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On geometric structure of generalized projections in C^*-algebras

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Abstract Let $\mathcal{H}$ be a Hilbert space and $\mathcal{A} \subseteq B(\mathcal{H})$ be a C^*-subalgebra. This paper is devoted to studying the set $\mathcal{GP}$ of generalized projections in $\mathcal{A}$ from a differential geometric point of view, and mainly focuses on geodesic curves. We prove that the space $\mathcal{GP}$ is a C^∞ Banach submanifold of $\mathcal{A}$, and a homogeneous reductive space under the action of Banach Lie group $U_{\mathcal{A}}$ of $\mathcal{A}$. Moreover, we compute the geodesics of $\mathcal{GP}$ in a standard fashion, and prove that any generalized projection in a prescribed neighbourhood of $p \in \mathcal{GP}$ can be joined with p by a unique geodesic curve in $\mathcal{GP}$.

Keywords generalized projections, Banach manifold, geodesics

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1 Introduction

The problem of unitary equivalence of operators on a Hilbert space has been studied in past decades. Kato [14,15] proved that two orthogonal projections p and q are unitarily equivalent whenever $\|p-q\| < 1$. Halmos and Mclaughlin [13] considered the unitary equivalence of partial isometries and proved that if x and y are partial isometries with $\|x - y\| < 1$, then there exist the unitary operators u and v such that $y = uxv^*$. Lately, Corach et al. [10], and Porta and Recht [18, 19] paid attention to the study of the differential geometry of the set of all the orthogonal projections (i.e., Grassmann manifold), and particularly, they are concerned with the problems of existence and uniqueness of geodesics joining two given projections (see also [2]). Andruchow et al. [5], and Andruchow and Corach [4] proved that the set of all the partial isometries is a C^∞ differential manifold, and studied the differential geometry of the set. The set $\mathcal{GP}$ (which we define below) of generalized projections contains the set of orthogonal projections, and is included in the set of partial isometries. Andruchow et al. [6] studied the local smooth structure of the set of generalized projections. In this paper, we proceed with the study of the differential geometry of the set $\mathcal{GP}$. In order to describe the results, we suppose $\mathcal{H}$ is a Hilbert space, and denote by $B(\mathcal{H})$ the C^*-algebra of all the bounded linear operators on $\mathcal{H}$. Fix a unital C^*-subalgebra $\mathcal{A}$ of $B(\mathcal{H})$, and $\mathcal{A}_{ah}$ is the real Banach space of anti-self adjoint elements of $\mathcal{A}$. Denote by $G_{\mathcal{A}}$ the group of invertible elements

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of $\mathcal{A}$, $U_{\mathcal{A}}$ the subgroup of unitary elements of $\mathcal{A}$ and $\mathcal{P}$ the set of all self-adjoint projections of $\mathcal{A}$, i.e., $\mathcal{P}=\{p\in\mathcal{A}:p^2=p=p^*\}$. The set $\mathcal{I}$ of partial isometries of $\mathcal{A}$ is defined by $\mathcal{I}=\{x\in\mathcal{A}:xx^*\in\mathcal{P}\}$, and the set $\mathcal{GP}$ of generalized projections of $\mathcal{A}$ is given by $\mathcal{GP}=\{p\in\mathcal{A}:p^2=p^*\}$. The differential geometry of $\mathcal{P}$ is well-known by now, and we often use this knowledge in order to obtain results on $\mathcal{GP}$. The main link between $\mathcal{GP}$ and $\mathcal{P}$ is provided by functional calculus.

Let us describe now the content of this paper. In Section 2, we prove that the action of Banach-Lie group $U_{\mathcal{A}}$ over $\mathcal{GP}$ is locally transitive, which makes $\mathcal{GP}$ a C^∞ submanifold of $\mathcal{A}$ in combination with the fact that the unitary orbit U_p is a C^∞ submanifold of $\mathcal{A}$. Hence, $\mathcal{GP}$ is a homogeneous space. In Section 3, we give the horizontal lifting differential equations, which enable one to perform the parallel transport of tangent spaces of $\mathcal{GP}$, and therefore a linear connection. We show that the horizontal liftings satisfy a linear differential equation, which implies that geodesics of this connection in $\mathcal{GP}$ exist for all $t\in\mathbb{R}$. In Section 4, we consider the problem of geodesics joining two given endpoints in $\mathcal{GP}$.

2 The differential structure of $\mathcal{GP}$

Consider the action $L:U_{\mathcal{A}}\times\mathcal{GP}\to\mathcal{GP}$, $L(u,p)=L_u(p)=upu^*$, $u\in U_{\mathcal{A}}$, $p\in\mathcal{GP}$. In fact, for every $p\in\mathcal{GP}$, p^3 is the orthogonal projection onto $R(p)$. The unitary orbit of p is

$$U_p=\{L_u(p):u\in U_{\mathcal{A}}\}=\{upu^*:u\in U_{\mathcal{A}}\}.$$

Note that $U_p\subseteq\mathcal{GP}$, i.e., $L_u(\mathcal{GP})\subseteq\mathcal{GP}$, for all $u\in U_{\mathcal{A}}$.

This action is locally transitive, i.e., there is an $r>0$ such that for any $p,q\in\mathcal{GP}$ satisfying the distance between p and q (measured with the norm of $\mathcal{A}$) is less than r, we can find a unitary operator u such that $L_u(p)=q$. Next, we shall give a proof and obtain the radius r is 1.

Generalized projections are normal elements, and thus have norm 1. Suppose $p,q\in\mathcal{GP}$. We know that

$$\sigma(p),\ \sigma(q)\subseteq\{0,1,\omega,\omega^2\}=\{\mu_0,\mu_1,\mu_2,\mu_3\},\quad\text{where}\quad\omega=\mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}.$$

Then

$$p=\sum_{\lambda\in\sigma(p)}\lambda E(\lambda)\quad\text{and}\quad q=\sum_{\lambda\in\sigma(q)}\lambda F(\lambda),$$

respectively (see [12, Theorem 2]), where $E(\lambda)$ (resp. $F(\lambda)$) is the spectral projection of p (resp. q) associated with the point $\lambda\in\sigma(p)$ (resp. $\sigma(q)$).

Theorem 2.1. *Suppose that p, $q\in\mathcal{GP}$ and $\|p-q\|<1$. Then there exists a unitary operator $u\in\mathcal{A}$ such that $q=upu^*$.*

Proof. Let

$$p=\sum_{\lambda\in\sigma(p)}\lambda E(\lambda)\quad\text{and}\quad q=\sum_{\lambda\in\sigma(q)}\lambda F(\lambda).$$

Note that

$$\|p^2-q^2\|=\|p^*-q^*\|=\|p-q\|<1.\tag{2.1}$$

By the Krein-Krasnoselski-Milman formula (see, for example, [1]),

$$\|p^3-q^3\|=\max\{\|p^3(1-q^3)\|,\|q^3(1-p^3)\|\}\leqslant 1.\tag{2.2}$$

For any $j=0,1,2,3$, put

$$f_j(z)=\prod_{i\neq j}\frac{z-\mu_i}{\mu_j-\mu_i},\quad\forall\,z\in\mathbb{C}.\tag{2.3}$$

Then f_j is a polynomial of degree 3 for $0\leqslant j\leqslant 3$ such that $f_j(\mu_j)=1$ and $f_j(\mu_i)=0$ for any $i\neq j$. In fact, $f_0(z)=1-z^3$ and

$$f_j(z)=\frac{1}{3}(a_{3j}z^3+a_{2j}z^2+a_{1j}z),\tag{2.4}$$

where $a_{ij} \in \mathbb{T}$, the unit circle of $\mathbb{C}$, for all $i, j = 1, 2, 3$. If $\mu_j \in \sigma(p)$ (resp. $\sigma(q)$), then $f_j(p)$ (resp. $f_j(q)$) is the spectral projection $E(\mu_j)$ (resp. $F(\mu_j)$) of p (resp. q) associated with $\{\mu_j\}$. Otherwise $f_j(p) = 0$ (resp. $f_j(q) = 0$).

