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The LYZ centroid conjecture for star bodies

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Abstract Lutwak et al. (2010) established the Orlicz centroid inequality for convex bodies and conjectured that their Orlicz centroid inequality could be extended to star bodies. Zhu (2012) confirmed the conjectured Lutwak, Yang and Zhang (LYZ) Orlicz centroid inequality and solved the equality condition for the case that ϕ is strictly convex. Without the condition that ϕ is strictly convex, this paper studies the equality condition of the conjectured LYZ Orlicz centroid inequality for star bodies.

Keywords Brunn-Minkowski theory, Orlicz centroid body, Orlicz centroid inequality, star body, Steiner symmetrization

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1 Introduction and preliminaries

The centroid body operator is one of the central notions in convex geometry which rooted back at least to Dupin (see also [28]). The classical affine isoperimetric inequality that relates the volume of a convex body with that of its centroid body was conjectured by Blaschke (see also [10, 28, 37, 46]) and established in a landmark work of Petty [36]. Since Petty "reinterpreted" and made critical use of Busemann's random simplex inequality (see [3]) in establishing his inequality, Petty's theorem is known as the Busemann-Petty centroid inequality (see [10, 37]).

In [25], Lutwak et al. extended the notion of centroid body to the L_p analogues of centroid body, and established the L_p analogues of centroid inequality. An alternative proof of the L_p centroid inequality was provided by Campi and Gronchi [4]. The L_p Busemann-Petty centroid inequality became a central focus in the L_p Brunn-Minkowski theory and its dual (see [10, 23, 24, 37] for more references). Furthermore, the L_p centroid bodies quickly became objects of interest in asymptotic geometric analysis (see [8, 9, 20, 31–34]) and the theory of stable distributions (see [30]). The literature is large and continues to grow (see, for example, [5, 7, 14, 26, 29, 38, 42]). For more references, see [2, 19, 39–41, 47–49].

Recently, as an extension of the L_p Brunn-Minkowski theory, the Orlicz Brunn-Minkowski theory emerged in three landmark works by Haberl et al. [15] and Lutwak et al. [27, 28]. This extension is motivated by asymmetric concepts within the L_p Brunn-Minkowski theory developed by Haberl and

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Schuster [16, 17], Haberl et al. [18], Ludwig and Reitzner [22] and Ludwig [21]. The new Orlicz Brunn-Minkowski theory has attracted considerable interest (see, for example, [1, 6, 11, 43, 44, 50]). See [12, 45] for its dual theory.

In [28], Lutwak et al. introduced the concept of Orlicz centroid body that is a natural extension of the centroid body and its L_p extension. The fundamental result in [28] is the Orlicz Busemann-Petty centroid inequalities for convex bodies. After that, many works have been done for inequalities and reverse inequalities (see [6, 35, 46]). Among those, Zhu [46] developed an important tool—the Steiner symmetrization for star bodies. By applying his new tool, Zhu settled the conjectured LYZ Orlicz centroid inequality. Zhu [46] also solved the equality condition for the case where ϕ is strictly convex. It is the aim of this paper to extend the method used by Zhu [46] and to study the equality condition for the Orlicz centroid inequality without the condition that ϕ is strictly convex.

In order to keep the paper self-contained, we first collect notation, definitions and basic facts about convex bodies and star bodies. More detailed theories and references are included in books of Gardner [10], Gruber [13] and Schneider [37].

Let $\mathbb{R}^n$ be the Euclidean space with the usual inner product $x \cdot y$ and standard Euclidean norm $|x|$ for $x, y \in \mathbb{R}^n$. We write $e_1, \ldots, e_n$ for the standard unit vector basis of $\mathbb{R}^n$. When we write $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$, we always assume that e_n is associated with the last factor. The unit sphere is denoted by S^{n-1}. We will use x, y for vectors in $\mathbb{R}^n$ and x', y' for vectors in $\mathbb{R}^{n-1}$.

If K is a Borel subset of $\mathbb{R}^n$ and is contained in an i-dimensional affine subspace of $\mathbb{R}^n$ but not in any affine subspace of lower dimension, let $|K|$ denote the i-dimensional Lebesgue measure of K.

For $A \in GL(n)$ we write A^t for the transpose of A, A^{-t} for the inverse of the transpose of A, and $|A|$ for the absolute value of the determinant of A.

Let K be a convex body (compact convex subset with nonempty interiors) in $\mathbb{R}^n$. Its *support function* $h_K = h(K, \cdot) : \mathbb{R}^n \to [0, +\infty)$, is defined by $h(K, x) = \max\{x \cdot y : y \in K\}$. When considering the convex body $K \subset \mathbb{R}^{n-1} \times \mathbb{R}$, we usually write $h(K; x', t)$ rather than $h(K; (x', t))$. Let $\mathcal{K}^n$ be the set of all convex bodies, and $\mathcal{K}^n_o$ be the set of convex bodies that contain the origin in their interiors.

The *Hausdorff distance* between two convex bodies K and L is

$$\delta(K, L) = \max_{u \in S^{n-1}} |h_K(u) - h_L(u)|.$$

If $K, L \in \mathcal{K}^n$, and h_K and h_L are support functions of K and L, respectively, then

$$h_K \leqslant h_L \quad \text{if and only if} \quad K \subset L.$$

For $c > 0$, $u \in S^{n-1}$,

$$h_{cK}(u) = c h_K(u), \quad h_K(cu) = c h_K(u),$$

where $cK = \{cx : x \in K\}$. More generally, by the definition of the support function, we have

$$\begin{aligned} h_{K+L}(u) &= h_K(u) + h_L(u), \\ h_{AK}(u) &= h_K(A^t u), \quad \text{for} \quad A \in GL(n), \end{aligned}$$

where $K + L = \{x + y : x \in K, y \in L\}$ and $AK = \{Ax : x \in K\}$.

Let K be a convex body in $\mathbb{R}^n$. For $u \in S^{n-1}$, let K_u be the image of the orthogonal projection of K onto the hyperplane $u^\perp$. If $\underline{l}_u(K, y')$ and $\bar{l}_u(K, y')$ are two concave real functions on K_u such that

$$K = \{y' + tu : -\underline{l}_u(K, y') \leqslant t \leqslant \bar{l}_u(K, y'), y' \in K_u\},$$

then we call $\underline{l}_u(K, y')$ the *undergraph function* of K in the direction u, and $\bar{l}_u(K, y')$ the *overgraph function*, respectively.

Let $S_u K$ denote the *Steiner symmetral* of K with respect to $u^\perp$,

$$S_u K = \left\{ y' + tu : |t| \leqslant \frac{\underline{l}_u(K, y') + \bar{l}_u(K, y')}{2} \right\}.$$

This means

$$\underline{l}_u(S_uK, y') = \bar{l}_u(S_uK, y') = (\underline{l}_u(K, y') + \bar{l}_u(K, y'))/2.$$

Fubini's theorem yields $|S_uK| = |K|$. If we iterate Steiner symmetrizations of K through a suitable sequence of unit directions, the successive Steiner symmetrals of K will approach a Euclidean ball in the Hausdorff topology on compact (in particular, convex) subsets of $\mathbb{R}^n$ (see, e.g., [13]).

Let K be a compact star-shaped set (about the origin) in $\mathbb{R}^n$. Its *radial function*, $\rho_K = \rho(K, \cdot) : \mathbb{R}^n \to [0, +\infty)$, is defined by $\rho_K(x) = \max\{\lambda \geqslant 0 : \lambda x \in K\}$. If ρ is strictly positive and continuous, then we call K a star body. Let $\mathcal{S}_o^n$ denote the set of star bodies with respect to the origin in $\mathbb{R}^n$.

Let K be a star body, and $u \in S^{n-1}$. We use $\bar{l}_u(K, y')$ and $\underline{l}_u(K, y')$ to denote $\max\{\lambda : y' + \lambda u \in K\}$ and $\max\{\lambda : y' - \lambda u \in K\}$, respectively. Note that if K is a convex body, they are just the overgraph and undergraph functions of K.

