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The Selberg-Delange method in short intervals with some applications

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Abstract In this paper, we establish a quite general mean value result of arithmetic functions over short intervals with the Selberg-Delange method and give some applications. In particular, we generalize Selberg's result on the distribution of integers with a given number of prime factors and Deshouillers-Dress-Tenenbaum's arcsin law on divisors to the short interval case.

Keywords asymptotic results on arithmetic functions, Selberg-Delange method, arithmetic functions, distribution of integers

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1 Introduction

This is the second paper of our series on the Selberg-Delange method for short intervals (see [1]). The method was initially introduced by Selberg [19] to study the distribution of integers having a given number of prime factors, and subsequently further developed by Delange [2, 3]. Roughly speaking, it applies to evaluating mean values of arithmetic functions whose associated Dirichlet series are close to *complex* powers of the Riemann ζ-function. An excellent exposition of the theory and applications can be found in [20, Chapters II.5 and II.6]. Recently, Cui and Wu [1] generalized this method to short interval when the power is *positive real*. In this paper we shall consider the complex power case which cannot be plainly treated with the method in [1]. Our aim is two-fold. First, we establish a quite general mean value result of arithmetic functions over short intervals, which generalizes and improves the main result of [1]. Second, we provide four arithmetic applications of our mean value result on:

- Distribution of integers having a given number of prime factors in short intervals.
- Deshouillers-Dress-Tenenbaum arcsin law on divisors in short intervals.
- Divisor problem for $\tau_k(n)$ in short intervals.
- Mean values of $1/\tau_k(n)$ over short intervals.

We shall proceed along the same line of argument as in [1]. Its origin can be found in [20, Chapters II.5 and II.6].

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1.1 Statement of main results

Let $f(n)$ be an arithmetic function and let its Dirichlet series be defined by

$$\mathcal{F}(s) := \sum_{n=1}^{\infty} f(n)n^{-s}. \tag{1.1}$$

Let $z \in \mathbb{C}$, $w \in \mathbb{C}$, $\alpha > 0$, $\delta \geqslant 0$, $A \geqslant 0$, $B > 0$, $C > 0$ and $M > 0$ be some constants. A Dirichlet series $\mathcal{F}(s)$ defined as in (1.1) is said to be of type $\mathcal{P}(z, w, \alpha, \delta, A, B, C, M)$ if the following conditions are verified:

(a) for any $\varepsilon > 0$, we have

$$|f(n)| \ll_{\varepsilon} Mn^{\varepsilon}, \quad n \geqslant 1, \tag{1.2}$$

where the implied constant depends only on ε;

(b) we have

$$\sum_{n=1}^{\infty} |f(n)|n^{-\sigma} \leqslant M(\sigma - 1)^{-\alpha}, \quad \sigma > 1;$$

(c) the Dirichlet series

$$\mathcal{G}(s; z, w) := \mathcal{F}(s)\zeta(s)^{-z}\zeta(2s)^{-w} \tag{1.3}$$

can be analytically continued to a holomorphic function in (some open set containing) $\sigma \geqslant \frac{1}{2}$ and, in this region, $\mathcal{G}(s; z, w)$ satisfies the bound

$$|\mathcal{G}(s; z, w)| \leqslant M(|\tau| + 1)^{\max\{\delta(1-\sigma),0\}} \log^A(|\tau| + 1) \tag{1.4}$$

uniformly for $|z| \leqslant B$ and $|w| \leqslant C$, where in the sequel we implicitly define the real numbers σ and τ by the relation $s = \sigma + \mathrm{i}\tau$ and choose the principal value of the complex logarithm.

Our first aim of this paper is to establish, under the previous assumptions, an asymptotic formula of

$$\sum_{x<n\leqslant x+x^{\theta}} f(n) \tag{1.5}$$

with the error term as good as that for the prime number theorem (PNT) and $\theta \in (0, 1]$ as small as possible. In view of the zero-free region of Vinogradov for $\zeta(s)$ (see [20, p. 161]), which gives the best known error estimate for the PNT, it seems rather difficult to prove such a result. One of our principal tools is Huxley's estimation on the zero density of the Riemann ζ-function. We denote by $N(\sigma, T)$ the number of zeros of $\zeta(s)$ in the region $\Re e\, s \geqslant \sigma$ and $|\Im m\, s| \leqslant T$. It is well known that there are two constants ψ and η such that

$$N(\sigma, T) \ll T^{\psi(1-\sigma)}(\log T)^{\eta} \tag{1.6}$$

for $\frac{1}{2} \leqslant \sigma \leqslant 1$ and $T \geqslant 2$. Huxley [7] showed that

$$\psi = \frac{12}{5} \quad \text{and} \quad \eta = 9 \tag{1.7}$$

are admissible. The zero density hypothesis is stated as

$$\psi = 2. \tag{1.8}$$

Combining (1.7) with the explicit formula (see [20, p. 177]), Huxley [7] derived his well-known prime number theorem in short intervals: for any $\theta \in (\frac{7}{12}, 1]$ and $y = x^{\theta}$, the asymptotic formula

$$\sum_{x<p\leqslant x+y} 1 \sim \frac{y}{\log x} \tag{1.9}$$

holds as $x \to \infty$. Motohashi [13] proved the following result for the Möbius function $\mu(n)$ corresponding to (1.9): for any $\theta > \frac{7}{12}$ and $y = x^\theta$, the inequality

$$\sum_{x<n\leqslant x+y} \mu(n) = o(y) \tag{1.10}$$

holds as $x \to \infty$. Independently, Ramachandra [14] obtained a better result

$$\sum_{x<n\leqslant x+y} \mu(n) \ll_{A,\theta} \frac{y}{(\log x)^A} \tag{1.11}$$

for each $A > 0$. Their methods are similar. Our approach is a generalization and refinement of Motohashi's method (see [13]). The first key point of this method is to construct a contour $\mathscr{M}_T$ (see Section 2 below for its precise definition) in the critical strip such that for any $\varepsilon > 0$ we have

$$(|\tau| + 1)^{-\varepsilon(1-\sigma)} \ll_\varepsilon |\zeta(s)| \ll_\varepsilon (|\tau| + 1)^{\varepsilon(1-\sigma)} \tag{1.12}$$

for $s \in \mathscr{M}_T$. The second key point is a very good bound for the density of "*small value points*" (i.e., satisfying (2.6) below), which was established by adapting Montgomery's new method to study the zero-densities of the Riemann ζ-function and of the Dirichlet L-functions (see [12]). With these two nice ideas and Huxley's zero density estimation, we establish a general asymptotic formula for the summatory function (1.5) (see Theorem 1.1 below). It is worth to point out that Theorem 1.1 allows us to unify the treatment of (1.9) and (1.10); indeed the latter is a particular case of the former.

In order to state our main result, it is necessary to introduce some more notation. From [20, Theorem II.5.1], the function[1)] $Z(s;z) := \{(s-1)\zeta(s)\}^z$ ($z \in \mathbb{C}$) is holomorphic in the disc $|s-1| < 1$, and admits, in the same disc, the Taylor series expansion

$$Z(s;z) = \sum_{j=0}^{\infty} \frac{\gamma_j(z)}{j!}(s-1)^j,$$

where the $\gamma_j(z)$'s are entire functions of z satisfying the estimate

$$\frac{\gamma_j(z)}{j!} \ll_{B,\varepsilon} (1+\varepsilon)^j, \quad j \geqslant 0, \quad |z| \leqslant B \tag{1.13}$$

for all $B > 0$ and $\varepsilon > 0$. Under our hypothesis, the function $\mathcal{G}(s;z,w)\zeta(2s)^w Z(s;z)$ is holomorphic in the disc $|s-1| < \frac{1}{2}$ and

$$|\mathcal{G}(s;z,w)\zeta(2s)^w Z(s;z)| \ll_{A,B,C,\delta,\varepsilon} M \tag{1.14}$$

for $|s-1| \leqslant \frac{1}{2} - \varepsilon$, $|z| \leqslant B$ and $|w| \leqslant C$. Thus for $|s-1| < \frac{1}{2}$, we can write

$$\mathcal{G}(s;z,w)\zeta(2s)^w Z(s;z) = \sum_{\ell=0}^{\infty} g_\ell(z,w)(s-1)^\ell, \tag{1.15}$$

where

$$g_\ell(z,w) := \frac{1}{\ell!}\sum_{j=0}^{\ell} \binom{\ell}{j} \frac{\partial^{\ell-j}(\mathcal{G}(s;z,w)\zeta(2s)^w)}{\partial s^{\ell-j}}\bigg|_{s=1} \gamma_j(z). \tag{1.16}$$

The main result of this paper is as follows.

Theorem 1.1. *Let $z \in \mathbb{C}$, $w \in \mathbb{C}$, $\alpha > 0$, $\delta \geqslant 0$, $A \geqslant 0$, $B > 0$, $C > 0$ and $M > 0$ be some constants. Suppose that the Dirichlet series defined as in* (1.1) *is of type $\mathcal{P}(z,w,\alpha,\delta,A,B,C,M)$. Then for any $\varepsilon > 0$, we have*

$$\sum_{x<n\leqslant x+y} f(n) = y(\log x)^{z-1}\left\{\sum_{\ell=0}^{N} \frac{\lambda_\ell(z,w)}{(\log x)^\ell} + O(MR_N(x,y))\right\} \tag{1.17}$$

1) In [20], $Z(s;z)$ is defined as $s^{-1}\{(s-1)\zeta(s)\}^z$ but obviously the argument of the proof there works for our $Z(s;z)$.

uniformly for $x \geqslant 3$, $x^{1-1/(\psi+\delta)+\varepsilon} \leqslant y \leqslant x$, $N \geqslant 0$, $|z| \leqslant B$ *and* $|w| \leqslant C$, *where*

$$\lambda_\ell(z,w) := g_\ell(z,w)/\Gamma(z-\ell)$$

and

$$R_N(x,y) := \frac{(c_1 N+1)^{N+1}}{(\log x)^{N+1} + \mathrm{e}^{c_2(\log x)^{1/3}(\log_2 x)^{-1/3}}} \tag{1.18}$$

for some constants $c_1 > 0$ *and* $c_2 > 0$ *depending only on* B, C, δ *and* ε. *The implied constant in the* O*-term depends only on* A, B, C, α, δ *and* ε. *In particular* $\psi = \frac{12}{5}$ *is admissible.*

The admissible length of short intervals in Theorem 1.1 depends only on the zero density constant ψ of $\zeta(s)$ and δ in (1.4) (for which we can take $\delta = 0$ in most applications). Its independence from the power z of $\zeta(s)$ in the representation of $\mathcal{F}(s)$ seems interesting. Theorem 1.1 generalizes and improves [1, Theorem 1] to the case of complex powers and intervals of shorter length.

Taking $N = 0$ in Theorem 1.1, we obtain readily the following corollary.

Corollary 1.2. *Under the conditions of Theorem* 1.1, *for any* $\varepsilon > 0$, *we have*

$$\sum_{x<n\leqslant x+y} f(n) = y(\log x)^{z-1}\bigg\{\lambda_0(z,w) + O\bigg(\frac{M}{\log x}\bigg)\bigg\} \tag{1.19}$$

uniformly for $x \geqslant 2$, $x^{1-1/(\psi+\delta)+\varepsilon} \leqslant y \leqslant x$, $|z| \leqslant B$ *and* $|w| \leqslant C$, *where*

$$\lambda_0(z,w) := \frac{\mathcal{G}(1;z,w)\zeta(2)^w}{\Gamma(z)}$$

and the implied constant in the O*-term depends only on* A, B, C, α, δ *and* ε. *Note that* $\psi = \frac{12}{5}$ *is admissible.*

Taking $f(n) = \mu(n)$ in Theorem 1.1, we have $z = -1$, $w = 0$ and $\mathcal{G}(s;z,w) \equiv 1$, $\delta = 0$, $\psi = \frac{12}{5}$, $\lambda(-1,\ell) = 0$ for all integers $\ell \geqslant 0$. Thus we can choose $N = [c'(\log x)^{1/3}(\log_2 x)^{-4/3}]$ with some small constant $c' > 0$ to obtain an improvement of Motohashi's result (1.10): for any $\theta > \frac{7}{12}$, we have

$$\sum_{x<n\leqslant x+y} \mu(n) \ll y\,\mathrm{e}^{-c(\log x)^{1/3}(\log_2 x)^{-1/3}}$$

uniformly for $x \geqslant 2$ and $x^\theta \leqslant y \leqslant x$, where $c > 0$ is a constant depending on θ.

