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Backward stochastic differential equations with rank-based data

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Abstract In this paper, we investigate Markovian backward stochastic differential equations (BSDEs) with the generator and the terminal value that depend on the solutions of stochastic differential equations with rank-based drift coefficients. We study regularity properties of the solutions of this kind of BSDEs and establish their connection with semi-linear backward parabolic partial differential equations in simplex with Neumann boundary condition. As an application, we study the European option pricing problem with capital size based stock prices.

Keywords backward stochastic differential equations, ranked particles, named particles, reflected Brownian motion, partial differential equations, viscosity solution

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1 Introduction

Linear backward stochastic differential equations (BSDEs for short) were introduced by Bismut [3] in 1973, while the general BSDEs were introduced and studied by Pardoux and Peng [27] in 1990. The solutions of BSDEs consist of a pair of adapted processes (Y, Z) taking values in $\mathbb{R} \times \mathbb{R}^n$ and satisfying

$$-dY(t) = f(t, Y(t), Z(t))dt - Z(t) \cdot dW(t), \quad Y(T) = \xi,$$

where $W(t)$ is an n-dimensional Brownian motion, f is a function on $[0, \infty) \times \mathbb{R} \times \mathbb{R}^n$, and ξ is an $\mathcal{F}_T$-measurable random variable. Here, $\{\mathcal{F}_t, t \geqslant 0\}$ is the minimal augmented filtration generated by W. In BSDEs, ξ is called terminal value and the function f is called generator. In [10], El Karoui et al. used BSDEs to determine the price of a contingent claim $\xi \geqslant 0$ of maturity T, which is a contract that pays an amount of ξ at time T. They showed the problem is well-posed, i.e., there exist a unique price and a unique hedging portfolio. The interest in BSDEs also comes from its connections with partial differential equations (PDEs for short). Pardoux [26] and Pardoux and Peng [28] studied Markovian BSDEs and gave a Feynman-Kac representation for the solutions to some nonlinear parabolic partial differential

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equations. For more information and applications on BSDEs, the interested readers are referred to Chen and Epstein [5], Cvitanic and Ma [8], Pardoux and Zhang [29] and the references therein.

Recently, systems of rank-based stochastic differential equations (rank-based SDEs for short), also called competing Brownian particles, have received lots of attention, where the drift and diffusion coefficients of each component are determined by its rank in the system. Rank-based SDEs are introduced by Karatzas and Fernholz [20] as a model in stochastic portfolio theory for analyzing portfolio behavior and equity market structure. It can be used to model capital distribution in financial market. The importance of rank-based models stems from the fact that they match the data of capital distribution curve. In [2], Bass and Pardoux showed that SDEs with piecewise constant coefficients have a weak solution, which is unique in law. Recently, Shkolnikov [32] obtained the existence and uniqueness of weak solution for SDEs driven by independent identically distributed Lévy processes with rank-based coefficients. In [11], Fernholz et al. established the existence and pathwise uniqueness of strong solution for two-dimensional SDEs with rank-based coefficients. It is extended to finite and countably infinite systems in [15]. For ranked particles from SDEs with rank-based coefficients, the collisions are symmetric, i.e., if two adjacent particles collide, they are pushed apart and the push, which is the local time of the collision, is split evenly between them. Karatzas et al. [21] and Sarantsev [30] studied systems of Brownian particles with asymmetric collisions. In these systems, the local time of collision between two particles can be split unevenly between them and the parameters of the collisions are decided by the ranks of the particles involved in the collisions. For more information on competing Brownian particles and their applications, the readers are referred to Chatterjee and Pal [4], Ichiba et al. [16], Jourdain and Reygner [18] and Karatzas and Sarantsev [22].

In this paper, we use BSDEs method to study the European option pricing problem under the scenario that the prices of stocks depend on their market capital size, i.e., the coefficients of the price processes are rank-based. This is motivated by the fact that stock price of a company with large capital asset tends to move differently than that of a company with small capital asset, and thus it is reasonable to model stock prices using SDEs with rank-based coefficients. The outline of the paper is as follows. In Section 2, we introduce SDEs with rank-based coefficients. We thus study Markovian BSDEs in which the generator and terminal value depend on the solutions from SDEs with rank-based drift coefficients in Section 3. More specifically, we study the following Markov type BSDEs:

$$Y^{t,\widetilde{x}}(s) = g(\widetilde{X}^{t,\widetilde{x}}(T)) + \int_s^T f(r, X^{t,\widetilde{x}}(r), Y^{t,\widetilde{x}}(r), Z^{t,\widetilde{x}}(r))dr - \int_s^T Z^{t,\widetilde{x}}(r) \cdot dW(r), \tag{1.1}$$

where $\widetilde{X}^{t,\widetilde{x}}(s) := (X^{t,\widetilde{x}}_{(1)}(s), \ldots, X^{t,\widetilde{x}}_{(n)}(s))$ with $X^{t,\widetilde{x}}_{(1)}(s) \geqslant X^{t,\widetilde{x}}_{(2)}(s) \geqslant \cdots \geqslant X^{t,\widetilde{x}}_{(n)}(s)$ are the ranked particles from the solution of the following SDEs with rank-based drift coefficient:

$$X_i^{t,\widetilde{x}}(s) = \widetilde{x}_i + \int_t^{s\vee t} \sum_{j=1}^n b_j \mathbf{1}_{\{X_i^{t,\widetilde{x}}(r) = X_{(j)}^{t,\widetilde{x}}(r)\}} dr + W_i(s\vee t) - W_i(t), \quad i = 1, \ldots, n.$$

We further assume that there is a function $h(t, \widetilde{x}, y, z)$ so that

$$f(t, x, y, z) = h(t, \widetilde{x}, y, \bar{z}),$$

where

$$\widetilde{x} = (\widetilde{x}_1, \ldots, \widetilde{x}_n) = (x_{(1)}, \ldots, x_{(n)}) \in \Gamma^n$$

and

$$\bar{z}_j := \sum_{i=1}^n z_i \mathbf{1}_{\{x_i = x_{(j)}\}}.$$

This allows us to give an equivalent form of BSDEs (1.1). Under this assumption, in Section 4, we establish a nonlinear Feynman-Kac formula which shows that $u(t, \widetilde{x}) := Y^{t,\widetilde{x}}(t)$ obtained from BSDEs

(1.1) is the unique viscosity solution of the following nonlinear PDEs with Neumann boundary condition:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,\widetilde{x}) = -\mathcal{L}u(t,\widetilde{x}) - h(t,\widetilde{x},u(t,\widetilde{x}),\nabla u(t,\widetilde{x})), & t\in[0,T], \quad \widetilde{x}\in\Pi^n, \\ u(T,\widetilde{x}) = g(x), & \widetilde{x}\in\Gamma^n, \\ \dfrac{\partial u}{\partial \widetilde{x}_{i+1}}(t,\widetilde{x}) = \dfrac{\partial u}{\partial \widetilde{x}_i}(t,\widetilde{x}), & t\in[0,T), \quad \widetilde{x}\in F_i, \quad i=1,\dots,n-1. \end{cases}$$

Here, $\Pi^n := \{\widetilde{x}\in\mathbb{R}^n : \widetilde{x}_1 > \widetilde{x}_2 > \cdots > \widetilde{x}_n\}$, $\mathcal{L} = \frac{1}{2}\sum_{i=1}^n \frac{\partial^2}{\partial\widetilde{x}_i^2} + \sum_{i=1}^n b_i\frac{\partial}{\partial\widetilde{x}_i}$ and

$$F_i := \{\widetilde{x}\in\partial\Pi^n : \widetilde{x}_1 > \widetilde{x}_2 > \cdots > \widetilde{x}_i = \widetilde{x}_{i+1} > \cdots > \widetilde{x}_n\}, \quad \Gamma^n := \Pi^n \cup \left(\bigcup_{i=1}^{n-1} F_i\right). \tag{1.2}$$

Observe that the simplex Π^n is unbounded with Lipschitz boundary. In Section 5, we study European option pricing in which the drift coefficients of stock prices processes are rank-based and show that there exists a unique hedging portfolio and unique price which is the unique viscosity solution of nonlinear PDEs with Neumann boundary condition. In this case, the generator of the wealth process is not continuous with respect to the ranked price processes (or price processes). Finally, we study BSDEs associated with ordered Brownian particles but with asymmetric collisions in Section 6.

We end the introduction with some notational conventions. Denote by $\mathbb{R}^n$ the n-dimensional Euclidean space with Euclidean norm $|\cdot|$. The Euclidean inner product between two vectors $x,y\in\mathbb{R}^n$ will be denoted by $x\cdot y$. Let $(\mathbb{R}_+)^n$ be the set of n-dimensional vectors whose components are all positive. Let $\mathbb{R}^{n\times n}$ be the Hilbert space of all $n\times n$ matrices with the inner product $\langle A,B\rangle := \mathrm{Tr}[AB^{\mathrm{tr}}]$ for every $A,B\in\mathbb{R}^{n\times n}$, where, the superscript tr denotes the transpose of vectors or matrices while $\mathrm{Tr}(A)$ denotes the trace of a matrix A. Denote by $S(n)$ the set of $n\times n$ symmetric matrix and I_n the $n\times n$ identity matrix. For a subset D of $\mathbb{R}^n$, denote the interior of D by $\mathrm{int}(D)$ and the Euclidean boundary of D by ∂D. Denote by $C([0,T]\times\mathbb{R}^n;\mathbb{R})$ the set of continuous functions $u(t,x):[0,T]\times\mathbb{R}^n\to\mathbb{R}$ and $C^{1,2}([0,T)\times\Gamma^n;\mathbb{R})$ the set of functions $u(t,x):[0,T)\times\Gamma^n\to\mathbb{R}$ such that $u(t,x)$ is first order continuously differentiable in t and second order continuously differentiable in x. For $u\in C([0,T]\times\mathbb{R}^n;\mathbb{R})$, denote by $\bar{D}_u^{2,+}(t,x)$ the parabolic superset of u at (t,x), i.e., $\bar{D}_u^{2,+}(t,x)$ is the set of triple $(p,q,A)\in\mathbb{R}\times\mathbb{R}^n\times S(n)$ such that for $(s,y)\in[0,T]\times\mathbb{R}^n$,

$$u(s,y) \leqslant u(t,x) + p(s-t) + q\cdot(y-x) + \frac{1}{2}(y-x)^{\mathrm{tr}}\cdot A(y-x) + o(|s-t|+|y-x|^2).$$

Here, the notation $o(\delta)$ means a quantity $f(\delta)$ such that $\lim_{\delta\to 0} f(\delta)/\delta = 0$. Similarly, denote by $\bar{D}_u^{2,-}(t,x)$ the parabolic subset of u at (t,x), i.e., $\bar{D}_u^{2,-}(t,x)$ is the set of triple $(p,q,A)\in\mathbb{R}\times\mathbb{R}^n\times S(n)$ such that for $(s,y)\in[0,T]\times\mathbb{R}^n$,

$$u(s,y) \geqslant u(t,x) + p(s-t) + q\cdot(y-x) + \frac{1}{2}(y-x)^{\mathrm{tr}}\cdot A(y-x) + o(|s-t|+|y-x|^2).$$

Denote by $C([0,T];\mathbb{R}^n)$ the set of continuous functions $u:[0,T]\to\mathbb{R}^n$. $C([0,T];\mathbb{R}^n)$ is a Banach space with the norm

$$\|u\|_{L^\infty([0,T])} := \max_{0\leqslant t\leqslant T}|u(t)|.$$

We will use C and c to denote positive constants whose values may change from line to line. For $a,b\in\mathbb{R}$, $a\vee b := \max\{a,b\}$ and $a\wedge b := \min\{a,b\}$. We use $\langle X,Y\rangle$ to denote the quadratic covariation process of two continuous semimartingales X and Y.

2 SDEs with rank-based coefficients

In this section, we give a brief introduction of rank-based stochastic differential equations. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space on which defines a standard n-dimensional Brownian motion $W =$

$(W_1, W_2, \ldots, W_n)$. Denote by $\{\mathcal{F}_t^W\}$ the minimal argument filtration generated by $\{W(t)\}$ so $\mathcal{F}_0^W$ contains all the $\mathbb{P}$-null subsets of $\mathcal{F}_\infty^W$. Fix $T > 0$, for $n \in \mathbb{N}$ and $p \geqslant 1$. We introduce the following spaces of random variables or stochastic processes:

- $L^p(\mathcal{F}_T^W; \mathbb{R}^n)$: the space of $\mathbb{R}^n$-valued, $\mathcal{F}_T^W$-measurable random variables ξ such that $\mathrm{E}[|\xi|^p] < +\infty$.
- $M^p([0,T]; \mathbb{R}^n)$: the space of n-dimensional progressively measurable processes $\{\varphi_t, 0 \leqslant t \leqslant T\}$ such that $\mathrm{E}[\int_0^T |\varphi_t|^p dt] < +\infty$.
- $S^p([0,T]; \mathbb{R}^n)$: the space of n-dimensional progressively measurable processes $\{\varphi_t, 0 \leqslant t \leqslant T\}$ such that $\mathrm{E}[\sup_{0 \leqslant t \leqslant T} |\varphi_t|^p] < +\infty$.

Consider the following system of SDEs:

$$dX_i(t) = \sum_{j=1}^n b_j \mathbf{1}_{\{X_i(t) = X_{(j)}(t)\}} dt + \sum_{j=1}^n \sigma_j \mathbf{1}_{\{X_i(t) = X_{(j)}(t)\}} dW_i(t), \quad i = 1, \ldots, n, \tag{2.1}$$

where b_j, $j = 1, \ldots, n$, are real constants; σ_j, $j = 1, \ldots, n$, are strictly positive real constants and

$$X_{(1)}(t) \geqslant X_{(2)}(t) \geqslant \cdots \geqslant X_{(n)}(t) \tag{2.2}$$

are the ordered particles for $(X_1(t), \ldots, X_n(t))$. Ties are resolved by resorting to the lowest index. For example, we set

$$X_{(i)}(t) = X_i(t), \quad i = 1, \ldots, n \quad \text{whenever} \quad X_1(t) = \cdots = X_n(t).$$

In addition, let the initial condition be deterministic and satisfy $X(0) = (X_1(0), X_2(0), \ldots, X_n(0)) \in \Pi^n$. We call the processes X_i, $i = 1, \ldots, n$, named particles and $X_{(j)}$, $j = 1, \ldots, n$, ranked particles.

Definition 2.1. (i) A finite sequence $(a_1, \ldots, a_n)$ is called concave, if for every three consecutive elements a_i, a_{i+1} and a_{i+2}, we have

$$a_{i+1} \geqslant \frac{1}{2}(a_i + a_{i+2}), \quad i = 1, \ldots, n-2.$$

(ii) A triple collision at time t occurs if there exists a rank $j \in \{2, \ldots, n-1\}$ so that

$$X_{(j-1)}(t) = X_{(j)}(t) = X_{(j+1)}(t).$$

Theorem 2.2. *Suppose the sequence $(0, \sigma_1^2, \ldots, \sigma_n^2, 0)$ is concave. Then with probability one, there are no triple collisions at any time $t > 0$ and there exists a unique strong solution of (2.1) defined for all $t \geqslant 0$.*

Proof. It is proved in [15, Theorem 2] that if $(0, \sigma_1^2, \ldots, \sigma_n^2, 0)$ is concave, the strong existence and pathwise uniqueness of (2.1) hold until the first time of a triple collision. In [31, Theorem 1.4], Sarantsev showed that there are a.s. no triple collisions if and only if $(0, \sigma_1^2, \ldots, \sigma_n^2, 0)$ is concave. Therefore, the proof is completed. □

Under the condition of Theorem 2.2, the ranked process $(X_{(1)}(t), X_{(2)}(t), \ldots, X_{(n)}(t))$ takes values in Γ^n (see (1.2)), a proper subset of $\overline{\Pi^n}$.

Denote by $\Lambda^{j,j+1}(t)$, $j = 1, \ldots, n-1$, the semimartingale local time at the origin over the time interval $[0,t]$ for $G_j(\cdot) = X_{(j)}(\cdot) - X_{(j+1)}(\cdot)$, $j = 1, \ldots, n-1$, i.e.,

$$\Lambda^{j,j+1}(t) := G_j(t) - G_j(0) - \int_0^t 1_{\{G_s > 0\}} dG_s.$$

Set $\Lambda^{0,1}(\cdot) = \Lambda^{n,n+1}(\cdot) \equiv 0$. We know from [1] that the ranked particles have the following representation: for $j = 1, \ldots, n$,

$$dX_{(j)}(t) = b_j dt + \sigma_j d\beta_j(t) + \frac{1}{2}(d\Lambda^{j,j+1}(t) - d\Lambda^{j-1,j}(t)), \quad t \geqslant 0,$$

where

$$\beta_j(\cdot) := \sum_{i=1}^{n} \int_0^{\cdot} \mathbf{1}_{\{X_i(t)=X_{(j)}(t)\}} dW_i(t). \tag{2.3}$$

Note that $(\beta_1(t), \ldots, \beta_n(t))$ is a standard n-dimensional Brownian motion. The process of ranked particles is a (normally) reflected Brownian motion (RBM for short) in the Weyl chamber

$$\mathbb{W} := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 \geqslant \cdots \geqslant x_n\}$$

with reflection matrix

$$R := \begin{pmatrix} \frac{1}{2} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \cdots & 0 & -\frac{1}{2} \end{pmatrix},$$

i.e., the process $(X_{(1)}, \ldots, X_{(n)})$ behaves like an n-dimensional Brownian motion with constant drift and covariance matrix in the interior of the wedge $\mathbb{W}$, and is normally reflected on the faces $\{x_i = x_{i+1}\}$, $i = 1, \ldots, n-1$, of $\mathbb{W}$. The directions of reflection are specified by the columns of the reflection matrix R. Occasionally, it will be more convenient to consider the following process of spacings (or gaps), instead of the process of the ranked particles $(X_{(1)}, \ldots, X_{(n)})$,

$$G := (X_{(1)} - X_{(2)}, \ldots, X_{(n-1)} - X_{(n)}),$$

where, for $j = 1, \ldots, n-1$ and $t \geqslant 0$,

$$dG_j(t) = (b_j - b_{j+1})dt + \sigma_j d\beta_j(t) - \sigma_{j+1} d\beta_{j+1}(t) - \frac{1}{2}(d\Lambda^{j-1,j}(t) + d\Lambda^{j+1,j+2}(t)) + d\Lambda^{j,j+1}(t).$$

The process G is a obliquely reflected Brownian motion in the $(n-1)$-dimensional non-negative orthant $(\mathbb{R}_+)^{n-1}$ with the reflection matrix

$$R := \begin{pmatrix} 1 & -\frac{1}{2} & 0 & \cdots & 0 & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & \cdots & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & \cdots & -\frac{1}{2} & 1 \end{pmatrix}.$$

We refer the readers to [9] and the survey [33] on semimartingale reflected Brownian motions.