By the proof of [5, Proposition 3.1], we know that $p^3 = p^*p$ and $q^3 = q^*q$ are unitarily equivalent, which implies that there exists a unitary operator $u_0 \in \mathcal{A}$ such that

$$F(0) = f_0(q) = u_0 f_0(p) u_0^* = u_0 E(0) u_0^*.$$

Moreover, it now follows from (2.1), (2.2) and (2.4) that

$$\|f_j(p) - f_j(q)\| \leqslant \frac{1}{3}(\|p^3 - q^3\| + \|p^2 - q^2\| + \|p - q\|) < 1 \tag{2.5}$$

for $j = 1, 2, 3$, which means that $\mu_j \in \sigma(p)$ if and only if $\mu_j \in \sigma(q)$ and

$$\|E(\mu_j) - F(\mu_j)\| = \|f_j(p) - f_j(q)\| < 1 \quad \text{if} \quad \mu_j \in \sigma(p)$$

for $j = 1, 2, 3$. It follows from [2, Lemma 1.1] that $E(\mu_j)$ and $F(\mu_j)$ are unitarily equivalent, i.e., there exists a unitary operator $u_j \in \mathcal{A}$ such that

$$F(\mu_j) = u_j E(\mu_j) u_j^*$$

for $j = 1, 2, 3$ if $\mu_j \in \sigma(p)$.

Then

$$u = F(\mu_0)u_0E(\mu_0) + F(\mu_1)u_1E(\mu_1) + F(\mu_2)u_2E(\mu_2) + F(\mu_3)u_3E(\mu_3)$$

is a unitary operator such that $q = upu^*$. □

Remark 2.2. (1) We have that

$$\|p^3 - q^3\| \leqslant \|p - q\|. \tag{2.6}$$

Indeed, we know that

$$\|p^3(1 - q^3)\| \leqslant \|p(1 - q^3)\| = \|p - pq^3 - q + qq^3\| = \|(p - q)(1 - q^3)\| \leqslant \|p - q\|$$

and

$$\|q^3(1 - p^3)\| \leqslant \|q(1 - p^3)\| = \|q - qp^3 - p + pp^3\| = \|(p - q)(1 - p^3)\| \leqslant \|p - q\|.$$

Then $\|p^3 - q^3\| \leqslant \|p - q\|$ from (2.2).

(2) From (2.6) and the proof of Theorem 2.1, we get that

$$\|f_i(p) - f_i(q)\| \leqslant \|p - q\|, \quad i = 0, 1, 2, 3.$$

For $p \in \mathcal{GP}$, let $\pi_p : U_{\mathcal{A}} \to U_p$, $\pi_p(u) = L_u(p) = upu^*$. Next, we consider a geometric-topological problem. A continuous local cross section for π_p is a pair $(s_p, \mathfrak{B})$ such that $\mathfrak{B}$ is a relatively open subset of U_p that contains p and $s_p : \mathfrak{B} \to U_{\mathcal{A}}$ is a norm continuous map such that $s_p(p) = 1$ and $\pi_p \circ s_p = 1_{\mathfrak{B}}$, i.e., $s_p(q)ps_p(q)^* = q$ for each q in $\mathfrak{B}$ (see [8]).

Corollary 2.3. *Suppose that $p \in \mathcal{GP}$. Then the map $\pi_p : U_{\mathcal{A}} \to U_p$ admits continuous local cross sections.*

Proof. It is apparent that π_p is continuous in norm topology and surjective. Put

$$\omega_p = \{q \in \mathcal{GP} : \|p - q\| < 1\}.$$

We define a continuous map $s_p : \omega_p \to U_{\mathcal{A}}$, which is a local cross section of π_p. By Theorem 2.1, we know that $f_i(p)$ and $f_i(q)$ are unitarily equivalent for all $0 \leqslant i \leqslant 3$. In fact, it follows from the proof of [5, Proposition 3.1] that the unitary operator $v_p^0(q)$ satisfying $f_0(q) = v_p^0(q)f_0(p)v_p^0(q)^*$ may be chosen as

$$v_p^0(q) = f_0(q)[f_0(p)f_0(q)f_0(p)]^{-\frac{1}{2}} + (1 - f_0(q))[(1 - f_0(p))(1 - f_0(q))(1 - f_0(p))]^{-\frac{1}{2}},$$

where the inverse of $f_0(p)f_0(q)f_0(p)$ (resp. $(1-f_0(p))(1-f_0(q))(1-f_0(p))$) is taken in $f_0(p)\mathcal{A}f_0(p)$ (resp. $(1-f_0(p))\mathcal{A}(1-f_0(p))$). Put

$$s_p^0(q) = f_0(q)v_p^0(q)f_0(p).$$

Then $s_p^0(q)$ is a partial isometry with initial space $f_0(p)$ and final space $f_0(q)$ for any $q \in \omega_p$, and s_p^0 is continuous on ω_p. On the other hand, it is also known that a unitary operator $v_p^i(q)$ such that $f_i(q) = v_p^i(q)f_i(p)v_p^i(q)^*$ for $1 \leqslant i \leqslant 3$ is given by

$$v_p^i(q) = [f_i(q)f_i(p) + (1-f_i(q))(1-f_i(p))]|f_i(q)f_i(p) + (1-f_i(q))(1-f_i(p))|^{-1}$$

(see [9, (2.6)]). We set

$$s_p^i(q) = f_i(q)v_p^i(q)f_i(p)$$

again for $1 \leqslant i \leqslant 3$. Then $s_p^i(q)$ is a partial isometry with initial space $f_i(p)$ and final space $f_i(q)$ for any $q \in \omega_p$. Putting

$$s_p(q) = s_p^0(q) + s_p^1(q) + s_p^2(q) + s_p^3(q),$$

we get $s_p(q) \in U_{\mathcal{A}}$ and s_p is continuous. Moreover,

$$(\pi_p \circ s_p)(q) = \pi_p(s_p(q)) = s_p(q)ps_p(q)^* = q.$$

One may obtain local cross sections at other points p_0 of U_p by translating this one. In fact, if $p_0 = u_0pu_0^*$ for some $u_0 \in U_{\mathcal{A}}$, we may consider $s_{p_0} = l_{u_0} \circ s_p \circ L_{u_0^*}$, where $l_{u_0} : U_{\mathcal{A}} \to U_{\mathcal{A}}$ is the left multiplication by u_0, i.e.,

$$s_{p_0}(q) = u_0s_p(u_0^*qu_0)$$

defined on the open set

$$\{q \in \mathcal{GP} : \|q - p_0\| = \|u_0^*qu_0 - p\| < 1\}.$$

It follows that π_p has a local cross section defined on a neighbourhood of any point in U_p. □

Corollary 2.4. *Suppose that $p \in \mathcal{GP}$ and the isotropy group of p by the action of $U_{\mathcal{A}}$ is defined by $I_p = \{u \in U_{\mathcal{A}} : upu^* = p\}$. Then the metric space U_p is homeomorphic to the quotient space $U_{\mathcal{A}}/I_p$, where the quotient topology is considered.*

We next recall some definitions and results on Banach-Lie groups (see [16] and the references therein).

