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Lipschitz continuity for solutions of the $\bar{\alpha}$-Poisson equation

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Abstract In this paper, we study the Lipschitz continuity for solutions of the $\bar{\alpha}$-Poisson equation. After characterizing the boundary conditions for the Lipschitz continuity of $\bar{\alpha}$-harmonic mappings, we present four equivalent conditions for the (K, K')-quasiconformal solutions of the $\bar{\alpha}$-Poisson equation with a nonhomogeneous term to be Lipschitz continuous.

Keywords weighted Laplacian operator, Lipschitz continuity, Dirichlet boundary problem, harmonic mapping, (K, K')-quasiconformal mapping

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1 Introduction

Let D and Ω be two simply-connected subdomains of the complex plane $\mathbb{C}$ and u a map from D to Ω. Two Wirtinger derivatives $\partial_w u$ and $\bar{\partial}_w u$ of u are defined by

$$\partial_w u = u_w = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \mathrm{i}\frac{\partial u}{\partial y}\right) \quad \text{and} \quad \bar{\partial}_w u = u_{\bar{w}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial u}{\partial y}\right),$$

respectively, wherein $w = x + \mathrm{i}y$. We write

$$|\nabla u| = |u_w| + |u_{\bar{w}}|, \quad l(\nabla u) = ||u_w| - |u_{\bar{w}}||, \quad \text{and} \quad J_u = |u_w|^2 - |u_{\bar{w}}|^2.$$

1.1 The $\bar{\alpha}$-Poisson equation and the integral representation theorem

The *weighted Laplacian operator* is defined by

$$\overline{L_\rho} = 4\partial_w \rho \bar{\partial}_w,$$

where ρ is a continuously differentiable function in a proper domain $D \subset \mathbb{C}$. The weighted Laplacian operator $\overline{L_\rho}$ and its adjoint operator $L_\rho = 4\bar{\partial}_w \rho \partial_w$ were first systematically studied by Garabedian [12].

If the weight function ρ is chosen to be $(1 - |w|^2)^{-\alpha}$ in the unit disk $\mathbb{D}$, we call $\overline{L_{(1-|w|^2)^{-\alpha}}}$ the *standard weighted Laplacian operator* and denote it by $\overline{L_\alpha}$ for simplicity, where α is a real number with $\alpha > -1$. Note that whenever α is nonzero, the differential operator $\overline{L_\alpha}$ has a singular or degenerate behavior on the boundary $\mathbb{T} = \partial\mathbb{D}$. The standard weight functions $(1-|w|^2)^{-\alpha}$'s are closely related to the study of weighted Bergman spaces of the unit disk $\mathbb{D}$. (See the monograph [14] by Hedenmalm et al.)

If a function u in $C^2(\mathbb{D})$ satisfies the $\bar{\alpha}$-*Poisson equation*

$$-\overline{L_\alpha} u = 0,$$

we call it an $\bar{\alpha}$-*harmonic mapping*. If α is taken by 0, then we have $\overline{L_\alpha} = \Delta$, where Δ is the classical Laplacian operator. In this case, $\bar{\alpha}$-harmonic mappings are just Euclidean harmonic mappings. One can refer to [9] or [10] for basic theories of Euclidean harmonic mappings. Several recent papers [3, 6–8, 26, 28, 29] have attracted much attention on $\bar{\alpha}$-harmonic mappings in the unit disk $\mathbb{D}$.

Write

$$P_r^\alpha(\theta) = \frac{1}{2\pi}\frac{(1-|w|^2)^{\alpha+1}}{(1-\bar{w})^{\alpha+1}(1-w)}, \quad w = re^{i\theta}, \tag{1.1}$$

and we call it the $\bar{\alpha}$-*Poisson kernel*. In the special case wherein α is 0, $P_r^0(\theta)$ is the classical Poisson kernel

$$P_r(\theta) = \frac{1}{2\pi}\frac{1-|w|^2}{|1-w|^2}, \quad w = re^{i\theta}.$$

The integral representation given by the Poisson kernel of a Euclidean harmonic mapping plays a vital role in the theory of harmonic mappings and its applications [10]. Olofsson and Wittsten [29] used power series to represent the $\bar{\alpha}$-Poisson kernel and generalized the classical integral representation theorem to the case of $\bar{\alpha}$-harmonic mappings. They presented the following theorem.

Theorem A (See [29]). *If $u \in C^2(\mathbb{D})$ is a solution of PDE $-\overline{L_\alpha}u = 0$ satisfying the condition that $u(re^{i\theta})$ tends to a function $f \in L^1(\mathbb{T})$ as r tends to 1, then for every $w \in \mathbb{D}$,*

$$u(w) = P_r^\alpha * f(e^{i\theta}) = \frac{1}{2\pi}\int_{\mathbb{T}} \frac{(1-|w|^2)^{\alpha+1}}{(1-z\bar{w})^{\alpha+1}(1-\bar{z}w)} f(z)d\varphi,$$

where $z = e^{i\varphi}$ and $\varphi \in [0, 2\pi)$.

The *associated Dirichlet boundary value problem* of the $\bar{\alpha}$-Poisson equation is the following problem:

$$\begin{cases} -\overline{L_\alpha} v = g & \text{in } \mathbb{D}, \\ v = f & \text{on } \mathbb{T}, \end{cases}$$

where $g \in C(\mathbb{D})$, $f \in L^1(\mathbb{T})$, and the boundary condition is to be understood as $u(re^{i\theta}) \to f$ in $L^1(\mathbb{T})$ when $r \to 1$.

For the case wherein the boundary function f vanishes, Behm [3] solved the above Dirichlet boundary value problem of the $\bar{\alpha}$-Poisson equation. Utilizing the generalized Green's theorem [20, pp. 148–150], Chen and Kalaj [8] combined the representation theorem given by Olofsson and Wittsten [29] with the one given by Behm [3]. They obtained the following theorem.