1.2 Integers having a fixed number of prime factors

Denote by $\omega(n)$ (resp. $\Omega(n)$) the number of distinct (resp. all) prime factors of n. For each positive integer $k \geqslant 1$, consider

$$\pi_k(x) := |\{n \leqslant x : \omega(n) = k\}|, \tag{1.20}$$

$$N_k(x) := |\{n \leqslant x : \Omega(n) = k\}|. \tag{1.21}$$

In 1909, Landau [11] proved by induction that for each *fixed* positive integer k, the following asymptotic formulas:

$$\pi_k(x),\ N_k(x) \sim \frac{x}{\log x}\frac{(\log_2 x)^{k-1}}{(k-1)!}, \quad x \to \infty$$

hold, where $\log_\ell$ denotes the ℓ-fold iterated logarithm. However, if we allow k to grow with x, the method by induction will become too technical (see [16, 17]). In [19], Selberg proposed a new and very elegant approach to attack this problem—identifying $\pi_k(x)$ with the coefficient of z^k in the expression $\sum_{n\leqslant x} z^{\omega(n)}$ and then applying Cauchy's integral formula. Through a detailed study of the sum over z, he proved that for any fixed constant $B > 0$ the asymptotic formula

$$\pi_k(x) = \frac{x}{\log x}\frac{(\log_2 x)^{k-1}}{(k-1)!}\bigg\{\lambda\bigg(\frac{k-1}{\log_2 x}\bigg) + O_B\bigg(\frac{k}{(\log_2 x)^2}\bigg)\bigg\} \tag{1.22}$$

holds uniformly for $x \geqslant 3$ and $1 \leqslant k \leqslant B \log_2 x$, where

$$\lambda(z) := \frac{1}{\Gamma(z+1)} \prod_p \left(1 + \frac{z}{p-1}\right)\left(1 - \frac{1}{p}\right)^z \tag{1.23}$$

and the implied constant depends only on B. In the same fashion, Selberg [19] also proved that for any $\delta \in (0, 2)$, the asymptotic formula

$$N_k(x) = \frac{x}{\log x} \frac{(\log_2 x)^{k-1}}{(k-1)!} \left\{ \nu\left(\frac{k-1}{\log_2 x}\right) + O_\delta\left(\frac{k}{(\log_2 x)^2}\right) \right\} \tag{1.24}$$

holds uniformly for $x \geqslant 3$ and $1 \leqslant k \leqslant (2-\delta) \log_2 x$, where

$$\nu(z) := \frac{1}{\Gamma(z+1)} \prod_p \left(1 - \frac{z}{p}\right)^{-1}\left(1 - \frac{1}{p}\right)^z \tag{1.25}$$

and the implied constant depends only on δ.

As the first application of Theorem 1.1, we shall generalize Selberg's results (1.22) and (1.24) to the short interval case.

Theorem 1.3. *Let $B > 0$ and $\varepsilon > 0$. There exist positive constants $c_1 = c_1(B, \varepsilon)$ and $c_2 = c_2(B, \varepsilon)$ such that we have*

$$\pi_k(x+y) - \pi_k(x) = \frac{y}{\log x} \left\{ \sum_{j=0}^N \frac{P_{j,k}(\log_2 x)}{(\log x)^j} + O_{B,\varepsilon}\left(\frac{(\log_2 x)^k}{k!} R_N(x, y)\right) \right\} \tag{1.26}$$

uniformly for $x \geqslant 3$, $x^{1-1/\psi+\varepsilon} \leqslant y \leqslant x$ and $1 \leqslant k \leqslant B \log_2 x$, where $P_{j,k}(X)$ is a polynomial of degree at most $k-1$ and $R_N(x, y)$ is defined as in (1.18)*. Here the implied constant depends on B and ε only. In particular, we have*

$$P_{0,k}(X) = \sum_{m+\ell=k-1} \frac{\lambda^{(m)}(0)}{\ell! m!} X^\ell.$$

Moreover, under the same conditions, we have

$$\pi_k(x+y) - \pi_k(x) = \frac{y}{\log x} \frac{(\log_2 x)^{k-1}}{(k-1)!} \left\{ \lambda\left(\frac{k-1}{\log_2 x}\right) + O\left(\frac{k}{(\log_2 x)^2}\right) \right\}. \tag{1.27}$$

In particular $\psi = \frac{12}{5}$ is admissible in both assertions (1.26) *and* (1.27)*.*

Theorem 1.4. *Let $\varepsilon > 0$ be an arbitrarily small positive number. There exist absolute positive constants c_1 and c_2 such that we have*

$$N_k(x+y) - N_k(x) = \frac{y}{\log x} \left\{ \sum_{j=0}^N \frac{Q_{j,k}(\log_2 x)}{(\log x)^j} + O_{B,\varepsilon}\left(\frac{(\log_2 x)^k}{k!} R_N(x, y)\right) \right\} \tag{1.28}$$

uniformly for $x \geqslant 3$, $x^{1-1/\psi+\varepsilon} \leqslant y \leqslant x$ and $1 \leqslant k \leqslant \log_2 x$, where $Q_{j,k}(X)$ is a polynomial of degree at most $k-1$ and $R_N(x, y)$ is defined as in (1.18)*. Here, the implied constant depends on B and ε only. In particular, we have*

$$Q_{0,k}(X) = \sum_{m+\ell=k-1} \frac{\nu^{(m)}(0)}{\ell! m!} X^\ell.$$

Moreover, under the same conditions, we have

$$N_k(x+y) - N_k(x) = \frac{y}{\log x} \frac{(\log_2 x)^{k-1}}{(k-1)!} \left\{ \nu\left(\frac{k-1}{\log_2 x}\right) + O\left(\frac{k}{(\log_2 x)^2}\right) \right\}. \tag{1.29}$$

In particular $\psi = \frac{12}{5}$ is admissible in both assertions (1.28) *and* (1.29)*.*

Remark 1.5. Kátai [9] applied Ramachandra's theorem (see [14]) to obtain

$$\pi_k(x+y)-\pi_k(x)=\{1+o(1)\}\frac{y}{\log x}\frac{(\log_2)^{k-1}}{(k-1)!}$$

uniformly for any $k\leqslant \log_2 x+c_x\sqrt{\log_2 x}$, where $c_x\to\infty$ sufficiently slowly, and $y\geqslant x^{1-1/\psi+\varepsilon}$. Clearly, Theorem 1.3 improves Kátai's result in two directions: get a more precise asymptotic formula and extend domain of k.

Taking $k=1$, we obtain Huxley's well-known prime number theorem in short intervals (see (1.9)).

1.3 The Deshouillers-Dress-Tenenbaum arcsin law on divisors

For each positive integer n, denote by $\tau(n)$ the number of divisors of n and define the random variable D_n which takes the value $(\log d)/\log n$, as d runs through the set of the $\tau(n)$ divisors of n, with the uniform probability $1/\tau(n)$. The distribution function F_n of D_n is given by

$$F_n(t)=\mathrm{Prob}(D_n\leqslant t)=\frac{1}{\tau(n)}\sum_{d\,|\,n,\,d\leqslant n^t}1,\quad 0\leqslant t\leqslant 1.$$

It is clear that the sequence $\{F_n\}_{n\geqslant 1}$ does not converge pointwisely on $[0,1]$. However, Deshouillers et al. [4] (see also [20, Theorem II.6.7]) proved that its Cesàro mean converges uniformly to the arcsin law. More precisely, they showed that the asymptotic formula

$$\frac{1}{x}\sum_{n\leqslant x}F_n(t)=\frac{2}{\pi}\arcsin\sqrt{t}+O\bigg(\frac{1}{\sqrt{\log x}}\bigg)$$

holds uniformly for $x\geqslant 2$ and $0\leqslant t\leqslant 1$, and that the error term is optimal. Recently, Cui and Wu [1, Theorem 2] established a short interval version of this result: for $\varepsilon>0$, we have

$$\frac{1}{y}\sum_{x<n\leqslant x+y}F_n(t)=\frac{2}{\pi}\arcsin\sqrt{t}+O_\varepsilon\bigg(\frac{1}{\sqrt{\log x}}\bigg)\tag{1.30}$$

uniformly for $0\leqslant t\leqslant 1$, $x\geqslant 2$ and $x^{62/77+\varepsilon}\leqslant y\leqslant x$, where the implied constant depends only on ε.

Our third application of Theorem 1.1 is to improve the exponent in (1.30).

Theorem 1.6. *For any $\varepsilon>0$, the asymptotic formula* (1.30) *holds uniformly for $0\leqslant t\leqslant 1$, $x\geqslant 2$ and $x^{19/24+\varepsilon}\leqslant y\leqslant x$, where the implied constant depends on ε only.*

For comparison, we have $\frac{62}{77}=0.805\cdots$ and $\frac{19}{24}=0.791\cdots$.

1.4 Divisor problem for $\tau_k(n)$ on short intervals

As usual, denote by $\Delta_k(x)$ the error term in the asymptotic formula for the k-dimension divisor problem

$$D_k(x):=\sum_{n\leqslant x}\tau_k(n)=xP_{k-1}(\log x)+\Delta_k(x),$$

where $P_{k-1}(t)$ is a polynomial of degree $k-1$ with leading coefficient $1/(k-1)!$. The best known result for $\Delta_k(x)$ for $k\geqslant 4$ is as follows:

$$\Delta_4(x)\ll x^{1/2}(\log x)^5,\quad \Delta_k(x)\ll_{k,\varepsilon}x^{\theta_k+\varepsilon},\quad k\geqslant 5\tag{1.31}$$

with $\theta_k=\frac{3}{4}-\frac{1}{k}$ $(5\leqslant k\leqslant 8)$, $\theta_9=\frac{35}{54}$, $\theta_{10}=\frac{41}{60}$, $\theta_{11}=\frac{7}{10}$, $\theta_k=\frac{k-1}{k+2}$ $(12\leqslant k\leqslant 25)$, $\theta_k=\frac{k-1}{k+4}$ $(26\leqslant k\leqslant 50)$, $\theta_k=\frac{31k-98}{32k}$ $(51\leqslant k\leqslant 57)$ and $\theta_k=\frac{7k-34}{7k}$ $(k\geqslant 58)$, where ε is an arbitrarily small positive number (see [5, (1.3)] for $k=4$ and [8, Theorem 12.3] for $k\geqslant 5$.) In 2006, Garaev et al. [5] considered the divisor problem for $\tau_4(n)$ in short intervals and proved that

$$\sum_{x<n\leqslant x+y}\tau_4(n)=\frac{1}{6}y(\log x)^3\bigg\{1+O\bigg(\bigg(\frac{\sqrt{x}\log x}{y}\bigg)^{2/3}\bigg)\bigg\}$$

for $x \geqslant 3$ and $x^{1/2}\log x \leqslant y \leqslant x^{1/2}(\log x)^{5/2}$. They also emphasized that for no other dimension $k \neq 4$ short interval results are known for the sum over $\tau_k(n)$ that are sharper than what is immediate from the ("long interval") asymptotics for $D_k(x)$ (see [5, Remark]). The next theorem gives such a result for all integers $k \geqslant 7$.