Definition 2.5. Given a Banach-Lie group G, a subgroup H of G is regular if it is a Banach-Lie group and if $(TH)_1$ is a closed and complemented subspace of $(TG)_1$, where $(TH)_1$ (resp. $(TG)_1$) is the tangent space of H (resp. G) at 1.

Theorem 2.6. *Let G be a Banach-Lie group, and $H \subseteq G$ be a regular subgroup. Then*

(1) *G/H has a unique structure of differential manifold such that $G \to G/H$ is a submersion*;

(2) *$G \to G/H$ is a principle bundle with structure group H*;

(3) *the action $G \times G/H \to G/H$ is smooth.*

From [20, Theorem 8.91], we know that the quotient manifold G/H in Theorem 2.6 is a Banach manifold. In order to provide $U_p \simeq U_{\mathcal{A}}/I_p$ with a Banach manifold structure using Theorem 2.6, we need to prove that I_p is a regular subgroup of $U_{\mathcal{A}}$.

Proposition 2.7. *Let $p \in \mathcal{GP}$. Then I_p is a closed regular subgroup of $U_{\mathcal{A}}$.*

Proof. From the definition of I_p, it is clear that I_p is a closed Banach-Lie subgroup of $U_{\mathcal{A}}$. We claim that $(TI_p)_1 = \{x \in \mathcal{A}_{ah} : xp = px\}$. Indeed, suppose that $u(t)$, $t \in [0,1]$ is a smooth curve in $U_{\mathcal{A}}$ such that

$$u(t)pu(t)^* = p, \quad u(0) = 1, \quad \dot{u(0)} = x.$$

Differentiating $u(t)pu(t)^* = p$, we arrive at

$$\dot{u(t)}pu(t)^* + u(t)p\dot{u(t)}^* = 0.$$

Then the derivative of $u(t)pu(t)^* = p$ at $t = 0$ is $xp + px^* = 0$. It is well known that the tangent space of $U_{\mathcal{A}}$ at 1 can be identified with $\mathcal{A}_{ah}$. Hence

$$(TI_p)_1 \subseteq \{x \in \mathcal{A}_{ah} : xp = px\}.$$

Conversely, suppose $x \in \mathcal{A}_{ah},\ xp = px$. If $u(t) = \mathrm{e}^{tx}$, we get that $p = u(t)pu(t)^*$ with $u(0) = 1$, $\dot{u(0)} = x$. This completes our claim.

Note that the elements of $(TI_p)_1$ are diagonal operators in $\mathcal{A}_{ah}$ with respect to the spectral decomposition of p, which follows from the above claim. Therefore, $(TI_p)_1$ is a closed and complemented subspace of $\mathcal{A}_{ah}$. □

In fact, the result that unitary orbit U_p is a C^∞ Banach manifold of $B(\mathcal{H})$ has been given in [6]. Indeed, Corollary 2.4, Theorem 2.6 and Proposition 2.7 tell us that the unitary orbit U_p of generalized projection p has a Banach manifold structure. Hence, we get the following result from Theorem 2.1.

Corollary 2.8. *Let $p \in \mathcal{GP}$. Then the set $\mathcal{GP}$ of generalized projections is a Banach submanifold of $\mathcal{A}$.*

Proof. Since generalized projections at distance less than 1 are unitarily equivalent, the set of generalized projections is a discrete union of unitary orbits. Therefore, the whole set of generalized projections is a Banach submanifold, and a homogeneous space. □

3 The transport equation

The following facts on parallel transport, horizontal liftings and geodesics follow from the general theory of homogeneous reductive spaces. We state them here, in the case of the space $\mathcal{GP}$ in order to make our paper more readable.

In fact, the manifold $\mathcal{GP}$ is a homogeneous reductive space. Indeed, for any $p \in \mathcal{GP}$, one has that

$$(T\mathcal{GP})_p = \{xp - px : x \in \mathcal{A}_{ah}\}.$$

Let $\delta_1^p = d(\pi_p)_1$, i.e.,

$$\delta_1^p : \mathcal{A}_{ah} \to (T\mathcal{GP})_p, \quad \delta_1^p(x) = xp - px,$$

which implies that the Lie algebra $(TI_p)_1 = V_1^p$ is equal to the kernel of δ_1^p. Set

$$H_1^p = \{x - f_1(p)xf_1(p) - f_2(p)xf_2(p) - f_3(p)xf_3(p) - f_0(p)xf_0(p) : x \in \mathcal{A}_{ah}\}.$$

It is easy to check that H_1^p satisfies the following two propositions:

(1) $H_1^p \oplus V_1^p = \mathcal{A}_{ah}$;

(2) $ad(u)(H_1^p) = H_1^p$, $\forall\, u \in I_p$, where $ad(u)(x) = uxu^*$.

It follows that the manifold $\mathcal{GP}$ is a homogeneous reductive space.

In order to define connection on $\mathcal{GP}$, we consider the linear map

$$\Sigma_1^p : (T\mathcal{GP})_p \to \mathcal{A}_{ah}, \quad \Sigma_1^p(y) = \varepsilon_1(y) - \varepsilon_1(y)^*,$$

where

$$\begin{aligned}\varepsilon_1(y) = {} & \frac{1}{\omega - 1} f_1(p)yf_2(p) + \frac{1}{\omega^2 - 1} f_1(p)yf_3(p) - f_1(p)yf_0(p) \\ & + \frac{1}{\omega^2 - \omega} f_2(p)yf_3(p) - \omega^2 f_2(p)yf_0(p) - \omega f_3(p)yf_0(p).\end{aligned}$$

We conclude that $\delta_1^p \circ \Sigma_1^p \circ \delta_1^p = \delta_1^p$. Indeed, if $x \in \mathcal{A}_{ah}$, then $\delta_1^p(x) = xp - px$ and

$$(\delta_1^p \circ \Sigma_1^p \circ \delta_1^p)(x) = \delta_1^p(\Sigma_1^p(xp - px)) = xp - px = \delta_1^p(x).$$

Note that, the range of $\Sigma_1^p \circ \delta_1^p$ is the linear subspace H_1^p of $\mathcal{A}_{ah}$, and then the map $\Sigma_1^p \circ \delta_1^p$, whose kernel is equal to the kernel of δ_1^p, is an idempotent in the Banach algebra of real linear bounded operators on the space $\mathcal{A}_{ah}$.

Define the horizontal space at 1 as H_1^p and the vertical space at 1 as V_1^p. From the above, we get δ_1^p is a linear isomorphism from the horizontal space H_1^p to the tangent space $(T\mathcal{GP})_p$, which provides a way to introduce a connection in this principal bundle. For every $u \in U_{\mathcal{A}}$, set $H_u^p = uH_1^p$ as the horizontal space at u and $V_u^p = uV_1^p$ the vertical space at u. Thus $(TU_{\mathcal{A}})_u = H_u^p \oplus V_u^p$.