Theorem B (See [8]). *Let g be continuous in the unit disk $\mathbb{D}$ such that $(1-|z|^2)^{\alpha+1}g$ belongs to $L^1(\mathbb{D})$, wherein $\alpha > -1$. If $v \in C^2(\mathbb{D})$ is a solution of the PDE $-\overline{L_\alpha}v = g$ satisfying the condition that $v(re^{i\theta})$ tends to a function $f \in L^1(\mathbb{T})$ as r tends to 1, then for every $w \in \mathbb{D}$ we have*

$$v(w) = u(w) + G_\alpha[g](w),$$

where

$$u(w) = \frac{1}{2\pi}\int_{\mathbb{T}} \frac{(1-|w|^2)^{\alpha+1}}{(1-z\bar{w})^{\alpha+1}(1-\bar{z}w)} f(z)d\theta, \quad G_\alpha[g](w) = \int_{\mathbb{D}} G_\alpha(z,w)g(z)dxdy,$$

and the Green function $G_\alpha(z,w)$ of the adjoint Laplacian operator L_α is given by

$$G_\alpha(z,w) = \frac{(1-\bar{z}w)^\alpha h \circ q}{2\pi}, \quad \text{with} \quad z \neq w, \tag{1.2}$$

$$h(r) = \frac{1}{2}\int_0^{1-r^2} \frac{t^\alpha}{1-t}dt, \quad q = q(z,w) = \left|\frac{z-w}{1-\bar{w}z}\right|.$$

1.2 (K,K')-quasiconformal $\bar{\alpha}$-harmonic mapping

A sense-preserving homeomorphism u from D to Ω is (K,K')-*quasiconformal* if it satisfies the following conditions:

(1) u is absolutely continuous on the lines in D and

(2) there exist two constants $K \geqslant 1$ and $K' \geqslant 0$ such that

$$|\nabla u|^2 \leqslant K J_u + K'.$$

If $K' = 0$, then u is said to be a K-*quasiconformal mapping*. For basic theories of quasiconformal mappings, one can see the monograph [1].

If an $\bar{\alpha}$-harmonic mapping is also a (K,K')-quasiconformal mapping, then it is called a (K,K')-*quasiconformal* $\bar{\alpha}$-*harmonic mapping*. Particularly, if $\alpha = 0$, we call it a (K,K')-*quasiconformal harmonic mapping*. One can see the papers [4, 5, 7, 11, 18, 21, 32] for recent progress on (K,K')-quasiconformal mappings.

1.3 Lipschitz continuity for certain classes of mappings

A mapping $u : D \to \Omega$ is said to be in Lip_β if there exist a constant L_β and an exponent $\beta \in (0,1]$ such that for all z and w in D, we have

$$|u(z) - u(w)| \leqslant L_\beta |z-w|^\beta.$$

Such mappings are also called β-*Hölder continuous*. In a special case wherein β is 1, the mapping is called *Lipschitz continuous*. The constants L_β and L_1 are called a Hölder constant and a Lipschitz constant, respectively.

Morrey [25] obtained a local Hölder continuity for K-quasiconformal mappings in the plane and showed that the exponent $\beta = 1/K$ is optimal (see [2, pp. 80–83]). After simplifying and improving the result given by Morrey [25], Nirenberg [27] developed a rather complete theory for second-order elliptic equations with two variables. Finn and Serrin [11] and Simon [32] showed that (K,K')-quasiconformal mappings are locally Hölder continuous. Kalaj and Mateljević [18] obtained a sufficient condition for the global Hölder continuity of (K,K')-quasiconformal mappings.

The class of K-quasiconformal harmonic mappings of the unit disk $\mathbb{D}$ onto itself was first studied by Martio [23]. After Pavlović [31] presented the bi-Lipschitz characteristic and an explicit Lipschitz constant in K for all K-quasiconformal harmonic mappings of $\mathbb{D}$ onto itself, recent studies [15–19, 22, 24, 30] improved and generalized the results obtained by Pavlović [31]. Lipschitz continuity for (K,K')-quasiconformal mappings satisfying certain second-order differential inequalities has been studied in papers [4, 5, 18, 21]. Summarizing the results in [4, 18], we can obtain the Lipschitz continuity of a (K,K')-quasiconformal mapping u between two smooth Jordan domains satisfying the partial differential inequality

$$|\Delta u| \leqslant A|\nabla u|^2 + B, \tag{1.3}$$

where the two constants A and B satisfy that $A \geqslant 0$ and $B \geqslant 0$, respectively.

1.4 Statement of the main result

In this paper, we aim to study the Lipschitz continuity of $\bar{\alpha}$-harmonic mappings. Example 2.1 says that it is proper to assume that $\alpha \geqslant 0$ when we consider the Lipschitz continuity of $\bar{\alpha}$-harmonic mappings. We note that a mapping need not be $\bar{\alpha}$-harmonic to satisfy the partial differential inequality (1.3) (see Example 2.2 in Section 2).

Let $V_{\mathbb{D}\to\Omega}[g]$ denote the family of solutions of $v : \mathbb{D} \to \Omega \in C^2(\mathbb{D})$ of the $\bar{\alpha}$-Poisson equation

$$\begin{cases} -\overline{L_\alpha} v = g & \text{in } \mathbb{D}, \\ v = f & \text{on } \mathbb{T}, \end{cases}$$

where $g \in C(\overline{\mathbb{D}})$, f is the limit of $v(re^{i\theta})$ as r tends to 1^-, and v is a sense-preserving diffeomorphism. We provide the boundary characterizations of a Lipschitz continuous $\bar{\alpha}$-harmonic mapping as follows:

Theorem 1.1. *Assume that $v \in V_{\mathbb{D}\to\Omega}[g]$. If f is absolutely continuous on the unit circle $\mathbb{T}$ satisfying that $f' \in L^\infty(\mathbb{T})$ and the integral*

$$\frac{1}{2\pi}\int_{\mathbb{T}} \frac{(1-|w|^2)^\alpha}{(1-z\bar{w})^\alpha} \frac{(w\bar{z}-\bar{w}z)/\mathrm{i}}{|z-w|^2}[f(e^{i\theta})]'_\theta d\theta$$

is bounded (here $z = e^{i\theta}$). Then v is Lipschitz continuous on the unit disk $\mathbb{D}$ when $\alpha \geqslant 0$. Particularly, if $\alpha = 0$, the boundedness of the above integral is equivalent to the boundedness of the Hilbert transform of f'; herein, $f' = [f(e^{i\theta})]'_\theta$.

Utilizing the above theorem, we obtain the main result of this paper.