Remark 2.3. As in [21, 31], if we start from ranked particles $\{X_{(j)}(t)\}_{j=1}^n$, we can define the corresponding named particles $\{X_i(t)\}_{i=1}^n$ such that

$$\max_{1 \leqslant i \leqslant n} X_i(t) = X_{(1)}(t) \geqslant \cdots \geqslant X_{(n)}(t) \geqslant \min_{1 \leqslant i \leqslant n} X_i(t).$$

Since the main motivation of our paper is to study European option pricing with rank-based stock prices and the stock prices are more easily observed and commonly used in the real stock market, we first study named particles, which corresponds to the stock price processes. Then switch to ranked particles.

Next, we study the continuous dependence on the initial conditions by the strong solution of (2.1). For this purpose, we consider a different system in which the diffusion coefficients are 1. For every $(t,\widetilde{x})\in[0,T]\times\Gamma^n$ and $i=1,\dots,n$, let $\{X_i^{t,\widetilde{x}}(s),s\geqslant 0\}$ be the solution to the following SDEs:

$$X_i^{t,\widetilde{x}}(s)=\widetilde{x}_i+\sum_{j=1}^n\int_t^{s\vee t}b_j\mathbf{1}_{\{X_i^{t,\widetilde{x}}(r)=X_{(j)}^{t,\widetilde{x}}(r)\}}dr+W_i(s\vee t)-W_i(t). \tag{2.4}$$

Since $\{b_j\}$ are bounded, it is easy to check that there exists a constant C depending on $(p,T,\{b_j\})$ such that

$$\mathrm{E}\Big[\sup_{0\leqslant s\leqslant T}|X_i^{t,\widetilde{x}}(s)|^p\Big]\leqslant C(1+|\widetilde{x}|^p),\quad i=1,\dots,n. \tag{2.5}$$

Moreover, we have the following theorem.

Theorem 2.4 (See [25, Corollary 13])**.** *For every $p\geqslant 1$, there exists a constant C depending on $(p,n,\{b_j\})$ such that for every $\widetilde{x},\widetilde{x}'\in\Gamma^n$ and $t,t'\in[0,T]$, we have*

$$\mathrm{E}[|X_i^{t,\widetilde{x}}(s)-X_i^{t',\widetilde{x}'}(s)|^p]\leqslant C(|\widetilde{x}-\widetilde{x}'|^p+|t-t'|^{p/2}),\quad s\in[0,T],\quad i=1,\dots,n. \tag{2.6}$$

Before we derive the corresponding properties for ranked particles $X_{(j)}$, $j=1,\dots,n$, we need the following lemma.

Lemma 2.5. *For two real number sequences $(x_1,\dots,x_n)$ and $(x_1',\dots,x_n')$ and $k=1,\dots,n$, let y_k and y_k' be the k-largest number in the two sequences, respectively. Then*

$$|y_k-y_k'|\leqslant\max_{1\leqslant i\leqslant n}|x_i-x_i'|,\quad k=1,\dots,n. \tag{2.7}$$

Proof. Suppose $y_1=x_i$ and $y_1'=x_j'$. If $i=j$, then $y_1-y_1'=x_i-x_i'$. If $i\neq j$, without loss of generality, assume that $y_1\geqslant y_1'$, then

$$0\leqslant y_1-y_1'=x_i-x_j'\leqslant x_i-x_i'.$$

So (2.7) holds for $k=1$. Similarly, (2.7) is true for $k=n$.

For $1<k<n$, note that $y_k=\max_{1\leqslant i_1<\cdots<i_k\leqslant n}\{\min(x_{i_1},\dots,x_{i_k})\}$. Thus we have by (2.7) for $k=1$ and $k=n$ that

$$\begin{aligned}|y_k-y_k'|&=\Big|\max_{1\leqslant i_1<\cdots<i_k\leqslant n}\{\min(x_{i_1},\dots,x_{i_k})\}-\max_{1\leqslant i_1<\cdots<i_k\leqslant n}\{\min(x_{i_1}',\dots,x_{i_k}')\}\Big|\\&\leqslant\max_{1\leqslant i_1<\cdots<i_k\leqslant n}|\{\min(x_{i_1},\dots,x_{i_k})\}-\{\min(x_{i_1}',\dots,x_{i_k}')\}|\\&\leqslant\max_{1\leqslant i_1<\cdots<i_k\leqslant n}\ \max_{i=i_1,\dots,i_k}|x_i-x_i'|\\&\leqslant\max_{1\leqslant i\leqslant n}|x_i-x_i'|.\end{aligned}$$

This completes the proof. □

Theorem 2.6. *For every $p\geqslant 1$, there exist positive constants C_1 and C_2 depending on $(p,T,n,\{b_j\})$ and $(p,n,\{b_j\})$, respectively, so that for every $\widetilde{x},\widetilde{x}'\in\Gamma^n$ and $t,t'\in[0,T]$, we have*

$$\mathrm{E}\Big[\sup_{0\leqslant s\leqslant T}|X_{(j)}^{t,\widetilde{x}}(s)|^p\Big]\leqslant C_1(1+|\widetilde{x}|^p),\quad j=1,\dots,n, \tag{2.8}$$

and

$$\mathrm{E}[|X_{(j)}^{t,\widetilde{x}}(s)-X_{(j)}^{t',\widetilde{x}'}(s)|^p]\leqslant C_2(|\widetilde{x}-\widetilde{x}'|^p+|t-t'|^{p/2}),\quad s\in[0,T],\quad j=1,\dots,n. \tag{2.9}$$

Proof. By the definition of $X_{(j)}^{t,\widetilde{x}}(s)$ and (2.5), we have

$$\mathrm{E}\Big[\sup_{0\leqslant s\leqslant T}|X_{(j)}^{t,\widetilde{x}}(s)|^p\Big]\leqslant\mathrm{E}\Big[\sum_{i=1}^n\sup_{0\leqslant s\leqslant T}|X_i^{t,\widetilde{x}}(s)|^p\Big]\leqslant C_1(1+|\widetilde{x}|^p).$$

Similarly, by Lemma 2.5 and Theorem 2.4, we have

$$\mathrm{E}[|X_{(j)}^{t,\widetilde{x}}(s) - X_{(j)}^{t',\widetilde{x}'}(s)|^p] \leqslant \mathrm{E}\bigg[\sum_{i=1}^n |X_i^{t,\widetilde{x}}(s) - X_i^{t',\widetilde{x}'}(s)|^p\bigg] \leqslant C_2(|\widetilde{x} - \widetilde{x}'|^p + |t-t'|^{p/2}).$$

This completes the proof. □

3 BSDEs with rank-based data

Denote the ranked particles defined in Section 2 by $\widetilde{X}^{t,\widetilde{x}}(s) := (X_{(1)}^{t,\widetilde{x}}(s), \dots, X_{(n)}^{t,\widetilde{x}}(s))$. By Theorem 2.2, we know that the ranked particles live on $\Gamma^n := \Pi^n \cup (\bigcup_{i=1}^{n-1} F_i)$, a proper subset of $\overline{\Pi^n}$. Moreover, it follows from [21, Theorem 5] that

$$\sum_{i \neq j} \int_t^\infty \mathbf{1}_{\{X_i^{t,\widetilde{x}}(r) = X_j^{t,\widetilde{x}}(r)\}} dr = 0 \quad \text{a.s.} \tag{3.1}$$

For $(t, \widetilde{x}) \in [0,T] \times \Gamma^n$, consider the following BSDEs with s running from t to T:

$$Y^{t,\widetilde{x}}(s) = g(\widetilde{X}^{t,\widetilde{x}}(T)) + \int_s^T f(r, X^{t,\widetilde{x}}(r), Y^{t,\widetilde{x}}(r), Z^{t,\widetilde{x}}(r))dr - \int_s^T Z^{t,\widetilde{x}}(r) \cdot dW(r), \tag{3.2}$$

where $f : [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and $g : \Gamma^n \to \mathbb{R}$ are two measurable functions.

Consider the following conditions on f and g:

(H1) $f(t,x,y,z)$ is jointly continuous on $[0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ and uniformly continuous in x, and there exists a constant $c > 0$ so that

$$|f(t,x,y,z) - f(t,x,y',z')| \leqslant c(|y-y'| + |z-z'|), \tag{3.3}$$

and

$$|f(t,x,0,0)| \leqslant c(1+|x|). \tag{3.4}$$

Furthermore, there exists a function $h : [0,T] \times \Gamma^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ such that for every $t \in [0,T]$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ with its projection $\widetilde{x} = (\widetilde{x}_1, \dots, \widetilde{x}_n) = (x_{(1)}, \dots, x_{(n)}) \in \Gamma^n$,

$$f(t,x,y,z) = h(t, \widetilde{x}, y, \bar{z}), \quad \text{where} \quad \bar{z}_j := \sum_{i=1}^n z_i \mathbf{1}_{\{x_i = x_{(j)}\}}. \tag{3.5}$$

(H2) There exists a constant c so that

$$|g(\widetilde{x}) - g(\widetilde{x}')| \leqslant c|\widetilde{x} - \widetilde{x}'| \quad \text{for} \quad \widetilde{x}, \widetilde{x}' \in \Gamma^n \tag{3.6}$$

and

$$|g(\widetilde{x})| \leqslant c(1+|\widetilde{x}|) \quad \text{for} \quad \widetilde{x} \in \Gamma^n. \tag{3.7}$$

Remark 3.1. From the relation (3.5) between f and h and (3.3) and (3.4), we obtain that $h : [0,T] \times \Gamma^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is jointly continuous and linear growth with $\widetilde{x}$. Furthermore, h is Lipshitz continuous in $(y, \bar{z})$, and uniformly continuous in $\widetilde{x}$.

Remark 3.2. Typical solutions (Y_t, Z_t) of BSDEs are of the form $Y_t = u(t, X_t)$ and $Z_t = \nabla_x u(t, X_t)$ for some suitable deterministic function $u(t,x)$. Based on the relation between x and $z = \nabla_x u(t,x)$, if we want to consider a function h on Π^n and symmetrically extend it to $\mathbb{R}^n$, we need to change the order of the subscripts of z at the same time, i.e., to change $\bar{z}$ to z. Thus, (3.5) is a natural assumption.

It follows from [27, Theorem 4.1] that BSDEs (3.2) have a unique solution $(Y^{t,\widetilde{x}}, Z^{t,\widetilde{x}}) \in S^2([t,T];\mathbb{R}) \times M^2([t,T];\mathbb{R}^n)$. Moreover, (3.2) can be rewritten as

$$Y^{t,\widetilde{x}}(s) = g(\widetilde{X}^{t,\widetilde{x}}(T)) + \int_s^T h(r, \widetilde{X}^{t,\widetilde{x}}(r), Y^{t,\widetilde{x}}(r), \bar{Z}^{t,\widetilde{x}}(r))dr - \int_s^T \bar{Z}^{t,\widetilde{x}}(r) \cdot d\beta(r), \tag{3.8}$$

where for $i = 1, \ldots, n$,

$$\bar{Z}_j^{t,\widetilde{x}}(r) := \sum_{i=1}^n Z_i^{t,\widetilde{x}}(r)\mathbf{1}_{\{X_i^{t,\widetilde{x}}(r)=X_{(j)}^{t,\widetilde{x}}(r)\}}$$

and

$$\beta_j(t) := \sum_{i=1}^n \int_0^t \mathbf{1}_{\{X_i^{t,\widetilde{x}}(r)=X_{(j)}^{t,\widetilde{x}}(r)\}} dW_i(r), \tag{3.9}$$

which is a Brownian motion on $\mathbb{R}^n$. Clearly, $\bar{Z}^{t,\widetilde{x}}(s) = (\bar{Z}_1^{t,\widetilde{x}}(s), \ldots, \bar{Z}_n^{t,\widetilde{x}}(s)) \in M^2([t,T];\mathbb{R}^n)$.

Theorem 3.3. *Suppose* (H1) *and* (H2) *hold. Let* $W(t)$ *be Brownian motion on* $\mathbb{R}^n$ *and let* $\beta(t)$ *be defined by* (3.9). *There exists a unique pair of the solution* $(Y, \bar{Z}) \in S^2([t,T];\mathbb{R}) \times M^2([t,T];\mathbb{R}^n)$ *for* (3.8).

Proof. It remains to show the solutions to (3.8) are unique. For simplicity, we omit the superscript (t, x). Suppose $(Y, \bar{Z})$ and $(Y', \bar{Z}')$ are the solutions of (3.8), from the above proof we know that (Y, Z) and (Y', Z') are the solutions of (3.2), where,

$$Z_i(r) := \sum_{j=1}^n \bar{Z}_j(r)\mathbf{1}_{\{X_i(r)=X_{(j)}(r)\}}, \quad i = 1, \ldots, n,$$

and

$$Z_i'(r) := \sum_{j=1}^n \bar{Z}_j'(r)\mathbf{1}_{\{X_i(r)=X_{(j)}(r)\}}, \quad i = 1, \ldots, n.$$

Since (3.2) has a unique solution, i.e.,

$$\mathrm{E}\bigg[\int_t^T |Y(s) - Y'(s)|^2 ds\bigg] + \mathrm{E}\bigg[\int_t^T |Z(s) - Z'(s)|^2 ds\bigg] = 0,$$

we have

$$\mathrm{E}\bigg[\int_t^T |Y(s) - Y'(s)|^2 ds\bigg] + \mathrm{E}\bigg[\int_t^T |\bar{Z}(s) - \bar{Z}'(s)|^2 ds\bigg] = 0.$$

This proves that (3.8) has a unique solution too. □

Remark 3.4. The existence and uniqueness of BSDEs (3.2) follow from the classical results on BSDEs. However, the equivalence between BSDEs (3.2) and BSDEs (3.8) allows us to connect the solution of BSDEs (3.2) with parabolic PDEs in simplex Π^n with Neumann boundary condition, which is different from the classical results.

Next, we establish regularity properties of the solutions of BSDEs (3.2). Note that we extend the definitions of $X^{t,\widetilde{x}}(s)$, $Y^{t,\widetilde{x}}(s)$ and $Z^{t,\widetilde{x}}(s)$ to every $(s,t) \in [0,T]$ by setting $X^{t,\widetilde{x}}(s) = X^{t,\widetilde{x}}(s \vee t)$, $Y^{t,\widetilde{x}}(s) = Y^{t,\widetilde{x}}(s \vee t)$, and $Z^{t,\widetilde{x}}(s) = 0$ for $s < t$. Based on Theorems 2.4 and 2.6, by classical estimation for BSDEs (see [10, Proposition 4.1]), we have the following theorem.

Theorem 3.5. *Suppose* (H1) *and* (H2) *hold, and* $p \geqslant 2$. *There exists a constant* C *depending on* $(p, T, n, \{b_j\})$ *such that for every* $\widetilde{x}, \widetilde{x}' \in \Gamma^n$ *and* $t, t' \in [0,T]$, *we have*

$$\mathrm{E}\Big[\sup_{0\leqslant s\leqslant T} |Y^{t,\widetilde{x}}(s)|^p\Big] \leqslant C(1 + |\widetilde{x}|^p), \tag{3.10}$$

and

$$\mathrm{E}\Big[\sup_{0\leqslant s\leqslant T} |Y^{t,\widetilde{x}}(s) - Y^{t',\widetilde{x}'}(s)|^p\Big] \to 0 \quad as \quad t' \to t, \quad \widetilde{x}' \to \widetilde{x}. \tag{3.11}$$

Define

$$u(t, \widetilde{x}) := Y^{t,\widetilde{x}}(t), \quad (t, \widetilde{x}) \in [0,T] \times \Gamma^n. \tag{3.12}$$

(3.11) shows that $(s, t, \widetilde{x}) \to Y^{t,\widetilde{x}}(s)$ is mean-square continuous. Since $Y^{t,\widetilde{x}}(t)$ is deterministic, this implies that $u(t, \widetilde{x})$ is jointly continuous in $(t, \widetilde{x})$.