To obtain the parallel transport of tangent spaces of $\mathcal{GP}$, we need some results.

Remark 3.1. Given $u \in U_{\mathcal{A}}$, the differential map of π_p at u is the map

$$\delta_u^p : (TU_{\mathcal{A}})_u \to (T\mathcal{GP})_q, \quad \delta_u^p(x) = xpu^* + upx^*,$$

where $q = upu^*$. Naturally, we define a linear map Σ_u^p on $(T\mathcal{GP})_q$, which is given by

$$\Sigma_u^p : (T\mathcal{GP})_q \to H_u^p, \quad \Sigma_u^p(y) = (\varepsilon_u(y) - \varepsilon_u^*(y))u,$$

where

$$\begin{aligned}\varepsilon_u(y) = {}& \frac{1}{\omega-1} uf_1(p)u^*yuf_2(p)u^* + \frac{1}{\omega^2-1} uf_1(p)u^*yuf_3(p)u^* \\ & - uf_1(p)u^*yuf_0(p)u^* + \frac{1}{\omega^2-\omega} uf_2(p)u^*yuf_3(p)u^* \\ & - \omega^2 uf_2(p)u^*yuf_0(p)u^* - \omega uf_3(p)u^*yuf_0(p)u^*.\end{aligned}$$

Note that V_u^p is the kernel of δ_u^p. Therefore,

$$\delta_u^p|_{H_u^p} : H_u^p \to (T\mathcal{GP})_q$$

is a linear isomorphism. □

Given a smooth curve $\gamma \subseteq \mathcal{GP}$, we say a smooth curve $\Gamma \subseteq U_{\mathcal{A}}$ is a lifting of γ if $\gamma = \pi_p(\Gamma) = \Gamma p \Gamma^*$. Moreover, if $\dot{\Gamma} \in H_{\Gamma}^p$, then we say that Γ is a horizontal lifting of γ.

The reductive structure on $\mathcal{GP}$ induces a linear connection in $\mathcal{GP}$. Next, we shall compute the horizontal lifting differential equation of this connection.

Remark 3.2. Let $q = upu^*$ and a smooth curve $\gamma(t) \subseteq \mathcal{GP}, t \in [0,1]$, with $\gamma(0) = q$. We find a smooth curve $\Gamma(t) \subseteq U_{\mathcal{A}}$ such that Γ is a horizontal lifting of γ, i.e.,

$$\pi_p(\Gamma(t)) = \Gamma(t)p\Gamma(t)^* = \gamma(t) \tag{3.1}$$

and

$$\dot{\Gamma(t)} \in H_{\Gamma(t)}^p, \quad t \in [0,1]. \tag{3.2}$$

In fact, we get $\delta_{\Gamma(t)}^p(\dot{\Gamma(t)}) = \dot{\gamma(t)}$ by differentiating (3.1), which implies that $\Gamma(t)$ satisfies the differential equation

$$\dot{\Gamma(t)} = \Sigma_{\Gamma(t)}^p(\dot{\gamma(t)}).$$

Omitting the parameter t,

$$\dot{\Gamma} = \Sigma_{\Gamma}^p(\dot{\gamma}) = (\varepsilon_{\Gamma}(\dot{\gamma}) - \varepsilon_{\Gamma}^*(\dot{\gamma}))\Gamma,$$

where

$$\begin{aligned}\varepsilon_{\Gamma}(\dot{\gamma}) = {}& \frac{1}{\omega-1} \Gamma f_1(p)\Gamma^*\dot{\gamma}\Gamma f_2(p)\Gamma^* + \frac{1}{\omega^2-1} \Gamma f_1(p)\Gamma^*\dot{\gamma}\Gamma f_3(p)\Gamma^* \\ & - \Gamma f_1(p)\Gamma^*\dot{\gamma}\Gamma f_0(p)\Gamma^* + \frac{1}{\omega^2-\omega} \Gamma f_2(p)\Gamma^*\dot{\gamma}\Gamma f_3(p)\Gamma^* \\ & - \omega^2 \Gamma f_2(p)\Gamma^*\dot{\gamma}\Gamma f_0(p)\Gamma^* - \omega \Gamma f_3(p)\Gamma^*\dot{\gamma}\Gamma f_0(p)\Gamma^*.\end{aligned}$$

If we suppose that Γ lifts γ, (3.2) may be rewritten in the form

$$\dot{\Gamma} = (\varepsilon_{\Gamma}(\dot{\gamma}) - \varepsilon_{\Gamma}^*(\dot{\gamma}))\Gamma, \tag{3.3}$$

where

$$\varepsilon_\Gamma(\dot\gamma) = \frac{1}{\omega-1} f_1(\gamma)\dot\gamma f_2(\gamma) + \frac{1}{\omega^2-\omega} f_1(\gamma)\dot\gamma f_3(\gamma) - f_1(\gamma)\dot\gamma f_0(\gamma) + \frac{1}{\omega^2-\omega} f_2(\gamma)\dot\gamma f_3(\gamma) - \omega^2 f_2(\gamma)\dot\gamma f_0(\gamma) - \omega f_3(\gamma)\dot\gamma f_0(\gamma).$$

Note that (3.3) is linear, and therefore the existence and uniqueness of its solutions under the initial conditions are guaranteed. In what follows, we shall prove that the solutions of (3.3) lift γ horizontally. To do this, we need the following result (see [17, Theorem 31.A]). If $\dot\Omega = \Sigma\Omega$ is a linear differential equation in $\mathcal{A}$ such that $\Omega(t_0) \in U_{\mathcal{A}}$ and $\Sigma \in \mathcal{A}_{ah}$, then the equation $\dot\Omega = \Sigma\Omega$ has a solution in $U_{\mathcal{A}}$.

The next lemma is proved by a simple computation.

Lemma 3.3. *Let γ be a smooth curve in $\mathcal{GP}$. Then*

$$\varepsilon_\Gamma(\dot\gamma) - \varepsilon_\Gamma^*(\dot\gamma)$$

lies in $\mathcal{A}_{ah}$.

Theorem 3.4. *Let γ be a smooth curve in $\mathcal{GP}$ with $\gamma(0) = p$. Suppose that Γ is the unique solution of*

$$\dot\Gamma = (\varepsilon_\Gamma(\dot\gamma) - \varepsilon_\Gamma^*(\dot\gamma))\Gamma,$$

where

$$\varepsilon_\Gamma(\dot\gamma) = \frac{1}{\omega-1} f_1(\gamma)\dot\gamma f_2(\gamma) + \frac{1}{\omega^2-1} f_1(\gamma)\dot\gamma f_3(\gamma) - f_1(\gamma)\dot\gamma f_0(\gamma) + \frac{1}{\omega^2-\omega} f_2(\gamma)\dot\gamma f_3(\gamma) - \omega^2 f_2(\gamma)\dot\gamma f_0(\gamma) - \omega f_3(\gamma)\dot\gamma f_0(\gamma),$$

with initial condition $\Gamma(0) = 1$. Then Γ is the horizontal lifting of γ in $U_{\mathcal{A}}$ with $\Gamma(0) = 1$.

