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On a class of two-dimensional Finsler manifolds of isotropic S-curvature

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Abstract For an (α,β)-metric (non-Randers type) of isotropic S-curvature on an n-dimensional manifold with non-constant norm $\|\beta\|_\alpha$, we first show that $n=2$, and then we characterize such a class of two-dimensional (α,β)-manifolds with some PDEs, and also construct some examples for such a class.

Keywords (α,β)-metric, Randers metric, S-curvature

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1 Introduction

The S-curvature is one of the most important non-Riemannian quantities in Finsler geometry, which was originally introduced for the volume comparison theorem (see [6]). Recent studies show that the S-curvature plays a very important role in Finsler geometry (see [1, 2, 7–10]). It is proved that, if an n-dimensional Finsler metric F is of *isotropic* S-curvature $\boldsymbol{S}=(n+1)c(x)F$ for a scalar function $c(x)$ and of scalar flag curvature $\boldsymbol{K}=\boldsymbol{K}(x,y)$, then the flag curvature $\boldsymbol{K}$ can be given by

$$\boldsymbol{K}=\frac{3c_{x^m}y^m}{F}+\tau(x),$$

where $\tau(x)$ is a scalar function (see [2]).

An (α,β)-metric is defined by a Riemann metric $\alpha=\sqrt{a_{ij}(x)y^iy^j}$ and a 1-form $\beta=b_i(x)y^i$ as follows:

$$F=\alpha\phi(s),\quad s=\beta/\alpha,$$

where $\phi(s)$ satisfies certain conditions such that F is regular (positively definite on $TM-0$). A special class of (α,β)-metrics are Randers metrics defined by $F=\alpha+\beta$. With the help of navigation technique, we can characterize and determine the local structures of Randers metrics with isotropic S-curvature (see [5, 8–10]).

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For a pair of α and β, let $b := \|\beta\|_\alpha$ denote the norm of β with respect to α. Define

$$r_{ij} := \frac{1}{2}(b_{i\,|\,j} + b_{j\,|\,i}), \quad s_{ij} := \frac{1}{2}(b_{i\,|\,j} - b_{j\,|\,i}),$$
$$r_j := b^i r_{ij}, \quad s_j := b^i s_{ij}, \quad s^i := a^{ik} s_k,$$

where $b_{i\,|\,j}$'s denote the covariant derivatives of β with respect to α and $b^i := a^{ij} b_j$ and $(a^{ij}) := (a_{ij})^{-1}$. For a C^∞ function $\phi(s) > 0$ on $(-b_o, b_o)$, define

$$\Phi := -(Q - sQ')(n\Delta + sQ + 1) - (b^2 - s^2)(1 + sQ)Q'', \tag{1.1}$$
$$\Delta := 1 + sQ + (b^2 - s^2)Q', \quad Q := \phi'/(\phi - s\phi').$$

It is known that a Randers metric $F = \alpha + \beta$ is of isotropic S-curvature, $\boldsymbol{S} = (n+1)c(x)F$, if and only if (see [3]) $r_{ij} = 2c(a_{ij} - b_i b_j) - b_i s_j - b_j s_i$.

In this paper, we mainly prove the following theorem.

Theorem 1.1. *Let $F = \alpha\phi(s)$ and $s = \beta/\alpha$, be an (α, β)-metric on an n $(\geqslant 2)$-dimensional manifold M, where $\phi(0) = 1$ and $\phi(s) \neq \sqrt{1 + \epsilon s^2} + ks$ for any constants ϵ and k. Suppose $b = \|\beta\|_\alpha \neq$ constant in any domain in M and F is of isotropic S-curvature. Then the following statements hold*:

(i) *the dimension $n = 2$, and*

(ii) *β satisfies*

$$r_{ij} = \frac{3k_1 + k_2 + 4k_1 k_2 b^2}{4 + (k_1 + 3k_2)b^2}(b_i s_j + b_j s_i), \tag{1.2}$$

and $\phi = \phi(s)$ is given by

$$\phi(s) = \{(1 + k_1 s^2)(1 + k_2 s^2)\}^{\frac{1}{4}} \mathrm{e}^{\int_0^s \tau(s) ds}, \tag{1.3}$$

where $\tau(s)$ is defined by

$$\tau(s) := \frac{\pm\sqrt{k_2 - k_1}}{2(1 + k_1 s^2)\sqrt{1 + k_2 s^2}}, \tag{1.4}$$

and k_1 and k_2 are constants with $k_2 > k_1$. In this case, the S-curvature $\boldsymbol{S} = 0$.

Note that we have used the assumption that $b \neq$ constant in Theorem 1.1. For the case that b is a constant, see [4]. In order to derive Theorem 1.1(i) and (1.3), the condition $b = \|\beta\|_\alpha \neq$ constant in any domain in M can be weakened to $db \neq 0$ at a point on M. Furthermore, letting $k_1 = k_2$ in (1.3) and (1.4) yields $\phi(s) = \sqrt{1 + k_1 s^2}$. So the case $k_1 = k_2$ is excluded.

Taking $k_1 = 0$ and $k_2 = 4$, by (1.2) and (1.3) we obtain

$$r_{ij} = \frac{1}{1 + 3b^2}(b_i s_j + b_j s_i), \tag{1.5}$$
$$F(\alpha, \beta) = (\alpha^2 + 4\beta^2)^{\frac{1}{4}} \sqrt{2\beta + \sqrt{\alpha^2 + 4\beta^2}}. \tag{1.6}$$

Theorem 1.1 shows that the metric (1.6) in the two-dimensional case is of isotropic S-curvature if and only if β satisfies (1.5). In the following example, we show a pair α and β satisfying (1.5). For more examples, see Example 6.2 below.