Next, consider the following semi-linear backward parabolic PDEs with Neumann boundary condition:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,\widetilde{x}) = -\mathcal{L}u(t,\widetilde{x}) - h(t,\widetilde{x},u(t,\widetilde{x}),\nabla u(t,\widetilde{x})), & t\in[0,T], \quad \widetilde{x}\in\Pi^n, \\ u(T,\widetilde{x}) = g(\widetilde{x}), & \widetilde{x}\in\Gamma^n, \\ \dfrac{\partial u}{\partial \widetilde{x}_{i+1}}(t,\widetilde{x}) = \dfrac{\partial u}{\partial \widetilde{x}_i}(t,\widetilde{x}), & t\in[0,T), \quad \widetilde{x}\in F_i, \quad i=1,\dots,n-1, \end{cases} \tag{3.13}$$

where

$$\mathcal{L} = \frac{1}{2}\sum_{i=1}^n \frac{\partial^2}{\partial\widetilde{x}_i^2} + \sum_{i=1}^n b_i \frac{\partial}{\partial \widetilde{x}_i}. \tag{3.14}$$

Theorem 3.6. *Suppose PDEs* (3.13) *has a solution* $u(t,\widetilde{x})\in C^{1,2}([0,T]\times\Gamma^n;\mathbb{R})$ *and there exist some* $c,p>0$ *such that*

$$|\nabla u(t,\widetilde{x})| \leqslant c(1+|\widetilde{x}|^p).$$

Then the solution of (3.13) *is unique.*

Proof. Let $u(t,\widetilde{x})$ be a solution of (3.13) in $C^{1,2}([0,T]\times\Gamma^n;\mathbb{R})$ with $|\nabla u(t,\widetilde{x})| \leqslant c(1+|\widetilde{x}|^p)$. We have by Itô's formula and (3.1),

$$\begin{aligned} du(s,\widetilde{X}^{t,\widetilde{x}}(s)) &= \frac{\partial u}{\partial s}(s,\widetilde{X}^{t,\widetilde{x}}(s))ds + \sum_{i=1}^n \frac{\partial u}{\partial \widetilde{x}_i}(s,\widetilde{X}^{t,\widetilde{x}}(s))dX^{t,\widetilde{x}}_{(i)}(s) \\ &\quad + \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n \frac{\partial u}{\partial \widetilde{x}_i}(s,\widetilde{X}^{t,\widetilde{x}}(s))\frac{\partial u}{\partial \widetilde{x}_j}(s,\widetilde{X}^{t,\widetilde{x}}(s))d\langle X^{t,\widetilde{x}}_{(i)}, X^{t,\widetilde{x}}_{(j)}\rangle_s \\ &= \frac{\partial u}{\partial s}(s,\widetilde{X}^{t,\widetilde{x}}(s))ds + \sum_{i=1}^n b_i\frac{\partial u}{\partial \widetilde{x}_i}(s,\widetilde{X}^{t,\widetilde{x}}(s))ds + \frac{1}{2}\sum_{i=1}^n \frac{\partial^2 u}{\partial \widetilde{x}_i^2}(s,\widetilde{X}^{t,\widetilde{x}}(s))ds \\ &\quad + \sum_{i=1}^n \frac{\partial u}{\partial \widetilde{x}_i}(s,\widetilde{X}^{t,\widetilde{x}}(s))d\beta_i(s) - \frac{1}{2}\sum_{i=1}^n \frac{\partial u}{\partial \widetilde{x}_i}(s,\widetilde{X}^{t,\widetilde{x}}(s))d\Lambda^{i-1,i}(s) \\ &\quad + \frac{1}{2}\sum_{i=1}^n \frac{\partial u}{\partial \widetilde{x}_i}(s,\widetilde{X}^{t,\widetilde{x}}(s))d\Lambda^{i,i+1}(s) \\ &= -h(s,\widetilde{X}^{t,\widetilde{x}}(s),u(s,\widetilde{X}^{t,\widetilde{x}}(s)),\nabla u(s,\widetilde{X}^{t,\widetilde{x}}(s)))ds + \sum_{i=1}^n \frac{\partial u}{\partial \widetilde{x}_i}(s,X^{t,\widetilde{x}}(s))d\beta_i(s). \end{aligned}$$

Let $Y^{t,\widetilde{x}}(s) = u(s,\widetilde{X}^{t,\widetilde{x}}(s))$ and $\bar{Z}^{t,\widetilde{x}}(s) = (\nabla u)(s,\widetilde{X}^{t,\widetilde{x}}(s))$. Set

$$Z_j^{t,\widetilde{x}}(s) := \sum_{i=1}^n \bar{Z}_i^{t,\widetilde{x}}(s)1_{\{X_i(s)=X_{(j)}(s)\}} = \sum_{i=1}^n \frac{\partial u}{\partial \widetilde{x}_i}(s,X^{t,\widetilde{x}}(s))1_{\{X_i(s)=X_{(j)}(s)\}}.$$

Note that $Y\in S^2([0,T];\mathbb{R})$ and $\bar{Z}\in M^2([0,T];\mathbb{R}^n)$ in view of Theorem 2.6. Then

$$\begin{aligned} dY^{t,\widetilde{x}}(s) &= -h(s,\widetilde{X}^{t,\widetilde{x}}(s),Y^{t,\widetilde{x}}(s),\bar{Z}^{t,\widetilde{x}}(s))ds + \bar{Z}^{t,\widetilde{x}}(s)\cdot d\beta(s) \\ &= -f(s,X^{t,\widetilde{x}}(s),Y^{t,\widetilde{x}}(s),Z^{t,\widetilde{x}}(s))ds + Z^{t,\widetilde{x}}(s)\cdot dW(s), \end{aligned} \tag{3.15}$$

and

$$Y^{t,\widetilde{x}}(T) = u(T,\widetilde{X}^{t,\widetilde{x}}(T)) = g(\widetilde{X}^{t,\widetilde{x}}(T)).$$

Hence, $(Y^{t,\widetilde{x}}(s), Z^{t,\widetilde{x}}(s))$ is the solution of BSDEs (3.2) and (3.12) holds. The uniqueness of the solution to (3.13) follows from the uniqueness of the solution of BSDEs (3.2). □

4 Connection with PDEs

In this section, we will study nonlinear Feynman-Kac formula which shows that $u(t,\widetilde{x})$ defined in (3.12) is the unique viscosity solution of the following PDEs:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,\widetilde{x}) = -\mathcal{L}u(t,\widetilde{x}) - h(t,\widetilde{x},u(t,\widetilde{x}),\nabla u(t,\widetilde{x})), & t\in[0,T], \quad \widetilde{x}\in\Pi^n, \\ u(T,\widetilde{x}) = g(\widetilde{x}), & \widetilde{x}\in\Gamma^n, \\ \dfrac{\partial u}{\partial \widetilde{x}_{i+1}}(t,\widetilde{x}) = \dfrac{\partial u}{\partial \widetilde{x}_i}(t,\widetilde{x}), & t\in[0,T), \quad \widetilde{x}\in F_i, \quad i=1,\ldots,n-1. \end{cases} \tag{4.1}$$

First, we recall the definition of a viscosity solution of PDEs.

Definition 4.1. (i) A function $u\in C([0,T]\times\Gamma^n;\mathbb{R})$ is said to be a *viscosity subsolution* of (4.1) if

$$u(T,\widetilde{x}) \leqslant g(\widetilde{x}) \quad \text{for} \quad \widetilde{x}\in\Gamma^n,$$

and whenever $\varphi\in C^{1,2}([0,T)\times\Gamma^n;\mathbb{R})$ and $(t,\widetilde{x})\in[0,T)\times\Gamma^n$ is a local minimum of $\varphi-u$, we have

$$-\frac{\partial\varphi}{\partial t}(t,\widetilde{x}) - \mathcal{L}\varphi(t,\widetilde{x}) - h(t,\widetilde{x},u(t,\widetilde{x}),\nabla\varphi(t,\widetilde{x})) \leqslant 0 \quad \text{if} \quad \widetilde{x}\in\Pi^n$$

and

$$\bigg(\frac{\partial\varphi}{\partial\widetilde{x}_{i+1}}(t,\widetilde{x}) - \frac{\partial\varphi}{\partial\widetilde{x}_i}(t,\widetilde{x})\bigg) \wedge \bigg(-\frac{\partial\varphi}{\partial t}(t,\widetilde{x}) - \mathcal{L}\varphi(t,\widetilde{x}) - h(t,\widetilde{x},u(t,\widetilde{x}),\nabla\varphi(t,\widetilde{x}))\bigg) \leqslant 0$$

if $\widetilde{x}\in F_i$ for some $i=1,\ldots,n-1$.

(ii) A function $u\in C([0,T]\times\Gamma^n;\mathbb{R})$ is said to be a *viscosity supersolution* of (4.1) if

$$u(T,\widetilde{x}) \geqslant g(\widetilde{x}) \quad \text{for} \quad \widetilde{x}\in\Gamma^n,$$

and whenever $\varphi\in C^{1,2}([0,T)\times\Gamma^n;\mathbb{R})$ and $(t,\widetilde{x})\in[0,T)\times\Gamma^n$ is a local maximum of $\varphi-u$, we have

$$-\frac{\partial\varphi}{\partial t}(t,\widetilde{x}) - \mathcal{L}\varphi(t,\widetilde{x}) - h(t,\widetilde{x},u(t,\widetilde{x}),\nabla\varphi(t,\widetilde{x})) \geqslant 0 \quad \text{if} \quad \widetilde{x}\in\Pi^n$$

and

$$\bigg(\frac{\partial\varphi}{\partial\widetilde{x}_{i+1}}(t,\widetilde{x}) - \frac{\partial\varphi}{\partial\widetilde{x}_i}(t,\widetilde{x})\bigg) \vee \bigg(-\frac{\partial\varphi}{\partial t}(t,\widetilde{x}) - \mathcal{L}\varphi(t,\widetilde{x}) - h(t,\widetilde{x},u(t,\widetilde{x}),\nabla\varphi(t,\widetilde{x}))\bigg) \geqslant 0$$

if $\widetilde{x}\in F_i$ for some $i=1,\ldots,n-1$.

(iii) A function $u\in C([0,T]\times\Gamma^n;\mathbb{R})$ is said to be a *viscosity solution* of (4.1) if it is both a viscosity subsolution and supersolution.

For more information on viscosity solutions, see Crandall et al. [7].

4.1 Existence of viscosity solution

Theorem 4.2. *Suppose* (H1) *and* (H2) *hold. The function* u *defined by* (3.12) *is a viscosity solution of* (4.1).

Proof. We only show that u is a viscosity subsolution of (4.1). A similar argument will show that it is also a supersolution. First, obviously, $u(T,\widetilde{x})=g(\widetilde{x})$. From the uniqueness of the solution to BSDEs (3.2), we have

$$Y^{t,\widetilde{x}}(s) = Y^{s,\widetilde{X}^{t,\widetilde{x}}(s)}(s) = u(s,\widetilde{X}^{t,\widetilde{x}}(s)), \quad \text{for} \quad t\leqslant s\leqslant T.$$

Let $\varphi\in C^{1,2}([0,T)\times\Gamma^n;\mathbb{R})$ and $(t,\widetilde{x})\in[0,T)\times\Gamma^n$ be a local minimum of $\varphi-u$. Without loss of generality, assume that $\varphi(t,\widetilde{x})=u(t,\widetilde{x})$.

First, we consider the case $\widetilde{x}\in\Pi^n$. Suppose that

$$-\frac{\partial\varphi}{\partial t}(t,\widetilde{x}) - \mathcal{L}\varphi(t,\widetilde{x}) - h(t,\widetilde{x},u(t,\widetilde{x}),\nabla\varphi(t,\widetilde{x})) > 0.$$

Let $0 < \alpha \leqslant T - t$ be such that $A := \{y : |y - \widetilde{x}| \leqslant \alpha\} \subset \Pi^n$ and for all $t \leqslant s \leqslant t + \alpha$, $y \in A$,

$$-\frac{\partial \varphi}{\partial s}(s, y) - \mathcal{L}\varphi(s, y) - h(s, y, u(s, y), \nabla\varphi(s, y)) > 0.$$

Define $\tau := \inf\{s > t : |\widetilde{X}^{t,\widetilde{x}}(s) - \widetilde{x}| \geqslant \alpha\} \wedge (t + \alpha)$.

Note that

$$(Y'(s), \bar{Z}'(s)) := (Y^{t,\widetilde{x}}(s \wedge \tau), \mathbf{1}_{[t,\tau]}\bar{Z}^{t,\widetilde{x}}(s)), \quad t \leqslant s \leqslant t + \alpha,$$

is the solution of BSDEs

$$\begin{aligned} Y'(s) &= u(\tau, \widetilde{X}^{t,\widetilde{x}}(\tau)) + \int_{s\wedge\tau}^{\tau} \mathbf{1}_{[t,\tau]}(r) h(r, \widetilde{X}^{t,\widetilde{x}}(r), u(r, \widetilde{X}^{t,\widetilde{x}}(r)), \bar{Z}'(r))dr - \int_{s\wedge\tau}^{\tau} \bar{Z}'(r) \cdot d\beta(r) \\ &= u(\tau, \widetilde{X}^{t,\widetilde{x}}(\tau)) + \int_{s}^{t+\alpha} \mathbf{1}_{[t,\tau]}(r) h(r, \widetilde{X}^{t,\widetilde{x}}(r), u(r, \widetilde{X}^{t,\widetilde{x}}(r)), \bar{Z}'(r))dr - \int_{s}^{t+\alpha} \bar{Z}'(r) \cdot d\beta(r). \end{aligned}$$

Using Itô's formula to $\varphi(s, \widetilde{X}^{t,\widetilde{x}}(s))$, we have

$$\begin{aligned} d\varphi(s, \widetilde{X}^{t,\widetilde{x}}(s)) &= \frac{\partial \varphi}{\partial s}(s, \widetilde{X}^{t,\widetilde{x}}(s))ds + \sum_{i=1}^{n} \frac{\partial \varphi}{\partial \widetilde{x}_i}(s, \widetilde{X}^{t,\widetilde{x}}(s))dX_{(i)}^{t,\widetilde{x}}(s) \\ &\quad + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial \varphi}{\partial \widetilde{x}_i}(s, \widetilde{X}^{t,\widetilde{x}}(s)) \frac{\partial \varphi}{\partial \widetilde{x}_j}(s, \widetilde{X}^{t,\widetilde{x}}(s)) d\langle X_{(i)}^{t,\widetilde{x}}, X_{(j)}^{t,\widetilde{x}}\rangle_s \\ &= \frac{\partial \varphi}{\partial s}(s, \widetilde{X}^{t,\widetilde{x}}(s))ds + \sum_{i=1}^{n} b_i \frac{\partial \varphi}{\partial \widetilde{x}_i}(s, \widetilde{X}^{t,\widetilde{x}}(s))ds + \frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2 \varphi}{\partial \widetilde{x}_i^2}(s, \widetilde{X}^{t,\widetilde{x}}(s))ds \\ &\quad + \sum_{i=1}^{n} \frac{\partial \varphi}{\partial \widetilde{x}_i}(s, \widetilde{X}^{t,\widetilde{x}}(s))d\beta_i(s) - \frac{1}{2}\sum_{i=1}^{n} \frac{\partial \varphi}{\partial \widetilde{x}_i}(s, \widetilde{X}^{t,\widetilde{x}}(s))d\Lambda^{i-1,i}(s) \\ &\quad + \frac{1}{2}\sum_{i=1}^{n} \frac{\partial \varphi}{\partial \widetilde{x}_i}(s, \widetilde{X}^{t,\widetilde{x}}(s))d\Lambda^{i,i+1}(s). \end{aligned}$$

Integrating from $s \wedge \tau$ to τ, we have

$$\begin{aligned} &\varphi(\tau, \widetilde{X}^{t,\widetilde{x}}(\tau)) - \varphi(s \wedge \tau, \widetilde{X}^{t,\widetilde{x}}(s \wedge \tau)) \\ &\quad = \int_{s\wedge\tau}^{\tau} \left[\frac{\partial \varphi}{\partial r}(r, \widetilde{X}^{t,\widetilde{x}}(r)) + \mathcal{L}\varphi(r, \widetilde{X}^{t,\widetilde{x}}(r))\right]dr + \sum_{i=1}^{n}\int_{s\wedge\tau}^{\tau} \frac{\partial \varphi}{\partial \widetilde{x}_i}(r, \widetilde{X}^{t,\widetilde{x}}(r))d\beta_i(r). \end{aligned}$$

Therefore,

$$(Y''(s), Z''(s)) := (\varphi(s \wedge \tau, \widetilde{X}^{t,\widetilde{x}}(s \wedge \tau)), \mathbf{1}_{[t,\tau]}(s)\nabla\varphi(s, \widetilde{X}^{t,\widetilde{x}}(s))), \quad t \leqslant s \leqslant t + \alpha$$

is the solution of BSDEs

$$Y''(s) = \varphi(\tau, \widetilde{X}^{t,\widetilde{x}}(\tau)) - \int_{s}^{t+\alpha} \mathbf{1}_{[t,\tau]}(r)\left(\frac{\partial \varphi}{\partial r} + \mathcal{L}\varphi\right)(r, \widetilde{X}^{t,\widetilde{x}}(r))dr - \int_{s}^{t+\alpha} Z''(r) \cdot d\beta(r).$$

Define

$$\begin{aligned} &\hat{Y}(r) := Y''(r) - Y'(r), \quad \hat{Z}(r) := Z''(r) - \bar{Z}'(r), \\ &\hat{h}(r) := -\left(\frac{\partial \varphi}{\partial r} + \mathcal{L}\varphi\right)(r, \widetilde{X}^{t,\widetilde{x}}(r)) - h(r, \widetilde{X}^{t,\widetilde{x}}(r), u(r, \widetilde{X}^{t,\widetilde{x}}(r)), Z''(r)), \end{aligned}$$

and

$$a(r) := h(r, \widetilde{X}^{t,\widetilde{x}}(r), u(r, \widetilde{X}^{t,\widetilde{x}}(r)), Z''(r)) - h(r, \widetilde{X}^{t,\widetilde{x}}(r), u(r, \widetilde{X}^{t,\widetilde{x}}(r)), \bar{Z}'(r)).$$

Note that $a(r) \leqslant C|Z''(r) - \bar{Z}'(r)|$ for some constant $C > 0$. So there exists a bounded $\mathcal{F}_t^W$-adapted process $b(r)$ such that

$$a(r) = b(r) \cdot (Z''(r) - \bar{Z}'(r)).$$

Therefore,

$$\hat{Y}(s)=\varphi(\tau,\widetilde{X}^{t,\widetilde{x}}(\tau))-u(\tau,\widetilde{X}^{t,\widetilde{x}}(\tau))+\int_{s\wedge\tau}^{\tau}(\hat{h}(r)+b(r)\cdot\hat{Z}(r))dr-\int_{s\wedge\tau}^{\tau}\hat{Z}(r)\cdot d\beta(r).$$

Let $\{M(s),s\geqslant t\}$ be the unique solution of

$$dM(s)=b(s)M(s)\cdot d\beta(s)\quad\text{with}\quad M(t)=1;$$

i.e.,

$$M(s)=\exp\bigg(\int_t^s b(r)\cdot d\beta(r)-\frac{1}{2}\int_t^s|b(r)|^2dr\bigg)>0\quad\text{for}\quad s\geqslant t.$$

By applying Itô's formula to $\hat{Y}(s)M(s)$, one can easily verify that

$$\varphi(t,\widetilde{x})-u(t,\widetilde{x})=\hat{Y}(t)=\mathrm{E}\bigg[\hat{Y}(\tau)M(\tau)+\int_t^\tau\hat{h}(r)M(r)dr\,\bigg|\,\mathcal{F}_t\bigg]>0.$$

This contracts the assumption that $\varphi(t,\widetilde{x})=u(t,\widetilde{x})$.

It remains to prove the case that $\widetilde{x}\in F_i$, $i=1,\dots,n-1$. Suppose

$$\frac{\partial\varphi}{\partial\widetilde{x}_{i+1}}(t,\widetilde{x})-\frac{\partial\varphi}{\partial\widetilde{x}_i}(t,\widetilde{x})>0\quad\text{and}\quad-\frac{\partial\varphi}{\partial t}(t,\widetilde{x})-\mathcal{L}\varphi(t,\widetilde{x})-h(t,\widetilde{x},u(t,\widetilde{x}),\nabla\varphi(t,\widetilde{x}))>0.$$

Let $0<\alpha\leqslant T-t$ be such that

$$\inf_{t\leqslant s\leqslant t+\alpha,|y-\widetilde{x}|\leqslant\alpha}\bigg(\frac{\partial\varphi}{\partial y_{i+1}}(s,y)-\frac{\partial\varphi}{\partial y_i}(s,y)\bigg)>0,$$

and

$$\inf_{t\leqslant s\leqslant t+\alpha,|y-\widetilde{x}|\leqslant\alpha}\bigg(-\frac{\partial\varphi}{\partial s}(s,y)-\mathcal{L}\varphi(s,y)-h(s,y,u(s,y),\nabla\varphi(s,y))\bigg)>0.$$

Define $\tau:=\inf\{s>t:|\widetilde{X}^{t,\widetilde{x}}(s)-\widetilde{x}|\geqslant\alpha\}\wedge\inf\{s>t:\widetilde{X}^{t,\widetilde{x}}(s)\in F_j,j\neq i\}\wedge(t+\alpha)$.

