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Simple Kac-Moody groups with trivial Schur multipliers

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Abstract We obtain new examples of simple Kac-Moody groups with trivial Schur multipliers.

Keywords Kac-Moody group, simple group, Schur multiplier

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1 Introduction

Using the terminology of central extensions, we may construct a family of perfect groups, more precisely called a CE (central extension) family as follows:

$$\left\{ \begin{array}{c} \nearrow \cdots \searrow \cdots \searrow \\ \nearrow \cdots \searrow \cdots \searrow \cdots \searrow \\ \hat{\Gamma} \to \cdots \to \ \Gamma \ \to \cdots \to \Gamma' \\ \searrow \cdots \nearrow \cdots \nearrow \cdots \nearrow \\ \searrow \cdots \nearrow \cdots \nearrow \end{array} \right\}.$$

All arrows in the diagram are central extensions of perfect groups. Note that a group Γ is called perfect if $\Gamma = [\Gamma, \Gamma]$. A perfect group Γ has a universal central extension, called $\hat{\Gamma}$, which is uniquely determined up to isomorphism (see [18, Section 7, Central extensions]).

Let Γ be a perfect group in this paragraph. If Γ is simple, then there is no nontrivial quotient of Γ. If the Schur multiplier of Γ is trivial, then there is no nontrivial central extension of Γ. Hence, if Γ is a simple group with trivial Schur multiplier, then the corresponding CE family is $\{\Gamma\}$, consisting of a single object. Note that one of the most famous simple groups with trivial Schur multipliers is the Monster group $\mathbb{M}$, which is known as the largest sporadic finite simple group.

In this note, we provide examples of several simple Kac-Moody groups with trivial Schur multipliers. Recently, the simplicity of some Kac-Moody groups has been established (see [2, 3, 16]). Here, we mainly discuss Kac-Moody groups over certain infinite fields. Then we will discuss their Schur multipliers, and we will find new examples of simple Kac-Moody groups with trivial Schur multipliers.

2 Results

Let $A = (a_{ij})$ be an $n \times n$ generalized Cartan matrix (GCM), i.e., A is an integer matrix satisfying (1) $a_{ii} = 2$, (2) $a_{ij} \leqslant 0$ $(i \neq j)$, (3) $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$. Using a GCM A, we can construct the associated Kac-Moody Lie algebra $\mathfrak{g}$. A GCM $A = (a_{ij})$ is called indecomposable if there is no permutation τ of indices such that

$$A^\tau = (a_{\tau(i),\tau(j)}) = \left(\begin{array}{c|c} A' & O \\ \hline O & A'' \end{array}\right),$$

where A' and A'' are non-trivial square blocks. An indecomposable GCM A is called of finite type if $\mathfrak{g}$ is a finite dimensional simple Lie algebra, or the corresponding root system is finite and irreducible (see [1,5,7]). An indecomposable GCM A is called of affine type if there are positive integers $b_1, \ldots, b_n$ such that $(b_1, \ldots, b_n)A = \mathbf{0}$. An indecomposable GCM A is called of indefinite type if A is neither of finite type nor of affine type. We call an indecomposable GCM A of non-affine type if A is either of finite type or of indefinite type. *In this note, we always assume that a GCM A is both indecomposable and of non-affine type unless otherwise stated.*

Let $\mathfrak{g} = \mathfrak{g}_{\mathbb{C}}(A)$ be the Kac-Moody Lie algebra associated with A over the field, $\mathbb{C}$, of complex numbers, which is generated by the Cartan subalgebra $\mathfrak{h}$ and Chevalley generators $e_1, \ldots, e_n, f_1, \ldots, f_n$ (see [8,10, 13]). Using $\mathfrak{g}$, one can construct the so-called Tits group functor $\mathcal{G} = \mathcal{G}_A$ from the category of commutative rings with identities to the category of groups (see [20]). For a field F, let G be the subgroup of $\mathcal{G}_A(F)$ generated by $x_\alpha(t) = \exp te_\alpha \in \mathcal{G}_A(F)$ for all $t \in F$ and all real roots α, where e_α is a Chevalley basis for α. Note that $G = [G, G]$ if $|F| \geqslant 4$ (see [6]). Let $Z = Z(G)$ be the center of G, and put $G' = G/Z$. We may call all of $\mathcal{G}$, G and G' Kac-Moody groups (see [9,17]). Let $\mathbb{F}_p$ be the prime field of characteristic $p > 0$, and we denote by $\mathbb{F}_{p^k}$ and $\overline{\mathbb{F}_p}$ the field with p^k elements and the algebraic closure of $\mathbb{F}_p$, respectively. Then, the following result is known (see [2,3,16,18]).