Example 1.2. Let F be an (α, β)-metric on a two-dimensional manifold defined by (1.6). Define α and β by $\alpha = \mathrm{e}^\sigma \sqrt{(y^1)^2 + (y^2)^2}$ and $\beta = \mathrm{e}^\sigma(\xi y^1 + \eta y^2)$, where ξ, η and σ are scalar functions which are given by

$$\xi = x^2, \quad \eta = -x^1, \quad \sigma = -\frac{1}{4}\ln(1 + 4|x|^2), \quad |x|^2 := (x^1)^2 + (x^2)^2.$$

Then α and β satisfy (1.5), and therefore, F is of isotropic S-curvature, $\boldsymbol{S} = 0$, by Theorem 1.1. Furthermore, we have $b^2 = \|\beta\|_\alpha^2 = |x|^2 \neq$ constant.

Taking $k_1 = -1$ and $k_2 = 0$ in (1.3), the metric F in Theorem 1.1 becomes $F = \sqrt{\alpha(\alpha + \beta)}$, which is a square-root metric. We can show in [11] that a square-root metric F on a two-dimensional manifold is an

Einstein metric if and only if F is of vanishing S-curvature, and in this case, F is generally not Ricci-flat (non-zero isotropic flag curvature).

The paper is organized as follows. In Section 2, we give some definitions and notation which are necessary for the present paper, and a lemma is contained. In Section 3, we will derive some results about (2.6), which are necessary for the proof of Theorem 1.1. Furthermore, in Section 4, under the assumptions that $b \neq$ constant in any domain and $\phi(s) \neq k_1\sqrt{1+k_2s^2}+k_3s$ for any constants $k_1>0, k_2$ and k_3, we are going to show that (2.8) has the non-trivial solutions only in the case of dimension $n=2$. Based on the above discussions, the proof of Theorem 1.1 is given in Section 5. Finally, some examples for the metric F satisfying (1.2)–(1.4) are given in Section 6. Besides, we write an appendix which introduces the formulas for some coefficients occurring in (3.1), (3.2), (3.17), (4.1), (4.9) and (4.15).

2 Preliminaries

Let F be a Finsler metric on an n-dimensional manifold M with the standard local coordinate (x^i, y^i) in TM. The Finsler metric F induces a vector field $G = y^i\frac{\partial}{\partial x^i} - 2G^i\frac{\partial}{\partial y^i}$ on TM defined by

$$G^i = \frac{1}{4}g^{il}\{[F^2]_{x^ky^l}y^k - [F^2]_{x^l}\}.$$

The Hausdorff-Busemann volume form $dV = \sigma_F(x)dx^1\wedge\cdots\wedge dx^n$ is defined by

$$\sigma_F(x) := \frac{\mathrm{Vol}(B^n)}{\mathrm{Vol}\{(y^i)\in\mathbb{R}^n \mid F(y^i\frac{\partial}{\partial x^i}|_x) < 1\}}.$$

Furthermore, the S-curvature is defined by

$$\boldsymbol{S} := \frac{\partial G^m}{\partial y^m} - y^m\frac{\partial}{\partial x^m}(\ln\sigma_F).$$

$\boldsymbol{S}$ is said to be *isotropic* if there is a scalar function $c(x)$ on M such that $\boldsymbol{S} = (n+1)c(x)F$. If $c(x)$ is a constant, then we call F is of *constant S-curvature*.

An (α,β)-metric is expressed in the following form:

$$F = \alpha\phi(s), \quad s = \beta/\alpha,$$

where $\phi(s)>0$ is a C^∞ function on an open interval $(-b_o, b_o)$. It is known that F is regular if

$$\phi(s) - s\phi'(s) > 0, \quad \phi(s) - s\phi'(s) + (b^2-s^2)\phi''(s) > 0, \quad |s| \leqslant b < b_o.$$

For an n-dimensional (α,β)-metric $F=\alpha\phi(s)$ and $s=\beta/\alpha$, it has been shown in [4] that the S-curvature is given by

$$\boldsymbol{S} = \left\{2\Psi - \frac{f'(b)}{bf(b)}\right\}(r_0+s_0) - \alpha^{-1}\frac{\Phi}{2\Delta^2}(r_{00} - 2\alpha Qs_0), \tag{2.1}$$

where Φ is defined by (1.1) and

$$\begin{aligned} &r_0 := r_iy^i, \quad s_0 := s_iy^i, \quad r_{00} := r_{ij}y^iy^j, \\ &\Psi := \frac{Q'}{2\Delta}, \quad \Delta := 1+sQ+(b^2-s^2)Q', \quad Q := \frac{\phi'}{\phi - s\phi'}, \\ &f(b) := \frac{\int_0^\pi \sin^{n-2}tdt}{\int_0^\pi \frac{\sin^{n-2}t}{\phi(b\cos t)^n}dt}. \end{aligned} \tag{2.2}$$