First, note that

$$(Y'(s),\bar{Z}'(s)):=(Y^{t,\widetilde{x}}(s\wedge\tau),\mathbf{1}_{[t,\tau]}\bar{Z}^{t,\widetilde{x}}(s)),\quad t\leqslant s\leqslant t+\alpha$$

is the solution of BSDEs

$$\begin{aligned}Y'(s)&=u(\tau,\widetilde{X}^{t,\widetilde{x}}(\tau))+\int_{s\wedge\tau}^{\tau}\mathbf{1}_{[t,\tau]}(r)h(r,\widetilde{X}^{t,\widetilde{x}}(r),u(r,\widetilde{X}^{t,\widetilde{x}}(r)),\bar{Z}'(r))dr-\int_{s\wedge\tau}^{\tau}\bar{Z}'(r)\cdot d\beta(r)\\&=u(\tau,\widetilde{X}^{t,\widetilde{x}}(\tau))+\int_s^{t+\alpha}\mathbf{1}_{[t,\tau]}(r)h(r,\widetilde{X}^{t,\widetilde{x}}(r),u(r,\widetilde{X}^{t,\widetilde{x}}(r)),\bar{Z}'(r))dr-\int_s^{t+\alpha}\bar{Z}'(r)\cdot d\beta(r).\end{aligned}$$

By Itô's formula, we have

$$(Y''(s),Z''(s)):=(\varphi(s\wedge\tau,\widetilde{X}^{t,\widetilde{x}}(s\wedge\tau)),\mathbf{1}_{[t,\tau]}(s)\nabla\varphi(s,\widetilde{X}^{t,\widetilde{x}}(s))),\quad t\leqslant s\leqslant t+\alpha$$

satisfies

$$\begin{aligned}dY''(s)=&-\frac{1}{2}\bigg(\frac{\partial\varphi}{\partial\widetilde{x}_{i+1}}(s,\widetilde{X}^{t,\widetilde{x}}(s))-\frac{\partial\varphi}{\partial\widetilde{x}_i}(s,\widetilde{X}^{t,\widetilde{x}}(s))\bigg)d\Lambda^{i,i+1}(s)\\&+\bigg(\frac{\partial\varphi}{\partial s}+\mathcal{L}\varphi\bigg)(s,\widetilde{X}^{t,\widetilde{x}}(s))ds+Z''(s)\cdot d\beta(s).\end{aligned}$$

Therefore, we can use the same notation and method as in the case $\widetilde{x}\in\Pi^n$ to obtain that

$$\begin{aligned}\varphi(t,\widetilde{x})-u(t,\widetilde{x})&=\hat{Y}(t)\\&=\mathrm{E}\bigg[\bigg(\hat{Y}(\tau)M(\tau)+\int_t^\tau\hat{h}(r)M(r)dr\bigg)\,\bigg|\,\mathcal{F}_t\bigg]\\&\quad+\frac{1}{2}\mathrm{E}\bigg[\int_t^\tau M(r)\bigg[\frac{\partial\varphi}{\partial\widetilde{x}_{i+1}}(r,\widetilde{X}^{t,\widetilde{x}}(r))-\frac{\partial\varphi}{\partial\widetilde{x}_i}(r,\widetilde{X}^{t,\widetilde{x}}(r))\bigg]d\Lambda^{i,i+1}(r)\,\bigg|\,\mathcal{F}_t\bigg]\\&>0.\end{aligned}$$

This contradiction completes the proof of the theorem. □

4.2 Uniqueness of viscosity solution

For the uniqueness of viscosity solution, we need the following condition on h:

(H3) For all $R>0$, there exists a positive function $\eta_R(\cdot)$ on $[0,+\infty)$ with $\lim_{r\to 0}\eta_R(r)=\eta_R(0)=0$ such that

$$|h(t,\widetilde{x},y,z)-h(t,\widetilde{x}',y,z)|\leqslant \eta_R(|\widetilde{x}-\widetilde{x}'|(1+|z|))$$

for $|\widetilde{x}|,|\widetilde{x}'|,|y|\leqslant R$, $t\in[0,T]$, $z\in\mathbb{R}^n$.

Theorem 4.3. *Suppose* (H1)–(H3) *hold. There exists at most one viscosity solution* u *of* (4.1) *such that*

$$\lim_{|\widetilde{x}|\to+\infty}|u(t,\widetilde{x})|\mathrm{e}^{-A\log^2|\widetilde{x}|}=0\quad \textit{uniformly in}\quad t\in[0,T] \tag{4.2}$$

for some $A>0$.

Suppose u and v are viscosity subsolution and viscosity supersolution of (4.1), respectively. Since both u and v are continuous, we only need to prove that

$$u(t,\widetilde{x})\leqslant v(t,\widetilde{x})\quad\text{on}\quad (0,T)\times\Pi^n.$$

First, we prove two lemmas that will be used in the proof of Theorem 4.3. See Appendix A for a detailed proof.

Lemma 4.4. *Suppose* u *and* v *are viscosity subsolution and viscosity supersolution of* (4.1), *respectively. Then the function* $w:=u-v$ *is a viscosity subsolution of the following equation*:

$$\begin{cases}\dfrac{\partial w}{\partial t}(t,\widetilde{x})=-\mathcal{L}w(t,\widetilde{x})-c(|w|+|\nabla w|)(t,\widetilde{x}), & (t,\widetilde{x})\in[0,T)\times\Pi^n,\\ w(T,\widetilde{x})=0, & \widetilde{x}\in\Pi^n,\end{cases} \tag{4.3}$$

where c *is the Lipschitz constant of* F *in* (y,z).

Lemma 4.5. *For every* $A>0$, *there exists* $C_1>0$ *such that the function*

$$\Psi(t,\widetilde{x})=\exp[(C_1(T-t)+A)\psi(\widetilde{x})]$$

satisfies

$$-\frac{\partial\Psi}{\partial t}(t,\widetilde{x})-\mathcal{L}\Psi(t,\widetilde{x})-c\Psi(t,\widetilde{x})-c|\nabla\Psi(t,\widetilde{x})|>0,\quad \textit{on}\quad [t_1,T]\times\Pi^\alpha,$$

where, $\psi(\widetilde{x})=[\log(\sqrt{|\widetilde{x}|^2+1})+1]^2$ *and* $t_1=(T-A/C_1)^+$.

Proof of Theorem 4.3. Suppose u and v are viscosity subsolution and viscosity supersolution of (4.1), respectively and define $w:=u-v$. From (4.2) we obtain that

$$\lim_{|\widetilde{x}|\to+\infty}|w(t,\widetilde{x})|\mathrm{e}^{-A[\log(\sqrt{|\widetilde{x}|^2+1})+1]^2}=0$$

uniformly for $t\in[0,T]$ and for some $A>0$. This implies that for $|\widetilde{x}|$ large enough and every $\beta>0$,

$$|w(t,\widetilde{x})|<\beta\Psi(t,\widetilde{x}).$$

Thus, $M:=\max_{[t_1,T]\times\Pi^n}(w-\beta\Psi)(t,\widetilde{x})\mathrm{e}^{c(t-T)}$ is achieved at some point $(t_0,\widetilde{x}_0)$ for any $\beta>0$.

We claim that $M\leqslant 0$ for any $\beta>0$.

When $t_0=T$, since

$$w(T,\widetilde{x})=u(T,\widetilde{x})-v(T,\widetilde{x})\leqslant 0,\quad \widetilde{x}\in\Gamma^n,$$

we have

$$w(t,\widetilde{x})-\beta\Psi(t,\widetilde{x})\leqslant 0\quad\text{on}\quad [t_1,T]\times\Pi^n,$$

i.e., $M\leqslant 0$.

When $t_0 < T$, suppose that $M = \rho > 0$. Then

$$w(t_0, \widetilde{x}_0) = \rho e^{c(T-t_0)} + \beta\Psi(t_0, \widetilde{x}_0) > 0.$$

Define

$$\phi(t, \widetilde{x}) := \beta\Psi(t, \widetilde{x}) + (w - \beta\Psi)(t_0, \widetilde{x}_0)e^{c(t_0-t)}.$$

From the definition of $(t_0, \widetilde{x}_0)$, we deduce that

$$w(t, \widetilde{x}) - \beta\Psi(t, \widetilde{x}) \leqslant (w - \beta\Psi)(t_0, \widetilde{x}_0)e^{c(t_0-t)},$$

i.e., $w - \phi$ attains a global maximum at $(t_0, \widetilde{x}_0)$ and $\phi(t_0, \widetilde{x}_0) = w(t_0, \widetilde{x}_0) > 0$. Since w is a viscosity subsolution of (4.3), we have

$$-\frac{\partial \phi}{\partial t}(t_0, \widetilde{x}_0) - \mathcal{L}\phi(t_0, \widetilde{x}_0) - c(w(t_0, \widetilde{x}_0) + |\nabla\phi(t_0, \widetilde{x}_0)|) \leqslant 0.$$

The left-hand side of the above inequality is equal to

$$\beta\bigg(-\frac{\partial \Psi}{\partial t}(t_0, \widetilde{x}_0) - \mathcal{L}\Psi(t_0, \widetilde{x}_0) - c(\Psi(t_0, \widetilde{x}_0) + |\nabla\Psi(t_0, \widetilde{x}_0)|)\bigg).$$

This leads to a contradiction in view of Lemma 4.5. Hence, $M \leqslant 0$ in the case $t_0 < T$. Therefore, we have proved the claim $M \leqslant 0$ for any $\beta > 0$. Since $\beta > 0$ is arbitrary, we have

$$w(t, \widetilde{x}) \leqslant 0 \quad \text{on} \quad [t_1, T] \times \Pi^n.$$

Applying the same argument for $[t_2, t_1]$, where, $t_2 = (t_1 - A/C_1)^+$ and if $t_2 > 0$, then repeat on $[t_3, t_2]$, where $t_3 = (t_2 - A/C_1)^+$. Finally, we have $w(t, \widetilde{x}) \leqslant 0$ on $(0, T) \times \Pi^n$. □

Corollary 4.6. *Suppose* (H1)–(H3) *hold. Then* $u(t, \widetilde{x}) := Y^{t,\widetilde{x}}(t)$ *is the unique viscosity solution of* (4.1) *in the class of viscosity solutions which satisfy* (4.2) *for some* $A > 0$.

Proof. By Theorem 3.5, we know that $u(t, \widetilde{x})$ has at most polynomial growth at infinity so it satisfies (4.2), therefore, it follows from Theorems 4.2 and 4.3 that $u(t, \widetilde{x})$ is the unique viscosity solution of (4.1) in the class of viscosity solutions which satisfy (4.2) for some $A > 0$. □

5 European option pricing

In this section, we study European option pricing problem. First, we fix some notation that will be used in this section. Define

$$(\Pi^n)^+ := \{\widetilde{x} \in \Pi^n : \widetilde{x}_1 > \widetilde{x}_2 > \cdots > \widetilde{x}_n > 0\},$$

and similarly for $(\Gamma^n)^+$, F_i^+, $i = 1, \ldots, n-1$.

Let us consider a financial market $\mathcal{M}$ that consists of one bond and n stocks. Fix $\widetilde{p} = (p_0, \widetilde{p}_1, \ldots, \widetilde{p}_n) \in \mathbb{R}_+ \times (\Gamma^n)^+$ and $T > 0$, let the prices $P_0^{t,\widetilde{p}}(s), P^{t,\widetilde{p}}(s) = \{P_i^{t,\widetilde{p}}(s)\}_{i=1}^n$ of these financial instruments evolve according to the following equations:

$$\begin{cases} P_0^{t,\widetilde{p}}(s) = p_0 + \displaystyle\int_t^{s\vee t} P_0^{t,\widetilde{p}}(u) r(u) du, \\ P_i^{t,\widetilde{p}}(s) = \widetilde{p}_i + \displaystyle\int_t^{s\vee t} P_i^{t,\widetilde{p}}(u)\bigg(\sum_{j=1}^n 1_{\{P_i^{t,\widetilde{p}}(u) = P_{(j)}^{t,\widetilde{p}}(u)\}} \delta_j du + dW_i(u)\bigg), \quad i = 1, \ldots, n. \end{cases} \tag{5.1}$$

Here, $r(s)$ (the interest rate) is assumed to be a bounded deterministic function, and $\delta_j, j = 1, \ldots, n$ are real numbers. By Hölder's inequality and Burkholder-Davis-Gundy's inequality, there exists a constant C depending on $(T, \{\delta_i\}, q)$ such that for $q \geqslant 2$ and $i = 1, \ldots, n$,

$$\mathrm{E}\Big[\sup_{0 \leqslant s \leqslant T} |P_i^{t,\widetilde{p}}(s)|^q\Big] \leqslant C(1 + |\widetilde{p}|^q).$$

Define $X_i^{t,\widetilde{x}}(s) := \log P_i^{t,\widetilde{p}}(s)$, $i = 1, \ldots, n$. By Itô's formula, we obtain for $i = 1, \ldots, n$,

$$\begin{cases} dX_i^{t,\widetilde{x}}(s) = \displaystyle\sum_{j=1}^n \mathbf{1}_{\{X_i^{t,\widetilde{x}}(s) = X_{(j)}^{t,\widetilde{x}}(s)\}} \left(\delta_j - \frac{1}{2}\right) ds + dW_i(s), & s \geqslant t, \\ X_i^{t,\widetilde{x}}(s) = \widetilde{x}_i = \log \widetilde{p}_i, & s \leqslant t. \end{cases} \tag{5.2}$$

From the existence and uniqueness of the strong solution of (5.2) we know that (5.1) has a unique strong solution. The ranked log-price processes satisfy the following equations (see [1]): for $s \geqslant t$,

$$dX_{(j)}^{t,\widetilde{x}}(s) = \left(\delta_j - \frac{1}{2}\right) ds + d\beta_j(s) + \frac{1}{2}(d\Lambda^{j,j+1}(s) - d\Lambda^{j-1,j}(s)), \quad j = 1, \ldots, n,$$

where, $\Lambda^{j,j+1}(s)$, $j = 1, \ldots, n-1$ are the local times accumulated at the origin by the non-negative semimartingales

$$G_j(\cdot) := X_{(j)}^{t,\widetilde{x}}(\cdot) - X_{(j+1)}^{t,\widetilde{x}}(\cdot), \quad j = 1, \ldots, n-1,$$

over the time interval $[0, s]$, $\Lambda^{0,1}(\cdot) = \Lambda^{n,n+1}(\cdot) \equiv 0$, and

$$\beta_j(\cdot) := \sum_{i=1}^n \int_0^{\cdot} \mathbf{1}_{\{X_i^{t,\widetilde{x}}(s) = X_{(j)}^{t,\widetilde{x}}(s)\}} dW_i(s), \quad j = 1, \ldots, n.$$

Therefore, the ranked price processes satisfy the following equations:

$$dP_{(j)}^{t,\widetilde{p}}(s) = P_{(j)}^{t,\widetilde{p}}(s) \left[\delta_j ds + d\beta_j(s) + \frac{1}{2}(d\Lambda^{j,j+1}(s) - d\Lambda^{j-1,j}(s))\right], \quad i = 1, \ldots, n,$$

and there exists a constant C depending on $(T, \{\delta_i\}, n, q)$ such that for $q \geqslant 2$ and $j = 1, \ldots, n$,

$$\mathrm{E}\Big[\sup_{0 \leqslant s \leqslant T} |P_{(j)}^{t,\widetilde{p}}(s)|^q\Big] \leqslant C(1 + |\widetilde{p}|^q).$$

Suppose that an economic agent will start with an initial endowment $y > 0$ and try to allocate his wealth into the bond and stocks, whose actions cannot affect market prices and decides to invest $\pi_i(s)$ amount of money in the i-th stock at time s. Thus the amount invested in the bond will be $Y(s) - \sum_{i=1}^n \pi_i(s)$, where Y is the wealth process. Of course, his decisions can only be based on the current information $\{\mathcal{F}_t^W\}$; i.e., the process $\pi = (\pi_1, \ldots, \pi_n)$ is predictable. A European contingent claim ξ settled at time T is an $\mathcal{F}_T^W$-measurable random variable. It can be thought of as a contract which pays ξ at maturity T. The arbitrage-free pricing of a positive contingent claim is based on the following principle: if we start with the price of the claim as initial endowment and invest in the bond and n stocks, the value of the portfolio at time T must match ξ. We now give a formal definition. We follow the presentation of Harrison and Pliska [13] and Karatzas and Shreve [23].

Definition 5.1. Let ξ be a positive contingent claim.

(i) A *self-financing trading strategy* is a pair of (Y, π), where Y is the wealth process and $\pi = (\pi_1, \ldots, \pi_n)$ is the portfolio process, such that (Y, π) satisfy the equation

$$Y(s) = Y(0) + \int_0^s \sum_{i=1}^n \pi_i(u) \frac{dP_i(u)}{P_i(u)} + \int_0^s \left(Y(u) - \sum_{i=1}^n \pi_i(u)\right) \frac{dP_0(u)}{P_0(u)}.$$

The strategy is called feasible if a.s. $Y(s) \geqslant 0$ for every $s \in [0, T]$.

(ii) A *hedging strategy* against ξ is a feasible self-financing strategy (Y, π) such that $Y(T) = \xi$. We denote by $\mathcal{H}(\xi)$ the set of hedging strategies against ξ. If $\mathcal{H}(\xi)$ is nonempty, ξ is called hedgeable.