Let $\mathcal{C}$ denote the set of convex functions $\phi : \mathbb{R} \to [0, \infty)$ such that $\phi(0) = 0$ and $\phi(t) + \phi(-t) \neq 0$ for $t \neq 0$. For a convex function ϕ, the condition "$\phi(0) = 0$" means that ϕ is monotone decreasing on $(-\infty, 0]$ and monotone increasing on $[0, \infty)$. Then the condition "$\phi(t) + \phi(-t) \neq 0$ for $t \neq 0$" is equivalent to that ϕ is either strictly monotone decreasing on $(-\infty, 0]$ or strictly monotone increasing on $[0, \infty)$.

A stronger condition "$\phi(t)\phi(-t) \neq 0$ for $t \neq 0$" can guarantee both strictly monotone decreasing on $(-\infty, 0]$ and strictly monotone increasing on $[0, \infty)$. Note that "$\phi(t)\phi(-t) \neq 0$ for $t \neq 0$" means "$\phi(t) \neq 0$ for $t \neq 0$". Moreover, the condition "ϕ is a strictly convex function" is stronger than both "$\phi(t) + \phi(-t) \neq 0$ for $t \neq 0$" and "$\phi(t)\phi(-t) \neq 0$ for $t \neq 0$".

Let K be a star body with respect to the origin in $\mathbb{R}^n$. The corresponding support function of the Orlicz centroid body $\Gamma_\phi K$ for $\phi \in \mathcal{C}$ and $x \in \mathbb{R}^n$ is defined by

$$h_{\Gamma_\phi K}(x) = \inf\left\{\lambda > 0 : \frac{1}{|K|}\int_K \phi\left(\frac{x \cdot y}{\lambda}\right) dy \leqslant 1\right\}, \tag{1.1}$$

where $|K|$ is the volume of $K \in \mathcal{S}_o^n$, $x \cdot y$ denotes the usual inner product of x and y in $\mathbb{R}^n$ and integration is with respect to Lebesgue measure in $\mathbb{R}^n$.

An important special case is when $\phi(t) = |t|^p$ for some $p \geqslant 1$. Then $\Gamma_\phi K$ is the L_p centroid body of K, whose support function is given by

$$h(\Gamma_p K, x) = \frac{1}{|K|}\int_K |x \cdot y|^p dy. \tag{1.2}$$

In particular, if $p = 1$, then the body $\Gamma_p K$ is the classical centroid body ΓK of K.

Unlike ΓK, the Orlicz centroid body $\Gamma_\phi K$ is not translation invariant for a general $\phi \in \mathcal{C}$, and may not be o-symmetric, while [28] showed the Orlicz centroid body operator retains continuity in Hausdorff metric and $GL(n)$ covariance. The property of $GL(n)$ covariance can be formulated as

$$\Gamma_\phi AK = A\Gamma_\phi K, \quad \text{for} \quad A \in GL(n), \quad K \in \mathcal{S}_o^n, \quad \phi \in \mathcal{C}. \tag{1.3}$$

Lutwak et al. [28] obtained the following Orlicz Busemann-Petty centroid inequality.

Theorem A. *If $\phi \in \mathcal{C}$ and K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior, then the volume ratio*

$$|\Gamma_\phi K|/|K|$$

is minimized if and only if K is an ellipsoid centered at the origin.

Lutwak et al. [28] posed the following open problem.

LYZ Conjecture. *If $\phi \in \mathcal{C}$ and K is a* **star** *body in $\mathbb{R}^n$ that contains the origin in its interior, then the volume ratio*

$$|\Gamma_\phi K|/|K| \tag{1.4}$$

is minimized if and only if K is an ellipsoid centered at the origin.

Zhu [46] extended the Orlicz Busemann-Petty centroid inequalities from convex to star bodies and solved the equality condition for the case that ϕ is strictly convex.

Theorem B. *If $\phi \in \mathcal{C}$ and K is a star body with respect to the origin, then the volume ratio*

$$|\Gamma_\phi K|/|K|$$

is minimized when K is an ellipsoid centered at the origin. If ϕ is a strictly convex function, then ellipsoids centered at the origin are the only minimizers.

Motivated by ideas of Lutwak et al. [28] and Zhu [46], we confirm the conjectured LYZ Orlicz centroid inequality and solve the equality condition for star bodies in $\tilde{\mathcal{S}}_o^n$ without the condition of ϕ's strict convexity.

If K is a star body that is not convex, then there exist $P_1, P_2 \in K$ such that the segment $\overline{P_1P_2}$ does not completely lie in K. Set $u = \overrightarrow{P_1P_2}/|\overrightarrow{P_1P_2}|$. Let U_K be the set of all of these u about K, and let $K' = K_u$ be the image of the projection of K onto $u^\perp$. By Lemma 3.6, for every $u \in U_K$, and $y' \in u^\perp$, there exist $x_1', x_2' \in u^\perp$ and λ_1, λ_2 satisfying that

$$\lambda_1 = h(\Gamma_\phi K, x_1', 1) = \bar{l}_{u_0}(\Gamma_\phi K, y') + x_1' \cdot y' \tag{1.5}$$

and

$$\lambda_2 = h(\Gamma_\phi K, x_2', -1) = \bar{l}_{u_0}(\Gamma_\phi K, y') + x_2' \cdot y'. \tag{1.6}$$

A star body K which is not convex is called a Φ-*star body* if there exist $u_0 \in U_K$, $y_0' \in K'$ such that there are at least three points in $(y_0' + \mathbb{R}u_0) \cap \partial K$, and $\bar{l}_{u_0}(K, y_0')$ (or $\underline{l}_{u_0}(K, y_0')$) $\notin [s_1(y_0'), s_2(y_0')]$. Here, $s_1(y')$ and $s_2(y')$ are, respectively, the left and right monotone points (see the note of Lemma 2.3) of

$$f_{y'}(s) = \frac{2\lambda_1}{\lambda_1 + \lambda_2}\phi\left(\frac{x_1' \cdot y' + s}{\lambda_1}\right) + \frac{2\lambda_2}{\lambda_1 + \lambda_2}\phi\left(\frac{x_2' \cdot y' - s}{\lambda_2}\right), \tag{1.7}$$

for some $x_1', x_2' \in u^\perp$ and λ_1, λ_2 which satisfy (1.5) and (1.6). Let $\tilde{\mathcal{S}}_o^n$ denote the union of $\mathcal{K}_o^n$ and the set of all Φ-star bodies.

We show that for any $\phi \in \mathcal{C}$, Orlicz Busemann-Petty centroid inequality holds for $K \in \tilde{\mathcal{S}}_o^n$ with its equality condition. This solves the uniqueness of the volume ratio $|\Gamma_\phi K|/|K|$ for arbitrary $\phi \in \mathcal{C}$ and confirms the LYZ conjecture for star bodies in $\tilde{\mathcal{S}}_o^n$. Our main work can be described as follows.

Theorem 1.1. *Let $\phi \in \mathcal{C}$ and K be a star body with respect to the origin. Then the volume ratio*

$$|\Gamma_\phi K|/|K|$$

is minimized when K is an ellipsoid centered at the origin. If $K \in \tilde{\mathcal{S}}_o^n$, then ellipsoids centered at the origin are the only minimizers.

In the case that ϕ is strictly convex, it follows from Lemma 2.2 that $s_1(y_0') = s_2(y_0')$ for (1.7). It is true that $\bar{l}_{u_0}(K, y_0')$ (or $\underline{l}_{u_0}(K, y_0')$) $\notin [s_1(y_0'), s_2(y_0')]$, so every non-convex body is Φ-star body. In this sense, our result is a generalization of Zhu's result (see Theorem B).