Theorem 1.2. *Assume that $v \in V_{\mathbb{D}\to\Omega}[g]$ with the representation $v(w) = u(w) + G_\alpha[g](w)$. If $\alpha \geqslant 0$, then the following conditions are equivalent*:

(a) *v is a (K, K')-quasiconformal mapping and $|\frac{\partial u}{\partial r}| \leqslant L$ on $\mathbb{D}$, and L is a constant.*
(b) *v is Lipschitz continuous with the Euclidean metric.*
(c) *u is Lipschitz continuous with the Euclidean metric.*
(d) *f is absolutely continuous on $\mathbb{T}$, $f' \in L^\infty(\mathbb{T})$ and the following integral*

$$\frac{1}{2\pi}\int_{\mathbb{T}} \frac{(1-|w|^2)^\alpha}{(1-z\bar{w})^\alpha} \frac{(w\bar{z}-\bar{w}z)/i}{|z-w|^2}[f(e^{i\theta})]'_\theta d\theta$$

is bounded; *herein, $z = e^{i\theta}$.*

We note that the above theorem gives a boundary characterizations of (K, K')-quasiconformal $\bar{\alpha}$-harmonic mappings, which is a kind of generalization of the result given by Pavlović [31].

2 Two auxiliary examples

The following example shows that it is necessary to assume that $\alpha \geqslant 0$ when considering the Lipschitz continuity of $\bar{\alpha}$-harmonic mappings.

Example 2.1. Let $f(e^{it}) = e^{it} + se^{-i2t}$, where $0 \leqslant s \leqslant 1/4$ and let $u(w)$ be the $\bar{\alpha}$-harmonic mapping of the unit disk $\mathbb{D}$ with the boundary function f. Then

$$u(w) = w + s\frac{1-(1+(\alpha+1)|w|^2)(1-|w|^2)^{1+\alpha}}{w^2}$$

and

$$|\nabla u| \to +\infty, \quad |w| \to 1^-$$

when $-1 < \alpha < 0$, which implies that u is not Lipschitz.

Proof. Note that $[f(e^{it})]'_t = -(\sin t + 2s\sin 2t) + i(\cos t - 2s\cos 2t)$. Let $\psi(t) = \arg\{[f(e^{it})]'_t\}$. Then

$$\tan'\psi(t) = \frac{1-8s^2+2s\cos 3t}{(2s\sin 2t+\sin t)^2}, \quad t \in [0, 2\pi).$$

Because $-1 \leqslant \cos 3t \leqslant 1$, it implies

$$1 - 8s^2 + 2s\cos 3t \geqslant s^2(1/s+2)(1/s-4).$$

Hence, $\tan'\psi(t) \geqslant 0$ if $0 \leqslant s \leqslant 1/4$, i.e., $f(\mathbb{T})$ is a convex Jordan curve.

Using Theorem A and the series expansion

$$\frac{1}{(1-x)^m} = 1 + mx + \frac{m(m+1)}{2!}x^2 + \cdots + \frac{m(m+1)\cdots(m+k-1)}{k!}x^k + \cdots,$$

we have

$$u(w)=w+s(1-|w|^2)^{1+\alpha}\left[\sum_{k=0}^{\infty}\frac{(\alpha+1)_{(k+2)}}{(k+2)!}|w|^{2k}\right]\bar{w}^2$$
$$=w+s\frac{1-(1+(\alpha+1)|w|^2)(1-|w|^2)^{1+\alpha}}{w^2}.$$

Differentiating $u(w)$ in $\bar{w}$ and w, we obtain

$$u_{\bar{w}}=s(\alpha+1)(\alpha+2)\bar{w}(1-|w|^2)^{\alpha} \tag{2.1}$$

and

$$u_w=1+\frac{s(1-|w|^2)^{\alpha}[2+2\alpha|w|^2+\alpha(\alpha+1)|w|^4]-2s}{w^3}. \tag{2.2}$$

Thus, it follows that

$$|\nabla u|\geqslant|u_{\bar{w}}|=\frac{s(\alpha+1)(\alpha+2)|w|}{(1-|w|^2)^{-\alpha}}\to\infty,\quad |w|\to 1^-,$$

when the condition $-1<\alpha<0$ is assumed. □

Next, we show that there exist some $\bar{\alpha}$-harmonic mappings which do not satisfy the inequality (1.3).

Example 2.2. Let $f(\mathrm{e}^{\mathrm{i}t})=\mathrm{e}^{\mathrm{i}t}+s\mathrm{e}^{-\mathrm{i}2t}$ where $0\leqslant s\leqslant 1/4$ and let $u(w)$ be the $\bar{\alpha}$-harmonic mapping of the unit disk $\mathbb{D}$ with the boundary function f. Then for $0<\alpha<1$, u does not satisfy the differential inequality

$$|\Delta u|\leqslant A|\nabla u|^2+B,$$

where A and B are two non-negative constants. Furthermore, ∇u is bounded; therefore, u is Lipschitz continuous in $\mathbb{D}$.

Proof. When α is strictly between zero and one, we have

$$|\Delta u|=|4u_{w\bar{w}}|=\left|-\frac{4\alpha\bar{w}}{1-|w|^2}u_{\bar{w}}\right|=\frac{4s\alpha(\alpha+1)(\alpha+2)|w|^2}{(1-|w|^2)^{1-\alpha}}\to\infty$$

as $|w|\to 1^-$, and

$$|\nabla u|=|u_w|+|u_{\bar{w}}|\to 1-2s,$$

as $|w|\to 1^-$. Thus, we see that there exists an $\bar{\alpha}$-harmonic mapping that does not satisfy the differential inequality

$$|\Delta u|\leqslant A|\nabla u|^2+B,$$

where A and B are two non-negative constants.

Moreover, when $|w|\to 0$, it implies

$$u_w=1-\frac{\alpha(\alpha+1)(\alpha+2)s}{3}\bar{w}^3+o(|w|^3).$$

Together with the relations (2.1) and (2.2), this shows that ∇u is bounded; thus, u is Lipschitz continuous. □

3 Boundary conditions for Lipschitz continuity of $\bar{\alpha}$-harmonic mappings

Lemma 3.1. *Suppose $u(w)$ is a C^1 function with a domain $\Omega\subset\mathbb{C}$ and let $w=r\mathrm{e}^{\mathrm{i}\varphi}$. Then u_φ and ru_r are bounded if and only if wu_w and $\bar{w}u_{\bar{w}}$ are bounded.*