(iii) The *fair price* of ξ at time 0 is the smallest initial endowment needed to hedge ξ, i.e.,

$$Y(0) = \inf\{y \geqslant 0 : \text{there exists } (Y, \pi) \in \mathcal{H}(\xi) \text{ such that } Y(0) = y\}.$$

In this context, the wealth process satisfies the following equations:

$$\begin{cases} dY^{t,\widetilde{p}}(s) = \Big(Y^{t,\widetilde{p}}(s)r(s) + \sum_{i=1}^n \pi_i(s) \sum_{j=1}^n \mathbf{1}_{\{P_i^{t,\widetilde{p}}(s)=P_{(j)}^{t,\widetilde{p}}(s)\}}(\delta_j - r(s))\Big)ds + \sum_{i=1}^n \pi_i(s)dW_i(s), \\ Y(t) = y, \end{cases} \tag{5.3}$$

where, the real number $y > 0$ represents the initial endowment. From the definition of $\beta_j,\ j = 1,\ldots,n$, we have

$$\begin{aligned} dY^{t,\widetilde{p}}(s) &= \Big(Y^{t,\widetilde{p}}(s)r(s) + \sum_{i=1}^n \pi_i(s) \sum_{j=1}^n \mathbf{1}_{\{P_i^{t,\widetilde{p}}(s)=P_{(j)}^{t,\widetilde{p}}(s)\}}(\delta_j - r(s))\Big)ds \\ &\quad + \sum_{i=1}^n \pi_i(s) \sum_{j=1}^n \mathbf{1}_{\{P_i^{t,\widetilde{p}}(s)=P_{(j)}^{t,\widetilde{p}}(s)\}} d\beta_j(s) \\ &= \Big(Y^{t,\widetilde{p}}(s)r(s) + \sum_{j=1}^n (\delta_j - r(s)) \sum_{i=1}^n \pi_i(s)\mathbf{1}_{\{P_i^{t,\widetilde{p}}(s)=P_{(j)}^{t,\widetilde{p}}(s)\}}\Big)ds \\ &\quad + \sum_{j=1}^n \Big(\sum_{i=1}^n \pi_i(s)\mathbf{1}_{\{P_i^{t,\widetilde{p}}(s)=P_{(j)}^{t,\widetilde{p}}(s)\}}\Big)d\beta_j(s). \end{aligned}$$

Define

$$\bar{\pi}_j(s) := \sum_{i=1}^n \pi_i(s)\mathbf{1}_{\{P_i^{t,\widetilde{p}}(s)=P_{(j)}^{t,\widetilde{p}}(s)\}}, \quad j = 1,\ldots,n.$$

Then BSDEs (5.3) are equivalent to

$$dY^{t,\widetilde{p}}(s) = \Big(Y^{t,\widetilde{p}}(s)r(s) + \sum_{j=1}^n (\delta_j - r(s))\bar{\pi}_j(s)\Big)ds + \sum_{j=1}^n \bar{\pi}_j(s)d\beta_j(s). \tag{5.4}$$

Consider a contingent claim $\xi = g(P_0^{t,\widetilde{p}}(T), \widetilde{P}^{t,\widetilde{p}}(T))$, where $\widetilde{P}^{t,\widetilde{p}}(T) = (P_{(1)}^{t,\widetilde{p}}(T),\ldots,P_{(n)}^{t,\widetilde{p}}(T))$ and a Lipschitz continuous function $g : \mathbb{R}^+ \times (\Gamma^n)^+ \to \mathbb{R}^+$ satisfying

$$|g(p_0,\widetilde{p}) - g(p_0',\widetilde{p}')| \leqslant c(|p_0 - p_0'| + |\widetilde{p} - \widetilde{p}'|) \quad \text{for} \quad p_0, p_0' \in \mathbb{R}^+, \quad \widetilde{p}, \widetilde{p}' \in (\Gamma^n)^+ \tag{5.5}$$

and

$$|g(p_0,\widetilde{p})| \leqslant c(1 + |p_0| + |\widetilde{p}|) \quad \text{for} \quad p_0 \in \mathbb{R}^+, \quad \widetilde{p} \in (\Gamma^n)^+. \tag{5.6}$$

It is easy to check that BSDEs (5.3) admit a unique solution $(Y^{t,\widetilde{p}}, \pi)$. Moreover, it follows from Theorem 3.3 that BSDE (5.4) has a unique solution $(Y^{t,\widetilde{p}}, \bar{\pi})$. Define $u(t,\widetilde{p}) := Y^{t,\widetilde{p}}(t)$, $(t,\widetilde{p}) \in [0,T] \times \mathbb{R}^+ \times (\Gamma^n)^+$.

Theorem 5.2. *Suppose* (5.5) *and* (5.6) *hold. We have* $u(t,\widetilde{p}) \in C([0,T] \times \mathbb{R}^+ \times (\Gamma^n)^+; \mathbb{R})$.

To prove this theorem, we need a stronger property of solutions of rank-based SDEs than (2.6). Thus, we first prepare a lemma on approximation of the solution of SDEs. Our proof borrows some idea from Kaneko and Nakao [19], with necessary modifications. See Appendix A for a detailed proof.

Lemma 5.3. *Let $b_0(x)$ be an $\mathbb{R}^n$-valued, Borel measurable, bounded function defined on $\mathbb{R}^n$. Fix $T > 0$ and $(t,x) \in [0,T] \times \mathbb{R}^n$. Suppose SDEs*

$$X^{t,x}(s) = x + \int_t^{s\vee t} b_0(X^{t,x}(u))du + W(s \vee t) - W(t) \tag{5.7}$$

has a strong solution and the pathwise uniqueness holds. Suppose also that there exists a sequence of $\mathbb{R}^n$-valued, uniformly bounded, Lipshictz continuous functions $b_m(x)$ such that $\lim_{m\to+\infty} b_m(x) = b_0(x)$ almost everywhere on $\mathbb{R}^n$ with respect to the Lebesgue measure. Denote by $X^{t,x,m}$ the unique strong solution of (5.7) *but with b_m in place of b_0. Then for each $q \geqslant 1$ and any compact subset $\mathcal{K}$ of $\mathbb{R}^n$,*

$$\lim_{m\to+\infty} \sup_{0\leqslant t\leqslant T} \sup_{x\in\mathcal{K}} \mathrm{E}\Big[\sup_{0\leqslant s\leqslant T} |X^{t,x,m}(s) - X^{t,x}(s)|^q\Big] = 0. \tag{5.8}$$

Proof of Theorem 5.1. For $i=1,\dots,n$ and $x\in\mathbb{R}^n$, define $b_i(x):=\sum_{j=1}^n \mathbf{1}_{\{x_i=x_{(j)}\}}(\delta_j-\frac{1}{2})$. Let $b_i^m:\mathbb{R}^n\to\mathbb{R}$ be a sequence of Borel measurable, uniformly bounded, smooth functions with compact support which approximate b_i almost everywhere with respect to the Lebesgue measure. From [2], we know that the uniqueness in law holds for SDEs (5.2) and [15] constructed a strong solution for SDEs (5.2), therefore, by [6, Theorem 3.2], we have that the pathwise uniqueness holds for SDEs (5.2). Therefore, by Lemma 5.3, we have for $i=1,\dots,n$, and $q\geqslant 1$,

$$\lim_{m\to+\infty}\sup_{0\leqslant t\leqslant T}\sup_{\widetilde{x}\in\mathcal{K}_2}\mathrm{E}\Big[\sup_{0\leqslant s\leqslant T}|X_i^{t,\widetilde{x},m}(s)-X_i^{t,\widetilde{x}}(s)|^q\Big]=0,$$

where $\mathcal{K}_2$ is any compact subset of Γ^n and $X_i^{t,\widetilde{x},m}$ is the unique solution of

$$X_i^{t,\widetilde{x},m}(s)=\widetilde{x}_i+\int_t^{s\vee t}b_i^m(X_i^{t,\widetilde{x},m}(u))du+\int_t^{s\vee t}dW_i(u).$$

Therefore, by Lemma 2.5, we have, for $0\leqslant s\leqslant T$,

$$\lim_{m\to+\infty}\sup_{0\leqslant t\leqslant T}\sup_{\widetilde{x}\in\mathcal{K}_2}\mathrm{E}[|X_{(j)}^{t,\widetilde{x},m}(s)-X_{(j)}^{t,\widetilde{x}}(s)|^q]\leqslant\lim_{m\to+\infty}\sup_{0\leqslant t\leqslant T}\sup_{\widetilde{x}\in\mathcal{K}_2}\sum_{i=1}^n\mathrm{E}[|X_i^{t,\widetilde{x},m}(s)-X_i^{t,\widetilde{x}}(s)|^q]=0.$$

Define $P_i^{t,\widetilde{p},m}(s):=\mathrm{e}^{X_i^{t,\widetilde{x},m}(s)}$, where $\widetilde{p}=(\widetilde{p}_1,\dots,\widetilde{p}_n)$ and $\widetilde{p}_i=\mathrm{e}^{\widetilde{x}_i}$, $i=1,\dots,n$. Denote by $X_{(j)}^{t,\widetilde{x},m}(s)$ and $P_{(j)}^{t,\widetilde{p},m}(s)$ the corresponding ordered particles. For any compact subset $\mathcal{K}_2$ of Γ^n, denote by $\bar{\mathcal{K}}_2$ its projection in $(\Gamma^n)^+$ through mapping: $\widetilde{p}_i=\mathrm{e}^{\widetilde{x}_i}$, $i=1,\dots,n$. For $0\leqslant s\leqslant T$, $j=1,\dots,n$, and any compact subset $\mathcal{K}=\mathcal{K}_1\times\bar{\mathcal{K}}_2$ in $\mathbb{R}^+\times(\Gamma^n)^+$, we have

$$\begin{aligned}
&\lim_{m\to+\infty}\sup_{0\leqslant t\leqslant T}\sup_{\widetilde{p}\in\mathcal{K}}\mathrm{E}[|P_{(j)}^{t,\widetilde{p},m}(s)-P_{(j)}^{t,\widetilde{p}}(s)|^2]\\
&\quad=\lim_{m\to+\infty}\sup_{0\leqslant t\leqslant T}\sup_{\widetilde{x}\in\mathcal{K}_2}\mathrm{E}[|\mathrm{e}^{X_{(j)}^{t,\widetilde{x},m}(s)}-\mathrm{e}^{X_{(j)}^{t,\widetilde{x}}(s)}|^2]\\
&\quad\leqslant\lim_{m\to+\infty}\sup_{0\leqslant t\leqslant T}\sup_{\widetilde{x}\in\mathcal{K}_2}\mathrm{E}[|P_{(j)}^{t,\widetilde{p},m}(s)+P_{(j)}^{t,\widetilde{p}}(s)|^2|X_{(j)}^{t,\widetilde{x},m}(s)-X_{(j)}^{t,\widetilde{x}}(s)|^2]\\
&\quad\leqslant\lim_{m\to+\infty}\sup_{0\leqslant t\leqslant T}\sup_{\widetilde{x}\in\mathcal{K}_2}C(\mathrm{E}[|X_{(j)}^{t,\widetilde{x},m}(s)-X_{(j)}^{t,\widetilde{x}}(s)|^4])^{\frac{1}{2}}\\
&\quad=0.
\end{aligned}$$

Similarly, we have, for $0\leqslant s\leqslant T$,

$$\lim_{m\to+\infty}\sup_{0\leqslant t\leqslant T}\sup_{\widetilde{p}\in\mathcal{K}}\mathrm{E}[|P_i^{t,\widetilde{p},m}(s)-P_i^{t,\widetilde{p}}(s)|^2]=0.$$

Denote by $(Y^{t,\widetilde{p},m}(s),\pi^m(s))$ the unique solution of the following BSDEs:

$$\begin{cases}dY^{t,\widetilde{p},m}(s)=\Big[Y^{t,\widetilde{p},m}(s)r(s)+\sum_{i=1}^n\pi_i^m(s)\Big(b_i^m(P^{t,\widetilde{p},m}(s))+\dfrac{1}{2}-r(s)\Big)\Big]ds+\sum_{i=1}^n\pi_i^m(s)dW_i(s),\\ Y^{t,\widetilde{p},m}(T)=g(P_0^{t,\widetilde{p}}(T),\widetilde{P}^{t,\widetilde{p},m}(T)).\end{cases}$$

By [10, Section 5], for $\alpha>2$, there exists a constant C depending on $(T,r(s),\{\delta_j\})$ such that

$$\begin{aligned}
&\mathrm{E}\Big[\sup_{0\leqslant s\leqslant T}|Y^{t,\widetilde{p}}(s)-Y^{t,\widetilde{p},m}(s)|^2\Big]\\
&\quad\leqslant C\mathrm{E}[(g(P_0^{t,\widetilde{p}}(T),\widetilde{P}^{t,\widetilde{p}}(T))-g(P_0^{t,\widetilde{p}}(T),\widetilde{P}^{t,\widetilde{p},m}(T)))^2]\\
&\qquad+\sum_{i=1}^n C\mathrm{E}\Big[\Big(\int_t^T\pi_i^m(s)[b_i(P^{t,\widetilde{p}}(s))-b_i^m(P^{t,\widetilde{p},m}(s))]ds\Big)^2\Big]\\
&\quad\leqslant C\mathrm{E}[(g(P_0^{t,\widetilde{p}}(T),\widetilde{P}^{t,\widetilde{p}}(T))-g(P_0^{t,\widetilde{p}}(T),\widetilde{P}^{t,\widetilde{p},m}(T)))^2]
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{n} C\mathrm{E}\bigg[\int_t^T |\pi_i^m(s)|^2 ds \int_t^T |b_i(P^{t,\widetilde{p}}(s)) - b_i^m(P^{t,\widetilde{p},m}(s))|^2 ds\bigg] \\
&\leqslant C\mathrm{E}[(g(P_0^{t,\widetilde{p}}(T), \widetilde{P}^{t,\widetilde{p}}(T)) - g(P_0^{t,\widetilde{p}}(T), \widetilde{P}^{t,\widetilde{p},m}(T)))^2] \\
&\quad + C\sum_{i=1}^{n}\bigg(\mathrm{E}\bigg[\bigg(\int_t^T |\pi_i^m(s)|^2 ds\bigg)^{\frac{\alpha}{2}}\bigg]\bigg)^{\frac{2}{\alpha}}\bigg(\mathrm{E}\bigg[\int_t^T |b_i(P^{t,\widetilde{p}}(s)) - b_i^m(P^{t,\widetilde{p},m}(s))|^{\frac{2\alpha}{\alpha-2}} dt\bigg]\bigg)^{\frac{\alpha-2}{\alpha}} \\
&\leqslant C\mathrm{E}[(\widetilde{P}^{t,\widetilde{p}}(T) - \widetilde{P}^{t,\widetilde{p},m}(T))^2] \\
&\quad + C(1+|\widetilde{p}|^2)\sum_{i=1}^{n}\bigg(\mathrm{E}\bigg[\int_t^T |b_i(P^{t,\widetilde{p}}(s)) - b_i^m(P^{t,\widetilde{p},m}(s))|^{\frac{2\alpha}{\alpha-2}} dt\bigg]\bigg)^{\frac{\alpha-2}{\alpha}}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\lim_{m\to+\infty}\sup_{0\leqslant t\leqslant T}\sup_{\widetilde{p}\in\mathcal{K}}\mathrm{E}\Big[\sup_{0\leqslant s\leqslant T}|Y^{t,\widetilde{p}}(s) - Y^{t,\widetilde{p},m}(s)|^2\Big] \\
&\quad\leqslant C\lim_{m\to+\infty}\sup_{0\leqslant t\leqslant T}\sup_{\widetilde{p}\in\mathcal{K}}\mathrm{E}[(\widetilde{P}^{t,\widetilde{p}}(T) - \widetilde{P}^{t,\widetilde{p},m}(T))^2] \\
&\qquad + C\sum_{i=1}^{n}\lim_{m\to+\infty}\sup_{0\leqslant t\leqslant T}\sup_{\widetilde{p}\in\mathcal{K}}\bigg(\mathrm{E}\bigg[\int_t^T |b_i(P^{t,\widetilde{p}}(s)) - b_i^m(P^{t,\widetilde{p},m}(s))|^{\frac{2\alpha}{\alpha-2}} dt\bigg]\bigg)^{\frac{\alpha-2}{\alpha}} \\
&\quad =: C\lim_{m\to+\infty}\sup_{0\leqslant t\leqslant T}\sup_{\widetilde{p}\in\mathcal{K}} I_1 + C\sum_{i=1}^{n}\lim_{m\to+\infty}\sup_{0\leqslant t\leqslant T}\sup_{\widetilde{p}\in\mathcal{K}}(I_2^i)^{\frac{\alpha-2}{\alpha}}.
\end{aligned}
$$

First,

$$
\lim_{m\to+\infty}\sup_{0\leqslant t\leqslant T}\sup_{\widetilde{p}\in\mathcal{K}} I_1 \leqslant \lim_{m\to+\infty}\sup_{0\leqslant t\leqslant T}\sup_{\widetilde{p}\in\mathcal{K}}\mathrm{E}\bigg[\sum_{j=1}^{n}|P_{(j)}^{t,\widetilde{p}}(T) - P_{(j)}^{t,\widetilde{p},m}(T)|^2\bigg] = 0.
$$

Let m_0 be a fixed positive integer. It holds that

$$
\begin{aligned}
I_2^i &\leqslant C\mathrm{E}\bigg[\int_t^T |b_i(P^{t,\widetilde{p}}(s)) - b_i^{m_0}(P^{t,\widetilde{p}}(s))|^{\frac{2\alpha}{\alpha-2}} ds\bigg] + C\mathrm{E}\bigg[\int_0^T |b_i^{m_0}(P^{t,\widetilde{p}}(s)) - b_i^{m_0}(P^{t,\widetilde{p},m}(s))|^{\frac{2\alpha}{\alpha-2}} ds\bigg] \\
&\quad + C\mathrm{E}\bigg[\int_t^T |b_i^{m_0}(P^{t,\widetilde{p},m}(s)) - b_i^m(P^{t,\widetilde{p},m}(s))|^{\frac{2\alpha}{\alpha-2}} ds\bigg] \\
&=: C(J_1^i + J_2^i + J_3^i).
\end{aligned}
$$