Proof. Since Γ is the unique solution of (3.3), it follows from Lemma 3.3 that Γ lies in $U_{\mathcal{A}}$. We next prove that Γ lifts γ, i.e., $\Gamma p \Gamma^* = \gamma$, or equivalently $\Gamma^*\gamma\Gamma = p$. Differentiating $\Gamma^*\gamma\Gamma$, we obtain

$$(\dot{\Gamma^*\gamma\Gamma}) = \dot{\Gamma}^*\gamma\Gamma + \Gamma^*\dot\gamma\Gamma + \Gamma^*\gamma\dot\Gamma = \Gamma^*(-\Delta\gamma + \dot\gamma + \gamma\Delta)\Gamma,$$

where $\Delta = \varepsilon_\Gamma(\dot\gamma) - \varepsilon_\Gamma^*(\dot\gamma)$. From $\gamma \subseteq \mathcal{GP}$, we have $\gamma^* = \gamma^2$, which implies

$$\dot\gamma^* = \dot\gamma\gamma + \gamma\dot\gamma.$$

Therefore, it is easy to see that

$$\Delta\gamma - \gamma\Delta = \dot\gamma.$$

Then $(\dot{\Gamma^*\gamma\Gamma}) = 0$ and $\Gamma^*(0)\gamma(0)\Gamma(0) = p$. Consequently, $\Gamma p \Gamma^* = \gamma$.

It remains to prove that Γ is horizontal. Since Γ lifts γ, we can reverse the argument leading to (3.3), and obtain that the equation is equivalent to the condition

$$\dot\Gamma = \Sigma_\Gamma^p(\dot\gamma) \in H_\Gamma^p,$$

i.e., Γ is horizontal. □

In order to compute the geodesic curves in $\mathcal{GP}$, we introduce the transport map

$$T_u^v : (TU_{\mathcal{A}})_u \to (TU_{\mathcal{A}})_v, \quad T_u^v(x) = vu^*x.$$

Note that this map has the following properties:

$$T_u^u = \mathrm{id}, \quad (T_u^v)^{-1} = T_v^u,$$

which enables one to define the parallel transport of tangent vectors along piecewise smooth curves of $\mathcal{GP}$. Suppose $\gamma(t)$ is a smooth curve of $\mathcal{GP}$, $t \in [0,1]$, with $\gamma(0) = p$, and $\Gamma(t) \subseteq U_{\mathcal{A}}$ is the horizontal lifting of $\gamma(t)$ with $\Gamma(0) = 1$. Then

$$\tau_{\gamma(t)} : (T\mathcal{GP})_p \to (T\mathcal{GP})_{\gamma(t)}, \quad \tau_{\gamma(t)}(y) = \delta^p_{\Gamma(t)}(T_1^{\Gamma(t)}(\Sigma^p_1(y))).$$

The covariant derivative of a vector field $X = X_{\gamma(t)}$ that is tangent along a curve $\gamma(t) \subseteq \mathcal{GP}$, $t \in [0,1]$, with $\gamma(0) = p$, is given by

$$\frac{DX}{dt}\bigg|_{t=0} = \frac{d}{dt}(\tau_{\gamma(t)})^{-1}(X_{\gamma(t)})\bigg|_{t=0}.$$

If Γ is the horizontal lifting of γ with $\Gamma(0) = 1$, then

$$(\tau_\gamma)^{-1}(X_\gamma) = \delta^p_1(T^1_\Gamma(\Sigma^p_\Gamma(X_\gamma))).$$

This completes the proof. □

Now, we give the main result of this section.

Theorem 3.5. *Let $y \in (T\mathcal{GP})_p$. Then the unique geodesic $\gamma(t)$, $t \in \mathbb{R}$, of this connection, with $\gamma(0) = p$ and $\dot{\gamma(0)} = y$, is given by*

$$\gamma(t) = \mathrm{e}^{t\Sigma^p_1(y)} p \mathrm{e}^{-t\Sigma^p_1(y)}, \quad t \in \mathbb{R}.$$

Proof. Let γ be the geodesic of $\mathcal{GP}$ satisfying $\gamma(0) = p$ and $\dot{\gamma(0)} = y$. Then γ satisfies

$$\frac{D\dot{\gamma}}{dt} = 0,$$

i.e.,

$$\frac{d}{dt}(\delta^p_1(T^1_\Gamma(\Sigma^p_\Gamma(\dot{\gamma})))) = 0,$$

where $\Gamma \subseteq U_{\mathcal{A}}$ is the horizontal lifting of γ with initial condition $\Gamma(0) = 1$. Recall that

$$\Sigma^p_\Gamma(\dot{\gamma}) = \dot{\Gamma},$$

so

$$0 = \frac{d}{dt}(\delta^p_1(T^1_\Gamma(\dot{\Gamma}))) = \delta^p_1\bigg(\frac{d}{dt}(T^1_\Gamma(\dot{\Gamma}))\bigg),$$

where the derivative is taken in the Banach space H^p_1, on which δ^p_1 is an isomorphism. It follows that

$$\frac{d}{dt}(T^1_\Gamma(\dot{\Gamma})) = 0,$$

which gives that $T^1_\Gamma(\dot{\Gamma})$ is a constant and equals

$$T^1_{\Gamma(0)}(\dot{\Gamma(0)}) = T^1_1(\Sigma^p_1(\dot{\gamma(0)})) = \Sigma^p_1(y).$$

Hence, using the fact that $(T^1_\Gamma)^{-1} = T^\Gamma_1$, we have that

$$\dot{\Gamma} = T^\Gamma_1(\Sigma^p_1(y)).$$

Let $x = \Sigma^p_1(y)$. According to $\dot{\Gamma} = T^\Gamma_1(x)$, we have

$$\dot{\Gamma} = \Gamma x$$

with $\Gamma(0) = 1$, which has the solution

$$\Gamma(t) = \mathrm{e}^{tx}.$$

Then the unique geodesic $\gamma(t)$, $t \in \mathbb{R}$, of this connection, satisfying $\gamma(0) = p$ and $\dot{\gamma(0)} = y$, is the form of

$$\gamma(t) = \mathrm{e}^{t\Sigma^p_1(y)} p \mathrm{e}^{-t\Sigma^p_1(y)}, \quad t \in \mathbb{R}.$$

This completes the proof. □

4 The problem of finding geodesics joining two given points

We endow $\mathcal{GP}$ with the Finsler metric consisting of the usual norm of $\mathcal{A}$ in each tangent space of $\mathcal{GP}$. If $\gamma(t)$, $t \in [a,b]$, is a smooth curve in $\mathcal{GP}$, we measure its length as follows:

$$\text{length}(\gamma) = \int_a^b \|\dot{\gamma(t)}\| dt.$$

From the results of geodesic curves of orthogonal projections, we can get the following proposition.

Proposition 4.1. *Suppose that generalized projections p and q have two spectral points. If $\|p-q\| < 1$, then there exists $x \in H_1^p$ with $\|x\| < \frac{\pi}{2}$ such that*

$$\gamma(t) = \mathrm{e}^{tx} p \mathrm{e}^{-tx}$$

is a minimal geodesic curve joining p and q in $\mathcal{GP}$.