Our proof of the LYZ conjecture is based on the methods used by Lutwak et al. [28] and Zhu [46] and Steiner symmetrization. The novel idea of the proof is to reduce the problem to show that if $K \in \tilde{\mathcal{S}}_o^n$ and is not convex, then the volume ratio is not minimized. This can be shown by Steiner symmetrization of $\Gamma_\phi K$ with its identity condition. To show the LYZ conjecture for any $\phi \in \mathcal{C}$, we will study the monotonicity and integral inequality of ϕ which will be strictly used in showing Steiner symmetrization of $\Gamma_\phi K$. After establishing Steiner symmetrization of Orlicz centroid bodies in Section 3, we prove the LYZ Orlicz centroid inequality for star bodies in $\tilde{\mathcal{S}}_o^n$ in the last section.

2 Properties of convex function $\phi \in \mathcal{C}$

Let $\mathcal{C}$ denote the set of convex functions $\phi : \mathbb{R} \to [0, \infty)$ such that $\phi(0) = 0$ and $\phi(t) + \phi(-t) \neq 0$ for $t \neq 0$. For any $\phi \in \mathcal{C}$, we have the following lemma (see [46]).

Lemma 2.1. *Let $\phi \in \mathcal{C}$. For real $a_i > 0$, $b_i, c_i > 0$ $(i = 1, 2)$, let $s_m = \min\{-\frac{b_1}{a_1}, \frac{b_2}{a_2}\}$, $s_M = \max\{-\frac{b_1}{a_1}, \frac{b_2}{a_2}\}$ and*

$$f(s) = c_1\phi(a_1 s + b_1) + c_2\phi(-a_2 s + b_2).$$

Then there exists an $s_0 \in [s_m, s_M]$ such that $f(s)$ is monotone decreasing on $(-\infty, s_0]$ and monotone increasing on $[s_0, \infty)$.

Lemma 2.2. *Let $\phi \in \mathcal{C}$. For real $a_i > 0$, $b_i, c_i > 0$ $(i = 1, 2)$, let $s_m = \min\{-\frac{b_1}{a_1}, \frac{b_2}{a_2}\}$, $s_M = \max\{-\frac{b_1}{a_1}, \frac{b_2}{a_2}\}$ and let*

$$f(s) = c_1\phi(a_1 s + b_1) + c_2\phi(-a_2 s + b_2).$$

Then, we have either

(I) *there exists a unique s_0 such that $f(s)$ is strictly monotone decreasing on $(-\infty, s_0]$ and strictly monotone increasing on $[s_0, \infty)$; or*

(II) *there exist $s_1, s_2 \in [s_m, s_M]$ with $s_1 < s_2$ such that $f(s)$ is strictly monotone decreasing on $(-\infty, s_1]$, strictly monotone increasing on $[s_2, \infty)$ and $f(s) = \text{const.}$ for all $s \in [s_1, s_2]$.*

Proof. Since (I) is the special case $s_1 = s_2$ in (II), it suffices to show (II). For $\phi(s) \in \mathcal{C}$, without loss of generality, let $\phi(s)$ be strictly monotone decreasing on $(-\infty, 0]$ and monotone increasing on $[0, \infty)$. Let $f_1(s) = c_1\phi(a_1 s + b_1), f_2(s) = c_2\phi(-a_2 s + b_2)$. Obviously, f_1 and f_2 are convex, by the convexity of ϕ.

By the monotone property of ϕ, f_1 is strictly monotone decreasing on $(-\infty, -\frac{b_1}{a_1}]$ and monotone increasing on $[-\frac{b_1}{a_1}, \infty)$, f_2 is monotone decreasing on $(-\infty, \frac{b_2}{a_2}]$ and strictly monotone increasing on $[\frac{b_2}{a_2}, \infty)$. Thus, $f = f_1 + f_2$ is strictly monotone decreasing on $(-\infty, s_m]$ and strictly monotone increasing on $[s_M, \infty,)$.

Next, we consider the monotone property of f on $[s_m, s_M]$. By the convexity of f_1 and f_2, f is convex on $[s_m, s_M]$.

Let us recall the differentiability of 1-dimensional convex functions (see, e.g., [37, Theorem 1.5.4]). If $f : \mathbb{R} \to \mathbb{R}$ is convex, then on the interior of the domain of f the right derivative f'_r and the left derivative f'_l exist and are monotonically increasing functions. Furthermore, $f'_l \leqslant f'_r$, and with the exception of at most countably many points, $f'_l = f'_r$ holds and hence f is differentiable.

If there does not exist $s_1, s_2 \in [s_m, s_M]$, such that $f(s) = \text{const.}$ for $s \in [s_1, s_2]$, then there exists a unique s_0 such that $f(s)$ is strictly monotone decreasing on $(-\infty, s_0]$ and strictly monotone increasing on $[s_0, \infty)$. Indeed, if there does not exist $s_1, s_2 \in [s_m, s_M]$, such that $f(s) = \text{const.}$ for $s \in [s_1, s_2]$, then it follows from the monotonically increasing property of f'_l and f'_r that there exists at most one point where $f'_l = f'_r = 0$. Therefore, there exists a unique s_0 such that $f(s)$ is strictly monotone decreasing on $(-\infty, s_0]$ and strictly monotone increasing on $[s_0, \infty)$, which is (I).

If there exist $s_1, s_2 \in [s_m, s_M]$ such that $f(s) = \text{const.}$ for $s \in [s_1, s_2]$, then choose the maximal interval $[s_1, s_2]$ such that the above holds.

First, we claim that if $f(s) = \text{const.}$ for $s \in [s_1, s_2]$, then f_1 and f_2 are linear functions and $f'_1 = -f'_2$ on $[s_1, s_2]$. In fact, if f_1 is not a linear function, the convexity of f_1 implies that f_1 is strictly convex on $[s_1, s_2]$. Then $f_2 = f - f_1 = \text{const.} - f_1$ is strictly concave on $[s_1, s_2]$, which obviously leads to a contradiction. We may assume $f_1 = ks + b_1$, $f_2 = k's + b_2$. Since $f_1 + f_2 = f$, we have $k' = -k$. Now, we prove that f is strictly monotone on $[s_m, s_1]$ and $[s_2, s_M]$. Suppose that there exists another maximal interval $[s_3, s_4] \subset [s_m, s_M]$ such that $f(s) = \text{const.}'$ for $s \in [s_3, s_4]$. Without loss of generalization, assume $s_3 > s_2$. Since f_1 and f_2 are convex then the (left) derivatives of f_1 and f_2 are monotonically increasing, namely for $s \in [s_3, s_4]$, $f'_{1l} > k, f'_{2l} > -k$. Then $f'_{1l} = -f'_{2l}$ is impossible on $[s_3, s_4]$. Therefore, there exists $[s_1, s_2] \subset [s_m, s_M]$ such that $f(s) = \text{const.}$ for $s \in [s_1, s_2]$.

Next, for any $s_0 \in [s_1, s_2]$, we have $f(s_0) = \min\{f(s) : s \in \mathbb{R}\}$. In fact, it is a corollary of the uniqueness of the maximal choice of $[s_1, s_2]$. The condition that $f(s) = \text{const.}$ for $s \in [s_1, s_2]$ implies that $f'(s) = 0$ for $s \in [s_1, s_2]$. Since f'_l is increasing, the uniqueness of the interval $[s_1, s_2]$ indicates that $f'_l(s) < 0$ on $s \in (-\infty, s_1]$ and $f'_l(s) > 0$ on $s \in (s_2, \infty]$.

Thus, we have proved that if there exist $s_1, s_2 \in [s_m, s_M]$ such that $f(s) = \text{const.}$ for $s \in [s_1, s_2]$, then (II) holds. □

The following lemma is a direct corollary of Lemma 2.2.