Theorem 1.7. *Let $k \geqslant 7$ be a positive integer and $\varepsilon > 0$ be an arbitrarily small positive number. Then there is a positive constant c depending on k and ε such that the asymptotic formula*

$$\sum_{x<n\leqslant x+y} \tau_k(n) = yQ_{k-1}(\log x)\{1 + O_{k,\varepsilon}(\mathrm{e}^{-c(\log x)^{1/3}(\log_2 x)^{-1/3}})\}$$

holds uniformly for $x \geqslant 2$ and $x^{1-1/\psi+\varepsilon} \leqslant y \leqslant x$, where $Q_{k-1}(t)$ is a polynomial of degree $k-1$ with leading coefficient $1/(k-1)!$ and the implied constant depends only on k and ε. In particular $\psi = \frac{12}{5}$ is admissible.

It is interesting to note that the exponent (1.31) tends to 1 as $k \to \infty$, and that the length of short intervals in Theorem 1.7 is independent of k.

1.5 The mean value of $1/\tau_k(n)$ on short intervals

Recently, Sedunova [18] considered mean values of the following arithmetic functions over short intervals: $\tau_k(n)^{-1}$, $\sigma(n)/\tau(n)$ and $r(n)^{-1}$, where

$$\tau_k(n) := \sum_{d\,|\,n} \tau_{k-1}(d), \quad \sigma(n) := \sum_{d\,|\,n} d, \quad r(n) := |\{(n_1, n_2) \in \mathbb{Z}^2 : n_1^2 + n_2^2 = n\}|.$$

In particular, she proved that for any fixed integer $N \geqslant 0$ the asymptotic formula

$$\sum_{x<n\leqslant x+y} \frac{1}{\tau_k(n)} = \frac{y}{\sqrt{\log x}}\left\{\sum_{\ell=0}^{N} \frac{a_\ell(k)}{(\log x)^\ell} + O_{k,N}\left(\frac{1}{(\log x)^{N+1}}\right)\right\} \tag{1.32}$$

holds uniformly for $x \geqslant 3$ and $x^{(21k+5)/(36k+5)}\mathrm{e}^{(\log x)^{0.1}} \leqslant y \leqslant x$, where the $a_\ell(k)$ are some constants depending on k (see [18, Theorem 1]).

The fourth application of Theorem 1.1 is the following result.

Theorem 1.8. *For any $\varepsilon > 0$, the asymptotic formula* (1.32) *holds uniformly for $x \geqslant 2$ and $x^{7/12+\varepsilon} \leqslant y \leqslant x$, where the implied constant depends only on k, N and ε.*

Since $\frac{21k+5}{36k+5}$ tends to $\frac{7}{12}$ decreasingly as $k \to \infty$, Theorem 1.8 improves Sedunova's (1.32) for all k. It is worth to note that our exponent is independent of k. Clearly, the other results in [18, Theorems 2–7] can also be improved by Theorem 1.1 or its method of proof.

2 Motohashi's method

This section is devoted to depicting Motohashi's method (see [13]). His original presentation is rather sketchy. Some key estimations (see Lemma 2.1 and Proposition 2.4 below) are outlined without many details. Here, we would give a complete and detailed presentation for the sake of readers' convenience and the importance of this method.

2.1 Hooley-Huxley-Motohashi's contour $\mathscr{M}_T$

Let ε be an arbitrarily small positive constant and let $T_0 = T_0(\varepsilon)$ be a large constant depending on ε only and is to be determined from the inequality $1 - \delta_T \leqslant 1 - \varepsilon$, where for $T > T_0$, we put

$$\delta_T := C_0(\log T)^{-2/3}(\log_2 T)^{-1/3}, \tag{2.1}$$

where C_0 is a suitable positive constant such that

$$(\log|\tau|)^{-2/3}(\log_2|\tau|)^{-1/3} \ll |\zeta(s)| \ll (\log|\tau|)^{2/3}(\log_2|\tau|)^{1/3} \tag{2.2}$$

for $\sigma \geqslant 1-100\delta_T$ and $1 \leqslant |\tau| \leqslant 100T$ (see [20, p. 162]).

For $T \geqslant T_0$, write

$$J_T := \left[\left(\frac{1}{2}-\delta_T\right)\log T\right] \quad \text{and} \quad K_T := [T(\log T)^{-1}]. \tag{2.3}$$

For each pair of integers (j,k) with $0 \leqslant j \leqslant J_T$ and $|k| \leqslant K_T$, we define

$$\Delta_{j,k} := \{s = \sigma + \mathrm{i}\tau : \sigma_j \leqslant \sigma < \sigma_{j+1} \text{ and } \tau_k \leqslant \tau < \tau_{k+1}\}, \tag{2.4}$$

where

$$\sigma_j := \frac{1}{2} + j(\log T)^{-1} \quad \text{and} \quad \tau_k := k\log T. \tag{2.5}$$

We divide $\Delta_{j,k}$ into two classes (W) and (Y) as follows:

- $\sigma_j \leqslant 1-\varepsilon$: Then $\Delta_{j,k} \in (W)$ if $\Delta_{j,k}$ contains at least one zero of $\zeta(s)$, and $\Delta_{j,k} \in (Y)$ otherwise;
- $1-\varepsilon < \sigma_j \leqslant 1-\delta_T$: $\Delta_{j,k} \in (W)$ if and only if $\exists$ at least one $s \in \Delta_{j,k}$ such that

$$|\zeta(s)M_{N_j}(s)| < \frac{1}{2} \tag{2.6}$$

with

$$\begin{cases} A' := \text{a fixed large integer}, \\ N_j := \Big(A'(\log T)^5 \max\limits_{\sigma \geqslant 4\sigma_j - 3,\, 1 \leqslant |\tau| \leqslant 4T} |\zeta(s)|\Big)^{1/2(1-\sigma_j)}, \\ M_x(s) := \sum\limits_{n \leqslant x} \mu(n) n^{-s} \end{cases} \tag{2.7}$$

and $\Delta_{j,k} \in (Y)$ if and only if for all $s \in \Delta_{j,k}$,

$$|\zeta(s)M_{N_j}(s)| \geqslant \frac{1}{2}. \tag{2.8}$$

For each k, we define

$$j_k := \begin{cases} \max\limits_{\Delta_{j,k} \in (W)} j, & \text{if } \exists\, j \text{ such that } \Delta_{j,k} \in (W), \\ 0, & \text{otherwise}. \end{cases}$$

Put

$$\begin{aligned} \mathscr{D}' &:= \bigcup_{0 \leqslant k \leqslant K_T} \bigcup_{0 \leqslant j \leqslant j_k} \Delta_{j,k}, \\ \mathscr{D}_0 &:= \bigcup_{0 \leqslant k \leqslant K_T} \bigcup_{j_k < j \leqslant j_T} \Delta_{j,k}. \end{aligned} \tag{2.9}$$

Clearly, $\mathscr{D}_0$ consists of $\Delta_{j,k}$ of type (Y) only.

Hooley-Huxley-Motohashi's contour $\mathscr{M}_T$ is symmetric about the real axis (see [6, 13]). Its upper part is the path in $\mathscr{D}_0$ consisting of horizontal and vertical line segments whose distances away from $\mathscr{D}'$ are, respectively, d_{h} and d_{v}, given by

$$d_{\mathrm{h}} := \log_2 T, \quad d_{\mathrm{v}} := \begin{cases} \varepsilon^2, & \text{if } \sigma \leqslant 1-\varepsilon, \\ (\log T)^{-1}, & \text{if } 1-\varepsilon < \sigma < 1-\delta_T \end{cases} \tag{2.10}$$

(see Figure 1).

2.2 Lower and upper bounds of $\zeta(s)$ on $\mathscr{M}_T$

In this subsection, we give bounds to $\zeta(s)$ on $\mathscr{M}_T$. The next two lemmas are essentially due to Motohashi [13, p. 478, lines 21–28]. For completeness we shall provide proofs.

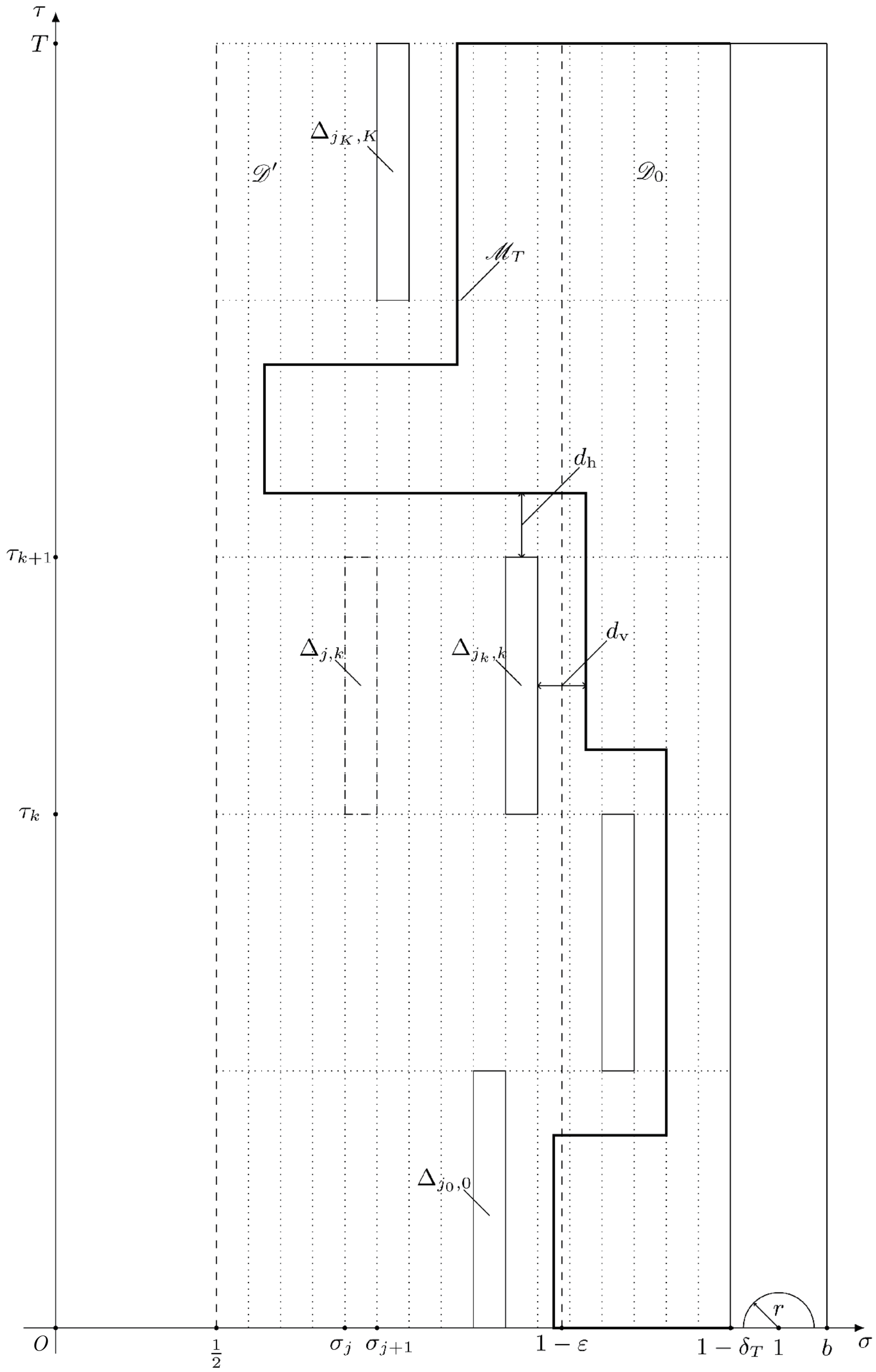

Figure 1 Upper part of the contour $\mathscr{M}_T$

Lemma 2.1. *Under the previous notation, we have*

$$\mathrm{e}^{-(\log T)^{1-\varepsilon^2}} \ll |\zeta(s)| \ll \mathrm{e}^{(\log T)^{1-\varepsilon^2}} \tag{2.11}$$

for $s \in \mathscr{M}_T$ with $\sigma \leqslant 1-\varepsilon$, or s (with $1-\varepsilon < \sigma \leqslant 1-\varepsilon+\varepsilon^2$) on the horizontal segments in $\mathscr{M}_T$ that intersect the vertical line $\Re\mathrm{e}\, s = 1-\varepsilon$. Here, the implied constant depends only on ε.