Theorem 1. *Let F be an infinite subfield of $\overline{\mathbb{F}_p}$. Suppose that A is of non-affine type. Then, G' is a simple group.*

Let $f : \Gamma' \to \Gamma$ be a homomorphism of groups. If f is surjective, then f is called an extension. If the kernel of an extension f is central, then f is called a central extension. If a central extension $\hat{f} : \hat{\Gamma} \to \Gamma$ uniquely dominates all other central extensions of Γ, then $\hat{f}$ is called a universal central extension. It is known that $\Gamma = [\Gamma, \Gamma]$ if and only if there exists a universal central extension $\hat{f}$ of Γ (see [18]). In this case, we put $M(\Gamma) = \mathrm{Ker}\hat{f}$ and call it the Schur multiplier of Γ. If a perfect group Γ is simple and $M(\Gamma) = 1$, then we call Γ a simple group with trivial Schur multiplier (see [18]).

For distinct prime numbers p and q, we define $\ell_q(p)$ by $\ell_q(p) = \min\{m > 0 \mid p^m \equiv 1 \pmod q\}$, which means that $\ell_q(p)$ is the order of p modulo q. Then, we obtain the following two results.

Theorem 2. *Let F be an infinite subfield of $\overline{\mathbb{F}_p}$. Suppose that $\det(A) \neq 0$ and $\mathbb{F}_{p^{\ell_q(p)}} \not\subset F$ for every prime number q satisfying $q \mid \det(A)$ and $q \neq p$. Then G' is a simple group with trivial Schur multiplier.*

Theorem 3. *If $\det(A) = \pm p^c$ for a prime number p and $c \geqslant 0$ and $F = \overline{\mathbb{F}_p}$, then G' is a simple group with trivial Schur multiplier.*

3 Proofs

Proof of Theorem 1 including remark. If $n > 2$, then the result can be obtained using Caprace and Rémy [2] (see also [4,18]). Suppose $n = 2$, i.e.,

$$A = \begin{pmatrix} 2 & a \\ b & 2 \end{pmatrix}.$$

If $ab < 4$, then $G' \simeq PSL_3(F), PSp_4(F), G_2(F)$ and the result is well-known (see [4,18]). If $ab > 4$ and one of a, b is -1, then the result can be also confirmed using Caprace and Rémy [3]. If $ab > 4$ as well as $a < -1$ and $b < -1$, then the result is established in [16]. Note that A is of affine type if $ab = 4$ (see [8, 13]). $\square$

Proof of Theorem 2 *including remark.* Let Δ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$, i.e.,

$$\Delta = \{\alpha \in \mathfrak{h}^* \mid \alpha \neq 0,\, \mathfrak{g}_\alpha \neq 0\},$$

where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \ (\forall\, h \in \mathfrak{h})\}$. Then, we can find simple roots $\alpha_1, \ldots, \alpha_n$ which have the corresponding root spaces $\mathfrak{g}_{\alpha_i} = \mathbb{C}e_i$ and $\mathfrak{g}_{-\alpha_i} = \mathbb{C}f_i$. For each $i = 1, \ldots, n$, we define $\sigma_i \in GL(\mathfrak{h}^*)$ to be the reflection defined by $\sigma_i(\mu) = \mu - \mu(h_i)\alpha_i \ (\forall\, \mu \in \mathfrak{h}^*)$. Then, the subgroup W of $GL(\mathfrak{h}^*)$ generated by $\sigma_1, \ldots, \sigma_n$ is called the Weyl group, and $\Delta^{\rm re} = \{\sigma(\alpha_i) \mid \sigma \in W,\ 1 \leqslant i \leqslant n\}$ is called the set of real roots. Let h_α be the coroot of $\alpha \in \Delta^{\rm re}$ (see [8, 13]).