Lemma 2.3. *Under the condition of Lemma 2.2, there exist $s_1, s_2 \in [s_m, s_M]$ with $s_1 \leqslant s_2$ such that $f(s)$ is strictly monotone decreasing on $(-\infty, s_1]$, strictly monotone increasing on $[s_2, \infty)$ and $f(s) =$ const. for all $s \in [s_1, s_2]$.*

Note that for convex function f in Lemma 2.3, s_1 and s_2 uniquely exist, and are called the *left monotone point and right one* of f, respectively.

Lemma 2.4. (I) *Let $f(s)$ be strictly monotone decreasing on $(-\infty, s_0]$ and strictly monotone increasing on $[s_0, \infty)$. If $E \subset \mathbb{R}$ is compact, then*

$$\int_E f(s)ds \geqslant \int_{s_0-\delta_{s_0}^-}^{s_0+\delta_{s_0}^+} f(s)ds,$$

where $\delta_{s_0}^+ = |E \cap [s_0, \infty)|, \delta_{s_0}^- = |E \cap (-\infty, s_0]|$. If there exists an s_0' not in E and $|E \cap (-\infty, s_0']| > 0$, $|E \cap [s_0', \infty)| > 0$, then equality cannot hold.

(II) *Suppose that $f(s)$ is strictly monotone decreasing on $(-\infty, s_1]$, strictly monotone increasing on $[s_2, \infty)$ and $f(s) =$ const. for $s \in [s_1, s_2]$. Let $E \subset \mathbb{R}$ be compact and $\delta_{s_i}^+ = |E \cap [s_i, \infty)|$, $\delta_{s_i}^- = |E \cap (-\infty, s_i]|, i = 1, 2$, and let $s_- = \min_{s \in E}\{s\}$ and $s_+ = \max_{s \in E}\{s\}$. Then, we have*

$$\int_E f(s)ds \geqslant \int_{s_1-\delta_{s_1}^-}^{s_1+\delta_{s_1}^+} f(s)ds, \tag{2.1}$$

$$\int_E f(s)ds \geqslant \int_{s_2-\delta_{s_2}^-}^{s_2+\delta_{s_2}^+} f(s)ds. \tag{2.2}$$

If there exists an $s_0' \in [s_-, s_+]$ but not in E and $|E \cap (-\infty, s_0']| > 0, |E \cap [s_0', \infty)| > 0$, then the equality in (2.1) holds if and only if $s_0' \in [s_1, s_2]$, $[s_-, s_1] \subset E$ and $s_+ \leqslant s_2$; the equality in (2.2) holds if and only if $s_0' \in [s_1, s_2]$, $[s_2, s_+] \subset E$ and $s_- \geqslant s_1$.

Therefore, if there exists an $s_0' \in [s_-, s_+]$ but not in E and $|E \cap (-\infty, s_0']| > 0, |E \cap [s_0', \infty)| > 0$, then both equalities hold only when $E \subset [s_1, s_2]$.

Proof. We only need to prove (II). In fact, (I) is a corollary of (II) when $s_1 = s_2$.

To prove the first inequality (2.1) in (II), it suffices to show that

$$\int_{E\cap(-\infty,s_1]} f(s)ds \geqslant \int_{s_1-\delta_{s_1}^-}^{s_1} f(s)ds, \tag{2.3}$$

$$\int_{E\cap[s_1,\infty)} f(s)ds \geqslant \int_{s_1}^{s_1+\delta_{s_1}^+} f(s)ds. \tag{2.4}$$

Since E is compact, we have $E \cap (-\infty, s_1]$ and $E \cap [s_1, \infty)$ are compact, and $s_- = \min_{s\in E}\{s\} > -\infty$ and $s_+ = \max_{s \in E}\{s\} < \infty$.

Now, we show (2.3).

Suppose that $[s_-, s_1] \subset E$. Then $s_- = s_1 - \delta_{s_1}^-$. Hence, $E \cap (-\infty, s_1] = E \cap [s_-, s_1] = [s_-, s_1] = [s_1 - \delta_{s_1}^-, s_1]$. Thus, the equality in (2.3) holds.

Suppose that $(s_-, s_1) \setminus E$ is a non-empty open set in $\mathbb{R}$, which is a union of at most countably many open intervals $\{U_i\}_{i=1}^{\infty}, U_i \cap U_j = \emptyset$ for $i \neq j$ and $i, j = 1, 2, \ldots, \infty$.

Let $U_i = (s_l^i, s_r^i)$ and $s_r^{i+1} \leqslant s_l^i, i = 1, 2, \ldots, \infty$. We may assume that $(s_-, s_r^1) \setminus (\bigcup_{i=1}^{\infty} U_i)$ is a non-zero 1-dimensional Lebesgue measurable set. Otherwise, the Lebesgue measurable set $(s_-, s_r^1) \setminus (\bigcup_{i=1}^{\infty} U_i)$ is measure zero. Then

$$\begin{aligned}\int_{E\cap(-\infty,s_1]} f(s)ds &= \int_{(s_-,s_r^1)\setminus(\bigcup_{i=1}^{\infty} U_i)} f(s)ds + \int_{(s_r^1,s_1)} f(s)ds \\ &= \int_{(s_r^1,s_1)} f(s)ds = \int_{s_1-\delta_{s_1}^-}^{s_1} f(s)ds,\end{aligned}$$

which is the equality of (2.3). We may also assume that $|\bigcup_{i=1}^{\infty} U_i| > 0$. Otherwise, $|\bigcup_{i=1}^{\infty} U_i| = 0$, then

$$\int_{E\cap(-\infty,s_1]} f(s)ds = \int_{(s_-,s^1)\setminus(\bigcup_{i=1}^{\infty} U_i)} f(s)ds = \int_{(s_-,s^1)\setminus(\bigcup_{i=1}^{\infty} U_i)} f(s)ds + \int_{\bigcup_{i=1}^{\infty} U_i} f(s)ds$$
$$= \int_{s_-}^{s^1} f(s)ds = \int_{s_1-\delta_{s_1}^-}^{s_1} f(s)ds,$$

as required.

Since $\bigcup_{i=1}^{\infty} U_i$ and $(s_-, s_r^1) \setminus (\bigcup_{i=1}^{\infty} U_i)$ are 1-dimensional non-zero Lebesgue measurable sets, without loss of generalization, we assume $|s_r^1 - s_l^1| > 0$ and $|s_l^1 - s_r^2| > 0$. Then $f(s) > f(s + (s_r^1 - s_l^1))$ for $s \in (s_r^2, s_l^1)$ by the monotonicity of f. Furthermore, for $k = 1, 2, \dots, \infty$, $f(s) > f(s + \sum_{i=1}^{k}(s_r^i - s_1^i))$ for $s \in (s_r^{k+1}, s_l^k)$. Therefore,