Proof. Let $w=re^{\mathrm{i}\varphi}$. Then

$$u_\varphi=u_w w_\varphi+u_{\bar{w}}\bar{w}_\varphi=\mathrm{i}(wu_w-\bar{w}u_{\bar{w}})$$

and

$$ru_r=r(u_w w_r+u_{\bar{w}}\bar{w}_r)=wu_w+\bar{w}u_{\bar{w}}.$$

Thus, the following identity

$$|wu_w|^2+|\bar{w}u_{\bar{w}}|^2=\frac{|u_\varphi|^2+|ru_r|^2}{2}$$

implies Lemma 3.1. □

Lemma 3.2. *Let f be absolutely continuous on the unit circle $\mathbb{T}$ such that $f'\in L^\infty(\mathbb{T})$. If u is an $\bar{\alpha}$-harmonic mapping with the boundary function f, then $u_\varphi(w)\in L^\infty(\mathbb{D})$, where $w=re^{\mathrm{i}\varphi}$. Moreover, $u_\varphi(w)$ is also $\bar{\alpha}$-harmonic.*

Proof. From Theorem A, we have

$$u(w)=\frac{1}{2\pi}\int_{\mathbb{T}}\frac{(1-|w|^2)^{\alpha+1}}{(1-z\bar{w})^{\alpha+1}(1-\bar{z}w)}f(z)d\theta,$$

where $z=e^{\mathrm{i}\theta}$ with $\theta\in[0,2\pi)$. Thus we observe the following:

$$\begin{aligned}u_\varphi(re^{\mathrm{i}\varphi})&=-\frac{1}{\pi}\int_{\mathbb{T}}\frac{\partial}{\partial\theta}\left(\frac{(1-r^2)^{\alpha+1}}{(1-re^{\mathrm{i}(\theta-\varphi)})^{\alpha+1}(1-re^{-\mathrm{i}(\theta-\varphi)})}\right)f(e^{\mathrm{i}\theta})d\theta\\&=-\frac{1}{2\pi}\frac{(1-|w|^2)^{\alpha+1}}{(1-z\bar{w})^{\alpha+1}(1-\bar{z}w)}f(z)\,|_{\mathbb{T}}+\frac{1}{2\pi}\int_{\mathbb{T}}\frac{(1-|w|^2)^{\alpha+1}}{(1-z\bar{w})^{\alpha+1}(1-\bar{z}w)}df(e^{\mathrm{i}\theta})\\&=\frac{1}{2\pi}\int_{\mathbb{T}}\frac{(1-|w|^2)^{\alpha+1}}{(1-z\bar{w})^{\alpha+1}(1-\bar{z}w)}[f(e^{\mathrm{i}\theta})]'_\theta d\theta=P_r^\alpha*\frac{d}{d\theta}f(e^{\mathrm{i}\theta}).\end{aligned}$$

We use the continuity of f on $\mathbb{T}$ to obtain the third equality above. Putting these equalities together show that u_φ is also $\bar{\alpha}$-harmonic.

Olofsson and Wittsten [29] showed that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|P_r^\alpha(\theta)|d\theta=\Gamma(\alpha+1)/\Gamma(\alpha/2+1)^2,$$

where $\Gamma(s)=\int_0^\infty t^{s-1}e^{-t}dt$ is the standard Gamma function for a positive s. Therefore, u_φ is bounded on $\mathbb{D}$ if f' is bounded, where $f'=\frac{d}{d\theta}f(e^{\mathrm{i}\theta})$. □

Olofsson and Wittsten [29, Lemma 2.3] used the Parseval formula to obtain the following lemma.

Lemma A (See [29])**.** *Let α be a positive real number. Then, whenever $0\leqslant r<1$ is satisfied, so is the inequality*

$$\frac{1}{2\pi}\int_{\mathbb{T}}\frac{(1-r^2)^\alpha}{|1-re^{\mathrm{i}\theta}|^{\alpha+1}}d\theta\leqslant\frac{\Gamma(\alpha)}{\Gamma(\frac{\alpha+1}{2})^2},$$

where Γ is the Gamma function given by $\Gamma(s)=\int_0^\infty t^{s-1}e^{-t}dt$.

Lemma 3.3. *Let f be absolutely continuous on the unit circle $\mathbb{T}$ such that $f'\in L^\infty(\mathbb{T})$. Assume that u is an $\bar{\alpha}$-harmonic mapping with the boundary function f. If there exists a constant M_α such that*

$$\left|\frac{1}{2\pi}\int_{\mathbb{T}}\frac{(1-|w|^2)^\alpha}{(1-z\bar{w})^\alpha}\frac{(w\bar{z}-\bar{w}z)/\mathrm{i}}{|z-w|^2}[f(e^{\mathrm{i}\theta})]'_\theta d\theta\right|\leqslant M_\alpha \tag{3.1}$$

wherein $z=e^{\mathrm{i}\theta}$, then $ru_r(w)\in L^\infty(\mathbb{D})$ and $w=re^{\mathrm{i}\varphi}$.

Proof. A direct calculation gives the identities

$$u_w=\frac{1}{2\pi}\int_{\mathbb{T}}\left(\frac{(1-|w|^2)^{\alpha+1}}{(1-z\bar{w})^{\alpha+1}(1-\bar{z}w)}\right)_w f(z)d\theta$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_{\mathbb{T}} \frac{(1-|w|^2)^\alpha[\bar{z}-(\alpha+1)\bar{w}+\alpha\bar{z}|w|^2]}{(1-z\bar{w})^{\alpha+1}(1-\bar{z}w)^2} f(z)d\theta \\
&= \frac{1}{2\pi}\int_{\mathbb{T}} \frac{(1-|w|^2)^\alpha[1-z\bar{w}-\alpha\bar{w}(z-w)]}{z^\alpha(\bar{z}-\bar{w})^{\alpha+1}(z-w)^2} f(z)d\theta
\end{aligned} \tag{3.2}
$$

and

$$
\begin{aligned}
u_{\bar{w}} &= \frac{1}{2\pi}\int_{\mathbb{T}} \left(\frac{(1-|w|^2)^{\alpha+1}}{(1-z\bar{w})^{\alpha+1}(1-\bar{z}w)}\right)_{\bar{w}} f(z)d\theta \\
&= \frac{1}{2\pi}\int_{\mathbb{T}} \frac{(\alpha+1)(1-|w|^2)^\alpha(z-w)}{(1-z\bar{w})^{\alpha+2}(1-\bar{z}w)} f(z)d\theta \\
&= \frac{1}{2\pi}\int_{\mathbb{T}} \frac{(\alpha+1)(1-|w|^2)^\alpha(1-\bar{z}w)}{z^\alpha(\bar{z}-\bar{w})^{\alpha+2}(z-w)} f(z)d\theta.
\end{aligned} \tag{3.3}
$$