Fix an arbitrary point $x\in M$ and take an orthonormal basis $\{e_i\}$ at x such that

$$\alpha = \sqrt{\sum_{i=1}^n (y^i)^2}, \quad \beta = by^1.$$

Then we change coordinates (y^i) to (s, y^A) such that

$$\alpha = \frac{b}{\sqrt{b^2 - s^2}}\bar{\alpha}, \quad \beta = \frac{bs}{\sqrt{b^2 - s^2}}\bar{\alpha},$$

where $\bar{\alpha} = \sqrt{\sum_{A=2}^{n}(y^A)^2}$. Let

$$\bar{r}_{10} := \sum_{A=2}^{n} r_{1A}y^A, \quad \bar{r}_{00} := \sum_{A,B=2}^{n} r_{AB}y^Ay^B, \quad \bar{s}_0 := \sum_{A=2}^{n} s_Ay^A.$$

By (2.1), it is shown in [4] that F is of isotropic S-curvature, $\boldsymbol{S} = (n+1)c(x)F$, if and only if the following two equations hold:

$$\frac{\Phi}{2\Delta^2}(b^2 - s^2)\bar{r}_{00} = -\left\{s\left[\frac{s\Phi}{2\Delta^2} - 2\Psi b^2 + \frac{bf'(b)}{f(b)}\right]r_{11} + (n+1)cb^2\phi\right\}\bar{\alpha}^2, \tag{2.3}$$

$$\left\{\frac{s\Phi}{\Delta^2} - 2\Psi b^2 + \frac{bf'(b)}{f(b)}\right\}r_{1A} = \left\{\left(\frac{\Phi Q}{\Delta^2} + 2\Psi\right)b^2 - \frac{bf'(b)}{f(b)}\right\}s_{1A}. \tag{2.4}$$

In [4], Cheng and Shen studied (2.3) and (2.4) by three steps: (i) $\Phi = 0$, (ii) $\Phi \neq 0$ and $\Upsilon = 0$ and (iii) $\Phi \neq 0$ and $\Upsilon \neq 0$, where Υ is defined by

$$\Upsilon := \frac{d}{ds}\left[\frac{s\Phi}{\Delta^2} - 2\Psi b^2\right].$$

For the two cases: (i) $\Phi = 0$, or (ii) $\Phi \neq 0$ and $\Upsilon = 0$ (in some neighborhood), it is proved in [4] that b must be a constant (in the neighborhood). For the third case $\Phi \neq 0$ and $\Upsilon \neq 0$, Lemma 2.1 is obtained (see [4, Lemma 6.1]), and our discussion (Sections 3 and 4) is based on such a lemma.

Lemma 2.1 (See [4])**.** *Let $F = \alpha\phi(s)$ and $s = \beta/\alpha$ be an (α, β)-metric on an n-dimensional manifold. Assume $\phi(s)$ satisfies $\Phi \neq 0$ and $\Upsilon \neq 0$, and F has isotropic S-curvature, $\boldsymbol{S} = (n+1)c(x)F$. Then*

$$r_{ij} = ka_{ij} - \epsilon b_ib_j - \lambda(b_is_j + b_js_i), \tag{2.5}$$

$$-2s(k - \epsilon b^2)\Psi + (k - \epsilon s^2)\frac{\Phi}{2\Delta^2} + (n+1)c\phi - s\nu = 0, \tag{2.6}$$

where $\lambda = \lambda(x), k = k(x)$ and $\epsilon = \epsilon(x)$ are some scalar functions and

$$\nu := -\frac{f'(b)}{bf(b)}(k - \epsilon b^2). \tag{2.7}$$

If in addition $s_0 \neq 0$, then

$$-2\Psi - \frac{Q\Phi}{\Delta^2} - \lambda\left(\frac{s\Phi}{\Delta^2} - 2\Psi b^2\right) = \delta, \tag{2.8}$$

where

$$\delta := -\frac{f'(b)}{bf(b)}(1 - \lambda b^2). \tag{2.9}$$

3 On (2.6)

In this section, we assume $b \neq$ constant (in any neighborhood) and $\phi(s) \neq k_1\sqrt{1 + k_2s^2} + k_3s$ for any constants $k_1 > 0, k_2$ and k_3. We are going to prove that $k = 0, c = 0, \epsilon = 0$ and $\nu = 0$ in (2.6). Before the discussion, we show a remark (needed in this section and Section 4).

Remark 3.1. Assume $b \neq$ constant in any neighborhood of the manifold M. Consider a polynomial

$$f(b) := c_0 + c_1b + \cdots c_mb^m,$$

where c_i's are constant and there is at least some c_i which is not zero. Let U be an open set of M, and $T := \{x \in U \mid f(b) = 0\}$. Then T is a closed and no-where dense set (since $b \neq$ constant in any neighborhood of M). So as an example, for a scalar function $\sigma = \sigma(x)$, if $\sigma = 0$ on $U - T$, then $\sigma = 0$ on U by continuity.

Thus without loss of generality, we can always assume $f(b) \neq 0$, or just have a restriction on $U - T$ in the following discussion, if c_i's are not all zero.