Proof. Let $s = \sigma + \mathrm{i}\tau$ satisfy the conditions in this lemma. Without loss of generality, we can suppose that $\tau \geqslant T_0(\varepsilon)$. Let us consider the four circles $\mathscr{C}_1$, $\mathscr{C}_2$, $\mathscr{C}_3$ and $\mathscr{C}_4$, all centered at $s_0 = \log_2 \tau + \mathrm{i}\tau$, with radii

$$r_1 := \log_2 \tau - 1 - \eta, \quad r_2 := \log_2 \tau - \sigma, \quad r_3 := \log_2 \tau - \sigma + \frac{1}{2}\varepsilon^2, \quad r_4 := \log_2 \tau - \sigma + \varepsilon^2,$$

respectively. Here, $\eta > 0$ is a parameter to be chosen later. We note that these four circles pass through the points $1 + \eta + \mathrm{i}\tau$, $\sigma + \mathrm{i}\tau$, $\sigma - \frac{1}{2}\varepsilon^2 + \mathrm{i}\tau$ and $\sigma - \varepsilon^2 + \mathrm{i}\tau$, respectively.

Clearly, $\zeta(s) \neq 0$ in a region containing the disc $|s - s_0| \leqslant r_4$. Thus we can unambiguously define $\log \zeta(s)$ in this region. We fix a branch of the logarithm throughout the remaining discussion.

Let M_i denote the maximum of $|\log \zeta(s)|$ on $\mathscr{C}_i$ relative to this branch. By using Hadamard's three circle theorem and the fact that $s = \sigma + \mathrm{i}\tau$ is on $\mathscr{C}_2$, we have

$$|\log \zeta(s)| \leqslant M_2 \leqslant M_1^{1-a} M_3^a, \tag{2.12}$$

where

$$\begin{aligned} a = \frac{\log(r_2/r_1)}{\log(r_3/r_1)} &= \frac{\log(1 + (1 + \eta - \sigma)/(\log_2 \tau - 1 - \eta))}{\log(1 + (1 + \eta - \sigma + \frac{1}{2}\varepsilon^2)/(\log_2 \tau - 1 - \eta))} \\ &= \frac{1 + \eta - \sigma}{1 + \eta - \sigma + \frac{1}{2}\varepsilon^2} + O((\log_2 \tau)^{-1}). \end{aligned}$$

On taking $\eta = \sigma - \frac{1}{2} - \frac{1}{2}\varepsilon^2 - \frac{\varepsilon^3}{2(1+\varepsilon^3)}$ ($\eta \geqslant \frac{1}{4}\varepsilon^2$, since $\sigma \geqslant \frac{1}{2} + \varepsilon^2$), we have

$$a = 1 - \varepsilon^2 - \varepsilon^5 + O((\log_2 \tau)^{-1}). \tag{2.13}$$

On the circle $\mathscr{C}_1$, we have

$$M_1 \leqslant \max_{\Re e s \geqslant 1+\eta} \sum_{n=2}^{\infty} \left| \frac{\Lambda(n)}{n^s \log n} \right| \leqslant \sum_{n=2}^{\infty} \frac{1}{n^{1+\eta}} \ll \frac{1}{\eta}, \tag{2.14}$$

where $\Lambda(n)$ is the von Mangoldt function.

In order to bound M_3, we shall apply the Borel-Carathéodory theorem to the function $\log \zeta(s)$ on the circles $\mathscr{C}_3$ and $\mathscr{C}_4$. On the circle $\mathscr{C}_4$, it is well known that $\Re e\,(\log \zeta(s)) = \log |\zeta(s)| \ll \log \tau$. Hence, the Borel-Carathéodory theorem gives

$$\begin{aligned} M_3 &\leqslant \frac{2r_3}{r_4 - r_3} \max_{|s-s_0| \leqslant r_4} \log |\zeta(s)| + \frac{r_4 + r_3}{r_4 - r_3} |\log \zeta(s_0)| \\ &\ll \frac{2(\log_2 \tau - \sigma + \frac{1}{2}\varepsilon^2)}{\frac{1}{2}\varepsilon^2} \log \tau + \frac{2\log_2 \tau - 2\sigma + \frac{1}{2}\varepsilon^2}{\frac{1}{2}\varepsilon^2} |\log \zeta(2 + \mathrm{i}\tau)| \\ &\ll (\log_2 \tau) \log \tau. \end{aligned} \tag{2.15}$$

From (2.12)–(2.15), we deduce that

$$|\log \zeta(s)| \ll (\eta^{-1})^{1-a} (\log_2 \tau \log \tau)^a \ll_\varepsilon (\log_2 \tau \log \tau)^{1-\varepsilon^2-\varepsilon^5} \leqslant (\log \tau)^{1-\varepsilon^2}.$$

This leads to the required estimates. □

Lemma 2.2. *Under the previous notation, we have*

$$T^{-400(1-\sigma_j)^{3/2}} (\log T)^{-4} \ll |\zeta(s)| \ll T^{100(1-\sigma_j)^{3/2}} (\log T)^4 \tag{2.16}$$

for $s \in \mathscr{M}_T$ *with* $1 - \varepsilon < \sigma_j \leqslant \sigma < \sigma_{j+1}$. *Here, the implied constants are absolute. In particular, we have*

$$T^{-400\sqrt{\varepsilon}(1-\sigma_j)} (\log T)^{-4} \ll |\zeta(s)| \ll T^{100\sqrt{\varepsilon}(1-\sigma_j)} (\log T)^4 \tag{2.17}$$

for $s \in \mathscr{M}_T$ *with* $1 - \varepsilon < \sigma_j \leqslant \sigma < \sigma_{j+1}$. *All the implied constants are absolute.*

Proof. According to [15, p. 98], we have

$$|\zeta(s)| \ll \tau^{100(1-\sigma)^{3/2}}(\log\tau)^{2/3}, \quad \frac{1}{2} \leqslant \sigma \leqslant 1, \quad \tau \geqslant 2. \tag{2.18}$$

This immediately implies the upper bound of (2.16) and

$$N_j^{1-\sigma_j} = \Big(A'(\log T)^5 \max_{\substack{\sigma\geqslant 4\sigma_j-3\\ 1\leqslant|\tau|\leqslant 4T}} |\zeta(s)|\Big)^{1/2} \ll T^{400(1-\sigma_j)^{3/2}}(\log T)^3. \tag{2.19}$$

Next, we consider the lower bound. Let $s \in \mathscr{M}_T$ with $1-\varepsilon < \sigma_j \leqslant \sigma < \sigma_{j+1}$. Then there is an integer k such that $s \in \Delta_{j,k}$. According to the definition of $\mathscr{M}_T$, this $\Delta_{j,k}$ must be in (Y) and (2.8) holds for all s of this $\Delta_{j,k}$. On the other hand, (2.19) allows us to deduce that for $\sigma_j \leqslant \sigma < \sigma_{j+1}$,

$$|M_{N_j}(s)| \leqslant \sum_{n\leqslant N_j} n^{-\sigma_j} \ll (1-\sigma_j)^{-1}N_j^{1-\sigma_j} \ll T^{400(1-\sigma_j)^{3/2}}(\log T)^4.$$

Combining this with (2.8) immediately yields $|\zeta(s)| \geqslant (2|M_{N_j}(s)|)^{-1} \gg T^{-400(1-\sigma_j)^{3/2}}(\log T)^{-4}$ for $s \in \mathscr{M}_T$ with $1-\varepsilon < \sigma_j \leqslant \sigma < \sigma_{j+1}$.

Finally, we note (2.17) is a simple consequence of (2.16) since $1-\varepsilon < \sigma_j \Rightarrow (1-\sigma_j)^{1/2} \leqslant \sqrt{\varepsilon}$. □

Proposition 2.3. *Under the previous notation, we have*

$$T^{-400\sqrt{\varepsilon}(1-\sigma)}(\log T)^{-4} \ll |\zeta(s)| \ll T^{400\sqrt{\varepsilon}(1-\sigma)}(\log T)^4 \tag{2.20}$$

for all $s \in \mathscr{M}_T$, where the implied constants depend only on ε.

Proof. Let $s \in \mathscr{M}_T$. Then there is a j such that $\sigma_j \leqslant \sigma < \sigma_{j+1}$. We consider the three possibilities.

• The case of $1-\varepsilon < \sigma_j$. The inequality (2.20) follows immediately from (2.17) of Lemma 2.2.

• The case of $\sigma_j \leqslant \sigma \leqslant 1-\varepsilon$. In this case, the first part of Lemma 2.1 shows that (2.20) holds again since $\sqrt{\varepsilon}(1-\sigma) \geqslant \varepsilon^{3/2} \geqslant (\log T)^{-\varepsilon^2}$ for $T \geqslant T_0(\varepsilon)$.

• The case of $\sigma_j \leqslant 1-\varepsilon < \sigma$. In this case, s must be on the horizontal segment in $\mathscr{M}_T$, because the vertical segment keeps the distance ε^2 from the line $\Re e\, s = \sigma_j$ and $\sigma_j < \sigma < \sigma_{j+1}$. Thus we can apply the second part of Lemma 2.1 to get (2.20) as before. □

2.3 Montgomery's method and Huxley's zero-density estimation

In [12], Montgomery developed a new method for studying zero-densities of the Riemann ζ-function and of the Dirichlet L-functions. Subsequently by modifying this method, Huxley [7] established his zero-density estimation (1.7) (see (2.21) below). In [13], Motohashi noted that Montgomery's method can be adapted to estimate the density of "*small value points*" (characterized by (2.6)). The estimation (2.22) below is due to Motohashi [13, (5)].

Proposition 2.4. *Under the previous notation, for $j = 0, 1, \dots, J_T$ we have*

$$|\{k \leqslant K_T : \Delta_{j,k} \in (W)\}| \ll T^{\psi(1-\sigma_j)}(\log T)^{\eta} \tag{2.21}$$

if $\sigma_j \leqslant 1-\varepsilon$; in addition,

$$|\{k \leqslant K_T : \Delta_{j,k} \in (W)\}| \ll T^{170(1-\sigma_j)^{3/2}}(\log T)^{13} \tag{2.22}$$

if $1-\varepsilon \leqslant \sigma_j \leqslant 1-\delta_T$. Here, $(\psi,\eta) = (\frac{12}{5}, 9)$ is admissible.

Proof. The case of $\sigma_j \leqslant 1-\varepsilon$ is very simple, because the number of (W) does not exceed the number of non-trivial zeros of $\zeta(s)$.

Next, we suppose $1-\varepsilon \leqslant \sigma_j \leqslant 1-\delta_T$.