Hence, the identities (3.2) and (3.3) imply that we have

$$
\begin{aligned}
ru_r &= wu_w + \bar{w}u_{\bar{w}} \\
&= \frac{1}{2\pi}\int_{\mathbb{T}} (1-|w|^2)^\alpha \left[\frac{(\alpha+1)\bar{z}\bar{w}}{z^\alpha(\bar{z}-\bar{w})^{\alpha+2}} \right. \\
&\quad \left. - \frac{\alpha|w|^2}{z^\alpha(\bar{z}-\bar{w})^{\alpha+1}(z-w)} + \frac{wz}{z^\alpha(\bar{z}-\bar{w})^\alpha(z-w)^2}\right] f(z)d\theta.
\end{aligned} \tag{3.4}
$$

Since we have that

$$
\frac{(\alpha+1)\bar{z}\bar{w}}{z^\alpha(\bar{z}-\bar{w})^{\alpha+2}} = \frac{1}{\mathrm{i}z^\alpha}\left[\frac{\bar{w}}{(\bar{z}-\bar{w})^{\alpha+1}}\right]_\theta
$$

and

$$
\frac{wz}{z^\alpha(\bar{z}-\bar{w})^\alpha(z-w)^2} - \frac{\alpha|w|^2}{z^\alpha(\bar{z}-\bar{w})^{\alpha+1}(z-w)} = -\frac{1}{\mathrm{i}}\left[\frac{w}{(z-w)(1-z\bar{w})^{\alpha+1}}\right]_\theta,
$$

it follows from the equation (3.4) that

$$
\begin{aligned}
ru_r &= \frac{1}{2\pi\mathrm{i}}\int_{\mathbb{T}} \left[\frac{\bar{w}(1-|w|^2)^\alpha}{(\bar{z}-\bar{w})^{\alpha+1}}\right]_\theta \frac{f(z)}{z^\alpha}d\theta - \frac{1}{2\pi\mathrm{i}}\int_{\mathbb{T}} \left[\frac{w(1-|w|^2)^\alpha}{(z-w)(1-z\bar{w})^\alpha}\right]_\theta f(z)d\theta \\
&= \frac{1}{2\pi}\int_{\mathbb{T}} \frac{(1-|w|^2)^\alpha}{(1-z\bar{w})^\alpha}\frac{(w\bar{z}-z\bar{w})/\mathrm{i}}{|z-w|^2}[f(e^{\mathrm{i}\theta})]'_\theta d\theta + \frac{\alpha}{2\pi}\int_{\mathbb{T}} \frac{z\bar{w}(1-|w|^2)^\alpha}{(1-z\bar{w})^{\alpha+1}} f(z)d\theta.
\end{aligned}
$$

As the assumption that f is absolutely continuous on the unit circle $\mathbb{T}$ implies that $f \in L^\infty(\mathbb{T})$, Lemma A gives that there is a constant N_α such that

$$
\left|\frac{\alpha}{2\pi}\int_{\mathbb{T}} \frac{z\bar{w}(1-|w|^2)^\alpha}{(1-z\bar{w})^{\alpha+1}} f(z)d\theta\right| \leqslant N_\alpha. \tag{3.5}
$$

Thus, by the assumption (3.1) and the inequality (3.5), we obtain the following inequality

$$
|ru_r| \leqslant M_\alpha + N_\alpha
$$

for non-negative α. □

Remark 3.1. We note that if $\alpha = 0$ in Lemma 3.3, then ru_r is the conjugate function of u_φ. Thus, ru_r is bounded if and only if the Hilbert transformation $H[f']$ of the boundary function f' is bounded (see, for example, [13, Chapter III]).

We will require an estimate of the definite integral given in Lemma B. The estimate below is given by Behm [3, Lemma 2].

Lemma B (See [3])**.** *Assume that* $\alpha > -1$ *and that* r *satisfies* $0 \leqslant r < 1$. *Define* $p(s)$ *by*

$$
p(s) = \int_0^s \frac{t^\alpha}{1-t}dt.
$$

Then the function $p(s)$ satisfies the estimate

$$p(s) \leqslant C_\alpha s^{\alpha+1}(1-\log(1-s)),$$

where C_α is a constant depending only on α.

Kalaj and Pavlović [19] used the Parseval formula to prove the following lemma.

Lemma C (See [19])**.** *If $z \in \mathbb{D}$, and*

$$I(z) = \frac{1}{2\pi}\iint_{\mathbb{D}} \frac{(1-|z|^2)(1-|\xi|^2)}{|\xi||1-\bar{z}\xi|^4}d\tau d\eta,$$

then

$$\frac{1}{2} \leqslant I(z) \leqslant \frac{2}{3},$$

wherein, $\xi = \tau + \mathrm{i}\eta$. Both inequalities are sharp. Moreover, the function $z \to I(z)$ is radial and decreasing for $|z| \in (0,1)$.

Lemma 3.4. *Let g be a bounded, continuous function on the unit disk $\mathbb{D}$. Assume that $G_\alpha[g]$ is the Green potential of g given by*

$$G_\alpha[g](w) = \iint_{\mathbb{D}} G_\alpha(z,w)g(z)dxdy.$$

Then $G_\alpha[g]_w$ and $G_\alpha[g]_{\bar{w}}$ are both bounded in the unit disk $\mathbb{D}$ when α is non-negative.

Proof. By Lemma B, we have the following equation:

$$\begin{aligned}2\pi|(G_\alpha)_w| &= \left|-\alpha\bar{z}(1-\bar{z}w)^{\alpha-1}h\circ q + \frac{1}{2}\frac{(1-|z|^2)^{\alpha+1}(1-|w|^2)^\alpha}{(1-z\bar{w})^\alpha(1-\bar{z}w)(z-w)}\right| \\ &\leqslant |\alpha|C_\alpha|1-\bar{w}z|^{\alpha-1}\left(1-\left|\frac{z-w}{1-\bar{w}z}\right|^2\right)^{\alpha+1}\left(1-\log\left|\frac{z-w}{1-z\bar{w}}\right|^2\right) \\ &\quad + \frac{(1-|z|^2)^{\alpha+1}(1-|w|^2)^\alpha}{2|1-z\bar{w}|^{\alpha+1}|z-w|}.\end{aligned}$$