We first transform (2.6) into a differential equation about $\phi(s)$ and then (2.6) $\times\, 2\phi[\phi - s\phi' + (b^2 - s^2)\phi'']^2$ yields

$$\begin{aligned}&-(b^2 - s^2)(k - \epsilon s^2)(\phi - s\phi')\phi\phi''' + \{s[(2\nu + 2\epsilon - n\epsilon)s^2 + 2(\epsilon - \nu)b^2 + k(n-4)]\\&\quad + 2(n+1)c(b^2 - s^2)\phi\}(b^2 - s^2)\phi(\phi'')^2 + \{(n+1)(b^2 - s^2)\big[4c\phi^2 - (k - \epsilon s^2)\phi'\big]\\&\quad - s[(n\epsilon + \epsilon - 4\nu)s^2 + 2(2\nu - \epsilon)b^2 - (n-1)k]\phi\}(\phi - s\phi')\phi'' + (\phi - s\phi')^2\\&\quad \times \{(n+1)[2c\phi^2 - (k - \epsilon s^2)\phi'] - 2\nu s\phi\} = 0. \end{aligned} \tag{3.1}$$

Express the power series of $\phi(s)$ at $s = 0$ as

$$\phi(s) = 1 + a_1 s + a_2 s^2 + a_3 s^3 + \cdots = 1 + \sum_{i=1}^{\infty} a_i s^i.$$

Let p_i be the coefficients of s^i in (3.1). The expressions of p_0, p_1, p_2, p_3, p_4 and p_5, which will be needed in the following discussion, are given in Remark A.1. All the equations $p_i = 0$ are homogeneous linear equations about k, c, ϵ and ν. The coefficient determinant of the linear system $p_0 = 0$, $p_1 = 0$, $p_2 = 0$ and $p_3 = 0$ is in the form

$$A_1 b^6 + A_2 b^4 + A_3 b^2 - (n+1)a_1[4(n+1)a_4 + 2(n+1)a_2^2 + (n-2)a_1 a_3], \tag{3.2}$$

where A_1, A_2 and A_3 are constant, and their expressions are given in Remark A.2. If

$$a_1 \neq 0, \quad 4(n+1)a_4 + 2(n+1)a_2^2 + (n-2)a_1 a_3 \neq 0,$$

then the above determinant is not zero (see Remark 3.1), and thus in this case we conclude that $k = 0$, $c = 0$, $\epsilon = 0$ and $\nu = 0$ from the linear system $p_0 = 0$, $p_1 = 0$, $p_2 = 0$ and $p_3 = 0$.

In the following, we further prove $k = 0, c = 0, \epsilon = 0$ and $\nu = 0$ if $a_1 = 0$, or $4(n+1)a_4 + 2(n+1)a_2^2 + (n-2)a_1 a_3 = 0$.

Case 1. Assume $a_1 = 0$. By $p_0 = 0, p_1 = 0$ and $a_1 = 0$, we obtain (assume $1 + 2a_2 b^2 \neq 0$ by Remark 3.1)

$$\nu = \frac{2[(18a_3^2 - 10a_2^3 - 12a_2 a_4)b^4 - (7a_2^2 + 6a_4)b^2 - a_2]k + 2a_2 b^2(1 + 2a_2 b^2)^2\epsilon}{(1 + 2a_2 b^2)^3}, \tag{3.3}$$

$$c = \frac{3a_3 b^2}{(n+1)(1 + 2a_2 b^2)^2} k. \tag{3.4}$$

Since $\phi(s) \neq \sqrt{1 + 2a_2 s^2} = \sum_{i=0}^{\infty} C_{\frac{1}{2}}^i (2a_2 s^2)^i$, there exists some minimal integer m such that

$$a_{2m+1} \neq 0, \quad m \geqslant 1, \quad \text{or} \quad a_{2m} \neq C_{\frac{1}{2}}^m (2a_2)^m, \quad m \geqslant 2, \tag{3.5}$$

where C_μ^i's are the generalized combination coefficients.

Case 1A. Assume $a_{2m+1} \neq 0$ in (3.5). First consider the case $m = 1$. Then $a_3 \neq 0$. Plug (3.3), (3.4) and $a_1 = 0$ into $p_2 = 0$ and $p_4 = 0$ and then we get a linear system about k and ϵ. The critical component of the determinant for this linear system is given by

$$(\cdots)b^8 + (\cdots)b^6 + (\cdots)b^4 + (\cdots)b^2 - 3(n-1)(n+3)a_3^2,$$

where the omitted terms are all constants. Now it is seen that $k = 0$ and $\epsilon = 0$ since $a_3 \neq 0$. Thus by (3.3) and (3.4) we have $c = 0$ and $\nu = 0$.

Now let $m > 1$. In this case, we have $a_3 = 0$. For our purpose to prove $k = 0$ and $\epsilon = 0$, we only need to compute p_{2m-2} and p_{2m}. Express $\phi(s)$ as

$$\phi(s) = g(s) + h(s), \tag{3.6}$$

where

$$g(s) := 1 + \sum_{i=1}^{\infty} a_{2i} s^{2i}, \quad h(s) := \sum_{i=m}^{\infty} a_{2i+1} s^{2i+1}.$$

Plug (3.6) into (3.1) and then we write the left-hand side of (3.1) as $P_1 + P_2$, where every term of P_1 includes at least h or its derivatives h', h'' and h'''', and P_2 is just the left-hand side of (3.1) with $\phi(s)$ being replaced with $g(s)$. Among h, h', h'' and h'''', the function h'''' has the power series of the least degree $2m - 2$. Since $m > 1$, we have $a_3 = 0$, and then we get $c = 0$ by (3.4). So the power series of P_2 has no term of even degree.