$$\begin{aligned}
&\int_{E\cap(-\infty,s_1]} f(s)ds\\
&= \lim_{k\to\infty}\int_{s_-}^{s_l^{k+1}} f(s)ds + \lim_{k\to\infty}\sum_{i=1}^{k}\int_{s_r^{i+1}}^{s_l^i} f(s)ds + \int_{s_r^1}^{s_1} f(s)ds\\
&= \lim_{k\to\infty}\int_{s_-}^{s_l^{k+1}} f(s)ds + \lim_{k\to\infty}\sum_{i=2}^{k}\int_{s_r^{i+1}}^{s_l^i} f(s)ds + \left(\int_{s_r^2}^{s_l^1} f(s)ds + \int_{s_r^1}^{s_1} f(s)ds\right)\\
&> \lim_{k\to\infty}\int_{s_-}^{s_l^{k+1}} f(s)ds + \lim_{k\to\infty}\sum_{i=2}^{k}\int_{s_r^{i+1}}^{s_l^i} f(s)ds + \left(\int_{s_r^2}^{s_l^1} f(s+(s_r^1-s_l^1))ds + \int_{s_r^1}^{s_1} f(s)ds\right)\\
&= \lim_{k\to\infty}\int_{s_-}^{s_l^{k+1}} f(s)ds + \lim_{k\to\infty}\sum_{i=2}^{k}\int_{s_r^{i+1}}^{s_l^i} f(s)ds + \int_{s_r^2+(s_r^1-s_l^1)}^{s_1} f(s)ds\\
&\geqslant \lim_{k\to\infty}\int_{s_-}^{s_l^{k+1}} f(s)ds + \lim_{k\to\infty}\sum_{i=3}^{k}\int_{s_r^{i+1}}^{s_l^i} f(s)ds + \int_{s_r^3+(s_r^2-s_l^2)+(s_r^1-s_l^1)}^{s_1} f(s)ds\\
&\geqslant \lim_{k\to\infty}\int_{s_-}^{s_l^{k+1}} f(s)ds + \lim_{k\to\infty}\int_{s_r^{k+1}+\sum_{i=1}^{k}(s_r^i-s_l^i)}^{s_1} f(s)ds\\
&\geqslant \lim_{k\to\infty}\int_{s_-}^{s_l^{k+1}} f\left(s+\sum_{i=1}^{k+1}(s_r^i-s_l^i)\right)ds + \lim_{k\to\infty}\int_{s_r^{k+1}+\sum_{i=1}^{k}(s_r^i-s_l^i)}^{s_1} f(s)ds\\
&\geqslant \int_{s_-+\sum_{i=1}^{\infty}(s_r^i-s_l^i)}^{s_1} f(s)ds = \int_{s_1-\delta_{s_1}^-}^{s_1} f(s)ds.
\end{aligned}$$

This proves (2.3).

The equality in (2.3) holds if and only if $|(s_-, s_1) \setminus E| = 0$, i.e., if $x \in (s_-, s_1)$ then $x \in E$ a.e. However, since E is compact, the equality in (2.3) holds if and only if $(s_-, s_1) \subset E$.

Similar to the proof of (2.3), we can obtain (2.4).

For $s_+ \leqslant s_2$, we have

$$\int_{E\cap[s_1,\infty)} f(s)ds = \int_{E\cap[s_1,s_2]} f(s)ds = \text{const.}|E\cap[s_1,s_2]|$$
$$= \int_{(s_1,s_1+\delta_{s_1}^+)} f(s)ds = \int_{s_1}^{s_1+\delta_{s_1}^+} f(s)ds.$$

Hence, the equality in (2.4) holds.

We assume $s_+ \geqslant s_2$. If $[s_1, s_+] \subset E$, then $s_+ = s_1 + \delta_{s_1}^+$. Hence, the equality in (2.4) holds. Suppose that $(s_1, s_+) \setminus E$ is a non-empty open set in $\mathbb{R}$, which is a union of at most countably many open intervals $\{U_i\}_{i=1}^{\infty}, U_i \cap U_j = \emptyset$ for $i \neq j, i, j = 1, 2, \dots, \infty$.

Let $U_i = (s_l^i, s_r^i)$ and $s_r^i \leqslant s_l^{i+1}, i = 1, 2, \ldots, \infty$. Assume that $|\bigcup_{i=1}^{\infty} U_i| > 0$. Indeed, if $|\bigcup_{i=1}^{\infty} U_i| = 0$, then the equality of (2.4) holds. Without loss of generalization, let $|(s_l^1, s_r^1)| > 0$. Furthermore, either $|(s_l^1, s_+) \setminus (\bigcup_{i=1}^{\infty} U_i)| = 0$ or $|(s_2, s_+) \setminus (\bigcup_{i=1}^{\infty} U_i)| = 0$ implies that the equality of (2.4) holds. Thus, we may also assume that $(\max\{s_l^1, s_2\}, s_+) \setminus (\bigcup_{i=1}^{\infty} U_i)$ is a non-zero Lebesgue measurable set.

Since $(\max\{s_l^1, s_2\}, s_+) \setminus (\bigcup_{i=1}^{\infty} U_i)$ is a non-zero Lebesgue measurable set, there exists a non-empty open set $(s_r^j, s_l^{j+1}) \subset E$, such that $(s_r^j, s_l^{j+1}) \subset (\max\{s_l^1, s_2\}, s_+)$. Hence, by the monotonicity of f, we have $f(s) > f(s - \sum_{i=1}^{j}(s_r^i - s_l^i))$ for (s_r^j, s_l^{j+1}). Moreover, for $k = 1, \ldots, j-1$, $f(s) \geqslant f(s - \sum_{i=1}^{k}(s_r^i - s_i^1))$ for $s \in (s_r^k, s_l^{k+1})$; for $k = j, j+1, \ldots, \infty$, $f(s) > f(s - \sum_{i=1}^{k}(s_r^i - s_i^1))$ for $s \in (s_r^k, s_l^{k+1})$. Therefore,

$$
\begin{aligned}
\int_{E\cap(s_1,\infty]} f(s)ds &= \int_{s_1}^{s_l^1} f(s)ds + \lim_{k\to\infty} \sum_{i=1}^{k} \int_{s_r^i}^{s_l^{i+1}} f(s)ds + \lim_{k\to\infty} \int_{s_r^{k+1}}^{s_+} f(s)ds \\
&\geqslant \int_{s_1}^{s_l^j + \sum_{i=1}^{j-1}(s_r^i - s_l^i)} f(s)ds + \lim_{k\to\infty} \sum_{i=j}^{k} \int_{s_r^i}^{s_l^{i+1}} f(s)ds + \lim_{k\to\infty} \int_{s_r^{k+1}}^{s_+} f(s)ds \\
&> \lim_{k\to\infty} \int_{s_1}^{s_l^{k+1} + \sum_{i=1}^{k}(s_r^i - s_l^i)} f(s)ds + \lim_{k\to\infty} \int_{s_r^{k+1}}^{s_+} f(s)ds \\
&> \lim_{k\to\infty} \int_{s_1}^{s_l^{k+1} + \sum_{i=1}^{k}(s_r^i - s_l^i)} f(s)ds + \lim_{k\to\infty} \int_{s_r^{k+1}}^{s_+} f\bigg(s + \sum_{i=1}^{k+1}(s_r^i - s_l^i)\bigg)ds \\
&\geqslant \int_{s_1}^{s_1 + \sum_{i=1}^{\infty}(s_r^i - s_l^i)} f(s)ds = \int_{s_1}^{s_1 + \delta_{s_1}^+} f(s)ds.
\end{aligned}
$$

Therefore, the inequality (2.4) is proved.

The equality in (2.4) holds if and only if $|(s_2, s_+) \setminus E| = 0$, i.e., if $x \in (s_2, s_+)$ then $x \in E$ a.e.. However, since E is compact, the equality in (2.4) holds if and only if $(s_2, s_+) \subset E$.

If there exists an $s_0' \in [s_-, s_+] \setminus E$ and $|E \cap (-\infty, s_0']| > 0, |E \cap [s_0', \infty)| > 0$, then the equality in (2.1) holds if and only if $s_0' \in [s_1, s_2]$, $[s_-, s_1] \subset E$ and $s_+ \leqslant s_2$.

The same argument in the proof of (2.1) can be used to show (2.2) with its equality condition. □

3 Steiner symmetrization of Orlicz centroid bodies

Now, we establish a sharp Steiner symmetrization of Orlicz centroid bodies for star bodies, which is critical in the proof of our main theorem. To get the sharp Steiner symmetrization, the following lemma (see [46]) is needed.

Lemma 3.1. *Let K be a nonempty compact set. Then K is a star body if and only if for each $u \in S^{n-1}$, all the points of $\{tu : 0 \leqslant t < \rho_K(u)\}$ are interior points of K.*

From the strictly monotone property of $\phi^* = \int_0^1 \phi(ts)ds^n$, we have the following useful lemma (see [28]).

Lemma 3.2. *Let $\phi \in \mathcal{C}$ and $K \in \mathcal{S}_o^n, x_0 \in \mathbb{R}^n$. Then*

$$\frac{1}{|K|} \int_K \phi\bigg(\frac{x_0 \cdot y}{\lambda_0}\bigg) dy = 1,$$

if and only if

$$h_{\Gamma_\phi K}(x_0) = \lambda_0.$$

The following result for star bodies was proved in [46]. The version for convex bodies was proved in [28].