Let $\mathcal{K}_j(T)$ be a subset of the set $\{\log T \leqslant k \leqslant K_T : \Delta_{j,k} \in (W)\}$ such that the difference of two distinct integers of $\mathcal{K}_j(T)$ is at least $3A'$, where A' is the large integer specified in (2.7). Obviously,

$$|\{(\log T)^2 \leqslant k \leqslant K_T : \Delta_{j,k} \in (W)\}| \leqslant 3A'|\mathcal{K}_j(T)|.$$

Therefore, it suffices to show that

$$|\mathcal{K}_j(T)| \ll_\varepsilon T^{170(1-\sigma_j)^{3/2}} (\log T)^{13} \tag{2.23}$$

for $T \geqslant T_0(\varepsilon)$, where the implied constant and the constant $T_0(\varepsilon)$ depend only on ε.

Let $M_x(s)$ be defined as in (2.7) and let $a_{n,x}$ be the n-th coefficient of the Dirichlet series $\zeta(s)M_x(s)$. Then

$$a_{n,x} = \sum_{d \mid n,\, d\leqslant x} \mu(d). \tag{2.24}$$

By the Mellin inversion formula (see [22, Lemma, p. 151]), we can write

$$\sum_{n\geqslant 1} \frac{a_{n,x}}{n^s} \mathrm{e}^{-n/y} = \frac{1}{2\pi \mathrm{i}} \int_{2-\mathrm{i}\infty}^{2+\mathrm{i}\infty} \zeta(w+s) M_x(w+s)\Gamma(w) y^w dw$$

for $y > x \geqslant 3$ and $s = \sigma + \mathrm{i}\tau \in \mathbb{C}$ with $\frac{1}{2} < \sigma < 1$. We take the contour to the line $\Re e w = \alpha - \sigma < 0$ with $\alpha := 4\sigma_j - 3 \geqslant 1 - 4\varepsilon$, and in doing so we pass two simple poles at $w = 0$ and $w = 1 - s$. Our equation becomes

$$\sum_{n\geqslant 1} \frac{a_{n,x}}{n^s} \mathrm{e}^{-n/y} = \zeta(s)M_x(s) + M_x(1)\Gamma(1-s)y^{1-s} + I(s; x, y),$$

where

$$I(s; x, y) := \frac{1}{2\pi} \int_{-\infty}^{+\infty} \zeta(\alpha + \mathrm{i}\tau + \mathrm{i}u) M_x(\alpha + \mathrm{i}\tau + \mathrm{i}u)\Gamma(\alpha - \sigma + \mathrm{i}u) y^{\alpha-\sigma+\mathrm{i}u} du.$$

Obviously, (2.24) implies that $a_{1,x} = 1$, $a_{n,x} = 0$ for $2 \leqslant n \leqslant x$ and $|a_{n,x}| \leqslant \tau(n)$ for $n > x$. With the classical estimate $\sum_{n\leqslant t} \tau(n) \ll t \log t$ and a simple partial integration, we obtain

$$\begin{aligned}\bigg|\sum_{n>y^2} \frac{a_{n,x}}{n^s} \mathrm{e}^{-n/y}\bigg| &\leqslant \int_{y^2}^{\infty} t^{-\sigma} \mathrm{e}^{-t/y} d\bigg(\sum_{n\leqslant t} \tau(n)\bigg) \\ &\ll \mathrm{e}^{-y} y^{2-2\sigma} \log y + y^{-1} \int_{y^2}^{\infty} \mathrm{e}^{-t/y} t^{1-\sigma} (\log t) dt \\ &\ll \mathrm{e}^{-y/2}\end{aligned}$$

for $\sigma > \frac{1}{2}$. Inserting it into the preceding relation, we find that

$$\mathrm{e}^{-1/y} + \sum_{x<n\leqslant y^2} \frac{a_{n,x}}{n^s} \mathrm{e}^{-n/y} + O(\mathrm{e}^{-y/2}) = \zeta(s)M_x(s) + M_x(1)\Gamma(1-s)y^{1-s} + I(s; x, y) \tag{2.25}$$

for $s \in \mathbb{C}$ with $\frac{1}{2} < \sigma < 1$ and $y > x \geqslant 3$.

If $k \in \mathcal{K}_j(T)$, then there is at least an $s_k := v_k + \mathrm{i}t_k \in \Delta_{j,k}$ such that

$$|\zeta(s_k) M_{N_j}(s_k)| \leqslant \frac{1}{2}, \tag{2.26}$$

where $M_{N_j}(s) = M(s, N_j)$ is defined as in (2.7). By the definition of $\mathcal{K}_j(T)$, we have

$$\sigma_j \leqslant v_k \leqslant \sigma_{j+1}, \quad (\log T)^2 \leqslant t_k \leqslant T \quad \text{and} \quad |t_{k_1} - t_{k_2}| \geqslant 3A' \log T, \quad k_1 \neq k_2.$$

By the Stirling formula (see [21, p. 151]), we have

$$|\Gamma(s)| = \sqrt{2\pi}\, \mathrm{e}^{-(\pi/2)|\tau|} |\tau|^{\sigma-1/2} \bigg\{1 + O\bigg(\frac{|\tan(\frac{\vartheta}{2})|}{|\tau|} + \frac{|a|^2 + |b|^2}{|\tau|^2} + \frac{|a|^3 + |b|^3}{|\tau|^3}\bigg)\bigg\} \tag{2.27}$$

uniformly for $a, b \in \mathbb{R}$ with $a < b$, $a \leqslant \sigma \leqslant b$ and $|\tau| \geqslant 1$, where $\vartheta := \arg s$ and the implied O-constant is absolute.

Since $|t_k| \geqslant (\log T)^2$, the Stirling formula allows us to deduce

$$|M_x(1)\Gamma(1-s_k)y^{1-s_k}| \ll (\log x)y^{1-v_k}\mathrm{e}^{-(\pi/2)|t_k|}|t_k|^{1/2-v_k} \leqslant \frac{1}{10} \tag{2.28}$$

for all $3 \leqslant x \leqslant y \leqslant T^{100}$.

Similarly, using the estimates $\zeta(\alpha+\mathrm{i}t_k+\mathrm{i}u) \ll T+|u|$, $M_x(\alpha+\mathrm{i}t_k+\mathrm{i}u) \ll x^{1-\alpha}\log x \ll T$, and the Stirling formula (2.27), we derive that

$$\int_{|u|\geqslant A'\log T} |\zeta(\alpha+\mathrm{i}t_k+\mathrm{i}u)M_x(\alpha+\mathrm{i}t_k+\mathrm{i}u)\Gamma(\alpha-v_k+\mathrm{i}u)|y^{\alpha-v_k}du \leqslant \frac{1}{10} \tag{2.29}$$

for all $3 \leqslant x \leqslant y \leqslant T^{100}$.

Taking $(s, x) = (s_k, N_j)$ in (2.25) and combining with (2.26), (2.28) and (2.29), we easily see that

$$\bigg|\sum_{N_j<n\leqslant y^2} \frac{a_{n,N_j}}{n^{s_k}}\mathrm{e}^{-n/y}\bigg| \geqslant \frac{1}{6} \tag{2.30}$$

or

$$\bigg|\int_{-A'\log T}^{A'\log T} \zeta(\alpha+\mathrm{i}t_k+\mathrm{i}u)M_{N_j}(\alpha+\mathrm{i}t_k+\mathrm{i}u)\Gamma(\alpha-v_k+\mathrm{i}u)y^{\alpha-v_k+\mathrm{i}u}du\bigg| \geqslant \frac{1}{6} \tag{2.31}$$

or both.

Let $\mathcal{K}'_j(T)$ and $\mathcal{K}''_j(T)$ be the subsets of $\mathcal{K}_j(T)$ for which (2.30) and (2.31) hold, respectively. Then

$$|\mathcal{K}_j(T)| \leqslant |\mathcal{K}'_j(T)| + |\mathcal{K}''_j(T)|. \tag{2.32}$$

First, we bound $|\mathcal{K}'_j(T)|$. By a dyadic argument, there is a $U \in [N_j, y^2]$ such that

$$\bigg|\sum_{U<n\leqslant 2U} \frac{a_{n,N_j}}{n^{s_k}}\mathrm{e}^{-n/y}\bigg| \geqslant (18\log y)^{-1} \tag{2.33}$$

holds for $\gg |\mathcal{K}'_j(T)|(\log y)^{-1}$ integers $k \in \mathcal{K}'_j(T)$. Let $\mathcal{S}'$ be the set of corresponding points s_k. Using [12, Theorem 8.4] with $\theta = \alpha := 4\sigma_j - 3$ and the bound

$$\sum_{U<n\leqslant 2U} \frac{\tau(n)^2}{n^{2\sigma_j}}\mathrm{e}^{-2n/y} \ll U^{1-2\sigma_j}(\log T)^3\mathrm{e}^{-2U/y},$$

it follows that

$$\begin{aligned}
&\sum_{s_k\in\mathcal{S}'}\bigg|\sum_{U<n\leqslant 2U} \frac{a_{n,N_j}}{n^{s_k}}\mathrm{e}^{-n/y}\bigg|^2 \\
&\quad\ll \Big(U + |\mathcal{S}'| \max_{\substack{\sigma\geqslant\alpha\\ 1\leqslant|\tau|\leqslant 4T}} |\zeta(s)|U^{\alpha}\Big)U^{1-2\sigma_j}(\log T)^3\mathrm{e}^{-2U/y} \\
&\quad\ll U^{2(1-\sigma_j)}(\log T)^3\mathrm{e}^{-2U/y} + |\mathcal{S}'| \max_{\substack{\sigma\geqslant\alpha\\ 1\leqslant|\tau|\leqslant 4T}} |\zeta(s)|U^{-2(1-\sigma_j)}(\log T)^3\mathrm{e}^{-2U/y}.
\end{aligned} \tag{2.34}$$

Since $U \geqslant N_j$, we have

$$\max_{\substack{\sigma\geqslant\alpha\\ 1\leqslant|\tau|\leqslant 4T}} |\zeta(s)|U^{-2(1-\sigma_j)}(\log T)^3 \leqslant A'^{-1}(\log T)^{-2}.$$

On the other hand, (2.33) implies that the member on the left-hand side of (2.34) is greater than or equal to $|\mathcal{S}'|(18\log y)^{-2} \geqslant |\mathcal{S}'|(1800\log T)^{-2}$. Since A' is a fixed large integer, the last term on the right-hand side of (2.34) is smaller than this lower bound. Thus it can be simplified as $|\mathcal{S}'|(\log T)^{-2} \ll U^{2(1-\sigma_j)}(\log T)^3 \mathrm{e}^{-2U/y}$ for all $N_j \leqslant y \leqslant T^{100}$ and some $U \in [N_j, y^2]$. Noticing that

$$|\mathcal{S}'| \gg |\mathcal{K}_j'(T)|(\log T)^{-1},$$

we obtain

$$|\mathcal{K}_j'(T)| \ll y^{2(1-\sigma_j)}(\log T)^6 \tag{2.35}$$

for all $N_j \leqslant y \leqslant T^{100}$.