Thus by the above analysis we see that, to get p_{2m-2}, it is sufficient to put

$$g(s) = 1 + o(s), \quad h(s) = a_{2m+1} s^{2m+1} + o(s^{2m+2}),$$

and plug (3.6) into (3.1). Then by (3.3), (3.4), $a_1 = 0$ and $a_3 = 0$, the equation $p_{2m-2} = 0$ is reduced to

$$-2m(4m^2 - 1)b^2 a_{2m+1} k = 0. \tag{3.7}$$

By (3.7) we have $k = 0$. Similarly, to get p_{2m}, it is sufficient to put

$$g(s) = 1 + a_2 s^2 + o(s^3), \quad h(s) = a_{2m+1} s^{2m+1} + a_{2m+3} s^{2m+3} + o(s^{2m+4}),$$

and plug (3.6) into (3.1). Then from (3.3), (3.4), $a_1 = 0$, $a_3 = 0$ and $k = 0$, the equation $p_{2m} = 0$ is reduced to

$$2m(2m + 1)^2 a_{2m+1} b^2 \epsilon = 0. \tag{3.8}$$

By (3.8) we have $\epsilon = 0$. Thus by (3.3) and (3.4) we have $c = 0$ and $\nu = 0$.

Case 1B. Assume all $a_{2i+1} = 0$ $(i \geqslant 0)$, and assume $a_{2m} \neq C^m_{\frac{1}{2}}(2a_2)^m$ in (3.5). If $m = 2$, then $2a_4 + a_2^2 \neq 0$. Plug (3.3), (3.4), $a_1 = 0$ and $a_3 = 0$ into $p_3 = 0$ and $p_5 = 0$ and then we get a linear system about k and ϵ. The critical component of the determinant for this linear system is given by

$$(\cdots)b^4 + (\cdots)b^2 - (n + 1)(n + 4)(2a_4 + a_2^2)^2,$$

where the omitted terms are all constants. Now it is easy to see that $k = 0$ and $\epsilon = 0$ since $2a_4 + a_2^2 \neq 0$. Thus by (3.3) and (3.4) we have $c = 0$ and $\nu = 0$.

Now let $m > 2$. In this case, we have $a_4 = -a_2^2/2$. For our purpose to prove $k = 0$ and $\epsilon = 0$, we only need to compute p_{2m-3} and p_{2m-1}. Since $\sqrt{1 + 2a_2 s^2} = \sum_{i=0}^{\infty} C^i_{\frac{1}{2}}(2a_2 s^2)^i$, we may express $\phi(s)$ as

$$\phi(s) = g(s) + h(s), \tag{3.9}$$

where $g(s) := \sqrt{1 + 2a_2 s^2}$, $h(s) := \sum_{i=m}^{\infty} d_{2i} s^{2i}$ and $d_{2m} \neq 0$. Plug (3.9) into (3.1) and then we write the left-hand side of (3.1) as $P_1 + P_2$, where every term of P_1 includes at least h or its derivatives h', h'' and h'''', and P_2 which is just the left-hand side of (3.1) with $\phi(s)$ being replaced with $g(s)$, will vanish when we plug (3.3), (3.4) ($a_3 = 0$) and $a_4 = -a_2^2/2$ into it. Among h, h', h'' and h'''', the function h'''' has the power series of the least degree $2m - 3$.

By the above analysis, to get p_{2m-3}, it is sufficient to plug (3.9) and

$$g(s) = 1 + o(1), \quad h(s) = d_{2m} s^{2m} + o(s^{2m+1})$$

into (3.1). Then from (3.3), (3.4) and $a_4 = -a_2^2/2$, the equation $p_{2m-3} = 0$ is reduced to

$$-4m(2m-1)(m-1)(1+2a_2b^2)^2b^2d_{2m}k = 0. \tag{3.10}$$

By (3.10) we get $k = 0$. To get p_{2m-1}, it is sufficient to plug (3.9) and

$$g(s) = 1 + a_2s^2 + o(s^2), \quad h(s) = d_{2m}s^{2m} + d_{2m+2}s^{2m+2} + o(s^{2m+3})$$

into (3.1). Then from (3.3), (3.4), $a_4 = -a_2^2/2$ and $k = 0$, the equation $p_{2m-1} = 0$ is reduced to

$$4m^2(2m-1)b^2(1+2a_2b^2)^2d_{2m}\epsilon = 0. \tag{3.11}$$

By (3.11) we get $\epsilon = 0$. Thus by (3.3) and (3.4) we have $c = 0$ and $\nu = 0$.

Case 2. Assume $a_1 \neq 0$ and $4(n+1)a_4 + 2(n+1)a_2^2 + (n-2)a_1a_3 = 0$. In this case, the coefficient determinant of the linear system $p_0 = 0$, $p_1 = 0$, $p_2 = 0$ and $p_3 = 0$ is not zero if $A_1 \neq 0$ or $A_2 \neq 0$ or $A_3 \neq 0$ (see (3.2)). So if $A_1 \neq 0$ or $A_2 \neq 0$ or $A_3 \neq 0$, then immediately we get $k = 0$, $c = 0$, $\epsilon = 0$ and $\nu = 0$.