Lemma 3.3. *Let $K \in \mathcal{S}_o^n$, $\phi \in \mathcal{C}$. Then for $u \in S^{n-1}$ and $x_1', x_2' \in u^\perp$,*

$$h\bigg(\Gamma_\phi(S_u K); \frac{1}{2}x_1' + \frac{1}{2}x_2', 1\bigg) \leqslant \frac{1}{2} h(\Gamma_\phi K; x_1', 1) + \frac{1}{2} h(\Gamma_\phi K; x_2', -1),$$

$$h\left(\Gamma_\phi(S_uK);\frac{1}{2}x_1'+\frac{1}{2}x_2',-1\right)\leqslant\frac{1}{2}h(\Gamma_\phi K;x_1',1)+\frac{1}{2}h(\Gamma_\phi K;x_2',-1).$$

Now, we prove a critical lemma.

Lemma 3.4. *Let K be a Φ-star body, $\phi\in\mathcal{C}$. Then there exist $u\in S^{n-1}$ and $x_1',x_2'\in u^\perp$,*

$$h\left(\Gamma_\phi(S_uK);\frac{1}{2}x_1'+\frac{1}{2}x_2',1\right)<\frac{1}{2}h(\Gamma_\phi K;x_1',1)+\frac{1}{2}h(\Gamma_\phi K;x_2',-1),\tag{3.1}$$

$$h\left(\Gamma_\phi(S_uK);\frac{1}{2}x_1'+\frac{1}{2}x_2',-1\right)<\frac{1}{2}h(\Gamma_\phi K;x_1',1)+\frac{1}{2}h(\Gamma_\phi K;x_2',-1).\tag{3.2}$$

Proof. From (1.3), we may assume, without loss of generality, that $|K|=|S_uK|=1$.

Since K is a star body, and is not convex, we can choose P_1 and P_2 to be two interior points of K such that $\overline{P_1P_2}$ does not completely lie in K. To see this, choose P_3 and P_4 to be two boundary points of K such that $\overline{P_3P_4}$ does not completely lie in K, i.e., there exists $Q\in\overline{P_3P_4}$ but not in K. Since K is a compact set, there exists an open ball $B(Q,r_Q)$ centered at Q of radius r_Q such that $B(Q,r_Q)\cap K=\emptyset$. Using Lemma 3.1, we may choose two interior points $P_1\in\overline{OP_3}$, $P_2\in\overline{OP_4}$ such that $\overline{P_1P_2}\cap B(Q,r_Q)\neq\emptyset$.

Let $u=(P_1-P_2)/|P_1-P_2|$, and let $K'=K_u$ be the image of the projection of K onto $u^\perp$. For any $y'\in K'$, we write $\delta_{y'}(u)=\delta_{y'}=|K\cap(y'+\mathbb{R}u)|$ for one-dimensional Lebesgue measure of $K\cap(y'+\mathbb{R}u)$.

Let $x_1',x_2'\in u^\perp, x_0'=\frac{1}{2}x_1'+\frac{1}{2}x_2'$, and let $\lambda_1,\lambda_2\in\mathbb{R}^+,\lambda_0=\frac{1}{2}\lambda_1+\frac{1}{2}\lambda_2\in\mathbb{R}^+$. For $y'\in K', s\in\mathbb{R}$, we consider the function

$$f_{y'}(s)=\frac{\lambda_1}{\lambda_0}\phi\left(\frac{x_1'\cdot y'+s}{\lambda_1}\right)+\frac{\lambda_2}{\lambda_0}\phi\left(\frac{x_2'\cdot y'-s}{\lambda_2}\right).\tag{3.3}$$

By the convexity of ϕ, we have that $f_{y'}(s)$ is convex, and there exits an $s_0(y')$ such that $f(s)$ is monotone decreasing on $(-\infty,s_0(y')]$ and monotone increasing on $[s_0(y'),\infty)$. Thus, from Lemma 2.4, we get

$$\int_{K\cap(y'+\mathbb{R}u)}f_{y'}(s)ds\geqslant\int_{s_0(y')-\delta^-_{s_0(y')}}^{s_0(y')+\delta^+_{s_0(y')}}f_{y'}(s)ds,\tag{3.4}$$

where $\delta^+_{s_0(y')}=|K\cap(s_0(y')+\mathbb{R}^+u)|$ and $\delta^-_{s_0(y')}=|K\cap(s_0(y')+\mathbb{R}^+u)|$.

To obtain the strict inequalities of our result, we need consider the strictly monotone property of (3.4). Now, we consider the strictly monotone property of $f_{y'}$. It follows from Lemma 2.2 that either (i) that there exists a unique $s_0(y')\in\mathbb{R}$ with $|s_0(y')|<\infty$ such that $f_{y'}(s)$ is strictly monotone decreasing on $(-\infty,s_0(y')]$ and strictly monotone increasing on $[s_0(y'),\infty)$; or (ii) that there exist $s_1(y'),s_2(y')\in\mathbb{R}$ with $s_1(y')<s_2(y')$ and $|s_i(y')|<\infty, i=1,2$ such that $f_{y'}(s)$ is strictly monotone decreasing on $(-\infty,s_1(y')]$, strictly monotone increasing on $[s_2(y'),\infty)$ and $f_{y'}(s)=\text{const.}$ for all $s\in[s_1(y'),s_2(y')]$. Thus, according to the strictly monotone property of $f_{y'}$, Lemmas 2.2 and 2.4 indicate that we should distinguish two cases.

(i) For all $y'\in K'$, $f_{y'}(s)$ satisfies the following condition: there exists a unique $s_0(y')\in\mathbb{R}$ with $|s_0(y')|<\infty$ such that $f_{y'}(s)$ is strictly monotone decreasing on $(-\infty,s_0(y')]$ and strictly monotone increasing on $[s_0(y'),\infty)$.

Using Lemma 2.4, we obtain

$$\int_{K\cap(y'+\mathbb{R}u)}f_{y'}(s)ds\geqslant\int_{s_0(y')-\delta^-_{s_0(y')}}^{s_0(y')+\delta^+_{s_0(y')}}f_{y'}(s)ds.\tag{3.5}$$

Since P_1 and P_2 are two interior points of K, choose two open balls $B(P_i,r_i), i=1,2$ centered at P_i of radius r_i such that $B(P_i,r_i)\subset K$. Since $\overline{P_1P_2}$ does not completely lie in K, and since K is compact, we can choose $P\in\overline{P_1P_2}$ but not in K and $B(P,r)$ centered at P of radius r such that $B(P,r)\cap K=\emptyset$. Moreover, we can require that $(B(P,r))_u\subset(B(P_i,r_i))_u$. Let P_u be the image of the projection of P to $u^\perp$. For $y'\in(B(P,r/2))_u$, we can choose $s_0'(y')=y'+(P-P_u)$ not in K such that $|K\cap(-\infty,s_0'(y')]|>0$,

$|K \cap [s_0'(y'), \infty)| > 0$. Thus, by applying Lemma 2.4, we get that the equality in (3.5) cannot hold for $y' \in (B(P, r/2))_u$.