Using the Chevalley involution ω of $\mathfrak{g}$, which is defined by $\omega(e_i) = -f_i$, $\omega(f_i) = -e_i$ and $\omega(h) = -h$ $(1 \leqslant i \leqslant n,\ \forall\, h \in \mathfrak{h})$, we choose and fix a Chevalley basis $\mathcal{C} = \{e_\alpha \mid \alpha \in \Delta^{\rm re}\}$ for $\Delta^{\rm re}$, where $e_\alpha \in \mathfrak{g}_\alpha$, $[e_\alpha, e_{-\alpha}] = h_\alpha$ and $\omega(e_\alpha) + e_{-\alpha} = 0$. For each $\alpha \in \Delta^{\rm re}$, there is an exponential map $x_\alpha : t \mapsto \exp\, te_\alpha$ from the additive group F into $\mathcal{G}(F)$. By the definition, G is generated by $x_\alpha(t)$ for all $\alpha \in \Delta^{\rm re}$ and $t \in F$. Then, G is presented by the generators $x_\alpha(t)$ with $\alpha \in \Delta^{\rm re}$ and $t \in F$, and the following defining relations (A), (B), (B′) and (C) (see [19]):

(A) $x_\alpha(s)x_\alpha(t) = x_\alpha(s + t)$,

(B) $[x_\alpha(s), x_\beta(t)] = \prod x_{i\alpha+j\beta}(N_{\alpha,\beta,i,j}s^it^j)$,

(B′) $w_\alpha(u)x_\beta(t)w_\alpha(-u) = x_{\beta'}(t')$,

(C) $h_\alpha(u)h_\alpha(v) = h_\alpha(uv)$.

To understand (B), put $Q_{\alpha,\beta} = \{i\alpha + j\beta \mid i, j \in \mathbb{Z}_{>0}\} \cap \Delta$. Then, we have

$$\text{(B)} \quad [x_\alpha(s), x_\beta(t)] = \prod_{Q_{\alpha,\beta}} x_{i\alpha+j\beta}(N_{\alpha,\beta,i,j}s^it^j),$$

whenever $Q_{\alpha,\beta} \subset \Delta^{\rm re}$ (see [14]). In fact, there are five possible types of relations in (B) (see [4, 14]), i.e.,

$$\begin{aligned}
&[x_\alpha(s), x_\beta(t)] = 1,\\
&[x_\alpha(s), x_\beta(t)] = x_{\alpha+\beta}(\pm(r + 1)st),\\
&r = \max\{i \in \mathbb{Z} \mid \beta - i\alpha \in \Delta^{\rm re}\},\\
&[x_\alpha(s), x_\beta(t)] = x_{\alpha+\beta}(\pm st)x_{2\alpha+\beta}(\pm s^2t),\\
&[x_\alpha(s), x_\beta(t)] = x_{\alpha+\beta}(\pm 2st)x_{2\alpha+\beta}(\pm 3s^2t)x_{\alpha+2\beta}(\pm 3st^2),\\
&[x_\alpha(s), x_\beta(t)] = x_{\alpha+\beta}(\pm st)x_{2\alpha+\beta}(\pm s^2t)x_{3\alpha+\beta}(\pm s^3t)x_{3\alpha+2\beta}(\pm 2s^3t^2).
\end{aligned}$$

To understand (B′) and (C), for $u, v \in F^\times$, we put

$$w_\alpha(u) = x_\alpha(u)x_{-\alpha}(-u^{-1})x_\alpha(u), \quad h_\alpha(u) = w_\alpha(u)w_\alpha(-1).$$