Next, we bound $|\mathcal{K}_j''(T)|$. Let $u_k \in [-A' \log T, A' \log T]$ such that

$$|\zeta(s_k')M_{N_j}(s_k')| = \max_{|u| \leqslant A \log T} |\zeta(\alpha + \mathrm{i}t_k + \mathrm{i}u)M_{N_j}(\alpha + \mathrm{i}t_k + \mathrm{i}u)|,$$

where $s_k' := \alpha + \mathrm{i}t_k'$ and $t_k' := t_k + u_k$. Thus from (2.31) we deduce that

$$\begin{aligned}\frac{1}{6} &\leqslant \bigg| \int_{-A' \log T}^{A' \log T} \zeta(\alpha + \mathrm{i}t_k + \mathrm{i}u) M_{N_j}(\alpha + \mathrm{i}t_k + \mathrm{i}u)\Gamma(\alpha - v_k + \mathrm{i}u) y^{\alpha - v_k + \mathrm{i}u} du \bigg| \\ &\leqslant y^{\alpha - v_k} |\zeta(s_k')M_{N_j}(s_k')| \int_{-A' \log T}^{A' \log T} |\Gamma(\alpha - v_k + \mathrm{i}u)| du.\end{aligned}$$

Since $\Gamma(s)$ has a simple pole at $s = 0$ and $|\alpha - v_k| \gg (\log T)^{-1}$, we can derive, via (2.27), that

$$\int_{-A' \log T}^{A' \log T} |\Gamma(\alpha - v_k + \mathrm{i}u)| du \ll \log T$$

and thus

$$1 \ll y^{\alpha - \sigma_j} |M_{N_j}(s_k')| \max_{\substack{\sigma \geqslant \alpha \\ 1 \leqslant |\tau| \leqslant 8T}} |\zeta(s)| \log T,$$

or equivalently

$$|M_{N_j}(s_k')| \gg y^{\sigma_j - \alpha} \Big(\max_{\substack{\sigma \geqslant \alpha \\ 1 \leqslant |\tau| \leqslant 8T}} |\zeta(s)| \log T \Big)^{-1}.$$

Hence, there is a $V \in [1, N_j]$ such that

$$\bigg| \sum_{V < n \leqslant 2V} \mu(n) n^{-s_k'} \bigg| \gg y^{\sigma_j - \alpha} \Big(\max_{\substack{\sigma \geqslant \alpha \\ 1 \leqslant |\tau| \leqslant 8T}} |\zeta(s)| \Big)^{-1} (\log T)^{-2}$$

holds for $\gg |\mathcal{K}_j''(T)|(\log T)^{-1}$ integers $k \in \mathcal{K}_j''(T)$. Let $\mathcal{S}''$ be the corresponding set of points s_k'. We note $|t_k'| \leqslant 2T$ and $|t_{k_1}' - t_{k_2}'| \geqslant |t_{k_1} - t_{k_2}| - |u_{k_1} - u_{k_2}| \geqslant A' \log T$. Using [12, Theorem 8.4] with $\theta = \alpha = 4\sigma_j - 3$ and the bound

$$\sum_{V < n \leqslant 2V} n^{-2\alpha} \ll V^{1-2\alpha} \ll V^{7-8\sigma_j},$$

it follows that

$$\begin{aligned}\sum_{s_k' \in \mathcal{S}''} \bigg| \sum_{V < n \leqslant 2V} \mu(n) n^{-s_k'} \bigg|^2 &\ll \Big(V + |\mathcal{S}''| \max_{\substack{\sigma \geqslant \alpha \\ 1 \leqslant |\tau| \leqslant 8T}} |\zeta(s)| V^{4\sigma_j - 3} \Big) V^{7 - 8\sigma_j} \\ &\ll V^{8(1-\sigma_j)} + |\mathcal{S}''| \max_{\substack{\sigma \geqslant \alpha \\ 1 \leqslant |\tau| \leqslant 8T}} |\zeta(s)| V^{4(1-\sigma_j)}.\end{aligned} \tag{2.36}$$

Take y such that

$$y^{2(\sigma_j - \alpha)} = A' N_j^{4(1-\sigma_j)} \Big(\max_{\substack{\sigma \geqslant \alpha \\ 1 \leqslant |\tau| \leqslant 8T}} |\zeta(s)| \Big)^3 (\log T)^4. \tag{2.37}$$

The left-hand side of (2.36) is greater than or equal to

$$|\mathcal{S}''|y^{2(\sigma_j-\alpha)}\Big(\max_{\substack{\sigma\geqslant\alpha\\ 1\leqslant|\tau|\leqslant 8T}}|\zeta(s)|\Big)^{-2}(\log T)^{-4}.$$

Hence, (2.36) can be simplified as

$$|\mathcal{S}''|y^{2(\sigma_j-\alpha)}\Big(\max_{\substack{\sigma\geqslant\alpha\\ 1\leqslant|\tau|\leqslant 8T}}|\zeta(s)|\Big)^{-2}(\log T)^{-4}\ll N_j^{8(1-\sigma_j)}.$$

With $|\mathcal{S}''|\gg|\mathcal{K}_j''(T)|(\log T)^{-1}$, we deduce that

$$|\mathcal{K}_j''(T)|\ll N_j^{8(1-\sigma_j)}y^{2(\alpha-\sigma_j)}\Big(\max_{\substack{\sigma\geqslant\alpha\\ 1\leqslant|\tau|\leqslant 8T}}|\zeta(s)|\Big)^{2}(\log T)^{5}. \tag{2.38}$$

On combining (2.32), (2.35), (2.38) and (2.37), it follows that $|\mathcal{K}_j(T)|\ll N_j^{(10/3)(1-\sigma_j)}(\log T)^3$. Now the required inequality follows from (2.19). This completes the proof. □

3 Proof of Theorem 1.1

We shall conserve the notation of Section 2. First we prove a lemma.

Lemma 3.1. *Let $z\in\mathbb{C}$, $w\in\mathbb{C}$, $\alpha>0$, $\delta\geqslant 0$, $A\geqslant 0$, $B>0$, $C>0$ and $M>0$ be some constants. Suppose that the Dirichlet series*

$$\mathcal{F}(s):=\sum_{n=1}^{\infty}f(n)n^{-s}$$

is of type $\mathcal{P}(z,w,\alpha,\delta,A,B,C,M)$. Then there is an absolute positive constant D such that we have

$$\mathcal{F}(s)\ll MD^BT^{(100B\sqrt{\varepsilon}+\delta)(1-\sigma)}(\log T)^{A+4B} \tag{3.1}$$

for all $s\in\mathscr{M}_T$, where the implied constant depends only on ε.

Proof. Since we have chosen the principal value of complex logarithm, we can write

$$|\zeta(s)^z|=|\zeta(s)|^{\Re e z}\mathrm{e}^{-(\Im m\, z)\arg\zeta(s)}\leqslant \mathrm{e}^{\pi B}|\zeta(s)|^{\Re e z} \tag{3.2}$$

for all $s\in\mathbb{C}$ such that $\zeta(s)\neq 0$.

Invoking Proposition 2.3, we see that there is a suitable absolute constant D such that

$$|\zeta(s)^z|\ll_{\varepsilon} D^BT^{100B\sqrt{\varepsilon}(1-\sigma)}(\log T)^{4B} \tag{3.3}$$

for all $s\in\mathscr{M}_T$, where the implied constant depends only on ε.

Finally, the required bound (3.1) follows from (3.3), the hypothesis (1.4) and the trivial bound $|\zeta(2s)|\asymp 1$ for $s\in\mathscr{M}_T$. □

Now we are ready to prove Theorem 1.1.

Since the Dirichlet series $\mathcal{F}(s)$ is of type $\mathcal{P}(z,w,\alpha,\delta,A,B,C,M)$, we can apply [20, Corollary II.2.2.1] with the choice of parameters $\sigma_a=1$, $\alpha=\alpha$ and $\sigma=0$ to write

$$\sum_{x<n\leqslant x+y}f(n)=\frac{1}{2\pi\mathrm{i}}\int_{b-\mathrm{i}T'}^{b+\mathrm{i}T'}\mathcal{F}(s)\frac{(x+y)^s-x^s}{s}ds+O_{\varepsilon}\Big(M\frac{x^{1+\varepsilon}}{T}\Big),$$

where $b=1+1/\log x$, $\mathrm{e}^{\sqrt{\log x}}\leqslant T\leqslant x$ is a parameter to be chosen later and $T'=K_T\log T\sim T$.

Denote by Γ_T the path formed from the circle $|s-1| = r := 1/(2\log x)$ excluding the point $s = 1-r$, together with the segment $[1-\delta_T, 1-r]$ traced out twice with respective arguments $+\pi$ and $-\pi$. By the residue theorem, the path $[b - \mathrm{i}T', b + \mathrm{i}T']$ is deformed into

$$\Gamma_T \cup [1-\delta_T - \mathrm{i}T', 1-\delta_T + \mathrm{i}T'] \cup [1-\delta_T \pm \mathrm{i}T', b \pm \mathrm{i}T'].$$

In view of (2.2) and the hypothesis (c), the function $\mathcal{F}(s)$ is analytic and admits the estimate in the interior of this contour

$$\mathcal{F}(s) \ll MD^C T^{\max\{\delta(1-\sigma),\,0\}} (\log T)^{A+B}, \tag{3.4}$$

where the implied constant and the constant D are absolute. The integral over the horizontal segments $[1-\delta_T \pm \mathrm{i}T',\, b \pm \mathrm{i}T']$ is

$$\begin{aligned}\int_{1-\delta_T \pm \mathrm{i}T'}^{b\pm \mathrm{i}T'} \mathcal{F}(s) \frac{(x+y)^s - x^s}{s} ds &\ll \frac{MD^C (\log T)^{A+B}}{T} \int_{1-\delta_T}^{b} T^{\max\{\delta(1-\sigma),0\}} x^\sigma d\sigma \\ &\ll MD^C \frac{x}{T} (\log T)^{A+B} \bigg(\int_{1-\delta_T}^{1} \bigg(\frac{x}{T^\delta}\bigg)^{\sigma-1} d\sigma + 1 \bigg) \\ &\ll MD^C \frac{x}{T} (\log T)^{A+B-1}.\end{aligned}$$

Thus

$$\sum_{x<n\leqslant x+y} f(n) = I + O\bigg(MD^C \frac{x^{1+\varepsilon}}{T} \bigg), \tag{3.5}$$

where the implied constant depends on ε only and

$$I := \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_T \cup [1-\delta_T - \mathrm{i}T',\, 1-\delta_T + \mathrm{i}T']} \mathcal{F}(s) \frac{(x+y)^s - x^s}{s} ds.$$

Let $\mathscr{M}_T$ be the Motohashi contour defined as in Section 2. Consider the two symmetric simply connected regions bounded by $\mathscr{M}_T$, the segment $[1-\delta_T - \mathrm{i}T',\, 1-\delta_T + \mathrm{i}T']$ and the two line segments $[\sigma_{j_0+1} + d_{\mathrm{v}}, 1-\delta_T]$ with respective arguments $+\pi$ and $-\pi$ measured from the real axis on the right of $1-\delta_T$. It is clear that $\mathcal{F}(s)$ is analytic in these two simply connected regions. Denote by Γ_T^* the path joining (the two end-points of) Γ_T with the two line segments $[\sigma_{j_0+1} + d_{\mathrm{v}}, 1-\delta_T]$ of the symmetric regions. Thanks to the residue theorem, we can write

$$I = I_1 + I_2, \tag{3.6}$$

with

$$I_1 := \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_T^*} \mathcal{F}(s) \frac{(x+y)^s - x^s}{s} ds, \quad I_2 := \frac{1}{2\pi \mathrm{i}} \int_{\mathscr{M}_T} \mathcal{F}(s) \frac{(x+y)^s - x^s}{s} ds.$$

A. *Evaluation of* I_1.

According to our hypothesis, $\mathcal{G}(s;\kappa,w)\zeta(2s)^w Z(s;\kappa)$ is holomorphic and $O(M)$ in the disc $|s-1| \leqslant \frac{1}{2} - \varepsilon^3 =: c$; the Cauchy integral formula implies that

$$g_\ell(\kappa, w) \ll Mc^{-\ell}, \quad \ell \geqslant 0, \quad |z| \leqslant B, \quad |w| \leqslant C, \tag{3.7}$$

where $g_\ell(\kappa,w)$ is defined as in (1.16). From this and (1.15), we deduce that for any integer $N \geqslant 0$ and $|s-1| \leqslant \frac{1}{2} - \varepsilon^2$,

$$\mathcal{G}(s;\kappa,w)\zeta(2s)^w Z(s;\kappa) = \sum_{\ell=0}^{N} g_\ell(\kappa,w)(s-1)^\ell + O(M(|s-1|/c)^{N+1}).$$

