^2 + \frac{1}{2}g'(0)\int_{\mathbb{R}} G(\lambda)V^2(t-\tau,\xi)d\xi \\
&= \frac{1}{2}g'(0)G(\lambda)(\|V(t)\|_{L^2}^2 + \|V(t-\tau)\|_{L^2}^2). \qquad (3.27)
\end{aligned}
$$

Substituting (3.26) and (3.27) into (3.25) yields

$$
\frac{1}{2}\frac{d}{dt}\|V(t)\|_{L^2}^2 + [c\lambda - d(\mathrm{e}^{\lambda}+\mathrm{e}^{-\lambda}-2)+1]\|V(t)\|_{L^2}^2 - \frac{1}{2}g'(0)G(\lambda)(\|V(t)\|_{L^2}^2 + \|V(t-\tau)\|_{L^2}^2) \leqslant \mathcal{R}(t),
$$

namely,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|V(t)\|_{L^2}^2 + [c\lambda - d(\mathrm{e}^{\lambda}+\mathrm{e}^{-\lambda}-2)+1-g'(0)G(\lambda)]\|V(t)\|_{L^2}^2 \\
&\quad + \frac{1}{2}g'(0)G(\lambda)(\|V(t)\|_{L^2}^2 - \|V(t-\tau)\|_{L^2}^2) \leqslant \mathcal{R}(t),
\end{aligned}
$$

which immediately implies (3.24). The proof is completed. □

Lemma 3.6. *There exists* $0<\mu<\mu_2$ *such that*

$$
\|V(t)\|_{L^2}^2 + \int_0^t \mathrm{e}^{-2\mu(t-s)}\|V(s)\|_{L^2}^2 ds \leqslant C\mathrm{e}^{-2\mu t}\bigg(\|V_0(0)\|_{L^2}^2 + \int_{-\tau}^0 \|V_0(s)\|_{L^2}^2 ds\bigg), \qquad (3.28)
$$

provided $M_U(T) \ll 1$.

Proof. Multiplying (3.24) by $\mathrm{e}^{2\mu t}$ and integrating the resultant inequality with respect to t over $[0,t]$, where $\mu>0$ will be selected later, we have

$$
\begin{aligned}
&\mathrm{e}^{2\mu t}\|V(t)\|_{L^2}^2 + 2(\mu_2-\mu)\int_0^t \mathrm{e}^{2\mu s}\|V(s)\|_{L^2}^2 ds + 2C_3\int_0^t \mathrm{e}^{2\mu s}[\|V(s)\|_{L^2}^2 - \|V(s-\tau)\|_{L^2}^2]ds \\
&\leqslant \|V_0(0)\|_{L^2}^2 + 2\int_0^t \mathrm{e}^{2\mu s}\mathcal{R}(s)ds. \qquad (3.29)
\end{aligned}
$$

Notice that, by the change of variable $s-\tau \to s$,

$$\begin{aligned}\int_0^t \mathrm{e}^{2\mu s}\|V(s-\tau)\|_{L^2}^2 ds &= \int_{-\tau}^{t-\tau} \mathrm{e}^{2\mu(s+\tau)}\|V(s)\|_{L^2}^2 ds\\ &\leqslant \int_{-\tau}^{0} \mathrm{e}^{2\mu(s+\tau)}\|V_0(s)\|_{L^2}^2 ds + \int_0^t \mathrm{e}^{2\mu(s+\tau)}\|V(s)\|_{L^2}^2 ds. \end{aligned} \tag{3.30}$$

Substituting (3.30) into (3.29), we get

$$\begin{aligned}&\mathrm{e}^{2\mu t}\|V(t)\|_{L^2}^2 + 2[(\mu_2-\mu)+C_3(1-\mathrm{e}^{2\mu\tau})]\int_0^t \mathrm{e}^{2\mu s}\|V(s)\|_{L^2}^2 ds\\ &\quad\leqslant \|V_0(0)\|_{L^2}^2 + 2\int_0^t \mathrm{e}^{2\mu s}\mathcal{R}(s)ds + 2C_3\mathrm{e}^{2\mu\tau}\int_{-\tau}^0 \mathrm{e}^{2\mu s}\|V_0(s)\|_{L^2}^2 ds. \end{aligned} \tag{3.31}$$