Then, we have

(B′) $w_\alpha(u)x_\beta(t)w_\alpha(-u) = x_{\beta'}(t')$,

(C) $h_\alpha(u)h_\alpha(v) = h_\alpha(uv)$,

where $\beta' = \beta - \beta(h_\alpha)\alpha$, $t' = \pm u^{-\beta(h_\alpha)}t$ (see [18]). For each $\alpha \in \Delta^{\rm re}$, there is a group isomorphism

$$\varphi_\alpha : \langle x_\alpha(t),\, x_{-\alpha}(t) \mid t \in F\rangle \xrightarrow{\simeq} SL_2(F)$$

satisfying

$$x_\alpha(t) \mapsto \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad x_{-\alpha}(t) \mapsto \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix},$$
$$w_\alpha(u) \mapsto \begin{pmatrix} 0 & u \\ -u^{-1} & 0 \end{pmatrix}, \quad h_\alpha(u) \mapsto \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}.$$

We take an abstract symbol $\hat{x}_\alpha(t)$ for each $\alpha \in \Delta^{\mathrm{re}}$ and $t \in F$. The Steinberg group $\hat{G} = \mathrm{St}(A, F)$ over F of type A is defined to be the group generated by $\hat{x}_\alpha(t)$ for all $\alpha \in \Delta^{\mathrm{re}}$ and $t \in F$ with the defining relations corresponding to (A), (B) and (B′). It is known that $\hat{G}$ is a universal central extension of G' (see [15]), which is induced by $\pi' : \hat{x}_\alpha(t) \mapsto x_\alpha(t) \bmod Z$. We define $K_2(A, F)$ by

$$1 \to K_2(A, F) \to \hat{G} \xrightarrow{\pi} G \to 1,$$

where π is given by $\pi(\hat{x}_\alpha(t)) = x_\alpha(t)$. Then, $K_2(A, F)$ has a Matsumoto-type presentation (see [15]). This gives a lot of information on $K_2(A, F)$ and the corresponding Schur multiplier (see [11,12]). Exactly saying, $K_2(A, F)$ is isomorphic to the abelian group generated by $c_i(u, v)$ for all $1 \leqslant i \leqslant n$ and $u, v \in F^\times$ with the following defining relations (see [15]):

(M1) $c_i(t, u)c_i(tu, v) = c_i(t, uv)c_i(u, v)$;
(M2) $c_i(1, 1) = 1$;
(M3) $c_i(u, v) = c_i(u^{-1}, v^{-1})$;
(M4) $c_i(u, v) = c_i(u, (1 - u)v)$, $u \neq 1$;
(M5) $c_i(u, v^{a_{ji}}) = c_j(u^{a_{ij}}, v)$;
(M6) $c_i(tu, v^{a_{ji}}) = c_i(t, v^{a_{ji}})c_i(u, v^{a_{ji}})$.

Using these relations, we can show that $K_2(A, F)$ is trivial, which means that $\hat{G} = G$ (see [18]).

Now we want to consider the structure of $Z = Z(G)$. Using the fact that Z is in the standard maximal torus $T = \{h_{\alpha_1}(t_1) \cdots h_{\alpha_n}(t_n) \mid t_i \in F^\times\}$, we can establish that Z is isomorphic to $\mathrm{Hom}(\Xi, F^\times)$, where $\Xi = \mathbb{Z}^n/\langle \boldsymbol{a}_1, \ldots, \boldsymbol{a}_n\rangle$ and $A = (a_{ij}) = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n)$ (see [18]).

If q is a prime number such that $q \mid \det(A)$ and $q \neq p$, then we should consider $t^q = 1$ in $F^\times$. In this case, we can assume that $t \in \mathbb{F}_{p^k} \subset F$ for some k. Then, $q \mid (p^k - 1)$ and $p^k \equiv 1 \pmod q$, which implies $\ell_q(p) \mid k$ and $\mathbb{F}_{p^{\ell_q(p)}} \subset \mathbb{F}_{p^k} \subset F$. Hence, our assumption implies $\mathrm{Hom}(\mathbb{Z}/q\mathbb{Z}, F^\times) = 1$. Therefore, the principal divisor theorem shows that $\mathrm{Hom}(\Xi, F^\times) = 1$ and $Z = 1$. Thus, $M(G') = 1$, i.e., G' is a simple group with trivial Schur multiplier. □

Proof of Theorem 3 *including remark.* The proof of Theorem 2 shows this result. Note that if q is a prime number satisfying $q \mid \det(A)$ and $q \neq p$, then we can always find $Z \neq 1$ in this case. □

4 New examples

Here, we will show several typical examples, which are new.