Thus we only need to consider the case $A_1 = 0, A_2 = 0$ and $A_3 = 0$. By an analysis on the equations $A_1 = 0$, $A_2 = 0$ and $A_3 = 0$, it is enough for us to prove $k = 0$, $c = 0$, $\epsilon = 0$ and $\nu = 0$ under one of the following two conditions:

$$a_3 = 0, \quad a_4 = -\frac{1}{2}a_2^2, \quad a_6 = \frac{1}{6}[(n-2)a_1a_5 + 3a_2^3] \tag{3.12}$$

and

$$a_3 = -\frac{(4n^3+15n^2+16)a_1^3}{36(n^2-1)}, \quad a_4 = \frac{2(n+1)a_2^2 + (n-2)a_1a_3}{4(n+1)}, \tag{3.13}$$

$$a_5 = \frac{(n+4)(4n^2-n+4)}{1440(n+1)^3(1-n)}T_0, \quad a_6 = \frac{T}{60(n+1)^2}, \tag{3.14}$$

where

$$\begin{aligned} T_0 &:= a_1^3[2a_1^2n^3 + 5(3a_1^2 - 16a_2)n^2 + (6a_1^2 - 160a_2)n + 20(a_1^2 - 4a_2)], \\ T &:= a_1(10a_5 + 20a_2a_3 - 3a_1^2a_3)n^3 + (30a_2^3 - 120a_3^2 + 45a_1a_2a_3 - 6a_1^3a_3)n^2 \\ &\quad + (60a_2^3 + 15a_1^3a_3 - 30a_1a_5 - 276a_3^2 - 105a_1a_2a_3)n + 18a_1^3a_3 - 130a_1a_2a_3 \\ &\quad - 48a_3^2 + 30a_2^3 - 20a_1a_5. \end{aligned}$$

Case 2A. Assume (3.12). Solving $p_0 = 0$, $p_1 = 0$, $p_2 = 0$ and $p_4 = 0$ yields (assume $c \neq 0$)

$$k = \frac{2(1+2a_2b^2)c}{a_1}, \quad \epsilon = \frac{2(a_1^2 - 2a_2)(1+2a_2b^2)c}{a_1}, \tag{3.15}$$

$$a_5 = 0, \quad \nu = \frac{2[(1+n+2a_2b^2)a_1^2 - 2a_2(1+2a_2b^2)]c}{a_1}. \tag{3.16}$$

Plug (3.15) and (3.16) into (3.1) and then we get

$$c(f_0 + f_2b^2 + f_4b^4) = 0, \tag{3.17}$$

where f_0, f_2 and f_4 are some ODEs about $\phi(s)$, where the expressions of f_0, f_2 and f_4 are given in Remark A.3. If $c \neq 0$, then by (3.17), solving $f_0 = 0$, $f_2 = 0$ and $f_4 = 0$ with $\phi(0) = 1$ yields $\phi(s) = a_1s + \sqrt{1+2a_2s^2}$. This case is excluded. So $c = 0$. Then by (3.15) and (3.16) we get $k = 0, \epsilon = 0$ and $\nu = 0$.

Case 2B. Assume (3.13) and (3.14). Plug (3.13) and (3.14) into $p_0 = 0, p_1 = 0, p_2 = 0$ and $p_4 = 0$ and we obtain $k = 0$, $\epsilon = 0$, $\nu = 0$ and $c = 0$, since the coefficient determinant of the linear system $p_0 = 0$, $p_1 = 0$, $p_2 = 0$ and $p_4 = 0$ is not zero.

4 On (2.8)

In this section, we assume $b \neq$ constant (in any neighborhood) and $\phi(s) \neq k_1\sqrt{1+k_2s^2}+k_3s$ for any constants $k_1>0$, k_2 and k_3. We are going to show that (2.8) has the non-trivial solutions only in the case of dimension $n=2$. In the following discussion, we will also use Remark 3.1.

We first transform (2.8) into a differential equation about $\phi(s)$ and then $(2.8)\times\phi(-\phi+s\phi')[\phi-s\phi'+(b^2-s^2)\phi'']^2$ gives

$$\begin{aligned}&-(b^2-s^2)(\phi-s\phi')[(1-\lambda s^2)\phi'+\lambda s\phi]\phi\phi'''-\{[1+(\delta-\lambda)b^2\\&\quad+(n\lambda-2\lambda-\delta)s^2](\phi-s\phi')+(n-2)s\phi'\}(b^2-s^2)\phi(\phi'')^2\\&\quad-\{[1+(\delta-\lambda)b^2+(n\lambda-2\delta+\lambda)s^2](\phi-s\phi')^2+[2(n\lambda-\delta+\lambda)s^2\\&\quad-(n\lambda-2\delta+2\lambda)b^2-n-2]s\phi'(\phi-s\phi')-(n+1)(b^2-2s^2)(\phi')^2\}\\&\quad\times(\phi-s\phi')\phi''-[\delta(\phi-s\phi')^2-(n\lambda-\delta+\lambda)s\phi'(\phi-s\phi')-(n+1)(\phi')^2]\\&\quad\times(\phi-s\phi')^2=0.\end{aligned}\tag{4.1}$$

Express the power series of $\phi(s)$ at $s=0$ as

$$\phi(s)=1+a_1s+a_2s^2+a_3s^3+\cdots=1+\sum_{i=1}^{\infty}a_is^i.$$

Let p_i be the coefficients of s^i in (4.1). We need to compute p_0,p_1,p_2 and p_3 first, and their expressions are given in Remark A.4. In the following, we will solve λ and δ in two cases.

Case 1. Assume $a_1=0$ and $a_3=0$. We are going to show that this case is excluded.