We choose $0<\mu<\mu_2$ to be small such that

$$C_4 := (\mu_2-\mu)+C_3(1-\mathrm{e}^{2\mu\tau})>0.$$

Then (3.31) becomes

$$\begin{aligned}&\|V(t)\|_{L^2}^2 + 2C_4\int_0^t \mathrm{e}^{-2\mu(t-s)}\|V(s)\|_{L^2}^2 ds\\ &\quad\leqslant C\mathrm{e}^{-2\mu t}\bigg(\|V_0(0)\|_{L^2}^2 + \int_{-\tau}^0 \mathrm{e}^{2\mu s}\|V_0(s)\|_{L^2}^2 ds\bigg) + 2\int_0^t \mathrm{e}^{-2\mu(t-s)}\mathcal{R}(s)ds. \end{aligned} \tag{3.32}$$

We now estimate the nonlinear terms involving $\mathcal{R}(t)$. Since $U(t,\xi)\in X(0,T)$, namely, $U\in C(\mathbb{R})$, one has

$$|U(t,\xi+x_0)|\leqslant CM_U(T).$$

By (3.2), we get

$$|Q(U(s-\tau,\xi-c\tau-i+x_0))|\leqslant CU^2(s-\tau,\xi-c\tau-i+x_0).$$

Then noting $V(t,\xi)=\sqrt{w(\xi)}U(t,\xi+x_0)=\mathrm{e}^{-\lambda\xi}U(t,\xi+x_0)$ and

$$V(t,\xi-c\tau-i)=\sqrt{w(\xi-c\tau-i)}U(t,\xi-c\tau-i+x_0)=\mathrm{e}^{-\lambda(\xi-c\tau-i)}U(t,\xi-c\tau-i+x_0),$$

we can estimate

$$\begin{aligned}&2\int_0^t \mathrm{e}^{-2\mu(t-s)}\mathcal{R}(s)ds\\ &\quad= 2\int_0^t \mathrm{e}^{-2\mu(t-s)}\bigg(\int_{\mathbb{R}} V(s,\xi)\sum_{i=-m}^m K(i)\sqrt{w(\xi)}Q(U(s-\tau,\xi-c\tau-i+x_0))d\xi\bigg)ds\\ &\quad\leqslant C\int_0^t \mathrm{e}^{-2\mu(t-s)}\bigg(\int_{\mathbb{R}} |V(s,\xi)|\sum_{i=-m}^m K(i)\mathrm{e}^{-\lambda\xi}U^2(s-\tau,\xi-c\tau-i+x_0)d\xi\bigg)ds\\ &\quad= C\int_0^t \mathrm{e}^{-2\mu(t-s)}\int_{\mathbb{R}}|V(s,\xi)|\\ &\qquad\times\bigg(\sum_{i=-m}^m K(i)\mathrm{e}^{-\lambda(i+c\tau)}\mathrm{e}^{-\lambda(\xi-c\tau-i)}U^2(s-\tau,\xi-c\tau-i+x_0)\bigg)d\xi ds\\ &\quad= C\int_0^t \mathrm{e}^{-2\mu(t-s)}\int_{\mathbb{R}}|V(s,\xi)|\\ &\qquad\times\bigg(\sum_{i=-m}^m K(i)\mathrm{e}^{-\lambda(i+c\tau)}V(s-\tau,\xi-c\tau-i)U(s-\tau,\xi-c\tau-i+x_0)\bigg)d\xi ds\end{aligned}$$