Let $w(x)$ be a decreasing Lipschitz continuous function on $[0,+\infty)$ such that $w(0)=1$ and $w(x)=0$ for $x\geqslant 1$. Then for $R>0$, by Theorem 2.4 in Chapter 2 of [24],

$$
\begin{aligned}
J_3^i &\leqslant C\mathrm{E}\bigg[\int_t^T \bigg(1 - w\bigg(\frac{|X^{t,x,m}(s)|}{R}\bigg)\bigg) ds\bigg] \\
&\quad + \mathrm{E}\bigg[\int_t^T w\bigg(\frac{|X^{t,x,m}(s)|}{R}\bigg)|b_i^{m_0}(P^{t,x,m}(s)) - b_i^m(P^{t,x,m}(s))|^{\frac{2\alpha}{\alpha-2}} ds\bigg] \\
&\leqslant C\mathrm{E}\bigg[\int_0^T \bigg(1 - w\bigg(\frac{|X^{t,x}(s)|}{R}\bigg)\bigg) ds\bigg] + \sum_{i=1}^{n} C\mathrm{E}\bigg[\int_0^T |X_i^{t,x}(s) - X_i^{t,x,m}(s)| ds\bigg] \\
&\quad + CT^{\frac{1}{2}}\bigg(\int_{B(0,R)} |b_i^{m_0}(\mathrm{e}^{y_1},\ldots,\mathrm{e}^{y_n}) - b_i^m(\mathrm{e}^{y_1},\ldots,\mathrm{e}^{y_n})|^{\frac{4\alpha}{\alpha-2}} dy_1\ldots dy_n\bigg)^{\frac{1}{2}},
\end{aligned}
$$

where $B(0,R)$ is the ball with center 0 and radius R in $\mathbb{R}^n$. Therefore,

$$
\begin{aligned}
&\lim_{m\to+\infty}\sup_{0\leqslant t\leqslant T}\sup_{p\in\mathcal{K}} J_3^i \\
&\quad\leqslant \sup_{0\leqslant t\leqslant T}\sup_{x\in\mathcal{K}_2} C\mathrm{E}\bigg[\int_0^T \bigg(1 - w\bigg(\frac{|X^{t,x}(s)|}{R}\bigg)\bigg) ds\bigg]
\end{aligned}
$$

$$
\begin{aligned}
&+C\lim_{m\to+\infty}\bigg(\int_{B(0,R)}|b_i^{m_0}(\mathrm{e}^{y_1},\dots,\mathrm{e}^{y_n})-b_i^m(\mathrm{e}^{y_1},\dots,\mathrm{e}^{y_n})|^{\frac{4\alpha}{\alpha-2}}dy_1\dots dy_n\bigg)^{\frac12}\\
&+\lim_{m\to+\infty}\sup_{0\leqslant t\leqslant T}\sup_{x\in\mathcal{K}_2}C\sum_{i=1}^n\mathrm{E}\bigg[\int_0^T|X_i^{t,x}(s)-X_i^{t,x,m}(s)|ds\bigg]\\
\leqslant{}& C\mathrm{E}\bigg[\int_0^T\bigg(1-w\bigg(\frac{\sup_{0\leqslant t\leqslant T}\sup_{x\in\mathcal{K}_2}|X^{t,x}(s)|}{R}\bigg)\bigg)ds\bigg]\\
&+C\lim_{m\to+\infty}\bigg(\int_{B(0,R)}|b_i^{m_0}(\mathrm{e}^{y_1},\dots,\mathrm{e}^{y_n})-b_i^m(\mathrm{e}^{y_1},\dots,\mathrm{e}^{y_n})|^{\frac{4\alpha}{\alpha-2}}dy_1\dots dy_n\bigg)^{\frac12}\\
&+\lim_{m\to+\infty}\sup_{0\leqslant t\leqslant T}\sup_{x\in\mathcal{K}_2}C\sum_{i=1}^n\mathrm{E}\Big[\sup_{0\leqslant s\leqslant T}|X_i^{t,x,m}(s)-X_i^{t,x}(s)|\Big]\\
\leqslant{}& C\mathrm{E}\bigg[\int_0^T\bigg(1-w\bigg(\frac{\sup_{0\leqslant t\leqslant T}\sup_{x\in\mathcal{K}_2}|X^{t,x}(s)|}{R}\bigg)\bigg)ds\bigg]\\
&+C\lim_{m\to+\infty}\bigg(\int_{B(0,R)}|b_i^{m_0}(\mathrm{e}^{y_1},\dots,\mathrm{e}^{y_n})-b_i^m(\mathrm{e}^{y_1},\dots,\mathrm{e}^{y_n})|^{\frac{4\alpha}{\alpha-2}}dy_1\dots dy_n\bigg)^{\frac12}.
\end{aligned}
$$

Since b_i^m converges to b_i almost everywhere with respect to the Lebesgue measure, the last expression in the right-hand side of the above inequality tends to 0 as m_0 tends to $+\infty$. Next, let R go to $+\infty$. Then from the properties of the function $w(x)$ we have

$$\lim_{m\to+\infty}\sup_{0\leqslant t\leqslant T}\sup_{\widetilde{p}\in\mathcal{K}}J_3^i=0.$$

Similarly, we have $\lim_{m\to+\infty}\sup_{0\leqslant t\leqslant T}\sup_{\widetilde{p}\in\mathcal{K}}J_1^i=0$. Finally,

$$
\begin{aligned}
&\lim_{m\to+\infty}\sup_{0\leqslant t\leqslant T}\sup_{\widetilde{p}\in\mathcal{K}}\mathrm{E}[|b_i^{m_0}(P^{t,\widetilde{p}}(s))-b_i^{m_0}(P^{t,\widetilde{p},m}(s))|^{\frac{2\alpha}{\alpha-2}}]\\
&\quad\leqslant\lim_{m\to+\infty}\sup_{0\leqslant t\leqslant T}\sup_{\widetilde{p}\in\mathcal{K}}C\mathrm{E}[|P^{t,\widetilde{p}}(s)-P^{t,\widetilde{p},m}(s)|^{\frac{2\alpha}{\alpha-2}}]=0.
\end{aligned}
$$

Therefore,

$$\lim_{m\to+\infty}\sup_{0\leqslant t\leqslant T}\sup_{\widetilde{p}\in\mathcal{K}}(I_2^i)^{\frac{\alpha-2}{\alpha}}\leqslant C\lim_{m\to+\infty}\Big(\sup_{0\leqslant t\leqslant T}\sup_{\widetilde{p}\in\mathcal{K}}J_1^i+\sup_{0\leqslant t\leqslant T}\sup_{\widetilde{p}\in\mathcal{K}}J_2^i+\sup_{0\leqslant t\leqslant T}\sup_{\widetilde{p}\in\mathcal{K}}J_3^i\Big)^{\frac{\alpha-2}{\alpha}}=0.$$

Therefore, for any compact subset $\mathcal{K}$ of $\mathbb{R}^+\times(\Gamma^n)^+$,

$$\lim_{m\to+\infty}\sup_{0\leqslant t\leqslant T}\sup_{\widetilde{p}\in\mathcal{K}}\mathrm{E}\Big[\sup_{0\leqslant s\leqslant T}|Y^{t,\widetilde{p}}(s)-Y^{t,\widetilde{p},m}(s)|^2\Big]=0.$$

Define $u^m(t,\widetilde{p}):=Y^{t,\widetilde{p},m}(t)$, $(t,\widetilde{p})\in[0,T]\times\mathbb{R}^+\times(\Gamma^n)^+$, similar to Theorem 3.5, we obtain that $u^m(t,\widetilde{p})$ is continuous on $[0,T]\times\mathbb{R}^+\times(\Gamma^n)^+$. Since $u^m\to u$ on any compact subset of $[0,T]\times\mathbb{R}^+\times(\Gamma^n)^+$, we have that u is continuous on $[0,T]\times\mathbb{R}^+\times(\Gamma^n)^+$. □

Since BSDE (5.4) is linear, we obtain for $t\leqslant s\leqslant T$,

$$Y^{t,\widetilde{p}}(s)=\frac{1}{M(s)}\mathrm{E}[g(P_0^{t,\widetilde{p}}(T),\widetilde{P}^{t,\widetilde{p}}(T))M(T)\,|\,\mathcal{F}_s],$$

where

$$M(s)=\exp\bigg\{\int_t^s\bigg[-r(u)-\frac12\sum_{j=1}^n(\delta_j-r(u))^2\bigg]du-\sum_{j=1}^n\int_t^s(\delta_j-r(u))d\beta_j(u)\bigg\}>0.$$

Therefore, there exists a unique hedging strategy $(Y^{t,\widetilde{p}},\pi)$ against $g(P_0^{t,\widetilde{p}}(T),\widetilde{P}^{t,\widetilde{p}}(T))$ and $Y^{t,\widetilde{p}}(s)$ is the fair price of the contingent $g(P_0^{t,\widetilde{p}}(T),\widetilde{P}^{t,\widetilde{p}}(T))$ at time s. Then similar to Theorems 4.2 and 4.3, the value of the contingent claim ξ at time s is

$$Y(s)=u(s,P_0^{t,\widetilde{p}}(s),\widetilde{P}^{t,\widetilde{p}}(s)),$$

where, $u(t,\widetilde{p})$ is the unique viscosity solution of the following parabolic PDEs:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,\widetilde{p}) = -\widetilde{\mathcal{L}}u(t,\widetilde{p}) + r(t)u(t,\widetilde{p}), & t\in[0,T], \quad \widetilde{p}\in\mathbb{R}^+\times(\Pi^n)^+, \\ u(T,\widetilde{p}) = g(\widetilde{p}), & \widetilde{p}\in\mathbb{R}^+\times(\Gamma^n)^+, \\ \widetilde{p}_{i+1}\dfrac{\partial u}{\partial \widetilde{p}_{i+1}}(t,\widetilde{p}) - \widetilde{p}_i\dfrac{\partial u}{\partial \widetilde{p}_i}(t,\widetilde{p}) = 0, & \widetilde{p}\in F_i^+, \quad i=1,\dots,n-1, \end{cases}$$

where

$$\widetilde{\mathcal{L}}u(t,\widetilde{p}) = \frac{1}{2}\sum_{i=1}^n \widetilde{p}_i^2\frac{\partial^2 u}{\partial \widetilde{p}_i^2}(t,\widetilde{p}) + \sum_{i=0}^n r(t)\widetilde{p}_i\frac{\partial u}{\partial \widetilde{p}_i}(t,p).$$

6 BSDEs with Brownian particles with asymmetric collisions

In the previous sections, the ranked particles have symmetric collisions. In this section, we will extend the previous results to asymmetric collisions case.

6.1 Property of Brownian particles

In this subsection, we will study the continuity dependent on the initial conditions of Brownian particles with asymmetric collisions. Fix $T>0$, for every $(t,\widetilde{x})\in[0,T]\times\Gamma^n$ and $i=1,\dots,n$, consider the following ordered Brownian particles:

$$X_i^{t,\widetilde{x}}(s) = \begin{cases} \widetilde{x}_i, & 0\leqslant s<t, \\ \widetilde{x}_i + b_i(s-t) + \sigma_i(W_i(s)-W_i(t)) + q_i^-\Lambda^{i,i+1}(s) - q_i^+\Lambda^{i-1,i}(s), & t\leqslant s\leqslant T. \end{cases} \tag{6.1}$$

Here, the drifts $b_1,\dots,b_n$ are given real numbers; dispersions $\sigma_1,\dots,\sigma_n$ are given positive real numbers; the collision parameters $q_1^\pm,\dots,q_n^\pm$ are given positive real numbers satisfying

$$q_i^- + q_{i+1}^+ = 1, \quad i=1,\dots,n-1,$$

and

$$(q_{i-1}^- + q_{i+1}^+)\sigma_i^2 \geqslant q_i^-\sigma_{i+1}^2 + q_i^+\sigma_{i-1}^2, \quad i=2,\dots,n-1.$$

In asymmetric case, the local times are split unevenly between the two colliding particles, as if they had different mass. If we denote by m_i the mass of the $X_i^{t,\widetilde{x}}$ and $m_i : m_{i+1} = q_{i+1}^+ : q_i^-$, then the physical meaning of the collision parameters is that the push (local time of the collision) is split according to their mass. For $i=1,\dots,n-1$ and $s\geqslant 0$, $\Lambda^{i,i+1}(s)$ denotes the local time accumulated at the origin by $X_i^{t,\widetilde{x}}(\cdot) - X_{i+1}^{t,\widetilde{x}}(\cdot)$ on the interval $[0,s]$. It is easily to see that if $0\leqslant s\leqslant t$, $\Lambda^{i,i+1}(s)=0$, $i=1,\dots,n-1$. We set

$$\Lambda^{0,1}(\cdot)\equiv\Lambda^{n,n+1}(\cdot)\equiv 0.$$

In [21, Subsection 2.1], a strong solution $X^{t,\widetilde{x}}(s)$ is constructed and is shown to be pathwise unique. In [31, Theorem 1.9], it is showed that there are no triple collisions at any time $s>0$. It will be more convenient to consider the following process of spacings (or gaps):

$$G^{t,\widetilde{x}}(\cdot) := (X_1^{t,\widetilde{x}}(\cdot) - X_2^{t,\widetilde{x}}(\cdot),\dots,X_{n-1}^{t,\widetilde{x}}(\cdot) - X_n^{t,\widetilde{x}}(\cdot)),$$

which has the following representation: for $i=1,\dots,n-1$,

$$G_i^{t,\widetilde{x}}(s) = \begin{cases} \widetilde{x}_i - \widetilde{x}_{i+1}, & 0\leqslant s<t, \\ \widetilde{x}_i - \widetilde{x}_{i+1} + (b_i-b_{i+1})(s-t) + \sigma_i(W_i(s)-W_i(t)) & \\ \quad -\sigma_{i+1}(W_{i+1}(s)-W_{i+1}(t)) - q_i^+\Lambda^{i-1,i}(s) & \\ \quad +\Lambda^{i,i+1}(s) - q_{i+1}^-\Lambda^{i+1,i+2}(s), & t\leqslant s\leqslant T. \end{cases}$$

In other words,

$$G(s) = \xi(s) + (I - Q)\Lambda(s), \tag{6.2}$$

where $\xi(s)$ is an $(n-1)$-dimensional process whose i-th component is

$$\xi_i(s) = \begin{cases} \widetilde{x}_i - \widetilde{x}_{i+1}, & 0 \leqslant s < t, \\ \widetilde{x}_i - \widetilde{x}_{i+1} + (b_i - b_{i+1})(s-t) + \sigma_i(W_i(s) - W_i(t)) & \\ \quad -\sigma_{i+1}(W_{i+1}(s) - W_{i+1}(t)), & t \leqslant s \leqslant T, \end{cases}$$

$$\Lambda(s) = (\Lambda^{1,2}(s), \ldots, \Lambda^{n-1,n}(s)),$$

and

$$Q := \begin{pmatrix} 0 & q_2^- & 0 & \cdots & 0 & 0 \\ q_2^+ & 0 & q_3^- & \cdots & 0 & 0 \\ 0 & q_3^+ & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & q_{n-1}^- \\ 0 & 0 & 0 & \cdots & q_{n-1}^+ & 0 \end{pmatrix}.$$

The process G is a semimartingale reflected Brownian motion on $[0,\infty)^n$ with driving process ξ and reflection matrix $R = I - Q$. It is showed in [21] that the spectrum radius of Q is less than 1, by [14, Theorem 1] (see also [33, Theorem 2.1]), there exists a Lipschitz continuous function $\phi_{0,T} : C([0,T];\mathbb{R}^{n-1}) \to C([0,T];\mathbb{R}^{n-1})$ with Lipschitz constant L (depending only on reflection matrix R) such that,

$$\Lambda(\omega) = \phi_{0,T}(\xi(\omega)), \quad \text{a.s.}$$

From the Skorokhod mapping (see [33, Definition 2.2]), we know that $\phi_{0,T}(f(s)) = 0$ for all positive function $f(s)$ on $[0,T]$. Thus,

$$\|\Lambda(\omega)\|_{L^\infty([0,T])} = \|\phi_{0,T}(\xi(\omega))\|_{L^\infty([0,T])} \leqslant C(\|\xi(\omega)\|_{L^\infty([0,T])} + 1), \quad \text{a.s.}$$

Theorem 6.1. *For every $p \geqslant 1$, there exists a constant C depending on $(L, p, T, n, \{b_i\}, \{\sigma_i\})$ such that for every $\widetilde{x}, \widetilde{x}' \in \Gamma^n$ and $t, t' \in [0,T]$, we have*

$$\mathrm{E}\Big[\sup_{0\leqslant s\leqslant T} |X_i^{t,\widetilde{x}}(s)|^p\Big] \leqslant C(1 + |\widetilde{x}|^p), \quad i = 1, \ldots, n, \tag{6.3}$$

and

$$\mathrm{E}\Big[\sup_{0\leqslant s\leqslant T} |X_i^{t,\widetilde{x}}(s) - X_i^{t',\widetilde{x}'}(s)|^p\Big] \leqslant C(|\widetilde{x} - \widetilde{x}'|^p + |t - t'|^{\frac{p}{2}}), \quad i = 1, \ldots, n. \tag{6.4}$$

Proof. For any $i = 1, \ldots, n-1$ and $t \leqslant s \leqslant T$,

$$\begin{aligned} |\xi_i(s)|^p &= |\widetilde{x}_i - \widetilde{x}_{i+1} + (b_i - b_{i+1})(s-t) + \sigma_i(W_i(s) - W_i(t)) - \sigma_{i+1}(W_{i+1}(s) - W_{i+1}(t))|^p \\ &\leqslant C(|\widetilde{x}_i - \widetilde{x}_{i+1}|^p + |(b_i - b_{i+1})(s-t)|^p + |\sigma_i(W_i(s) - W_i(t))|^p \\ &\quad + |\sigma_{i+1}(W_{i+1}(s) - W_{i+1}(t))|^p). \end{aligned}$$

Consequently,

$$\begin{aligned} |X_i^{t,\widetilde{x}}(s)|^p &= |\widetilde{x}_i + b_i(s-t) + \sigma_i(W_i(s) - W_i(t)) + q_i^- \Lambda^{i,i+1}(s) - q_i^+ \Lambda^{i-1,i}(s)|^p \\ &\leqslant C(|\widetilde{x}_i|^p + |b_i(s-t)|^p + |\sigma_i(W_i(s) - W_i(t))|^p + |q_i^- \Lambda^{i,i+1}(s)|^p + |q_i^+ \Lambda^{i-1,i}(s)|^p) \\ &\leqslant C \sum_{i=1}^n \sup_{t\leqslant s\leqslant T} (|\widetilde{x}_i|^p + |b_i(s-t)|^p + |\sigma_i(W_i(s) - W_i(t))|^p + |\Lambda^{i,i+1}(s)|^p) \\ &\leqslant C \sum_{i=1}^n \sup_{t\leqslant s\leqslant T} (1 + |\widetilde{x}_i|^p + |b_i(s-t)|^p + |\sigma_i(W_i(s) - W_i(t))|^p + |\xi_i(s)|^p) \end{aligned}$$

$$\leqslant C\sum_{i=1}^{n}\Big(1+|\widetilde{x}_i|^p+|b_i(T-t)|^p+\sup_{t\leqslant s\leqslant T}|\sigma_i(W_i(s)-W_i(t))|^p\Big).$$

By Burkholder-Davis-Gundy's inequality, we have

$$\begin{aligned}\mathrm{E}\Big[\sup_{0\leqslant s\leqslant T}|X_i^{t,\widetilde{x}}(s)|^p\Big]&\leqslant C\sum_{i=1}^{n}\Big(1+|\widetilde{x}_i|^p+|b_i(T-t)|^p+\mathrm{E}\Big[\sup_{t\leqslant s\leqslant T}|\sigma_i(W_i(s)-W_i(t))|^p\Big]\Big)\\&\leqslant C(1+|\widetilde{x}|^p),\quad i=1,\ldots,n.\end{aligned}$$