A similar calculation gives the inequality

$$2\pi|(G_\alpha)_{\bar{w}}| \leqslant \frac{(1-|z|^2)^{\alpha+1}(1-|w|^2)^\alpha}{2|1-z\bar{w}|^{\alpha+1}|z-w|}.$$

We write

$$I_1 = \iint_{\mathbb{D}} \frac{(1-|z|^2)^{\alpha+1}(1-|w|^2)^\alpha}{2|1-z\bar{w}|^{\alpha+1}|z-w|}dxdy$$

and

$$I_2 = \iint_{\mathbb{D}} |1-\bar{w}z|^{\alpha-1}\left(1-\left|\frac{z-w}{1-\bar{w}z}\right|^2\right)^{\alpha+1}\left(1-\log\left|\frac{z-w}{1-z\bar{w}}\right|^2\right)dxdy.$$

Let $\xi = \varphi(z) = (w-z)/(1-\bar{w}z)$. Then, for each $w \in \mathbb{D}$, $\varphi(z)$ is a conformal mapping of $\mathbb{D}$ onto itself satisfying the following four identities:

$$z = \frac{w-\xi}{1-\bar{w}\xi}, \quad 1-|z|^2 = \frac{(1-|w|^2)(1-|\xi|^2)}{|1-\bar{w}\xi|^2},$$
$$1-\bar{w}z = \frac{1-|w|^2}{1-\bar{w}\xi}, \quad dz = -\frac{1-|w|^2}{(1-\bar{w}\xi)^2}d\xi.$$

These equations imply that the equation

$$I_1 = \iint_{\mathbb{D}} \frac{(1-|w|^2)^{\alpha+1}(1-|\xi|^2)^{\alpha+1}}{2|\xi||1-\bar{w}\xi|^{\alpha+4}}d\tau d\eta$$

is satisfied. Then, when $|w| < 1$ and $|\xi| < 1$, the following inequality

$$\frac{(1-|w|^2)(1-|\xi|^2)}{|1-\bar{w}\xi|} \leqslant \frac{4(1-|w|)(1-|\xi|)}{(1-|w||\xi|)} \leqslant 4, \tag{3.6}$$

together with Lemma C shows that

$$I_1 \leqslant \frac{4^\alpha}{2} \iint_{\mathbb{D}} \frac{(1-|w|^2)(1-|\xi|^2)}{|\xi||1-\bar{w}\xi|^4} |d\xi|^2 \leqslant \frac{2\cdot 4^\alpha}{3}\pi \tag{3.7}$$

when α is non-negative.

Similarly, the linear transformation $\xi = (w-z)/(1-\bar{w}z)$ of $\mathbb{D}$ onto itself gives

$$I_2 = \iint_{\mathbb{D}} \frac{(1-|w|^2)^{\alpha+1}(1-|\xi|)^{\alpha+1}}{|1-\bar{w}\xi|^{\alpha+3}} (1-\log|\xi|^2) d\tau d\eta.$$

Thus, for non-negative α, the inequality (3.6) implies that

$$\begin{aligned} I_2 &\leqslant 4^\alpha \iint_{\mathbb{D}} \frac{(1-|w|^2)(1-|\xi|^2)}{|1-\bar{w}\xi|^3}(1-\log|\xi|^2)|d\xi|^2 \\ &= \frac{4^\alpha}{2}\int_0^1 \left[2\frac{(1-|w|^2)(1-|\xi|^2)}{(1-|w|^2|\xi|^2)^2}\int_0^{2\pi}\frac{(1-|w|^2|\xi|^2)^2}{|1-|w||\xi|e^{i\varphi}|^{2+1}}d\varphi\right]|\xi|(1-\log|\xi|^2)d|\xi|. \end{aligned}$$

Furthermore, Lemma A together with the conditions $|\xi| < 1$ and $|w| < 1$ gives us the following bound:

$$\begin{aligned} I_2 &\leqslant \frac{4^\alpha \pi}{\Gamma(3/2)}\int_0^1 2\frac{(1-|w|^2)(1-|\xi|^2)}{(1-|w|^2|\xi|^2)^2}|\xi|(1-\log|\xi|^2)d|\xi| \\ &\leqslant \frac{4^\alpha \pi}{\Gamma(3/2)}\int_0^1 2|\xi|(1-\log|\xi|^2)d|\xi| = \frac{4^{\alpha+1}\pi}{3\Gamma(3/2)}. \end{aligned} \tag{3.8}$$

Hence, if g is a continuous and bounded function on the unit disk $\mathbb{D}$, then it follows from (3.7) and (3.8) that

$$|(G_\alpha[g])_w| \leqslant \left(\frac{|\alpha| 4^{\alpha+1} C_\alpha}{6\Gamma(3/2)} + \frac{4^\alpha}{3}\right)\|g\|_\infty, \quad |(G_\alpha[g])_{\bar{w}}| \leqslant \frac{4^\alpha}{3}\|g\|_\infty,$$

where $\|g\|_\infty$ is the essential upper bound of the function g on $\mathbb{D}$. Hence, the proof of Lemma 3.4 is completed. □

Recall Theorem 1.1, as stated in the introduction. This is the last result we need before proving the main result of this paper in the next section. The proof of Theorem 1.1 is based on the above lemmas.

Proof of Theorem 1.1. Assume that v is a solution of the PDE $-\overline{L_\alpha} v = g$ satisfying that $v(re^{i\theta})$ tends to the function f as r tends to 1. Theorem B provides

$$\begin{aligned} v(w) &= \frac{1}{2\pi}\int_{\mathbb{T}} \frac{(1-|w|^2)^{\alpha+1}}{(1-z\bar{w})^{\alpha+1}(1-\bar{z}w)} f(z) d\theta + \iint_{\mathbb{D}} G_\alpha(z,w) g(z) dxdy \\ &= u(w) + G_\alpha[g](w), \end{aligned}$$

where u is the $\bar{\alpha}$-harmonic mapping on $\mathbb{D}$ with boundary function f and $G_\alpha[g]$ is the Green potential on $\mathbb{D}$ of the function g.