Proof. Since $\|p-q\| < 1$, we know that p and q are unitarily equivalent from Theorem 2.1. Thus p and q have the same spectrum. We treat the case $\sigma(p) = \sigma(q) = \{\omega, \omega^2\}$. The other cases are dealt with in a similar way. Then

$$p = \omega E(\omega) + \omega^2 E(\omega^2) = \mathrm{i}\sqrt{3}E(\omega) + \omega^2 \quad \text{and} \quad q = \omega F(\omega) + \omega^2 F(\omega^2) = \mathrm{i}\sqrt{3}F(\omega) + \omega^2,$$

respectively (see [12, Theorem 2]), where $E(\lambda)$ (resp. $F(\lambda)$) is the spectral projection of p (resp. q) associated with the point $\lambda \in \sigma(p)$ (resp. $\sigma(q)$). According to Remark 2.2, we have that

$$\|E(\omega) - F(\omega)\| < 1.$$

From the main theorem in [18], we know that there exists $x \in H_1^p$ with $\|x\| \leqslant \frac{\pi}{2}$ such that

$$\gamma(t) = \mathrm{e}^{tx} p \mathrm{e}^{-tx}$$

is a minimal geodesic curve joining p and q in $\mathcal{GP}$. Moreover,

$$\|x\| = \arcsin(\|E(\omega) - F(\omega)\|) < \frac{\pi}{2}$$

from [7, Theorem 3.5]. □

The link between $\mathcal{GP}$ and $\mathcal{P}$ is provided by the maps

$$f_i : \mathcal{GP} \to \mathcal{P}, \quad i = 0,1,2,3.$$

For $p \in \mathcal{GP}$, it is well known that the tangent space of $\mathcal{P}$ at $f_0(p)$ is

$$(T\mathcal{P})_{f_0(p)} = (T\mathcal{P})_{p^3} = \{xp^3 - p^3x : x \in \mathcal{A}_{ah}\}.$$

We need the following lemma.

Lemma 4.2. *Let $\varphi : (T\mathcal{GP})_p \to (T\mathcal{P})_{p^3}$ be the linear map given by $\varphi(z) = zp^2 + p^2z + pzp$. Then φ is norm-decreasing.*

Proof. For $z \in (T\mathcal{GP})_p$, we can find an $x \in \mathcal{A}_{ah}$ such that $z = xp - px$. It follows from Theorem 3.5 that there exists a smooth curve $\mathrm{e}^{t\tilde{x}}$ in $U_{\mathcal{A}}$ such that $\mathrm{e}^{t\tilde{x}} p \mathrm{e}^{-t\tilde{x}} \subseteq \mathcal{GP}$ is a geodesic curve, where $\tilde{x} = \Sigma_1^p(z) \in H_1^p$, which yields

$$f_0(\mathrm{e}^{t\tilde{x}} p \mathrm{e}^{-t\tilde{x}}) = \mathrm{e}^{t\tilde{x}} p^3 \mathrm{e}^{-t\tilde{x}} \in \mathcal{P}.$$

Thus

$$\varphi(z) = y := \frac{d}{dt} f_0(\mathrm{e}^{t\tilde{x}} p \mathrm{e}^{-t\tilde{x}}) \bigg|_{t=0} = \tilde{x}p^3 - p^3\tilde{x}.$$

In fact,

$$\tilde{x} = \begin{pmatrix} 0 & x_{12} & x_{13} & x_{10} \\ -x_{12}^* & 0 & x_{23} & x_{20} \\ -x_{13}^* & -x_{23}^* & 0 & x_{30} \\ -x_{10}^* & -x_{20}^* & -x_{30}^* & 0 \end{pmatrix}$$

with respect to the spectral decomposition of p. We get

$$y = \begin{pmatrix} 0 & 0 & 0 & -x_{10} \\ 0 & 0 & 0 & -x_{20} \\ 0 & 0 & 0 & -x_{30} \\ -x_{10}^* & -x_{20}^* & -x_{30}^* & 0 \end{pmatrix}$$

and

$$z = \begin{pmatrix} 0 & (\omega-1)x_{12} & (\omega^2-1)x_{13} & -x_{10} \\ (\omega-1)x_{12}^* & 0 & (\omega^2-\omega)x_{23} & -\omega x_{20} \\ (\omega^2-1)x_{13}^* & (\omega^2-\omega)x_{23}^* & 0 & -\omega^2 x_{30} \\ -x_{10}^* & -\omega x_{20}^* & -\omega^2 x_{30}^* & 0 \end{pmatrix}$$

with respect to the spectral decomposition of p. From this, we have $\|z\| = \|vzv\|$, where $v = 1 \oplus \omega^2 \oplus \omega \oplus 1$. Then

$$\|z\| = \|vzv\| \geqslant \left\| \frac{1}{2}(vzv + (vzv)^*) \right\| \geqslant \|y\|$$

from [6, Lemma 5.2]. This establishes the lemma. □

Corollary 4.3. *If $\gamma(t) \subseteq \mathcal{GP}$ is a smooth curve, then* $\mathrm{length}(f_0(\gamma)) \leqslant \mathrm{length}(\gamma)$.

Remark 4.4. Given $p \in \mathcal{GP}$, denote

$$\mathcal{M}_p = \{q \in \mathcal{GP} : q^3 = p^3\}, \quad \mathcal{B} = \{\tilde{T} : \tilde{T} = T\,|_{R(p)},\, \tilde{T}(R(p)) \subset R(p),\, T \in \mathcal{A}\},$$
$$\mathcal{F} = \{\tilde{T} : \tilde{T} = T\,|_{N(p)},\, \tilde{T}(N(p)) \subset N(p),\, T \in \mathcal{A}\}$$

and $U_{\mathcal{B}}$ (resp. $U_{\mathcal{F}}$) is the unitary group of C^*-algebra $\mathcal{B}$ (resp. $\mathcal{F}$). We have the following facts:

1. The Banach Lie group $U_{\mathcal{B}} \oplus U_{\mathcal{F}}$ also acts on $\mathcal{M}_p$ by means of $(u, q) \mapsto uqu^*$. The unitary orbits of this action lie at distance greater than or equal to 1 from Theorem 2.1. Therefore each one of these orbits consists of a discrete union of connected components of the space of special generalized projection in $\mathcal{A}$.

2. Similar to the proof that $\mathcal{GP}$ has a Banach manifold structure, we get that $\mathcal{M}_p$ endowed with a quotient topology is a Banach submanifold of $\mathcal{A}$. Put $\mathcal{M}_{p,i} = \{q \in \mathcal{GP} : f_i(q) = f_i(p)\}$, $i = 1, 2, 3$. We also have that $\mathcal{M}_{p,i}$ is a Banach submanifold of $\mathcal{GP}$ from the same way, $i = 1, 2, 3$.

Theorem 4.5. *Suppose $p \in \mathcal{GP}$ and $\|p - q\| < 1$. Then there exists a geodesic curve of $\mathcal{GP}$ of the given connection of the form*

$$\gamma(t) = \mathrm{e}^{tx} p \mathrm{e}^{-tx}, \quad t \in [0, 1],$$

where $x \in H_1^p$ with $\|x\| < \frac{\pi}{2}$, with the following properties:

(1) *γ has minimal length in $\mathcal{GP}$ along its path*;

(2) *γ has minimal length among all smooth curves in $\mathcal{GP}$ joining p and $\mathcal{M}_q$.*

Proof. We have $\|p^3 - q^3\| \leqslant \|p - q\| < 1$ from Remark 2.2. Then p^3 and q^3 can be joined with a minimal geodesic of $\mathcal{P}$, which is given by a p^3-co-diagonal anti-self operator x with $\|x\| < \frac{\pi}{2}$. From the definition of H_1^p, we have $x \in H_1^p$. Let $\gamma(t) = \mathrm{e}^{tx} p \mathrm{e}^{-tx}$. Then γ is a geodesic curve of $\mathcal{GP}$ of the given connection.