Next, we prove (6.4). Without loss of generality, assume that $t'<t$. For simplicity, denote $X_i(s)=X_i^{t,\widetilde{x}}(s)$ and $X_i'(s)=X_i^{t',\widetilde{x}'}(s)$. For $j=1,\ldots,n-1$ and $s\geqslant 0$, denote by $\Lambda^{(i,i+1)'}(s)$ the local time accumulated at the origin by $X_i'(\cdot)-X_{i+1}'(\cdot)$ on the interval $[0,s]$. It is easy to obtain that for $0\leqslant s\leqslant t'$, $\Lambda^{(i,i+1)'}(s)=0$, $i=1,\ldots,n-1$. We also set

$$\Lambda^{(0,1)'}(\cdot)\equiv\Lambda^{(n,n+1)'}(\cdot)\equiv 0.$$

Define

$$\xi_i'(s):=\begin{cases}\widetilde{x}_i'-\widetilde{x}_{i+1}', & 0\leqslant s<t',\\ \widetilde{x}_i'-\widetilde{x}_{i+1}'+(b_i-b_{i+1})(s-t')+\sigma_i(W_i(s)-W_i(t')) & \\ \quad-\sigma_{i+1}(W_{i+1}(s)-W_{i+1}(t')), & t'\leqslant s\leqslant T.\end{cases}$$

Step 1. If $s\leqslant t'$, then

$$|X_i(s)-X_i'(s)|^p=|\widetilde{x}_i-\widetilde{x}_i'|^p\leqslant|\widetilde{x}-\widetilde{x}'|^p.$$

Step 2. If $t'<s\leqslant t$, define

$$\eta(s):=(\widetilde{x}_1'-\widetilde{x}_2',\ldots,\widetilde{x}_{n-1}'-\widetilde{x}_n'),\quad 0\leqslant s\leqslant T.$$

Hence,

$$\begin{aligned}|\Lambda'(s)|&\leqslant\sup_{t'<s\leqslant t}|\phi_{0,t}(\xi')(s)|\\&\leqslant\|\phi_{0,t}(\xi')-\phi_{0,t}(\eta)\|_{L^\infty([0,t])}\\&\leqslant C\sup_{0\leqslant s\leqslant t}|\xi'(s)-\eta(s)|\\&\leqslant C\sum_{i=1}^{n}\Big(|b_i(t-t')|+\sup_{t'\leqslant s\leqslant t}|\sigma_i(W_i(s)-W_i(t'))|\Big).\end{aligned}$$

Therefore,

$$\begin{aligned}|X_i(s)-X_i'(s)|^p&=|\widetilde{x}_i-\widetilde{x}_i'-b_i(s-t')-\sigma_i(W_i(s)-W_i(t'))-q_i^-\Lambda^{(i,i+1)'}(s)+q_i^+\Lambda^{(i-1,i)'}(s)|^p\\&\leqslant C\sum_{i=1}^{n}\sup_{t'\leqslant s\leqslant t}(|\widetilde{x}_i-\widetilde{x}_i'|^p+|b_i(t-t')|^p+|\sigma_i(W_i(s)-W_i(t'))|^p).\end{aligned}$$

By Burkholder-Davis-Gundy's inequality, we have

$$\mathrm{E}\Big[\sup_{t'<s\leqslant t}|X_i(s)-X_i'(s)|^p\Big]\leqslant C(|\widetilde{x}-\widetilde{x}'|^p+|t-t'|^{\frac{p}{2}}).$$

Step 3. If $t<s\leqslant T$,

$$\begin{aligned}X_i(s)-X_i'(s)&=X_i(t)-X_i'(t)+q_i^-(\Lambda^{i,i+1}(s)-\Lambda^{i,i+1}(t))-q_i^+(\Lambda^{i-1,i}(s)-\Lambda^{i-1,i}(t))\\&\quad-q_i^-(\Lambda^{(i,i+1)'}(s)-\Lambda^{(i,i+1)'}(t))+q_i^+(\Lambda^{(i-1,i)'}(s)-\Lambda^{(i-1,i)'}(t))\\&=X_i(t)-X_i'(t)+q_i^-(\Lambda^{i,i+1}(s)-\Lambda^{(i,i+1)'}(s))-q_i^-(\Lambda^{i,i+1}(t)-\Lambda^{(i,i+1)'}(t))\end{aligned}$$

$$- q_i^+(\Lambda^{i-1,i}(s) - \Lambda^{(i-1,i)'}(s)) + q_i^+(\Lambda^{i-1,i}(t) - \Lambda^{(i-1,i)'}(t)).$$

Since

$$\begin{aligned}
&\sup_{t<s\leqslant T} |\Lambda^{i,i+1}(s) - \Lambda^{(i,i+1)'}(s)| \\
&\quad \leqslant \|\Lambda - \Lambda'\|_{L^\infty([0,T])} \\
&\quad = \big\|\phi_{0,T}(\xi) - \phi_{0,T}(\xi')\big\|_{L^\infty([0,T])} \\
&\quad \leqslant C \sup_{0\leqslant s\leqslant T} |\xi(s) - \xi'(s)| \\
&\quad \leqslant C \sum_{i=1}^n \Big(|\widetilde{x}_i - \widetilde{x}'_i| + |b_i(t-t')| + \sup_{t'\leqslant s\leqslant t} |\sigma_i(W_i(s) - W_i(t'))| \Big),
\end{aligned}$$

we have

$$\begin{aligned}
&\mathrm{E}\Big[\sup_{t<s\leqslant T} |X_i(s) - X'_i(s)|^p\Big] \\
&\quad \leqslant C\mathrm{E}\Big[|X_i(t) - X'_i(t)|^p + \sum_{i=1}^{n-1} \sup_{t\leqslant s\leqslant T} |\Lambda^{i,i+1}(s) - \Lambda^{(i,i+1)'}(s)|^p\Big] \\
&\quad \leqslant C\mathrm{E}\Big[|X_i(t) - X'_i(t)|^p + \sum_{i=1}^{n} \Big(|\widetilde{x}_i - \widetilde{x}'_i|^p + |b_i(t-t')|^p + \sup_{t'\leqslant s\leqslant t} |\sigma_i(W_i(s) - W_i(t'))|^p \Big)\Big] \\
&\quad \leqslant C(|\widetilde{x} - \widetilde{x}'|^p + |t-t'|^{\frac{p}{2}}).
\end{aligned}$$

This completes the proof. □

Remark 6.2. $X_i^{t,\widetilde{x}}(s)$ in (6.1) is ranked systems for asymmetric collisions. Sarantsev [31] studied these systems and showed that named systems also exist until the first time of a triple collision.

6.2 BSDEs with Brownian particles

In this subsection, for each $(t,\widetilde{x}) \in [0,T] \times \Gamma^n$, consider the following BSDEs:

$$Y^{t,\widetilde{x}}(s) = g(X^{t,\widetilde{x}}(T)) + \int_s^T h(r, X^{t,\widetilde{x}}(r), Y^{t,\widetilde{x}}(r), Z^{t,\widetilde{x}}(r))dr - \int_s^T Z^{t,\widetilde{x}}(r) \cdot dW(r), \tag{6.5}$$

where, $h : [0,T] \times \Gamma^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and $g : \Gamma^n \to \mathbb{R}$ are measurable functions.

We need the following assumptions on h and g:

(H4) $h(t,\widetilde{x},y,z)$ is jointly continuous on $[0,T] \times \Gamma^n \times \mathbb{R} \times \mathbb{R}^n$, uniformly continuous in x, and there exists a constant c so that

$$|h(t,\widetilde{x},y,z) - h(t,\widetilde{x},y',z')| \leqslant c(|y-y'| + |z-z'|), \tag{6.6}$$

and

$$|h(t,\widetilde{x},0,0)| \leqslant c(1 + |\widetilde{x}|). \tag{6.7}$$

(H5) There exists a constant c such that

$$|g(\widetilde{x}) - g(\widetilde{x}')| \leqslant c|\widetilde{x} - \widetilde{x}'| \quad \text{for} \quad \widetilde{x}, \widetilde{x}' \in \Gamma^n \tag{6.8}$$

and

$$|g(\widetilde{x})| \leqslant c(1 + |\widetilde{x}|) \quad \text{for} \quad \widetilde{x} \in \Gamma^n. \tag{6.9}$$

Theorem 6.3. *Suppose* (H4) *and* (H5) *hold. Then* (6.5) *has a unique solution* $(Y,Z) \in S^2([t,T];\mathbb{R}) \times M^2([t,T];\mathbb{R}^n)$. *Furthermore, for any* $T > t \geqslant 0$ *and* $p \geqslant 2$, *there exists a constant* C *depending on* $(L,T,p,n,\{b_i\},\{\sigma_i\})$ *such that for any* $\widetilde{x}, \widetilde{x}' \in \Gamma^n$ *and any* $t,t' \in [0,T]$, *we have*

$$\mathrm{E}\Big[\sup_{0\leqslant s\leqslant T} |Y^{t,\widetilde{x}}(s)|^p\Big] \leqslant C(1 + |\widetilde{x}|^p), \tag{6.10}$$

and

$$\mathrm{E}\Big[\sup_{0\leqslant s\leqslant T}|Y^{t,\widetilde{x}}(s)-Y^{t',\widetilde{x}'}(s)|^p\Big]\to 0,\quad \text{as}\quad t'\to t,\quad \widetilde{x}'\to\widetilde{x}. \tag{6.11}$$

Proof. Under (H4) and (H5), one can construct the solution $(Y,Z)\in S^2([t,T];\mathbb{R})\times M^2([t,T];\mathbb{R}^n)$ of (6.5) in three steps as in [27, 28]. The rest of proof is similar to that of Theorem 3.5 so it is omitted here. □

Define

$$u(t,\widetilde{x}):=Y^{t,\widetilde{x}}(t),\quad (t,\widetilde{x})\in[0,T]\times\Gamma^n, \tag{6.12}$$

which is a deterministic quantity. (6.11) shows that $(s,t,\widetilde{x})\to Y^{t,\widetilde{x}}(s)$ is mean-square continuous. Since $Y^{t,\widetilde{x}}(t)$ is deterministic, we obtain that $u(t,\widetilde{x})$ is continuous with $(t,\widetilde{x})$. Consider the following semi-linear backward parabolic PDEs with Cauchy condition and Neumann boundary condition:

$$\begin{cases}\dfrac{\partial u}{\partial t}(t,\widetilde{x})=-\mathcal{L}u(t,\widetilde{x})-h(t,\widetilde{x},u(t,\widetilde{x}),(\nabla\varphi)(t,\widetilde{x})\sigma), & t\in[0,T],\quad \widetilde{x}\in\Pi^n,\\ u(T,\widetilde{x})=g(\widetilde{x}), & \widetilde{x}\in\Gamma^n,\\ q_{i+1}^+\dfrac{\partial u}{\partial\widetilde{x}_{i+1}}(t,\widetilde{x})=q_i^-\dfrac{\partial u}{\partial\widetilde{x}_i}(t,\widetilde{x}), & t\in[0,T),\quad \widetilde{x}\in F_i,\quad i=1,\dots,n-1,\end{cases} \tag{6.13}$$

where,

$$\mathcal{L}=\frac{1}{2}\sum_{i=1}^n\sigma_i^2\frac{\partial^2}{\partial\widetilde{x}_i^2}+\sum_{i=1}^n b_i\frac{\partial}{\partial\widetilde{x}_i}, \tag{6.14}$$

and σ is a diagonal matrix with diagonal elements $\sigma_1,\dots,\sigma_n$.

Similar to Theorem 3.6, we have the following result.

Theorem 6.4. *Suppose PDEs* (6.13) *have a solution* $u(t,\widetilde{x})\in C^{1,2}([0,T]\times\Gamma^n;\mathbb{R})$ *and there exist some* $c,p>0$ *such that*

$$|\nabla u(t,\widetilde{x})|\leqslant c(1+|\widetilde{x}|^p)\quad \textit{for}\quad t>0\quad \textit{and}\quad \widetilde{x}\in\Gamma^n.$$

Then the solution of (6.13) *is unique.*

6.3 Connection with PDEs

To study the viscosity solution of PDEs (6.13), we need another assumption on h.

(H6) For all $R>0$, there exists a positive function $\eta_R(\cdot)$ tending to 0 at $0+$ such that

$$|h(t,\widetilde{x},y,z)-h(t,\widetilde{x}',y,z)|\leqslant\eta_R(|\widetilde{x}-\widetilde{x}'|(1+|z|)),$$

if $|\widetilde{x}|,|\widetilde{x}'|,|y|\leqslant R$, $t\in[0,T]$, $z\in\mathbb{R}^n$.

Theorem 6.5. *Suppose* (H4)–(H6) *hold. Then the function* $u(t,\widetilde{x})$ *defined by* (6.12) *is the unique viscosity solution of* (6.13) *such that*

$$\lim_{|\widetilde{x}|\to+\infty}|u(t,\widetilde{x})|\mathrm{e}^{-A\log^2|\widetilde{x}|}=0, \tag{6.15}$$

uniformly for $t\in[0,T]$, *for some* $A>0$.

Proof. The proof of existence and uniqueness of the viscosity solution is similar to that of Theorems 4.2 and 4.3. Therefore, it is omitted here. It follows from Theorem 6.3 that $u(t,x)$ has at most polynomial growth at infinity. Thus, $u(t,\widetilde{x})$ defined by (6.12) is the unique viscosity solution of (6.13) that satisfies (6.15). □

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Appendix A

Proof of Lemma 4.4. Observe that

$$w(T,\widetilde{x}) = u(T,\widetilde{x}) - v(T,\widetilde{x}) \leqslant 0.$$

Let $\varphi \in C^{1,2}([0,T)\times \Pi^n;\mathbb{R})$ and $(t_0,\widetilde{x}_0)\in[0,T)\times\Pi^n$ be a maximum point of $w-\varphi$. Modifying φ if necessary, we may assume without loss of generality that $(t_0,\widetilde{x}_0)\in[0,T)\times\Pi^n$ is a strict global maximum point of $w-\varphi$. Define

$$\psi_\varepsilon(t,\widetilde{x},\widetilde{y}) := u(t,\widetilde{x}) - v(t,\widetilde{y}) - \varphi(t,\widetilde{x}) - \frac{(\widetilde{x}-\widetilde{y})^2}{\varepsilon^2},$$

where ε is a positive parameter that will be later taken to approach 0. Choose $R>0$ large enough and define

$$\Pi^{n,R} := B_R \cap \Pi^n$$

so that $(t_0,\widetilde{x}_0)\in[0,T)\times\Pi^{n,R}$, where B_R is the open ball in $\mathbb{R}^n$ centered at origin with radius R. Let $(t_\varepsilon,\widetilde{x}_\varepsilon,\widetilde{y}_\varepsilon)$ be a global maximum point of $\psi_\varepsilon(t,\widetilde{x},\widetilde{y})$ on $[0,T]\times\Pi^{n,R}$. Then by [7, Proposition 3.7], we have

(i) $(t_\varepsilon,\widetilde{x}_\varepsilon,\widetilde{y}_\varepsilon)\to(t_0,\widetilde{x}_0,\widetilde{x}_0)$ as $\varepsilon\to 0$;

(ii) $\varepsilon^{-2}|\widetilde{x}_\varepsilon-\widetilde{y}_\varepsilon|^2$ is bounded and tends to zero as $\varepsilon\to 0$.

Now for each fixed $\varepsilon>0$, it follows from [7, Theorem 8.3] that for any $\delta>0$, there exist $(X_\delta,Y_\delta)\in S(n)\times S(n)$ and $c_\delta\in\mathbb{R}$ so that

$$\left(c_\delta + \frac{\partial\varphi}{\partial t}(t_\varepsilon,\widetilde{x}_\varepsilon), p_\varepsilon + \nabla\varphi(t_\varepsilon,\widetilde{x}_\varepsilon), X_\delta\right) \in \bar{D}^{2,+}_u(t_\varepsilon,\widetilde{x}_\varepsilon),$$
$$(c_\delta, p_\varepsilon, Y_\delta)\in \bar{D}^{2,-}_v(t_\varepsilon,\widetilde{y}_\varepsilon),$$

and

$$\begin{pmatrix} X_\delta & 0 \\ 0 & -Y_\delta \end{pmatrix} \leqslant A + \delta A^2,$$

where,

$$p_\varepsilon = \frac{2(\widetilde{x}_\varepsilon-\widetilde{y}_\varepsilon)}{\varepsilon^2}, \quad A = \begin{pmatrix} D^2\varphi(t_\varepsilon,\widetilde{x}_\varepsilon)+\frac{2}{\varepsilon^2} & -\frac{2}{\varepsilon^2} \\ -\frac{2}{\varepsilon^2} & \frac{2}{\varepsilon^2} \end{pmatrix},$$

and $\bar{D}^{2,+}_u(t,\widetilde{x})$ (resp. $\bar{D}^{2,-}_v(t,\widetilde{y})$) is the parabolic superset (resp. parabolic subset) of u (resp. v) at $(t,\widetilde{x})$ (resp. $(t,\widetilde{y})$). For $\gamma = 1+\frac{4\delta}{\varepsilon^2}$,

$$A+\delta A^2 = \begin{pmatrix} D^2\varphi & 0 \\ 0 & 0 \end{pmatrix} + \frac{2}{\varepsilon^2}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \delta M(\varepsilon),$$

where

$$M(\varepsilon) = \begin{pmatrix} (D^2\varphi)^2 + \frac{4}{\varepsilon^2}D^2\varphi & -\frac{2}{\varepsilon^2}D^2\varphi \\ -\frac{2}{\varepsilon^2}D^2\varphi & 0 \end{pmatrix} + \frac{8}{\varepsilon^4}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Since u and v are viscosity subsolution and supersolution, respectively, we have

$$-c_\delta - \frac{\partial\varphi}{\partial t}(t_\varepsilon,\widetilde{x}_\varepsilon) - b\cdot(p_\varepsilon+\nabla\varphi(t_\varepsilon,\widetilde{x}_\varepsilon)) - \frac{1}{2}\mathrm{Tr}(X_\delta) - h(t_\varepsilon,\widetilde{x}_\varepsilon,u(t_\varepsilon,\widetilde{x}_\varepsilon),p_\varepsilon+\nabla\varphi(t_\varepsilon,\widetilde{x}_\varepsilon)) \leqslant 0,$$

and

$$-c_\delta - b\cdot p_\varepsilon - \frac{1}{2}\mathrm{Tr}(Y_\delta) - h(t_\varepsilon,\widetilde{y}_\varepsilon,v(t_\varepsilon,\widetilde{y}_\varepsilon),p_\varepsilon) \geqslant 0,$$

where $b=(b_1,\dots,b_n)$. Thus,

$$0\leqslant \frac{\partial\varphi}{\partial t}(t_\varepsilon,\widetilde{x}_\varepsilon)+b\cdot\nabla\varphi(t_\varepsilon,\widetilde{x}_\varepsilon)+\frac{1}{2}\mathrm{Tr}(X_\delta)-\frac{1}{2}\mathrm{Tr}(Y_\delta)$$
$$+\,h(t_\varepsilon,\widetilde{x}_\varepsilon,u(t_\varepsilon,\widetilde{x}_\varepsilon),p_\varepsilon+\nabla\varphi(t_\varepsilon,\widetilde{x}_\varepsilon))-h(t_\varepsilon,\widetilde{y}_\varepsilon,v(t_\varepsilon,\widetilde{y}_\varepsilon),p_\varepsilon).$$