Thus we have

$$I_1 = \sum_{\ell=0}^{N} g_\ell(\kappa,w) M_\ell(x,y) + O(Mc^{-N} E_N(x,y)), \tag{3.8}$$

where

$$M_\ell(x,y) := \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_T^*} (s-1)^{\ell-z} \frac{(x+y)^s - x^s}{s} ds,$$
$$E_N(x,y) := \int_{\Gamma_T^*} \left| (s-1)^{N+1-z} \frac{(x+y)^s - x^s}{s} \right| |ds|.$$

Firstly, we evaluate $M_\ell(x,y)$. Using the formula

$$\frac{(x+y)^s - x^s}{s} = \int_x^{x+y} t^{s-1} dt \tag{3.9}$$

and [20, Corollary II.5.2.1], we write

$$\begin{aligned} M_\ell(x,y) &= \int_x^{x+y} \left(\frac{1}{2\pi \mathrm{i}} \int_{\Gamma_T^*} (s-1)^{\ell-z} t^{s-1} ds \right) dt \\ &= \int_x^{x+y} (\log t)^{z-1-\ell} \left\{ \frac{1}{\Gamma(\kappa-\ell)} + O\left(\frac{(c_1\ell+1)^\ell}{t^{\delta_T/2}} \right) \right\} dt, \end{aligned}$$

where we have used the following inequality:

$$47^{|z-\ell|}\Gamma(1+|z-\ell|) \ll_B (c_1\ell+1)^\ell, \quad \ell \geqslant 0, \quad |z| \leqslant B.$$

The constant c_1 and the implied constant depend at most on B. Besides, for $|z| \leqslant B$, we have

$$\begin{aligned} \int_x^{x+y} (\log t)^{z-1-\ell} dt &= \int_0^y \log^{z-1-\ell}(x+t) dt \\ &= y(\log x)^{z-1-\ell} \left\{ 1 + O_B\left(\frac{(\ell+1)y}{x\log x} \right) \right\}. \end{aligned}$$

Inserting this into the preceding formula, we obtain

$$M_\ell(x,y) = y(\log x)^{z-1-\ell} \left\{ \frac{1}{\Gamma(z-\ell)} + O_B\left(\frac{(\ell+1)y}{\Gamma(z-\ell)x\log x} + \frac{(c_1\ell+1)^\ell}{x^{\delta_T/2}} \right) \right\} \tag{3.10}$$

for $\ell \geqslant 0$ and $|z| \leqslant B$.

Next, we estimate $E_N(x,y)$. In view of the trivial inequality

$$\left| \frac{(x+y)^s - x^s}{s} \right| \ll yx^{\sigma-1}, \tag{3.11}$$

which follows from (3.9) immediately, we deduce that

$$\begin{aligned} E_N(x,y) &\ll \int_{1/2+\varepsilon^2}^{1-1/\log x} (1-\sigma)^{N+1-\Re e\, z} x^{\sigma-1} y d\sigma + \frac{y}{(\log x)^{N+2-\Re ez}} \\ &\ll \frac{y}{(\log x)^{N+2-\Re ez}} \left(\int_1^\infty t^{N+1-\Re ez} \mathrm{e}^{-t} dt + 1 \right) \\ &\ll y(\log x)^{\Re ez-1} \left(\frac{c_1 N+1}{\log x} \right)^{N+1} \end{aligned} \tag{3.12}$$

uniformly for $x \geqslant y \geqslant 2$, $N \geqslant 0$ and $|z| \leqslant B$, where the constant $c_1 > 0$ and the implied constant depends only on B.

Inserting (3.10) and (3.12) into (3.8) and using (3.7), we find that

$$I_1 = y(\log x)^{z-1} \left\{ \sum_{\ell=0}^N \frac{\lambda_\ell(z,w)}{(\log x)^\ell} + O_B(E_N^*(x,y)) \right\}, \tag{3.13}$$

where

$$E_N^*(x,y) := \frac{y}{x}\sum_{\ell=1}^{N+1}\frac{\ell|\lambda_{\ell-1}(z,w)|}{(\log x)^\ell} + \frac{(c_1N+1)^{N+1}}{x^{\delta_T/2}} + M\left(\frac{c_1N+1}{\log x}\right)^{N+1}.$$

B. *Evaluation of* I_2.

Let $\mathscr{M}_T'$ be the union of those vertical line segments of $\mathscr{M}_T$ whose real part is equal to $\frac{1}{2}+\varepsilon^2$ (i.e., corresponding to those k such that $j_k=0$) and $\mathscr{M}_T'' := \mathscr{M}_T \setminus \mathscr{M}_T'$. Denote by I_2' and I_2'' the contribution of $\mathscr{M}_T'$ and $\mathscr{M}_T''$ to I_2, respectively. Using the trivial inequality

$$\left|\frac{(x+y)^s - x^s}{s}\right| \ll \frac{x^{1/2+\varepsilon^2}}{|\tau|+1}, \quad s\in\mathscr{M}_T'$$

and Lemma 3.1, we can deduce

$$\begin{aligned} I_2' &\ll MD^B x^{1/2+\varepsilon^2} T^{(\delta+100B\sqrt{\varepsilon})(1/2-\varepsilon^2)}(\log T)^{A+4B+1} \\ &\ll Mx^{1/2+\delta/(2\psi+2\delta)+\sqrt{\varepsilon}} \\ &\ll Mx^{1-1/(\psi+\delta)+\sqrt{\varepsilon}} \end{aligned} \tag{3.14}$$

with the value of T given by (3.16) below and $\psi \geqslant 2$.

Next, we bound I_2''. In view of (3.11), we can write that

$$I_2'' \ll y\int_{\mathscr{M}_T''}|\mathcal{F}(s)|x^{\sigma-1}|ds| \ll y\sum_{0\leqslant j\leqslant J_T}\ \sum_{\substack{0\leqslant k\leqslant K_T\\ \Delta_{j,k}\in(W)}}\int_{\mathscr{M}_T(j,k)}|\mathcal{F}(s)|x^{\sigma-1}|ds|, \tag{3.15}$$

where $\mathscr{M}_T(j,k)$ is the vertical line segment of $\mathscr{M}_T''$ around $\Delta_{j,k}$ and the horizontal line segments with $\sigma\leqslant\sigma_j+d_{\mathrm{v}}$. Clearly, the length of $\mathscr{M}_T(j,k)$ is $\ll \log T$. Thus with the help of Lemma 3.1, it is easy to see that

$$\int_{\mathscr{M}_T(j,k)}|\mathcal{F}(s)|x^{\sigma-1}|ds| \ll MD^B(\log T)^{A+4B+1}T^{(\delta+100B\sqrt{\varepsilon})(1-\sigma_j-d_{\mathrm{v}})}x^{\sigma_j+d_{\mathrm{v}}-1}$$

for all $0\leqslant k\leqslant K_T$. Inserting it into (3.15) and using Proposition 2.4, we can deduce, with the notation $J_{T,0} := [(\frac{1}{2}-\varepsilon)\log T]$, that $I_2'' \ll MD^By(\log T)^{A+4B+18+\eta}(I_{2,*}''+I_{2,\dagger}'')$, where

$$\begin{aligned} I_{2,*}'' &:= \sum_{0\leqslant j\leqslant J_{T,0}} T^{(\delta+100B\sqrt{\varepsilon})(1-\sigma_j-d_{\mathrm{v}})}x^{\sigma_j+d_{\mathrm{v}}-1}\cdot T^{\psi(1-\sigma_j)}, \\ I_{2,\dagger}'' &:= \sum_{J_{T,0}<j\leqslant J_T} T^{(\delta+100B\sqrt{\varepsilon})(1-\sigma_j-d_{\mathrm{v}})}x^{\sigma_j+d_{\mathrm{v}}-1}\cdot T^{100\sqrt{\varepsilon}(1-\sigma_j)}. \end{aligned}$$

Taking

$$T := x^{(1-\sqrt{\varepsilon})/(\psi+\delta+100B\sqrt{\varepsilon})} \tag{3.16}$$

and in view of (2.10), it is easy to check that

$$I_{2,*}'' \ll x^{\varepsilon^2}\sum_{0\leqslant j\leqslant J_{T,0}}(x/T^{\psi+\delta+100B\sqrt{\varepsilon}})^{-(1-\sigma_j)}\log x \ll x^{\varepsilon^2-\varepsilon^{3/2}}\log x \ll x^{-\varepsilon^2}$$

and

$$I_{2,\dagger}'' \ll \sum_{J_{T,0}<j\leqslant J_T}(x/T^{\delta+100(B+1)\sqrt{\varepsilon}})^{-(1-\sigma_j)} \ll \mathrm{e}^{-2c_2(\log x)^{1/3}(\log_2 x)^{-1/3}}.$$

Inserting it into the preceding estimate for I_2'', we conclude that

$$I_2'' \ll_B My\mathrm{e}^{-c_2(\log x)^{1/3}(\log_2 x)^{-1/3}}. \tag{3.17}$$

Now from (3.5), (3.6), (3.13), (3.14) and (3.17), we deduce that

$$\sum_{x<n\leqslant x+y} f(n) = y(\log x)^{z-1}\bigg\{\sum_{\ell=0}^{N} \frac{\lambda_\ell(z,w)}{(\log x)^\ell} + O_{A,B,C,\alpha,\delta,\varepsilon}(R_N^*(x,y))\bigg\}$$

uniformly for $x \geqslant 3$, $x^{1-1/(\psi+\delta)+\varepsilon} \leqslant y \leqslant x$, $N \geqslant 0$, $|z| \leqslant B$ and $|w| \leqslant C$, where

$$R_N^*(x,y) := \frac{y}{x}\sum_{\ell=1}^{N+1} \frac{\ell|\lambda_{\ell-1}(z,w)|}{(\log x)^\ell} + M\bigg\{\bigg(\frac{c_1N+1}{\log x}\bigg)^{N+1} + \frac{(c_1N+1)^{N+1}}{\mathrm{e}^{c_2(\log x)^{1/3}(\log_2 x)^{-1/3}}}\bigg\}$$

for some constants $c_1 > 0$ and $c_2 > 0$ depending only on B, C, δ and ε.

It remains to prove that the first term on the right-hand side can be absorbed by the third. In view of (1.14), the Cauchy formula allows us to write $g_\ell(z,w) \ll_{A,B,C,\delta} M3^\ell$ for $|z| \leqslant B$, $|w| \leqslant C$ and $\ell \geqslant 1$. Combining this with the Stirling formula, we easily derive $\lambda_\ell(z,w) \ll_{A,B,C,\delta} M(9/\ell)^\ell$ for $|z| \leqslant B$, $|w| \leqslant C$ and $\ell \geqslant 1$. This implies that

$$\frac{y}{x}\sum_{\ell=1}^{N+1} \frac{\ell|\lambda_{\ell-1}(z,w)|}{(\log x)^\ell} \ll_{A,B,C,\delta} M\frac{y}{x} \ll_{A,B,C,\delta,\varepsilon} \frac{M(c_1N+1)^{N+1}}{\mathrm{e}^{c_2(\log x)^{1/3}(\log_2 x)^{-1/3}}}$$

holds uniformly for $x \geqslant 3$, $x^{1-1/(\psi+\delta)+\varepsilon} \leqslant y \leqslant x$, $N \geqslant 0$, $|z| \leqslant B$ and $|w| \leqslant C$. This completes the proof.