Plugging $a_1=0$ and $a_3=0$ into $p_0=0$ yields

$$\delta=\frac{2a_2}{1+2a_2b^2}(\lambda b^2-1).\tag{4.2}$$

Since $\phi(s)\neq\sqrt{1+2a_2s^2}$, there exists some minimal integer m such that

$$a_{2m+1}\neq 0,\quad m\geqslant 2,\quad\text{or}\quad a_{2m}\neq C^m_{\frac{1}{2}}(2a_2)^m,\quad m\geqslant 2,\tag{4.3}$$

where C^i_μ's are the generalized combination coefficients. Then we will determine λ in the two cases of (4.3).

Case 1A. Assume $a_{2m+1}\neq 0$ in (4.3). In this case, we need to compute p_{2m-1}. For this, express $\phi(s)$ as

$$\phi(s)=g(s)+h(s),\tag{4.4}$$

where

$$g(s):=1+\sum_{i=1}^{\infty}a_{2i}s^{2i},\quad h(s):=\sum_{i=m}^{\infty}a_{2i+1}s^{2i+1}.$$

Plug (4.4) into (4.1) and then we write the left-hand side of (4.1) as P_1+P_2, where every term of P_1 includes at least h or its derivatives h',h'' and h''', and P_2 is just the left-hand side of (4.1) with $\phi(s)$ being replaced with $g(s)$. Among h,h',h'' and h''', the function h''' has the power series of the least degree $2m-2$. Furthermore, it is easy to see that the power series of P_2 has no term of odd degree.

Thus by the above analysis we see that, to get p_{2m-1}, it is sufficient to put

$$g(s)=1+a_2s^2+o(s^3),\quad h(s)=a_{2m+1}s^{2m+1}+o(s^{2m+2}),$$

and plug (4.4) into (4.1). Then by $p_{2m-1}=0$, $a_{2m+1}\neq 0$ and (4.2) we obtain

$$\lambda=\frac{1-2(2m-1)a_2b^2}{2mb^2}.\tag{4.5}$$

Case 1B. Assume all $a_{2i+1}=0$ ($i\geqslant 0$), and assume $a_{2m}\neq C_{\frac{1}{2}}^{m}(2a_2)^m$ in (4.3). Express $\phi(s)$ as

$$\phi(s)=g(s)+h(s), \tag{4.6}$$

where

$$g(s):=\sqrt{1+2a_2s^2},\quad h(s):=\sum_{i=m}^{\infty}d_{2i}s^{2i},\quad d_{2m}\neq 0.$$

Plug (4.6) into (4.1) and then we write the left-hand side of (4.1) as P_1+P_2, where every term of P_1 includes at least h or its derivatives h',h'' and h'''', and P_2 which is just the left-hand side of (4.1) with $\phi(s)$ being replaced with $g(s)$, will vanish when we plug (4.2) into it. Among h,h',h'' and h'''', the function h'''' has the power series of the least degree $2m-3$.

Now by the above analysis, to compute p_{2m-2} in (4.1), it is sufficient to put

$$g(s)=1+a_2s^2+o(s),\quad h(s)=d_{2m}s^{2m}+o(s^{2m+1})$$

in (4.6) and plug (4.6) into (4.1). Then using (4.2) and $d_{2m}\neq 0$, by $p_{2m-2}=0$ we obtain

$$\lambda=\frac{1-4(m-1)a_2b^2}{(2m-1)b^2}. \tag{4.7}$$

Now we have solved λ in the two cases of (4.3). It is easy to see that (4.5) and (4.7) can be written in the following form:

$$\lambda=\frac{1-2(k-1)a_2b^2}{kb^2}, \tag{4.8}$$

where $k\geqslant 3$ is an integer.

Plugging (4.2) and (4.8) into (4.1) yields

$$f_0+f_2b^2+f_4b^4=0, \tag{4.9}$$

where f_0,f_2 and f_4 are some ODEs about $\phi(s)$ given in Remark A.5. Then by (4.9), solving $f_0=0$, $f_2=0$ and $f_4=0$ with $\phi(0)=1$ yields $\phi(s)=\sqrt{1+2a_2s^2}$. This case is excluded.

Case 2. Assume $a_1\neq 0$ or $a_3\neq 0$. We are going to show that for one case, there are the non-trivial solutions for $\phi(s)$ in dimension $n=2$.

Case 2A. Assume $a_1=0$ and $a_3\neq 0$. It follows that $a_4=-\frac{1}{2}a_2^2$ from $p_0=0$, $p_1=0$, $p_2=0$ and $a_1=0$. Then by $p_0=0$, $p_1=0$, $p_3=0$, $a_1=0$ and $a_4=-\frac{1}{2}a_2^2$ we get a contradiction.

Case 2B. Assume $a_1\neq 0$. Solving λ and δ from $p_0=0$ and $p_1=0$ gives

$$\lambda=\frac{B_4b^4+B_2b^2+B_0}{T},\quad \delta=\frac{C_4b^4+C_2b^2+C_0}{T}, \tag{4.10}$$

where

$$\begin{aligned}
&B_4:=4(n+1)a_1^2a_2(a_1a_2+3a_3)-8(6a_4a_2-9a_3^2+na_2^3+4a_2^3)a_1-24a_2^2a_3,\\
&B_2:=(n+1)a_1^2(4a_1a_2+6a_3)-(8a_2^2n+20a_2^2+24a_4)a_1,\\
&B_0:=(n+1)a_1(a_1^2-2a_2)+6a_3,\\
&C_4:=-4(n+1)a_1^2a_2(a_1a_2+3a_3)+8(4a_2^3+6a_4a_2+a_2^3n-9a_3^2)a_1+24a_2^2a_3,\\
&C_2:=(n+1)a_1(-2(n+2)a_2a_1^2-18a_3a_1+8a_2^2)+12a_3a_2,\\
&C_0:=-(n+1)^2a_1^3+2(n+1)a_2a_1,\\
&T:=(2a_2b^2+1)[(12a_3+2a_2a_1(n+1))b^2+a_1(n+1)].
\end{aligned}$$