Let

$$I = \frac{\lambda_1}{\lambda_0} \int_K \phi\bigg(\frac{(x_1', 1) \cdot y}{\lambda_1}\bigg) dy + \frac{\lambda_2}{\lambda_0} \int_K \phi\bigg(\frac{(x_1', -1) \cdot y}{\lambda_2}\bigg) dy.$$

By applying (3.5), we have

$$\begin{aligned} I &= \int_{K'} \int_{K \cap (y' + \mathbb{R}u)} \frac{\lambda_1}{\lambda_0} \phi\bigg(\frac{x_1' \cdot y' + s}{\lambda_1}\bigg) + \frac{\lambda_2}{\lambda_0} \phi\bigg(\frac{x_2' \cdot y' - s}{\lambda_2}\bigg) dy' ds \\ &> \int_{K'} \int_{s_0(y') - \delta^-_{s_0(y')}}^{s_0(y') + \delta^+_{s_0(y')}} \frac{\lambda_1}{\lambda_0} \phi\bigg(\frac{x_1' \cdot y' + s}{\lambda_1}\bigg) + \frac{\lambda_2}{\lambda_0} \phi\bigg(\frac{x_2' \cdot y' - s}{\lambda_2}\bigg) dy' ds \\ &= \int_{K'} \int_{s_0(y') - \delta^-_{s_0(y')}}^{s_0(y') + \delta^+_{s_0(y')}} \frac{\lambda_1}{\lambda_0} \phi\bigg(\frac{x_1' \cdot y' + s}{\lambda_1}\bigg) dy' ds \\ &\quad + \int_{K'} \int_{s_0(y') - \delta^-_{s_0(y')}}^{s_0(y') + \delta^+_{s_0(y')}} \frac{\lambda_2}{\lambda_0} \phi\bigg(\frac{x_2' \cdot y' - s}{\lambda_2}\bigg) dy' ds. \end{aligned} \tag{3.6}$$

Note that the inequality in (3.6) is strict. The reason is that there exists a non-zero $(n-1)$-dimensional Lebesgue measurable set $(B(P, r/2))_u$ such that for all $y' \in (B(P, r/2))_u$, the equality in (3.5) cannot hold.

Let $m_{y'}$ be the midpoint of $[s_0(y') - \delta^-_{s_0(y')}, s_0(y') + \delta^+_{s_0(y')}]$, and let $\delta_{s_0(y')} = \delta^+_{s_0(y')} + \delta^-_{s_0(y')}$. Since $s_0(y') - \delta^-_{s_0(y')} = m_{y'} - \frac{1}{2}\delta_{s_0(y')}$, $s_0(y') + \delta^+_{s_0(y')} = m_{y'} + \frac{1}{2}\delta_{s_0(y')}$, we make the change of variables $s = m_{y'} + t$ for the first integral of the last equation in (3.6), and make the change of variables $s = m_{y'} - t$ for the second one. Then using the convexity of ϕ we have

$$\begin{aligned} I &> \int_{K'} \int_{-\frac{1}{2}\delta_{s_0(y')}}^{\frac{1}{2}\delta_{s_0(y')}} \frac{\lambda_1}{\lambda_0} \phi\bigg(\frac{x_1' \cdot y' + t + m_{y'}}{\lambda_1}\bigg) dy' dt \\ &\quad + \int_{K'} \int_{-\frac{1}{2}\delta_{s_0(y')}}^{\frac{1}{2}\delta_{s_0(y')}} \frac{\lambda_2}{\lambda_0} \phi\bigg(\frac{x_2' \cdot y' + t - m_{y'}}{\lambda_2}\bigg) dy' dt \\ &= \int_{S_u K} \bigg[\frac{\lambda_1}{\lambda_0} \phi\bigg(\frac{x_1' \cdot y' + t + m_{y'}}{\lambda_1}\bigg) + \frac{\lambda_2}{\lambda_0} \phi\bigg(\frac{x_2' \cdot y' + t - m_{y'}}{\lambda_2}\bigg)\bigg] dy' dt \\ &\geqslant 2 \int_{S_u K} \phi\bigg(\frac{\frac{x_1' + x_2'}{2} \cdot y' + t}{\frac{\lambda_1 + \lambda_2}{2}}\bigg) dy' dt. \end{aligned} \tag{3.7}$$

Therefore, it follows from (3.6) and (3.7) that

$$\begin{aligned} &\frac{\lambda_1}{\lambda_0} \int_K \phi\bigg(\frac{(x_1', 1) \cdot y}{\lambda_1}\bigg) dy + \frac{\lambda_2}{\lambda_0} \int_K \phi\bigg(\frac{(x_1', -1) \cdot y}{\lambda_2}\bigg) dy \\ &\quad > 2 \int_{S_u K} \phi\bigg(\frac{(x_0', 1) \cdot y}{\lambda_0}\bigg) dy. \end{aligned} \tag{3.8}$$

Choose

$$\lambda_1 = h(\Gamma_\phi K; x_1', 1) \quad \text{and} \quad \lambda_2 = h(\Gamma_\phi K; x_1', -1).$$

Recall that $|K| = 1$, it follows from Lemma 3.2 that

$$\int_K \phi\bigg(\frac{(x_1', 1) \cdot y}{\lambda_1}\bigg) dy = 1 \quad \text{and} \quad \int_K \phi\bigg(\frac{(x_1', -1) \cdot y}{\lambda_2}\bigg) dy = 1.$$

Hence, from (3.8), we obtain

$$\int_{S_u K} \phi\bigg(\frac{(x_0', 1) \cdot y}{\lambda_0}\bigg) dy < 1.$$

By the definition of the Orlicz centroid body, we conclude

$$h(\Gamma_\phi(S_uK);x_0',1) < \lambda_0,$$

which is (3.1).

Note that if we put $s = m_{y'} + t$ for the first integral of the last equation in (3.6), and put $s = m_{y'} - t$ for the second one, the same manner implies (3.2), as required.

(ii) There exists $y' \in K'$, such that $f_{y'}(s)$ satisfies that there exist $s_1(y'), s_2(y') \in \mathbb{R}$ with $s_1(y') < s_2(y')$ and $|s_i(y')| < \infty, i = 1,2$ such that $f_{y'}(s)$ is strictly monotone decreasing on $(-\infty, s_1(y')]$, strictly monotone increasing on $[s_2(y'),\infty)$ and $f_{y'}(s) = \text{const.}$ for all $s \in [s_1(y'), s_2(y')]$.

By Lemma 2.4, for those y' satisfying the above condition we have that

$$\int_{K\cap(y'+\mathbb{R}u)} f_{y'}(s)ds \geqslant \int_{s_1(y')-\delta^-_{s_1(y')}}^{s_1(y')+\delta^+_{s_1(y')}} f_{y'}(s)ds =: A_1, \tag{3.9}$$

$$\int_{K\cap(y'+\mathbb{R}u)} f_{y'}(s)ds \geqslant \int_{s_2(y')-\delta^-_{s_2(y')}}^{s_2(y')+\delta^+_{s_2(y')}} f_{y'}(s)ds =: A_2. \tag{3.10}$$

If there exists y_0' such that $A_1 \neq A_2$, without loss of generalization, let $A_1 < A_2$, then

$$\int_{K\cap(y_0'+\mathbb{R}u)} f_{y_0'}(s)ds > \int_{s_1(y_0')-\delta^-_{s_1(y_0')}}^{s_1(y_0')+\delta^+_{s_1(y_0')}} f_{y_0'}(s)ds. \tag{3.11}$$

Hence, there exists $B(y_0', r_{y_0'})$, such that for all $y' \in B(y_0', r_{y_0'})$, (3.11) holds. Together with the same argument of (i), we have (3.1) and (3.2).

Otherwise, for all y', $A_1 = A_2$. From our assumption there exists $y_0' \in K'$ such that there are at least three points in $(y_0' + \mathbb{R}u) \cap \partial K$ and $\bar{l}(K, y_0')$ or $\underline{l}(K, y_0') \notin [s_1(y_0'), s_2(y_0')]$. Lemma 2.4 implies that the equality of (3.9) or (3.10) cannot hold. Therefore, the same method implies that (3.1) and (3.2) hold for $x_i \in u^\perp, i = 1, 2$. □

Combining Lemma 3.3 and Lemma 3.4, we have the following corollary.