4 Proofs of Theorems 1.3 and 1.4

Since the proofs of Theorems 1.3 and 1.4 are very similar, we shall only prove the former. For $z \in \mathbb{C}$ and $\sigma > 1$, we can write

$$\begin{aligned}\mathcal{F}_1(s;z) &:= \sum_{n\geqslant 1} z^{\omega(n)} n^{-s} = \prod_p (1 + z(p^s-1)^{-1})\\ &= \zeta(s)^z \zeta(2s)^{z(1-z)/2} \mathcal{G}_1\bigg(s; z, \frac{z(1-z)}{2}\bigg),\end{aligned}$$

where

$$\mathcal{G}_1\bigg(s; z, \frac{z(1-z)}{2}\bigg) := \prod_p \bigg(1 + \frac{z}{p^s-1}\bigg)\bigg(1 - \frac{1}{p^s}\bigg)^z \bigg(1 - \frac{1}{p^{2s}}\bigg)^{z(1-z)/2}.$$

We expand $\mathcal{G}_1(s; z, \frac{z(1-z)}{2})$ into the Dirichlet series

$$\mathcal{G}_1\bigg(s; z, \frac{z(1-z)}{2}\bigg) = \sum_{n\geqslant 1} b_{1z}(n) n^{-s}.$$

Then $b_{1z}(n)$ is the multiplicative function whose values on prime powers are determined by the identity

$$1 + \sum_{\nu\geqslant 1} b_{1z}(p^\nu)\xi^\nu = \bigg(1 + \frac{z\xi}{1-\xi}\bigg)(1-\xi)^z(1-\xi^2)^{z(1-z)/2}, \quad |\xi| < 1.$$

In particular, $b_{1z}(p) = b_{1z}(p^2) = 0$ and the Cauchy integral formula gives

$$|b_{1z}(p^\nu)| \leqslant M(B)2^{\nu/2}, \quad \nu \geqslant 3, \quad |z| \leqslant B,$$

where

$$M(B) := \sup_{|z|\leqslant B,\, |\xi|\leqslant 1/\sqrt{2}} \bigg|\bigg(1 + \frac{z\xi}{1-\xi}\bigg)(1-\xi)^z(1-\xi^2)^{z(1-z)/2}\bigg|.$$

From these we deduce that for $\sigma > \frac{1}{3}$,

$$\sum_p \sum_{\nu \geqslant 1} \frac{|b_{1z}(p^\nu)|}{p^{\nu\sigma}} \leqslant \sum_p \sum_{\nu \geqslant 3} \frac{M(B)}{(p^\sigma/\sqrt{2})^\nu} \leqslant \sum_p \frac{2^{3/2} M(B)}{p^{2\sigma}(p^\sigma - \sqrt{2})} \ll_B \frac{1}{3\sigma - 1}.$$

So the Dirichlet series $\sum_{n=1}^\infty z^{\omega(n)} n^{-s}$ is of type $\mathcal{P}(z, \frac{z(1-z)}{2}, B, 0, 0, B, C(B), M(B))$, where $C(B)$ is a positive constant depending on B.

Define $g_\ell(z)$ by

$$\begin{aligned} \mathcal{F}_1(s)(s-1)^z &= Z(s;z)\zeta(2s)^{z(1-z)/2} \mathcal{G}_1\bigg(s; z, \frac{z(1-z)}{2}\bigg) \\ &= \sum_{\ell=0}^\infty g_\ell(z)(s-1)^\ell, \quad |s-1| < \frac{1}{6}. \end{aligned} \tag{4.1}$$

Applying Theorem 1.1 to the Dirichlet series $\sum_{n=1}^\infty z^{\omega(n)} n^{-s}$, we obtain the following result.

Lemma 4.1. *Let $B > 0$ be a constant. For any $\varepsilon > 0$, we have*

$$\sum_{x < n \leqslant x+y} z^{\omega(n)} = y(\log x)^{z-1} \Bigg\{ \sum_{\ell=0}^N \frac{\lambda_\ell(z)}{(\log x)^\ell} + O_{B,\varepsilon}(M R_N(x,y)) \Bigg\} \tag{4.2}$$

uniformly for

$$x \geqslant 3, \quad x \geqslant y \geqslant x^{1-1/\psi+\varepsilon}, \quad |z| \leqslant B, \quad N \geqslant 0,$$

where $\lambda_\ell(z) := g_\ell(z)/\Gamma(z-\ell)$ and $R_N(x,y)$ is defined as in (1.18)*. The constants c_1 and c_2 in $R_N(x,y)$ and the implied constant depends only on B and ε.*

Lemma 4.1 improves [10, Theorem 3] in two directions: get a more precise asymptotic formula and extend the domain $x^{7/12+\varepsilon} \leqslant y \leqslant x^{2/3-\varepsilon}$ to $x^{7/12+\varepsilon} \leqslant y \leqslant x$.

The next lemma is a short interval version of the asymptotic formula (13) of [20, Theorem II.6.3]. We omit the proof as it is very similar.

Lemma 4.2. *Let $B > 0$ and $0 < \theta \leqslant 1$ be two positive constants. For each integer $n \geqslant 1$, let*

$$a_z(n) = \sum_{k=0}^\infty c_k(n) z^k$$

be a holomorphic function for $|z| \leqslant B$. Let $N \geqslant 0$ be a non-negative integer. Suppose that there exist $N+1$ holomorphic functions $h_0(z), \ldots, h_N(z)$ for $|z| \leqslant B$ and a quantity $R_N(x,y)$ independent of z such that

$$\sum_{x < n \leqslant x+y} a_z(n) = y(\log x)^{z-1} \Bigg\{ \sum_{\ell=0}^N \frac{z h_\ell(z)}{(\log x)^\ell} + O_{B,\theta}(R_N(x,y)) \Bigg\} \tag{4.3}$$

holds uniformly for $x \geqslant 3$, $x \geqslant y \geqslant x^\theta$ and $|z| \leqslant B$. Then we have

$$\sum_{x < n \leqslant x+y} c_k(n) = \frac{y}{\log x} \Bigg\{ \sum_{j=0}^N \frac{R_{j,k}(\log_2 x)}{(\log x)^j} + O_{B,\theta}\bigg(\frac{(\log_2 x)^k}{k!} R_N(x,y) \bigg) \Bigg\} \tag{4.4}$$

uniformly for $x \geqslant 3$, $x \geqslant y \geqslant x^\theta$ and $1 \leqslant k \leqslant B \log_2 x$, where

$$R_{j,k}(X) := \sum_{\ell+m=k-1} \frac{h_j^{(m)}(0)}{\ell! m!} X^\ell \tag{4.5}$$

and the implied constants depend only on B and θ.

If, in addition, we suppose that $|h_0''(z)| \leqslant D$ $(|z| \leqslant B)$*, then we have*

$$\sum_{x<n\leqslant x+y} c_k(n) = \frac{y}{\log x}\frac{(\log_2 x)^{k-1}}{(k-1)!}\left\{h_0\left(\frac{k-1}{\log_2 x}\right) + O_{B,\theta}\left(\frac{D(k-1)}{(\log_2 x)^2} + \frac{\log_2 x}{k}R_0(x,y)\right)\right\}$$

uniformly for $x \geqslant 3$*,* $x \geqslant y \geqslant x^\theta$ *and* $1 \leqslant k \leqslant B\log_2 x$*. Here, the implied constants depend on* B *and* θ *only.*

Now we are ready to finish the proof of Theorem 1.3. According to Lemma 4.1, (4.3) of Lemma 4.2 is satisfied with the following choices:

$$a_z(n) = z^{\omega(n)}, \quad zh_\ell(z) = \lambda_\ell(z), \quad \theta = 1 - 1/\psi + \varepsilon,$$

$\lambda_\ell(z)$ and $R_N(x,y)$ are defined as in Lemma 4.1, and $c_k(n)$ is the characteristic function on the set of integers n such that $\omega(n) = k$. Thus (a) is an immediate consequence of this lemma.

5 Proofs of Theorems 1.6 and 1.7

The proof of Theorem 1.6 will be proceeded exactly as in [1]. The only difference is the use of Corollary 1.2 in place of [1, Theorem 1].

Since $\sum_{n\geqslant 1}\tau_k(n)n^{-s} = \zeta(s)^k$ for $\sigma > 1$, we can apply Theorem 1.1 with $z = k$, $w = 0$, $\mathcal{G}(s;k,0) \equiv 1$ and $A = \delta = 0$. Taking $N = [c'(\log x)^{1/3}(\log_2 x)^{-4/3}]$ with some small constant c' and noticing that $\lambda_\ell(k,0) = 0$ for all $\ell \geqslant k$, we obtain the result of Theorem 1.7.

6 Proof of Theorem 1.8

Since the function $\tau_k(n)$ is multiplicative and

$$\tau_k(p^\nu) = \binom{k+\nu-1}{\nu} = \frac{1}{\nu!}\prod_{j=0}^{\nu-1}(k+j),$$

we can write, for $\sigma > 1$,

$$\begin{aligned}\sum_{n\geqslant 1}\tau_k(n)^{-1}n^{-s} &= \prod_p\left(1 + \sum_{\nu\geqslant 1}\binom{k+\nu-1}{\nu}^{-1}p^{-\nu s}\right)\\ &= \zeta(s)^{\frac{1}{k}}\zeta(2s)^{-\frac{2k^3+2k^2+2k+1}{k^2}}\mathcal{G}_3\left(s;\frac{1}{k},-\frac{2k^3+2k^2+2k+1}{k^2}\right),\end{aligned}$$

where

$$\mathcal{G}_3(s;z,w) := \prod_p\left(\sum_{\nu\geqslant 0}\binom{k+\nu-1}{\nu}^{-1}\frac{1}{p^{\nu s}}\right)\left(1-\frac{1}{p^s}\right)^z\left(1-\frac{1}{p^{2s}}\right)^w.$$

As before, we expand $\mathcal{G}_3(s;\frac{1}{k},-\frac{2k^3+2k^2+2k+1}{k^2})$ as a Dirichlet series

$$\mathcal{G}_3\left(s;\frac{1}{k},-\frac{2k^3+2k^2+2k+1}{k^2}\right) = \sum_{n\geqslant 1}b_{3k}(n)n^{-s},$$

where $b_{3k}(n)$ is the multiplicative function for which the values on prime powers are determined by the identity

$$1 + \sum_{\nu\geqslant 1}b_{3k}(p^\nu)\xi^\nu = \left(\sum_{\nu\geqslant 0}\binom{k+\nu-1}{\nu}^{-1}\xi^\nu\right)(1-\xi)^{\frac{1}{k}}(1-\xi^2)^{-\frac{2k^3+2k^2+2k+1}{k^2}}.$$

It is easy to see that the right-hand side is an analytic function in $|\xi| < 1$ and $b_{3k}(p) = b_{3k}(p^2) = 0$. Again the Cauchy integral formula yields

$$|b_{3k}(p^\nu)| \ll_k 2^{\nu/2}, \quad \nu \geqslant 3, \quad \mathcal{G}_3\bigg(s; \frac{1}{k}, -\frac{2k^3+2k^2+2k+1}{k^2}\bigg) \ll_{k,\sigma} 1, \quad \sigma > \frac{1}{3}.$$

This shows that the Dirichlet series associated to $\tau_k(n)^{-1}$ is of type

$$\mathcal{P}\bigg(\frac{1}{k}, -\frac{2k^3+2k^2+2k+1}{k^2}, \frac{1}{k}, 0, \frac{1}{k}, \frac{2k^3+2k^2+2k+1}{k^2}, M(k)\bigg),$$

where $M(k)$ is a positive constant depending on k. Therefore, the required result follows immediately from Theorem 1.1 with any fixed positive integer N.

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