First,

$$\frac{1}{2}\mathrm{Tr}(X_\delta)-\frac{1}{2}\mathrm{Tr}(Y_\delta)\leqslant\frac{1}{2}\mathrm{Tr}(D^2\varphi(t_\varepsilon,\widetilde{x}_\varepsilon))+\frac{\delta}{2}R^\varepsilon,$$

where,

$$R^\varepsilon=\left(\begin{pmatrix}I_n\\I_n\end{pmatrix},M(\varepsilon)\begin{pmatrix}I_n\\I_n\end{pmatrix}\right).$$

Finally,

$$\begin{aligned}&h(t_\varepsilon,\widetilde{x}_\varepsilon,u(t_\varepsilon,\widetilde{x}_\varepsilon),p_\varepsilon+\nabla\varphi(t_\varepsilon,\widetilde{x}_\varepsilon))-h(t_\varepsilon,\widetilde{y}_\varepsilon,v(t_\varepsilon,\widetilde{y}_\varepsilon),p_\varepsilon)\\&\quad=h(t_\varepsilon,\widetilde{x}_\varepsilon,u(t_\varepsilon,\widetilde{x}_\varepsilon),p_\varepsilon+\nabla\varphi(t_\varepsilon,\widetilde{x}_\varepsilon))-h(t_\varepsilon,\widetilde{x}_\varepsilon,v(t_\varepsilon,\widetilde{y}_\varepsilon),p_\varepsilon)\\&\qquad+h(t_\varepsilon,\widetilde{x}_\varepsilon,v(t_\varepsilon,\widetilde{y}_\varepsilon),p_\varepsilon)-h(t_\varepsilon,\widetilde{y}_\varepsilon,v(t_\varepsilon,\widetilde{y}_\varepsilon),p_\varepsilon)\\&\quad\leqslant\eta(|\widetilde{x}_\varepsilon-\widetilde{y}_\varepsilon|(1+|p_\varepsilon|))+c|u(t_\varepsilon,\widetilde{x}_\varepsilon)-v(t_\varepsilon,\widetilde{y}_\varepsilon)|+c|\nabla\varphi(t_\varepsilon,\widetilde{x}_\varepsilon)|,\end{aligned}$$

where η is the modulus η_R that appeared in (H3) for R large enough. By first letting $\delta\to 0$ and then $\varepsilon\to 0$, we obtain

$$-\frac{\partial\varphi}{\partial t}(t_0,\widetilde{x}_0)-b\cdot\nabla\varphi(t_0,\widetilde{x}_0)-\frac{1}{2}\mathrm{Tr}(D^2\varphi(t_0,\widetilde{x}_0))-c|w(t_0,\widetilde{x}_0)|-c|\nabla\varphi(t_0,\widetilde{x}_0)|\leqslant 0.$$

This completes the proof. □

Proof of Lemma 4.5. It is easy to check that

$$|\nabla\psi(\widetilde{x})|=2\left|(\log(\sqrt{|\widetilde{x}|^2+1})+1)\frac{1}{\sqrt{|\widetilde{x}|^2+1}}\frac{\widetilde{x}}{\sqrt{|\widetilde{x}|^2+1}}\right|\leqslant\frac{2(\psi(\widetilde{x}))^{1/2}}{\sqrt{|\widetilde{x}|^2+1}}$$

and

$$\begin{aligned}|D^2\psi(\widetilde{x})|&=\bigg|\frac{2\widetilde{x}}{|\widetilde{x}|^2+1}\frac{\widetilde{x}}{|\widetilde{x}|^2+1}-2(\psi(\widetilde{x}))^{1/2}\frac{\widetilde{x}}{(|\widetilde{x}|^2+1)^{3/2}}\frac{\widetilde{x}}{\sqrt{|\widetilde{x}|^2+1}}\\&\qquad+2(\psi(\widetilde{x}))^{1/2}\frac{1}{\sqrt{|\widetilde{x}|^2+1}}\frac{1}{(|\widetilde{x}|^2+1)^{\frac{3}{2}}}\bigg|\\&\leqslant\frac{2}{|\widetilde{x}|^2+1}+2(\psi(\widetilde{x}))^{1/2}\frac{1}{|\widetilde{x}|^2+1}+2(\psi(\widetilde{x}))^{1/2}\frac{1}{|\widetilde{x}|^2+1}\\&\leqslant\frac{6\psi(\widetilde{x})}{|\widetilde{x}|^2+1}.\end{aligned}$$

Based on these estimates, we have for $t\in[t_1,T]$,

$$|\nabla\Psi(t,\widetilde{x})|\leqslant(C_1(T-t)+A)\Psi(t,\widetilde{x})|\nabla\psi(\widetilde{x})|\leqslant 4A\Psi(t,\widetilde{x})\psi(\widetilde{x}),$$

and

$$|D^2\Psi(t,\widetilde{x})|\leqslant 4A(4A+3)\Psi(t,\widetilde{x})\psi(\widetilde{x}).$$

Therefore, for some constant C, independent of C_1,

$$\begin{aligned}&-\frac{\partial\Psi}{\partial t}(t,\widetilde{x})-\mathcal{L}\Psi(t,\widetilde{x})-c\Psi(t,\widetilde{x})-c|\nabla\Psi(t,\widetilde{x})|\\&\quad\geqslant\Psi(t,\widetilde{x})[C_1\psi(\widetilde{x})-AC\psi(\widetilde{x})-A^2C\psi(\widetilde{x})-c-cAC\psi(\widetilde{x})].\end{aligned}$$

Since $\psi(\widetilde{x}) \geqslant 1$, we can choose the constant C_1 large enough such that the right-hand side of the inequality is positive. □

Proof of Lemma 5.3. Suppose the conclusion of the lemma is not true. Then there exist a positive constant ε and a subsequence of m (still denoted by m), a sequence $\{t_m\}$ contained in $[0,T]$ and a sequence $\{x_m\}$ contained in some compact subset of $\mathbb{R}^n$ such that

$$\inf_m \mathrm{E}\Big[\sup_{0\leqslant s\leqslant T}|X^{t_m,x_m,m}(s)-X^{t_m,x_m}(s)|^q\Big]\geqslant \varepsilon.$$

Without loss of generality, we may assume that $\{t_m\}$ converges to t in $[0,T]$ and $\{x_m\}$ converges to x in $\mathbb{R}^n$.

First, since b_0 and b_m are uniformly bounded, we have that there exists a constant C depending on $(T,\mathcal{K})$ such that for $0\leqslant r_1<r_2\leqslant T$,

$$\sup_{0\leqslant t\leqslant T}\sup_{x\in\mathcal{K}}\mathrm{E}\Big[\sup_{r_1\leqslant u_1,u_2\leqslant r_2}|X^{t,x}(u_2)-X^{t,x}(u_1)|^4\Big]\leqslant C|r_2-r_1|^2,$$

and

$$\sup_m\sup_{0\leqslant t\leqslant T}\sup_{x\in\mathcal{K}}\mathrm{E}\Big[\sup_{r_1\leqslant u_1,u_2\leqslant r_2}|X^{t,x,m}(u_2)-X^{t,x,m}(u_1)|^4\Big]\leqslant C|r_2-r_1|^2.$$

Then the family of the processes $\{X^{t_m,x_m}(s),X^{t_m,x_m,m}(s),W(s)\}_{m=1}^{+\infty}$ is tight (see [17, Theorems 1.4.2 and 1.4.3]). Therefore, there exist some probability space $(\hat{\Omega},\hat{\mathcal{F}},\hat{\mathbb{P}})$ and a sequence of continuous stochastic processes $\{\hat{X}_m(s),\hat{Y}_m(s),\hat{W}_m(s)\}_{m=0}^{+\infty}$ on it which enjoy the following properties:

(i) The probability law of $\{\hat{X}_m,\hat{Y}_m,\hat{W}_m\}$ coincides with the law of $\{X^{t_m,x_m},X^{t_m,x_m,m},W\}$ for each $m=1,2,\ldots$

(ii) There exists a subsequence $(m_j)_{j\geqslant 1}$ such that $\{\hat{X}_{m_j},\hat{Y}_{m_j},\hat{W}_{m_j}\}$ converges to $\{\hat{X}_0,\hat{Y}_0,\hat{W}_0\}$ uniformly on every finite time interval a.s.

Without loss of generality, we write $t_{m_j}=t_m$, $x_{m_j}=x_m$, $\hat{X}_{m_j}=\hat{X}_m$, $\hat{Y}_{m_j}=\hat{Y}_m$ and $\hat{W}_{m_j}=\hat{W}_m$. By virtue of uniformly integrability, we have

$$\begin{aligned}\varepsilon&\leqslant\liminf_{m\to+\infty}\mathrm{E}\Big[\sup_{0\leqslant s\leqslant T}|X^{t_m,x_m,m}(s)-X^{t_m,x_m}(s)|^q\Big]\\&=\liminf_{m\to+\infty}\hat{\mathrm{E}}\Big[\sup_{0\leqslant s\leqslant T}|\hat{X}_m(s)-\hat{Y}_m(s)|^q\Big]\\&=\hat{\mathrm{E}}\Big[\sup_{0\leqslant s\leqslant T}|\hat{X}_0(s)-\hat{Y}_0(s)|^q\Big].\end{aligned}\tag{A.1}$$

On the other hand, because of the coincidence (i) of probability law, we have for $m=1,2,\ldots,$

$$\hat{X}_m(s)=x_m+\int_{t_m}^{s\vee t_m}b_0(\hat{X}_m(u))du+\hat{W}_m(s\vee t_m)-\hat{W}_m(t_m),$$

and

$$\hat{Y}_m(s)=x_m+\int_{t_m}^{s\vee t_m}b_m(\hat{Y}_m(u))du+\hat{W}_m(s\vee t_m)-\hat{W}_m(t_m).$$

Now we are going to take the limit $m\to+\infty$ in the above two equations. First, we deal with the drift term. If $s=t$, clearly, $\int_t^{s\vee t}b_0(\hat{X}_0(u))du=0$. Since

$$\lim_{m\to+\infty}\Big|\int_{t_m}^{s\vee t_m}b_0(\hat{X}_m(u))du\Big|\leqslant C\lim_{m\to+\infty}|t\vee t_m-t_m|=0,$$

we thus have in this case

$$\lim_{m\to+\infty}\hat{\mathrm{E}}\Big[\Big|\int_{t_m}^{s\vee t_m}b_0(\hat{X}_m(u))du-\int_t^{s\vee t}b_0(\hat{X}_0(u))du\Big|\Big]=0.$$

If $s<t$, then $\int_t^{s\vee t} b_0(\hat{X}_0(u))du=0$. Since $\lim_{m\to+\infty} t_m=t$, there exists a sufficiently large $M\in\mathbb{N}$ such that for every $m\geqslant M$, $s\leqslant t_m$. Thus, $\int_{t_m}^{s\vee t_m} b_0(\hat{X}_m(u))du=0$. Therefore,

$$\lim_{m\to+\infty}\hat{\mathrm{E}}\bigg[\bigg|\int_{t_m}^{s\vee t_m} b_0(\hat{X}_m(u))du-\int_t^{s\vee t} b_0(\hat{X}_0(u))du\bigg|\bigg]=0.$$

If $s>t$, from $\lim_{m\to+\infty} t_m=t$, we know that there exists a sufficiently large $M\in\mathbb{N}$ such that for every $m\geqslant M$, $s\geqslant t_m$. Fixing m_0, we have

$$\begin{aligned}
&\hat{\mathrm{E}}\bigg[\bigg|\int_{t_m}^{s} b_0(\hat{X}_m(u))du-\int_t^{s} b_0(\hat{X}_0(u))du\bigg|\bigg]\\
&\quad=\hat{\mathrm{E}}\bigg[\bigg|\int_0^s[\mathbf{1}_{[t_m,s]}(u)b_0(\hat{X}_m(u))-\mathbf{1}_{[t,s]}(u)b_0(\hat{X}_0(u))]du\bigg|\bigg]\\
&\quad\leqslant\hat{\mathrm{E}}\bigg[\bigg|\int_0^s[\mathbf{1}_{[t_m,s]}(u)b_0(\hat{X}_m(u))-\mathbf{1}_{[t_m,s]}(u)b_{m_0}(\hat{X}_m(u))]du\bigg|\bigg]\\
&\qquad+\hat{\mathrm{E}}\bigg[\bigg|\int_0^s[\mathbf{1}_{[t_m,s]}(u)b_{m_0}(\hat{X}_m(u))-\mathbf{1}_{[t_m,s]}(u)b_{m_0}(\hat{X}_0(u))]du\bigg|\bigg]\\
&\qquad+\hat{\mathrm{E}}\bigg[\bigg|\int_0^s[\mathbf{1}_{[t_m,s]}(u)b_{m_0}(\hat{X}_0(u))-\mathbf{1}_{[t,s]}(u)b_{m_0}(\hat{X}_0(u))]du\bigg|\bigg]\\
&\qquad+\hat{\mathrm{E}}\bigg[\bigg|\int_0^s[\mathbf{1}_{[t,s]}(u)b_{m_0}(\hat{X}_0(u))-\mathbf{1}_{[t,s]}(u)b_0(\hat{X}_0(u))]du\bigg|\bigg]\\
&\quad\leqslant\hat{\mathrm{E}}\bigg[\int_{t_m}^s|b_0(\hat{X}_m(u))-b_{m_0}(\hat{X}_m(u))|du\bigg]+\hat{\mathrm{E}}\bigg[\int_0^s|b_{m_0}(\hat{X}_m(u))-b_{m_0}(\hat{X}_0(u))|du\bigg]\\
&\qquad+\hat{\mathrm{E}}\bigg[\int_0^s|b_{m_0}(\hat{X}_0(u))||\mathbf{1}_{[t_m,s]}(u)-\mathbf{1}_{[t,s]}(u)|du\bigg]+\hat{\mathrm{E}}\bigg[\int_t^s|b_{m_0}(\hat{X}_0(u))-b_0(\hat{X}_0(u))|du\bigg]\\
&\quad=:I_1+I_2+I_3+I_4.
\end{aligned}$$

Let $w(x)$ be a continuous function defined on $\mathbb{R}^n$ such that $w(0)=1$ and $w(x)=0$ for $|x|^2\geqslant 1$. Then for $R>0$, by [24, Theorem II.2.4], we have

$$\begin{aligned}
I_1&\leqslant C\hat{\mathrm{E}}\bigg[\int_{t_m}^s\bigg[1-w\bigg(\frac{\hat{X}_m(u)}{R}\bigg)\bigg]du\bigg]+\hat{\mathrm{E}}\bigg[\int_{t_m}^s w\bigg(\frac{\hat{X}_m(u)}{R}\bigg)|b_0(\hat{X}_m(u))-b_{m_0}(\hat{X}_m(u))|du\bigg]\\
&\leqslant C\hat{\mathrm{E}}\bigg[\int_{t_m}^s\bigg[1-w\bigg(\frac{\hat{X}_m(u)}{R}\bigg)\bigg]du\bigg]+C\bigg(\int_{B(0,R)}|b_0(y)-b_{m_0}(y)|^{n+1}dy\bigg)^{\frac{1}{n+1}},
\end{aligned}$$

where, $B(0,R)$ is the ball with center 0 and radius R in $\mathbb{R}^n$. Therefore,

$$\lim_{m\to+\infty} I_1\leqslant C\hat{\mathrm{E}}\bigg[\int_0^s\bigg[1-w\bigg(\frac{\hat{X}_0(u)}{R}\bigg)\bigg]du\bigg]+C\bigg(\int_{B(0,R)}|b_0(y)-b_{m_0}(y)|^{n+1}dy\bigg)^{\frac{1}{n+1}}.$$

First letting m_0 tend to $+\infty$ and then R go to $+\infty$, we have $\lim_{m\to+\infty} I_1=0$. Similarly, we have $\lim_{m\to+\infty} I_4=0$. From the convergence of $\hat{X}_m$ to $\hat{X}_0$, the continuity of b_{m_0} and dominated convergence theorem, we have $\lim_{m\to+\infty} I_2=0$. Finally,

$$\lim_{m\to+\infty} I_3\leqslant\lim_{m\to+\infty} C|t_m-t|=0.$$

Therefore, if $s>t$, we also have

$$\lim_{m\to+\infty}\hat{\mathrm{E}}\bigg[\bigg|\int_{t_m}^{s\vee t_m} b_0(\hat{X}_m(u))du-\int_t^{s\vee t} b_0(\hat{X}_0(u))du\bigg|\bigg]=0.$$

Similarly, we have

$$\lim_{m\to+\infty}\hat{\mathrm{E}}\bigg[\bigg|\int_{t_m}^{s\vee t_m} b_m(\hat{Y}_m(u))du-\int_t^{s\vee t} b_0(\hat{Y}_0(u))du\bigg|\bigg]=0.$$

On the other hand, it follows from [12, Lemma 5.2] that

$$\hat{W}_m(s \vee t_m) - \hat{W}_m(t_m) \to \hat{W}_0(s \vee t) - \hat{W}_0(t)$$

in probability. Therefore, we have that both $\hat{X}_0$ and $\hat{Y}_0$ are the solutions of

$$X(s) = x + \int_t^{s \vee t} b_0(X(u))du + \hat{W}_0(s \vee t) - \hat{W}_0(t).$$

From the pathwise uniqueness of solutions for the above SDEs, we must have $\hat{X}_0(s) = \hat{Y}_0(s)$ almost surely for $0 \leqslant s \leqslant T$. This contradicts (A.1) and the proof is completed. □