Next, we shall prove that γ has minimal length in $\mathcal{GP}$. Let $\tau(t) \subseteq \mathcal{GP}$ be a smooth curve which satisfies $\tau(0) = \gamma(0)$, $\tau(1) = \gamma(1)$, where $t \in [0, 1]$. It follows that length$(\tau^3) \leqslant$ length(τ) from Corollary 4.3. On the other hand, $\gamma(t)^3 = \mathrm{e}^{tx}p^3\mathrm{e}^{-tx}$, i.e., $\gamma(t)^3$ is a minimal geodesic in $\mathcal{P}$, which implies

$$\text{length}(\gamma^3) \leqslant \text{length}(\tau^3)$$

by the main theorem in [18]. We claim that length(γ) = length(γ^3), a fact which would conclude the proof. Indeed, note that

$$\text{length}(\gamma^3) = \int_0^1 \|\gamma(\dot{t})^3\| dt = \int_0^1 \|\mathrm{e}^{tx}xp^3\mathrm{e}^{-tx} - \mathrm{e}^{tx}p^3x\mathrm{e}^{-tx}\| dt = \|xp^3 - p^3x\|.$$

Likewise, length$(\gamma) = \|xp - px\|$. Thus we only need to show that

$$\|xp^3 - p^3x\| = \|xp - px\|.$$

We have that x has matrix representation

$$x = \begin{pmatrix} 0 & 0 & 0 & x_{10} \\ 0 & 0 & 0 & x_{20} \\ 0 & 0 & 0 & x_{30} \\ -x_{10}^* & -x_{20}^* & -x_{30}^* & 0 \end{pmatrix}$$

with respect to the spectral decomposition of p. Then $v(xp - px)v = xp^3 - p^3x$, where $v = 1 \oplus \omega^2 \oplus \omega \oplus 1$ is a unitary operator. Therefore, $\|xp - px\| = \|xp^3 - p^3x\|$. It follows that γ has minimal length in $\mathcal{GP}$.

What is left to prove is that γ has minimal length among all smooth curves in $\mathcal{GP}$ joining p and $\mathcal{M}_q$. Since $\gamma(t)^3 = \mathrm{e}^{tx}p^3\mathrm{e}^{-tx}$ is a geodesic curve in $\mathcal{P}$ joining p^3 and q^3, this makes $\mathrm{e}^{x}p\mathrm{e}^{-x} \in \mathcal{M}_q$. Then $\gamma(t)$ is a geodesic curve in $\mathcal{GP}$ joining p and $\mathcal{M}_q$. Suppose that $\tilde{\gamma}$ is another curve in $\mathcal{GP}$ with $\gamma(\tilde{0}) = p$, $\gamma(\tilde{1}) \in \mathcal{M}_q$. We have that $\tilde{\gamma}^3$ joins p^3 and q^3, which gives

$$\text{length}(\tilde{\gamma}^3) \geqslant \text{length}(\gamma^3) = \text{length}(\gamma)$$

from what has already been proved in the above paragraph. From Corollary 4.3, we know

$$\text{length}(\tilde{\gamma}^3) \leqslant \text{length}(\tilde{\gamma}),$$

which completes the proof. □

With a similar proof to that in Theorem 4.5, one has the following proposition.

Proposition 4.6. *Suppose $p \in \mathcal{GP}$ and $\|p - q\| < 1$. Then there exists $x_i \in H_1^p$ with $\|x_i\| < \frac{\pi}{2}$ such that the curve $\gamma_i(t) = \mathrm{e}^{tx_i}p\mathrm{e}^{-tx_i}$ of the given connection is a geodesic curve in $\mathcal{GP}$ joining p and manifold $\mathcal{M}_{q,i}, i = 1, 2, 3$.*

For $p \in \mathcal{GP}$, put $\mathcal{S}^{p^3} = \{v \in \mathcal{A} : v^*v = p^3\}$. The general theory shows the existence of a number $0 < R \leqslant 1$ with the property that two elements $v_1, v_2 \in \mathcal{S}^{p^3}$ such that $\|v_1 - v_2\| < R$ can be joined by a unique geodesic. From the proof of Proposition 3.1 in [4], we get that: If the partial isometry $v \in \mathcal{S}^{p^3}$ satisfies $\|v - p\| < R$, then there exists $x \in H_1^p$ with $\|x\| < R$ such that $\mathrm{e}^{x}p = v$. Next, we prove that the generalized projection in some neighbourhood of $p \in \mathcal{GP}$ can be joined with p by a unique geodesic curve in $\mathcal{GP}$.

Theorem 4.7. *Let $p \in \mathcal{GP}$ and suppose generalized projection q satisfies that $\|p - q\| < \frac{R}{4}$. Then there exists $x \in H_1^p$ with $\|x\| < R$ such that the curve $\gamma(t) = \mathrm{e}^{tx}p\mathrm{e}^{-tx}$ of the given connection is a unique geodesic curve in $\mathcal{GP}$ joining p and q.*

Proof. Since $\|p - q\| < \frac{R}{4}$, we know that $\|f_i(p) - f_i(q)\| \leqslant \|p - q\| < \frac{R}{4} < 1$ from Remark 2.2. Then there exist the unitary operators $u_i \in \mathcal{A}$ such that

$$\|u_i - 1\| \leqslant \|f_i(p) - f_i(q)\| < \frac{R}{4} \quad \text{and} \quad u_i f_i(p) u_i^* = f_i(q), \quad i = 0, 1, 2, 3$$

from the main theorem in [18]. Let $u=\sum_{i=0}^{3} f_i(q)u_i f_i(p)$. We get that $upu^*=q$. By computation, we have that

$$\|u-1\|=\bigg\|\sum_{i=0}^{3} f_i(q)u_i f_i(p)-\sum_{i=0}^{3} f_i(q)\bigg\|=\bigg\|\sum_{i=0}^{3} f_i(q)(u_i-1)\bigg\|\leqslant\sum_{i=0}^{3}\|(u_i-1)\|<R.$$

Set $\widehat{v}=up$. It is easy to see that $\widehat{v}\in\mathcal{S}^{p^3}$ and $\|\widehat{v}-p\|\leqslant\|u-1\|<R$. Then there exists a unique anti-self operator $x\in H_1^p$ such that $up=\widehat{v}=\mathrm{e}^x p$. Therefore $\mathrm{e}^x p\mathrm{e}^{-x}=q$. □

Consider the case $\mathcal{A}=B(H)$. We characterize the pairs of generalized projections with two-point spectra which can joined by (minimal) geodesics.