By the assumption that $v \in C^2(\mathbb{D})$, there exists a positive constant M_1 such that $|\nabla v| \leqslant M_1$ when $w \in D(0,1/2)$. When $w \in \mathbb{D}\backslash D(0,1/2)$, we obtain from Lemma 3.1 the following inequality:

$$\begin{aligned} |\nabla v| = |v_w| + |v_{\bar{w}}| &\leqslant (|u_w| + |u_{\bar{w}}|) + (|(G_\alpha[g])_w| + |(G_\alpha[g])_{\bar{w}}|) \\ &\leqslant 2\sqrt{2}(|u_\varphi| + r|u_r|) + (|(G_\alpha[g])_w| + |(G_\alpha[g])_{\bar{w}}|). \end{aligned}$$

Because $f' \in L^\infty(\mathbb{T})$, Lemma 3.2 proves that $|u_\varphi|$ is bounded. Furthermore, Lemma 3.3 implies that ru_r is also bounded if the integral

$$\frac{1}{2\pi}\int_{\mathbb{T}} \frac{(1-|w|^2)^\alpha}{(1-z\bar{w})^\alpha} \frac{(w\bar{z}-\bar{w}z)/i}{|z-w|^2}[f(e^{i\theta})]'_\theta d\theta$$

is bounded. Hence, there exists a constant M_2 such that $|u_\varphi| + r|u_r| \leqslant M_2$ for all non-negative α. Furthermore, Lemma 3.4 says there exists a constant M_3 satisfying the condition

$$|\nabla G_\alpha[g]| = |(G_\alpha[g])_w| + |(G_\alpha[g])_{\bar{w}}| \leqslant M_3$$

whenever α is non-negative. Hence, we have shown that the condition $|\nabla v| \leqslant \max\{M_1, 2\sqrt{2}M_2 + M_3\}$, which implies that v is Lipschitz continuous on $\mathbb{D}$, as desired. □

In particular, if g vanishes, then we obtain the following corollary.

Corollary 3.1. *Assume that $u \in C^2(\mathbb{D})$ is an $\bar{\alpha}$-harmonic mapping of the unit disk $\mathbb{D}$ satisfying that $u(re^{i\theta})$ tends to the function f as r tends to 1, where f is absolutely continuous on the unit circle $\mathbb{T}$. If f satisfies that $f' \in L^\infty(\mathbb{T})$ and the integral*

$$\frac{1}{2\pi}\int_{\mathbb{T}} \frac{(1-|w|^2)^\alpha}{(1-z\bar{w})^\alpha} \frac{(w\bar{z}-\bar{w}z)/\mathrm{i}}{|z-w|^2}[f(\mathrm{e}^{\mathrm{i}\theta})]'_\theta d\theta$$

is bounded, where $z = \mathrm{e}^{\mathrm{i}\theta}$, then u is Lipschitz continuous on the unit disk $\mathbb{D}$ when α is at least zero.

4 Proof of Theorem 1.2

In this section, we will give the proof of main result of this paper, which gives four equivalent conditions for the solutions of the $\bar{\alpha}$-Poisson equation with a nonhomogeneous term to be Lipschitz continuous.

Theorem 1.2. *Assume that $v \in V_{\mathbb{D}\to\Omega}[g]$ with the representation $v(w) = u(w) + G_\alpha[g](w)$. Then the following conditions are equivalent:*

(a) *v is a (K, K')-quasiconformal mapping and $|\frac{\partial u}{\partial r}| \leqslant L$ on $\mathbb{D}$, where L is a constant.*
(b) *v is Lipschitz continuous with the Euclidean metric.*
(c) *u is Lipschitz continuous with the Euclidean metric.*
(d) *f is absolutely continuous on $\mathbb{T}$, $f' \in L^\infty(\mathbb{T})$ and the following integral*

$$\frac{1}{2\pi}\int_{\mathbb{T}} \frac{(1-|w|^2)^\alpha}{(1-z\bar{w})^\alpha} \frac{(w\bar{z}-\bar{w}z)/\mathrm{i}}{|z-w|^2}[f(\mathrm{e}^{\mathrm{i}\theta})]'_\theta d\theta$$

is bounded, where $z = \mathrm{e}^{\mathrm{i}\theta}$.

Proof. We first prove that (a) ⇒ (b) holds. By the assumption that v is (K, K')-quasiconformal, we have the inequality

$$|\nabla v|^2 \leqslant KJ_v + K',$$

where J_v is the Jacobian of v. Thus, it follows that the inequality

$$|\nabla v| \leqslant Kl(\nabla v) + \sqrt{K'} \leqslant K|v_r| + \sqrt{K'}$$

is established. Furthermore, Lemmas 3.1 and 3.4 together imply

$$|v_r| \leqslant |u_r| + |(G_\alpha[g])_r| \leqslant L + (|(G_\alpha[g])_w| + |(G_\alpha[g])_{\bar{w}}|) \leqslant L + M_G,$$

where $M_G = (\frac{|\alpha|4^{\alpha+1}C_\alpha}{6\Gamma(3/2)} + \frac{4^{\alpha+1}}{6})\|g\|_\infty$. Hence,

$$|\nabla v| \leqslant K(L + M_G) + \sqrt{K'}$$

is satisfied, which implies that v is Lipschitz continuous in the Euclidean metric on $\mathbb{D}$.

By the assumption that v is Lipschitz continuous in the Euclidean metric, there exists a constant M_v satisfying

$$|\nabla v| \leqslant M_v.$$

Thus, the relation $|\nabla G_\alpha[g]| \leqslant M_G$ together with the representation $u = v - G_\alpha[g]$ imply that the relations

$$|\nabla u| \leqslant |\nabla v| + |\nabla G_\alpha[g]| \leqslant M_v + M_G$$

hold. Hence, u is also Lipschitz continuous in the Euclidean metric on $\mathbb{D}$. Thus we have completed the proof of $(b) \Rightarrow (c)$.

Since u is Lipschitz continuous in the Euclidean metric on $\mathbb{D}$, there exists a constant M_u satisfying

$$|\nabla u| \leqslant M_u.$$

Utilizing Lemma 3.1, we obtain

$$\left|\frac{\partial u}{\partial r}\right| \leqslant |\nabla u| \leqslant M_u.$$

Furthermore, we have the inequality

$$|\nabla v|^2 \leqslant J_v + 2(M_u + M_G)^2.$$

Thus, v is $(1, 2(M_u + M_G)^2)$-quasiconformal on $\mathbb{D}$. Therefore, we have established $(c) \Rightarrow (a)$.

Theorem 1.1 says that (d) $\Rightarrow$ (b) holds.