$$
\begin{aligned}
&\leqslant CM_U(T)\int_0^t \mathrm{e}^{-2\mu(t-s)}\int_{\mathbb{R}}\sum_{i=-m}^{m}K(i)\mathrm{e}^{-\lambda(i+c\tau)}|V(s,\xi)||V(s-\tau,\xi-c\tau-i)|d\xi ds\\
&\leqslant \frac{1}{2}CM_U(T)\int_0^t \mathrm{e}^{-2\mu(t-s)}\int_{\mathbb{R}}G(\lambda)V^2(s,\xi)d\xi ds\\
&\quad+\frac{1}{2}CM_U(T)\int_0^t \mathrm{e}^{-2\mu(t-s)}\int_{\mathbb{R}}\sum_{i=-m}^{m}K(i)\mathrm{e}^{-\lambda(i+c\tau)}V^2(s-\tau,\xi-c\tau-i)d\xi ds\\
&= CM_U(T)\int_0^t \mathrm{e}^{-2\mu(t-s)}\int_{\mathbb{R}}G(\lambda)V^2(s,\xi)d\xi ds\\
&\quad+CM_U(T)\int_0^t \mathrm{e}^{-2\mu(t-s)}\int_{\mathbb{R}}\sum_{i=-m}^{m}K(i)\mathrm{e}^{-\lambda(i+c\tau)}V^2(s-\tau,\xi)d\xi ds\\
&= CM_U(T)G(\lambda)\int_0^t \mathrm{e}^{-2\mu(t-s)}[\|V(s)\|_{L^2}^2+\|V(s-\tau)\|_{L^2}^2]ds\\
&\leqslant CM_U(T)\int_0^t \mathrm{e}^{-2\mu(t-s)}(\|V(s)\|_{L^2}^2+\mathrm{e}^{2\mu\tau}\|V(s)\|_{L^2}^2)ds\\
&\quad+CM_U(T)\int_{-\tau}^0 \mathrm{e}^{-2\mu(t-s-\tau)}\|V_0(s)\|_{L^2}^2ds\\
&\leqslant CM_U(T)\int_0^t \mathrm{e}^{-2\mu(t-s)}\|V(s)\|_{L^2}^2ds+C\mathrm{e}^{-2\mu t}\int_{-\tau}^0 \mathrm{e}^{2\mu s}\|V_0(s)\|_{L^2}^2ds.
\end{aligned}\tag{3.33}
$$

Substituting (3.33) into (3.32), we obtain

$$
\begin{aligned}
&\|V(t)\|_{L^2}^2+[2C_4-CM_U(T)]\int_0^t \mathrm{e}^{-2\mu(t-s)}\|V(s)\|_{L^2}^2ds\\
&\quad\leqslant C\mathrm{e}^{-2\mu t}\bigg(\|V_0(0)\|_{L^2}^2+\int_{-\tau}^0 \mathrm{e}^{2\mu s}\|V_0(s)\|_{L^2}^2ds\bigg).
\end{aligned}
$$

Let $M_U(T)\ll 1$. Then we immediately get (3.28). The proof is completed. □

Next, we establish the estimates for the one order derivatives $V_\xi(t,\xi)$ of the solution $V(t,\xi)$.

Lemma 3.7. *It holds that*

$$
\|V_\xi(t)\|_{L^2}^2+\int_0^t \mathrm{e}^{-2\mu(t-s)}\|V_\xi(s)\|_{L^2}^2ds\leqslant C\mathrm{e}^{-2\mu t}\bigg(\|V_0(0)\|_{H^1}^2+\int_{-\tau}^0\|V_0(s)\|_{H^1}^2ds\bigg),\tag{3.34}
$$

provided $M_U(T)\ll 1$.

Proof. Differentiating (3.23) with respect to ξ and multiplying it by $V_\xi(t,\xi)$, then integrating the resultant equation with respect to ξ and t over $\mathbb{R}\times[0,t]$ and applying Lemma 3.6, we can similarly prove (3.34) provided $M_U(T)\ll 1$. Thus, we omit the details. □

Finally, combining Lemmas 3.6 and 3.7, we can get the following a priori estimates.

Lemma 3.8. *It holds that*

$$
\|V(t)\|_{H^1}^2+\int_0^t \mathrm{e}^{-2\mu(t-s)}\|V(s)\|_{H^1}^2ds\leqslant C\mathrm{e}^{-2\mu t}\bigg(\|V_0(0)\|_{H^1}^2+\int_{-\tau}^0\|V_0(s)\|_{H^1}^2ds\bigg),\tag{3.35}
$$

namely,

$$
\begin{aligned}
&\|\sqrt{w}U(t)\|_{H^1}^2+\int_0^t \mathrm{e}^{-2\mu(t-s)}\|\sqrt{w}U(s)\|_{H^1}^2ds\\
&\quad\leqslant C\mathrm{e}^{-2\mu t}\bigg(\|\sqrt{w}U_0(0)\|_{H^1}^2+\int_{-\tau}^0\|\sqrt{w}U_0(s)\|_{H^1}^2ds\bigg),
\end{aligned}\tag{3.36}
$$

provided $M_U(T)\ll 1$.