Example 4. G' is a simple group with trivial Schur multiplier if

(1) $A = \left(\begin{smallmatrix} 2 & -2 \\ -3 & 2 \end{smallmatrix}\right)$, $F = \overline{\mathbb{F}_2}$;
(2) $A = \left(\begin{smallmatrix} 2 & -5 \\ -17 & 2 \end{smallmatrix}\right)$, $F = \overline{\mathbb{F}_3}$;
(3) $A = \left(\begin{smallmatrix} 2 & -3 \\ -3 & 2 \end{smallmatrix}\right)$, $F = \overline{\mathbb{F}_5}$;
(4) $A = \left(\begin{smallmatrix} 2 & -3 \\ -5 & 2 \end{smallmatrix}\right)$, $F = \overline{\mathbb{F}_{11}}$.

Example 5. Let $q = |\det(A)|$. Suppose that q is a prime number satisfying $q \neq p$ and $p \not\equiv 1 \pmod q$. In particular, $\ell_q(p) > 1$ and $\ell_q(p)$ divides $q - 1$, which means that $\ell_q(p)$ does not divide q^i. Set

$$F = \bigcup_{i \geqslant 0} \mathbb{F}_{p^{q^i}} \subsetneq \overline{\mathbb{F}_p}\ .$$

Then, we see $\mathbb{F}_{p^{\ell_q(p)}} \not\subset F$. Therefore, G' is a simple group with trivial Schur multiplier.

Example 6. G' is a simple group with trivial Schur multiplier in the following cases:

(1) Let $p = 7$ and $A = \left(\begin{smallmatrix} 2 & -3 \\ -3 & 2 \end{smallmatrix}\right)$, and we set $F = \bigcup_{i \geqslant 0} \mathbb{F}_{7^{5^i}} \subsetneq \overline{\mathbb{F}_7}$.
(2) Let $p = 7$ and $A = \left(\begin{smallmatrix} 2 & -3 \\ -5 & 2 \end{smallmatrix}\right)$, and we set $F = \bigcup_{i \geqslant 0} \mathbb{F}_{7^{11^i}} \subsetneq \overline{\mathbb{F}_7}$.
(3) Let $p = 7$ and $A = \left(\begin{smallmatrix} 2 & -3 \\ -7 & 2 \end{smallmatrix}\right)$, and we set $F = \bigcup_{i \geqslant 0} \mathbb{F}_{7^{17^i}} \subsetneq \overline{\mathbb{F}_7}$.
(4) Let $p = 7$ and $A = \left(\begin{smallmatrix} 2 & -5 \\ -7 & 2 \end{smallmatrix}\right)$, and we set $F = \bigcup_{i \geqslant 0} \mathbb{F}_{7^{31^i}} \subsetneq \overline{\mathbb{F}_7}$.

5 Remarks

Remark 7. Let $F = \mathbb{F}_{p^k}$. Suppose that $\det(A) \neq 0$ and $\ell_q(p) \nmid k$ for every prime number q satisfying $q \mid \det(A)$ and $q \neq p$. Then, we have $\mathrm{St}(A, F) = G = G'$.

Remark 8 (See [2,3])**.** About the simplicity of G', the following results are known for a finite field F:

(1) Suppose that A is of non-affine type, and $F = \mathbb{F}_{p^k}$ satisfying $p^k \geqslant n > 2$. Then G' is a simple group.

(2) Let

$$A = \begin{pmatrix} 2 & a \\ -1 & 2 \end{pmatrix}$$

be a 2×2 hyperbolic GCM, i.e., $a < -4$, and $F = \mathbb{F}_{p^k}$ satisfying $p^k > 3$. Then G' is a simple group.