Assume that v is Lipschitz continuous in the Euclidean metric on $\mathbb{D}$, and hence it is Lipschitz continuous on $\mathbb{T}$; thus, f and $f' \in L^\infty(\mathbb{T})$. Furthermore, from the proof of Lemma 3.3, we have

$$\left|\frac{1}{2\pi}\int_{\mathbb{T}} \frac{(1-|w|^2)^\alpha}{(1-z\bar{w})^\alpha} \frac{(w\bar{z}-\bar{w}z)/\mathrm{i}}{|z-w|^2}[f(\mathrm{e}^{\mathrm{i}\theta})]'_\theta d\theta\right| \leqslant |ru_r| + \left|\frac{\alpha}{2\pi}\int_{\mathbb{T}} \frac{z\bar{w}(1-|w|^2)^\alpha}{(1-z\bar{w})^{\alpha+1}} f(z) d\theta\right|.$$

Lemma A with the inequality $|u_r| \leqslant |v_r| + |(G_\alpha[g])_r| \leqslant (|\nabla v| + |\nabla G_\alpha[g]|)$ gives

$$\left|\frac{1}{2\pi}\int_{\mathbb{T}} \frac{(1-|w|^2)^\alpha}{(1-z\bar{w})^\alpha} \frac{(w\bar{z}-\bar{w}z)/\mathrm{i}}{|z-w|^2}[f(\mathrm{e}^{\mathrm{i}\theta})]'_\theta d\theta\right| \leqslant (M_v + M_G) + \alpha\frac{\Gamma(\alpha)}{\Gamma(\frac{\alpha+1}{2})^2}\|f\|_\infty.$$

Thus, the proof of (b) $\Rightarrow$ (d) is completed. □

Remark 4.1. When α is zero and $f(\mathbb{T})$ is a convex Jordan curve, Theorem 1.2 with the Radó-Choquet-Kneser theorem is the version of Theorem 3.2 of [21].

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References

1 Ahlfors L V. Lectures on Quasiconformal Mappings. University Lecture Series, vol. 38. Providence: Amer Math Soc, 2006

2 Astala K. Iwaniec T, Martin G. Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane. Princeton: Princeton University Press, 2009

3 Behm G. Solving Poisson equation for the standard weighted Laplacian in the unit disc. ArXiv:1306.2199v2, 2013

4 Chen J L, Li P J, Sahoo S K, et al. On the Lipschitz continuity of certain quasiregular mappings between smooth Jordan domains. Israel J Math, 2017, 220: 453–478

5 Chen M, Chen X D. (K, K')-quasiconformal harmonic mappings of the upper half plane onto itself. Ann Acad Sci Fenn Math, 2012, 37: 265–276

6 Chen S L, Vuorinen M. Some properties of a class of elliptic partial differential operators. J Math Anal Appl, 2015, 431: 1124–1137

7 Chen X D, Kalaj D. A representation theorem for standard weighted harmonic mappings with an integer exponent and its applications. J Math Anal Appl, 2016, 444: 1233–1241

8 Chen X D, Kalaj D. Representation theorem of standard weighted harmonic mappings in the unit disk. Https://www.researchgate.net/publication/330243622, 2018
9 Clunie J, Theil-Small T. Harmonic univalent functions. Ann Acad Sci Fenn Math, 1984, 9: 3–25
10 Duren P. Harmonic Mappings in the Plane. Cambridge: Cambridge University Press, 2004
11 Finn R, Serrin J. On the Hölder continuity of quasi-conformal and elliptic mappings. Trans Amer Math Soc, 1958, 89: 1–15
12 Garabedian P R. A partial differential equation arising in conformal mapping. Pacific J Math, 1951, 1: 485–524
13 Garnett J B. Bounded Analytic Functions, 1st ed. Graduate Texts in Mathematics. New York: Springer, 2007
14 Hedenmalm H, Korenblum B, Zhu K. Theory of Bergman spaces. Graduate Texts in Mathematics. New York: Springer-Verlag, 2000
15 Kalaj D. Quasiconformal mapping between Jordan domains. Math Z, 2008, 260: 237–252
16 Kalaj D. On quasi-conformal harmonic maps between surfaces. Int Math Res Not IMRN, 2015, 2015: 355–380
17 Kalaj D, Mateljević M. Inner estimate and quasiconformal harmonic maps between smooth domains. J Anal Math, 2006, 100: 117–132
18 Kalaj D, Mateljević M. (K, K')-quasiconformal harmonic mappings. Potential Anal, 2012, 36: 117–135
19 Kalaj D, Pavlović M. On quasiconformal self-mappings of the unit disk satisfying Poisson equation. Trans Amer Math Soc, 2011, 363: 4043–4061
20 Lehto O, Virtanen K I. Quasiconformal Mappings in the Plane. Berlin-Heidelberg-New York: Springer-Verlag, 1973
21 Li P J, Chen J L, Wang X T. Quasiconformal solution of Poisson equations. Bull Aust Math Soc, 2015, 92: 420–428
22 Manojlović V. Bi-Lipschicity of quasiconformal harmonic mappings in the plane. Filomat, 2009, 23: 85–89
23 Martio O. On harmonic quasiconformal mappings. Ann Acad Sci Fenn Ser A, 1968, 425: 3–10
24 Mateljević M. Quasiconformality of harmonic mappings between Jordan domains. Filomat, 2012, 26: 479–510
25 Morrey C B. On the solutions of quasi-linear elliptic partial differential equations. Trans Amer Math Soc, 1938, 43: 126–166
26 Mu J J, Chen X D. Landau-type theorems for solutions of a quasilinear differential equation. J Math Study, 2014, 47: 295–304
27 Nirenberg L. On nonlinear elliptic partial differential equations and Hölder continuity. Comm Pure Appl Math, 1953, 6: 103–156
28 Olofsson A. Differential operators for a scale of Poisson type kernels in the unit disc. J Anal Math, 2014, 123: 227–249
29 Olofsson A, Wittsten J. Poisson integrals for standard weighted Laplacians in the unit disc. J Math Soc Japan, 2013, 65: 447–486
30 Partyka D, Sakan K. On bi-Lipschitz type inequalities for quasiconformal hamonic mappings. Ann Acad Sci Fenn Math, 2007, 32: 579–594
31 Pavlović M. Boundary correspondence under harmonic quasiconformal homeomorphisms of the unit disk. Ann Acad Sci Fenn Math, 2002, 27: 365–372
32 Simon L. A Hölder estimate for quasiconformal maps between surfaces in Euclidean space. Acta Math, 1977, 139: 19–51