Corollary 3.5. *Let $K \in \mathcal{S}_o^n$, $\phi \in \mathcal{C}$. Then for $u \in S^{n-1}$ and $x_1', x_2' \in u^\perp$,*

$$h\left(\Gamma_\phi(S_uK); \frac{1}{2}x_1' + \frac{1}{2}x_2', 1\right) \leqslant \frac{1}{2}h(\Gamma_\phi K; x_1', 1) + \frac{1}{2}h(\Gamma_\phi K; x_2', -1),$$

$$h\left(\Gamma_\phi(S_uK); \frac{1}{2}x_1' + \frac{1}{2}x_2', -1\right) \leqslant \frac{1}{2}h(\Gamma_\phi K; x_1', 1) + \frac{1}{2}h(\Gamma_\phi K; x_2', -1).$$

If K is a Φ-star body, then there exists $u \in S^{n-1}$ such that either equality cannot hold.

We need the next well-known lemma (see, e.g., [25, 28]).

Lemma 3.6. *Let $K \in \mathcal{K}_o^n$ and $u \in S^{n-1}$. For $y' \in \operatorname{relint}(K_u)$, the overgraph and undergraph functions of K in direction u are given by*

$$\bar{l}_u(K, y') = \min_{x'\in u^\perp}\{h_K(x', 1) - x' \cdot y'\}$$

and

$$\underline{l}_u(K, y') = \min_{x'\in u^\perp}\{h_K(x', -1) - x' \cdot y'\}.$$

Corollary 3.5 and Lemma 3.6 imply the sharp Steiner symmetrization.

Theorem 3.7. *Let $K \in \mathcal{S}_o^n$, $\phi \in \mathcal{C}$ and $u \in S^{n-1}$. Then*

$$\Gamma_\phi(S_uK) \subset S_u(\Gamma_\phi K). \tag{3.12}$$

If K is a Φ-star body, then there exists $u \in S^{n-1}$ such that the identity cannot hold.

Proof. For $y' \in \mathrm{relint}(\Gamma_\phi K)_u$, Lemma 3.6 implies that there exist $x_1'(y')$ and $x_2'(y')$ such that

$$\bar{l}_u(\Gamma_\phi(K), y') = h_{\Gamma_\phi(K)}(x_1', 1) - x_1' \cdot y', \tag{3.13}$$

$$\underline{l}_u(\Gamma_\phi(K), y') = h_{\Gamma_\phi(K)}(x_2', -1) - x_2' \cdot y'. \tag{3.14}$$

From the definition of Steiner symmetrization, (3.13), (3.14) and Corollary 3.5 and Lemma 3.6, we obtain

$$\begin{aligned}\bar{l}_u(S_u(\Gamma_\phi K), y') &= \frac{1}{2}\bar{l}_u(\Gamma_\phi(K), y') + \frac{1}{2}\underline{l}_u(\Gamma_\phi(K), y')\\ &= \frac{1}{2}(h_{\Gamma_\phi(K)}(x_1', 1) - x_1' \cdot y') + \frac{1}{2}(h_{\Gamma_\phi(K)}(x_2', -1) - x_2' \cdot y')\\ &\geqslant h_{\Gamma_\phi(S_uK)}\left(\frac{1}{2}x_1' + \frac{1}{2}x_2', 1\right) - \left(\frac{1}{2}x_1' + \frac{1}{2}x_2'\right) \cdot y'\\ &\geqslant \min_{x' \in u^\perp}\{h_{\Gamma_\phi(S_uK)}(x', 1) - x' \cdot y'\}\\ &= \bar{l}_u(\Gamma_\phi(S_uK), y'), \end{aligned}\tag{3.15}$$

and

$$\begin{aligned}\underline{l}_u(S_u(\Gamma_\phi K), y') &= \frac{1}{2}\bar{l}_u(\Gamma_\phi(K), y') + \frac{1}{2}\underline{l}_u(\Gamma_\phi(K), y')\\ &= \frac{1}{2}(h_{\Gamma_\phi(K)}(x_1', 1) - x_1' \cdot y') + \frac{1}{2}(h_{\Gamma_\phi(K)}(x_2', -1) - x_2' \cdot y')\\ &\geqslant h_{\Gamma_\phi(S_uK)}\left(\frac{1}{2}x_1' + \frac{1}{2}x_2', -1\right) - \left(\frac{1}{2}x_1' + \frac{1}{2}x_2'\right) \cdot y'\\ &\geqslant \min_{x' \in u^\perp}\{h_{\Gamma_\phi(S_uK)}(x', -1) - x' \cdot y'\}\\ &= \underline{l}_u(\Gamma_\phi(S_uK), y'). \end{aligned}\tag{3.16}$$

Thus, the inclusion holds.

Assume that K is a Φ-star body. For $x_1', x_2' \in u^\perp$ there exists $y_0' \in K'$ such that there are at least three points in $(y_0' + \mathbb{R}u) \cap \partial K$ and $\bar{l}(K, y_0')$ or $\underline{l}(K, y_0') \notin [s_1(y_0'), s_2(y_0')]$. Corollary 3.5 also implies that there exists $u \in S^{n-1}$ such that the equalities in (3.15) and (3.16) cannot hold, then the identity in (3.12) cannot hold. □

4 Proof of the main theorem

We are now in a position to prove the LYZ conjecture for arbitrary ϕ. The core arguments in the proof is to use the Steiner symmetrization of Orlicz centroid bodies established in Section 3. The inequality for star bodies was proved by Zhu [46]. Our main purpose is to show the equality condition.

Proof of Theorem 1.1. We first prove that the centered ellipsoids are the minimizers of $|\Gamma_\phi K|/|K|$.

Suppose that there exists $K \in \mathcal{S}_o^n$ such that $|\Gamma_\phi K|/|K| < |\Gamma_\phi B|/|B|$. Since the case of convex bodies has been considered by Theorem A, we assume that $K \in \mathcal{S}_o^n$ is not convex. By Theorem 3.7, there exists $u_1 \in S^{n-1}$, such that $|\Gamma_\phi S_{u_1} K| \leqslant |\Gamma_\phi K|$. Then choose a suitable sequence of unit directions $\{u_i\}_{i=1}^\infty$ so that the sequence of convex bodies K_i defined by

$$K_i = S_{u_i} \cdots S_{u_1} K$$

converges to the centered closed ball $\bar{B}(r_K)$ with respect to the Hausdorff distance, where r_K is the volume radius of K, namely $r_K = (|K|/\omega_n)^{\frac{1}{n}}$.

Since $K \to \bar{B}(r_K)$ with respect to the Hausdorff distance, we have $\lim_{i\to\infty} |\Gamma_\phi K_i| \to |\Gamma_\phi \bar{B}(r_K)|$ (see [28]). Theorem 3.7 implies that

$$|\Gamma_\phi \bar{B}(r_K)| \leqslant \cdots \leqslant |\Gamma_\phi K_i| \leqslant \cdots \leqslant |\Gamma_\phi K_1| \leqslant |\Gamma_\phi K|.$$

Since the volume of $\{K_i\}_{i=1}^{\infty}$ is not changed in Stenier symmetrization, we have

$$\frac{|\Gamma_\phi B(r_K)|}{|B(r_K)|} \leqslant \cdots \leqslant \frac{|\Gamma_\phi K_1|}{|K_1|} \leqslant \frac{|\Gamma_\phi K|}{|K|}.$$

This leads to a contradiction with the hypothesis $|\Gamma_\phi K|/|K| < |\Gamma_\phi B|/|B|$.

Next, we prove the uniqueness of minimizers for $\phi(s) \in \mathcal{C}$.

From the argument of existence of minimizers, if K is a Φ-star body, then there exists a centered closed ball $\bar{B}(r_K)$ such that

$$\frac{|\Gamma_\phi B(r_K)|}{|B(r_K)|} < \frac{|\Gamma_\phi K|}{|K|}.$$

This means that if K is a Φ-star body, then K is not a minimizer. Hence, the minimizers need to be convex bodies. By Theorem A, ellipsoids centered at the origin are the only minimizers, as desired. □

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