From (3.35), by Sobolev's inequality $H^1(\mathbb{R}) \hookrightarrow C(\mathbb{R})$, we obtain

$$|V(t,\xi)| \leqslant C\|V(t)\|_{H^1} \leqslant C\delta_0 \mathrm{e}^{-\mu t}.$$

Noticing that

$$V(t,\xi) = \sqrt{w(\xi)}U(t,\xi+x_0) = \mathrm{e}^{-\lambda\xi}U(t,\xi+x_0),$$

and $\sqrt{w(\xi)} = \mathrm{e}^{-\lambda\xi} \geqslant 1$ for $\xi \in (-\infty, 0]$, we then have

$$\sup_{\xi\in(-\infty,0]} |U(t,\xi+x_0)| \leqslant C\delta_0 \mathrm{e}^{-\mu t}, \quad t > 0.$$

This derives the following estimate for the unshifted $U(t,\xi)$.

Lemma 3.9. *It holds that*

$$\|U(t)\|_{L^\infty(-\infty,x_0]} \leqslant C\delta_0 \mathrm{e}^{-\mu t}, \quad t > 0. \tag{3.37}$$

provided $M_U(T) \ll 1$.

Proof of Proposition 3.3. Combining (3.36), (3.37) and (3.17), we immediately prove (3.16), namely

$$\begin{aligned}
&\|U(t)\|_C^2 + \|\sqrt{w}U(t)\|_{H^1}^2 + \int_0^t \mathrm{e}^{-2\mu(t-s)}\|\sqrt{w}U(s)\|_{H^1}^2 ds \\
&\quad \leqslant C_2 \mathrm{e}^{-2\mu t}\left(\max_{s\in[-\tau,0]}\|U_0(s)\|_C^2 + \|\sqrt{w}U_0(0)\|_{H^1}^2 + \int_{-\tau}^0 \|\sqrt{w}U_0(s)\|_{H^1}^2 ds\right),
\end{aligned}$$

for some positive constant C_2, where μ is taken as $0 < \mu \leqslant \min\{\mu_1, \mu_2\}$. The proof of Proposition 3.3 is completed. □

4 Uniqueness of traveling waves

Proof of Corollary 2.5. Let $\phi_1(x+ct)$ and $\phi_2(x+ct)$ be two different traveling waves of (1.3) with the same speed $c > c_*$ and the same exponential decay at $-\infty$, i.e.,

$$\phi_1(\xi) = A\mathrm{e}^{-\lambda_1|\xi|} \quad \text{as} \quad \xi \to -\infty$$

and

$$\phi_2(\xi) = B\mathrm{e}^{-\lambda_1|\xi|} \quad \text{as} \quad \xi \to -\infty,$$

for some positive constant A and B, where $\lambda_1 = \lambda_1(c) > 0$ is defined in Lemma 2.1. We shift $\phi_2(x+ct)$ to $\phi_2(x+ct+\xi_0)$ with some constant shift ξ_0 such that

$$\xi_0 = \frac{1}{\lambda_1}\ln\frac{A}{B}.$$

Then by taking $\xi \to -\infty$, we obtain that $\xi + \xi_0 < 0$, and

$$\phi_2(\xi+\xi_0) = B\mathrm{e}^{-\lambda_1|\xi+\xi_0|} = B\mathrm{e}^{\lambda_1(\xi+\xi_0)} = B\mathrm{e}^{\lambda_1\xi_0}\mathrm{e}^{-\lambda_1|\xi|} = A\mathrm{e}^{-\lambda_1|\xi|} \quad \text{as} \quad \xi \to -\infty.$$

Hence, we get

$$|\phi_2(\xi+\xi_0) - \phi_1(\xi)| = O(1)\mathrm{e}^{-\alpha|\xi|} \quad \text{for} \quad \alpha > \lambda_1 \quad \text{as} \quad \xi \to -\infty,$$

which implies

$$\sqrt{w(\xi)}[\phi_2(\xi+\xi_0) - \phi_1(\xi)] \in C(\mathbb{R}) \cap H^1(\mathbb{R}).$$

If we take the initial data for (1.3) by

$$v_0(s,x) = \phi_2(x+cs+\xi_0), \quad x \in \mathbb{R}, \quad s \in [-r, 0],$$

then the corresponding solution to (1.3) is

$$v(t,x)=\phi_2(x+ct+\xi_0).$$

By applying Theorem 2.4, we obtain

$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}}|\phi_2(x+ct+\xi_0)-\phi_1(x+ct)|=0,$$

which means that $\phi_2(x+ct+\xi_0)=\phi_1(x+ct)$ for all $x\in\mathbb{R}$ as $t\gg 1$. This proves the uniqueness of the traveling waves up to a translation. □

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant No. 11401478). The authors are grateful to the referees for their valuable comments and suggestions which helped them improve the presentation of the paper.

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