Remark 9. About the simplicity of G' in the case when A is of indefinite type, the following table is known (see Theorem 1 and Remark 8):

char $= p > 0$	Finite field	Infinite field
rank $\geqslant 3$	$\mathbb{F}_{p^k}$ $(p^k \gg 0)$	$F \subset \overline{\mathbb{F}_p}$
$\begin{pmatrix} 2 & a \\ -1 & 2 \end{pmatrix}$	$\mathbb{F}_{p^k}$ $(p^k \gg 0)$	$F \subset \overline{\mathbb{F}_p}$
$\begin{pmatrix} 2 & a \\ b & 2 \end{pmatrix}$; $a, b < -1$	$\exists$ non-simple	$F \subset \overline{\mathbb{F}_p}$
char $= 0$	–	open

Remark 10. In the above table (see Remark 9), we have several questions.

(1) Is G' simple for a general infinite field F of characteristic $p > 0$?

(2) Is there any example for G' to be simple if F is of characteristic 0?

(3) Especially, is G' simple if $F = \mathbb{C}$?

(4) Let

$$A = \begin{pmatrix} 2 & a \\ b & 2 \end{pmatrix}$$

be a 2×2 hyperbolic GCM satisfying $a, b < -1$ and suppose that F is a finite field. There are many non-simple examples of G'. Is there any example for G' to be simple?

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References

1 Bourbaki N. "Groupes et algebrès de Lie", Chapters 4–6. Paris: Hermann, 1968
2 Caprace P-E, Rémy B. Simplicity and superrigidity of twin building lattices. Invent Math, 2009, 176: 169–221
3 Caprace P-E, Rémy B. Simplicity of twin tree lattices with non-trivial commutation relations. In: Davis M W, Fowler J, Lafont J-F, et al, eds. Proceedings of the special year on Topology and Geometric Group Theory at OSU 2011–2012. Topology and Geometric Group Theory. New York: Springer, 2016, 143–151
4 Carter R W. Simple Groups of Lie Type. New York: Wiley, 1972
5 Humphreys J F. Introduction to Lie Algebras and Representation Theory. New York: Springer, 1972
6 Hurrelbrink J, Morita J, Rehmann U. On the homological π_0 of Kac-Moody groups. Contemp Math, 1992, 126: 71–77
7 Jacobson N. Lie Algebras. New York: Interscience, 1962
8 Kac V G. Infinite Dimensional Lie Algebras. Boston: Birkhäuser, 1983
9 Kumar S. Kac-Moody Groups, Their Flag Varieties and Representation Theory. Boston: Birkhäuser, 2002
10 Lepowsky J. Lectures on Kac-Moody Lie Algebras. Paris: Paris University Lecture Notes, 1978
11 Matsumoto H. Sur les sous-groupes arithmétiques des groupes semisimples déployés. Ann Sci Éc Norm Supér (4), 1969, 2: 1–62

12 Milnor J. Introduction to Algebraic K-Theory. Princeton: Princeton University Press, 1971

13 Moody R V, Pianzola A. Lie Algebras with Triangular Decompositions. New York: Wiely, 1995

14 Morita J. Commutator relations in Kac-Moody groups. Proc Japan Acad Ser A Math Sci, 1987, 63: 21–22

15 Morita J, Rehmann U. A Matsumoto-type theorem for Kac-Moody groups. Tohoku Math J (2), 1990, 42: 537–560

16 Morita J, Rémy B. Simplicity of some twin tree automorphism groups with trivial commutation relations. Canad Math Bull, 2014, 57: 390–400

17 Peterson D H, Kac V G. Infinite flag varieties and conjugacy theorems. Proc Natl Acad Sci USA, 1983, 80: 1779–1782

18 Steinberg R. Lectures on Chevalley Groups. New Heaven: Yale University Lecture Notes, 1968

19 Tits J. Ensembles ordonnés, immeubles et sommes amalgamées. Bull Soc Math Belg Sér A, 1986, 38: 367–387

20 Tits J. Uniqueness and presentation of Kac-Moody groups over fields. J Algebra, 1987, 105: 542–573