Proposition 4.8. *Suppose that generalized projections p and q have two spectral points. Then the following statements are equivalent*:

(1) *There exists a self-adjoint unitary u such that $upu=q$.*

(2) *There exists a unitary u such that $upu^*=q$ and $uqu^*=p$.*

(3) *There exists $x\in H_1^p$ with $\|x\|\leqslant\frac{\pi}{2}$ such that $\gamma(t)=\mathrm{e}^{tx}p\mathrm{e}^{-tx}$ is a minimal geodesic curve in $\mathcal{GP}$ joining p and q.*

(4) *There exists $x\in H_1^p$ with $\|x\|\leqslant\frac{\pi}{2}$ such that $\gamma(t)=\mathrm{e}^{tx}p\mathrm{e}^{-tx}$ is a geodesic in $\mathcal{GP}$ joining p and q.*

Proof. We only need to prove that (2) ⇒ (3) and (4) ⇒ (1).

(2) ⇒ (3). If the unitary operator u satisfies that $upu^*=q$ and $uqu^*=p$, we know that p and q have the same spectrum. Consider the case $\sigma(p)=\sigma(q)=\{\omega,\omega^2\}$, and other cases are treated similarly. Then

$$p=\omega E(\omega)+\omega^2E(\omega^2)=\mathrm{i}\sqrt{3}E(\omega)+\omega^2 \quad\text{and}\quad q=\omega F(\omega)+\omega^2F(\omega^2)=\mathrm{i}\sqrt{3}F(\omega)+\omega^2,$$

respectively (see [12, Theorem 2]), where $E(\lambda)$ (resp. $F(\lambda)$) is the spectral projection of p (resp. q) associated with the point $\lambda\in\sigma(p)$ (resp. $\sigma(q)$). Consider the following subspaces:

$$\begin{aligned}&H_{11}=R(E(\omega))\cap R(F(\omega)),\quad H_{00}=R(E(\omega^2))\cap R(F(\omega^2)),\\&H_{10}=R(E(\omega))\cap R(F(\omega^2)),\quad H_{01}=R(E(\omega^2))\cap R(F(\omega)),\\&H_0=[R(E(\omega))\ominus(H_{11}\oplus H_{10})]\oplus[R(E(\omega^2))\ominus(H_{01}\oplus H_{00})].\end{aligned}$$

We get that (see [11, Theorem 1.4])

$$p=\omega\oplus\omega^2\oplus\omega\oplus\omega^2\oplus\begin{pmatrix}\omega&0\\0&\omega^2\end{pmatrix}$$

and

$$q=\omega\oplus\omega^2\oplus\omega^2\oplus\omega\oplus u_0\begin{pmatrix}\omega&0\\0&\omega^2\end{pmatrix}u_0^*$$

with respect to $\mathcal{H}=H_{11}\oplus H_{00}\oplus H_{10}\oplus H_{01}\oplus H_0$, where

$$u_0=\begin{pmatrix}c_0^{\frac{1}{2}}&-(I-c_0)^{\frac{1}{2}}v\\v^*(I-c_0)^{\frac{1}{2}}&v^*c_0^{\frac{1}{2}}v\end{pmatrix},$$

c_0 is a positive contraction operator satisfying that 0 and 1 are not in the point spectrum of c_0, and $v: R(E(\omega^2))\ominus(H_{01}\oplus H_{00})\rightarrow R(E(\omega))\ominus(H_{11}\oplus H_{10})$ is a unitary operator. From $upu^*=q$ and $uqu^*=p$, we get that

$$uE(\omega)u^*=uf_1(p)u^*=f_1(q)=F(\omega)\quad\text{and}\quad uF(\omega)u^*=uf_1(q)u^*=f_1(p)=E(\omega).$$

By computation, we have that

$$u=u_{11}\oplus u_{22}\oplus\begin{pmatrix}0&u_{34}\\u_{43}&0\end{pmatrix}\oplus\begin{pmatrix}u_{55}&u_{56}\\u_{65}&u_{66}\end{pmatrix},$$

where u_{34} (resp. u_{43}) is a unitary from H_{01} (resp. H_{10}) onto H_{10} (resp. H_{01}), and u_{11} (resp. u_{22}) is a unitary operator on H_{11} (resp. H_{00}). Since $upu^* = q$, we have

$$\begin{pmatrix} u_{55} & u_{56} \\ u_{65} & u_{66} \end{pmatrix} \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix} \begin{pmatrix} u_{55} & u_{56} \\ u_{65} & u_{66} \end{pmatrix}^* = u_{|H_0} p_{|H_0} u^*_{|H_0} = q_{|H_0} = u_0 \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix} u_0^* = u_0 p_{|H_0} u_0^*,$$

which implies that $u^*_{|H_0} u_0 \in \{p_{|H_0}\}'$, where $\{p_{|H_0}\}'$ is the commutant of $p_{|H_0}$. Then

$$u = u_{11} \oplus u_{22} \oplus \begin{pmatrix} 0 & u_{34} \\ u_{43} & 0 \end{pmatrix} \oplus u_0 \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix},$$

where s_1 (resp. s_2) is a unitary operator on $R(E(\omega)) \ominus (H_{11} \oplus H_{10})$ (resp. $R(E(\omega^2)) \ominus (H_{01} \oplus H_{00})$). Consider the operator

$$x = 0 \oplus 0 \oplus \frac{\pi}{2} \begin{pmatrix} 0 & -u_{34} \\ u_{34}^* & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & -(\operatorname{arccos} c_0^{\frac{1}{2}})v \\ v^*(\operatorname{arccos} c_0^{\frac{1}{2}}) & 0 \end{pmatrix}.$$

Then $\|x\| \leqslant \frac{\pi}{2}$ and $\mathrm{e}^x p \mathrm{e}^{-x} = q$. Thus $\mathrm{e}^x E(\omega) \mathrm{e}^{-x} = F(\omega)$. In fact, the curve $\tau(t) = \mathrm{e}^{tx} E(\omega) \mathrm{e}^{-tx}, t \in [0,1]$ is a minimal geodesic curve in $\mathcal{P}$ joining $E(\omega)$ and $F(\omega)$. This implies that the curve $\gamma(t) = \mathrm{e}^{tx} p \mathrm{e}^{-tx}, t \in [0,1]$ is a minimal geodesic curve in $\mathcal{GP}$ joining p and q, with $\text{length}(\gamma) = \sqrt{3}\text{length}(\tau)$.

(4) $\Rightarrow$ (1). If there exists $x \in H_1^p$ with $\|x\| \leqslant \frac{\pi}{2}$ such that $\gamma(t) = \mathrm{e}^{tx} p \mathrm{e}^{-tx}$ is a geodesic curve in $\mathcal{GP}$ joining p and q, then $\tau(t) = \mathrm{e}^{tx} E(\omega) \mathrm{e}^{-tx}$, and $t \in [0,1]$ is a geodesic curve in $\mathcal{P}$ joining $E(\omega)$ and $F(\omega)$. From [3, Theorem 3.1], we get $\dim H_{10} = \dim H_{01}$. Put

$$u = 1 \oplus 1 \oplus \begin{pmatrix} 0 & v \\ v^* & 0 \end{pmatrix} \oplus \widetilde{u_0},$$

where $v : H_{01} \to H_{10}$ is unitary and

$$\widetilde{u_0} = \begin{pmatrix} c_0^{\frac{1}{2}} & (I - c_0)^{\frac{1}{2}} v \\ v^*(I - c_0)^{\frac{1}{2}} & -v^* c_0^{\frac{1}{2}} v \end{pmatrix}.$$

It is easy to check that u is a self-adjoint unitary operator and $upu = q$. □

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