Then plugging (4.10) into $p_2=0$ yields

$$a_4=-\frac{1}{2}a_2^2-a_1a_3, \tag{4.11}$$

$$a_5 = -\frac{a_3[n^2a_1^3 + (3a_3 + 20a_1a_2 - 6a_1^3)n + 20a_1a_2 - 21a_3 - 7a_1^3]}{10(n+1)a_1}, \tag{4.12}$$

$$(n-7)a_3^2(na_1^3 + a_1^3 - 6a_3) = 0. \tag{4.13}$$

By (4.13), we break our discussion into the following three steps.

(I) If $n = 7$ and $a_3 \neq 0$, plugging (4.10) together with $n = 7$, (4.11) and (4.12) into $p_3 = 0$ yields

$$q_4b^4 + q_2b^2 + q_0 = 0,$$

where

$$\begin{aligned} q_4 &:= -24a_1(4a_2a_1 + 3a_3)a_6 - 4a_2(-12a_2^3a_1^2 - 9a_2^2a_1a_3 - 9a_2a_3^2 \\ &\quad - 56a_3a_2a_1^3 - 60a_1^2a_3^2 + 12a_1^5a_3), \\ q_2 &:= (36a_2 + 12a_1^2)a_3^2 + 8a_1^3(-3a_1^2 + 10a_2)a_3 + 24a_1^2(a_2^3 - 2a_6), \\ q_0 &:= a_3(9a_3 - 16a_1^3). \end{aligned}$$

So we have $q_0 = 0$, $q_2 = 0$ and $q_4 = 0$, which implies a contradiction since $a_1 \neq 0$ and $a_3 \neq 0$.

(II) If $a_3 = 0$, then plug (4.11) and $a_3 = 0$ into (4.10) and we can get

$$\lambda = a_1^2 - 2a_2, \quad \delta = \frac{na_1^2 + (1 + 2a_2b^2)(a_1^2 - 2a_2)}{1 + 2a_2b^2}. \tag{4.14}$$

Plugging (4.14) into (4.1) yields

$$f_0 + f_2b^2 + f_4b^4 = 0, \tag{4.15}$$

where f_0, f_2 and f_4 are some ODEs about $\phi(s)$, where the expressions of f_0, f_2 and f_4 are given in Remark A.6. Then by (4.15), solving $f_0 = 0, f_2 = 0$ and $f_4 = 0$ with $\phi(0) = 1$ yields

$$\phi(s) = a_1s + \sqrt{1 + 2a_2s^2}.$$

This case is excluded.

(III) Assume

$$a_3 = \frac{1}{6}(n+1)a_1^3. \tag{4.16}$$

Plugging (4.10) together with (4.11), (4.12) and (4.16) into $p_3 = 0$ yields

$$(\cdots)b^2 + (n+1)(n-2)a_1^4 = 0,$$

which implies $n = 2$. Plugging (4.10) together with (4.11), (4.16) and $n = 2$ into (4.1) yields

$$f_0 + f_2b^2 + f_4b^4 = 0, \tag{4.17}$$

where f_0, f_2 and f_4 are some ODEs about $\phi(s)$ given by

$$\begin{aligned} f_0 &:= [2(a_1^2 - a_2)s(\phi - s\phi') + \phi']s^2\phi\phi''' - s^2[1 + (2a_2 - 3a_1^2)s^2]\phi(\phi'')^2 \\ &\quad + \{(1 - 2a_2s^2)(\phi - s\phi')^2 + [4 + 2(3a_1^2 - 4a_2)s^2]s\phi'(\phi - s\phi') + 6s^2(\phi')^2\}\phi'' \\ &\quad + [(3a_1^2 - 2a_2)(\phi - s\phi')^2 + (4a_2 - 3a_1^2)s\phi'(\phi - s\phi') - 3(\phi')^2](\phi - s\phi'), \\ f_2 &:= \{[(2a_2 + a_1^2)(3a_1^2 - 2a_2)s^2 + 2(a_2 - a_1^2)]s(\phi - s\phi') - (1 - 2a_2s^2)\phi'\}\phi\phi''' \\ &\quad \times [1 - (2a_2 + a_1^2)s^2][1 + (2a_2 - 3a_1^2)s^2]\phi(\phi'')^2 + \{[(2a_2 + a_1^2)(3a_1^2 - 2a_2)s^2 \\ &\quad + 4a_1^2](\phi - s\phi')^2 + [4(2a_2 + a_1^2)(3a_1^2 - 2a_2)s^2 + 2(6a_2 - a_1^2)]s\phi'(\phi - s\phi') \\ &\quad + 3(4a_2s^2 - 1)(\phi')^2\}\phi'' + \{(2a_2 + a_1^2)(2a_2 - 3a_1^2)(3s\phi' - \phi)(\phi - s\phi') \\ &\quad - 6a_2(\phi')^2\}(\phi - s\phi') \end{aligned}$$

and

$$f_4 := [(2a_2 + a_1^2)(2a_2 - 3a_1^2)s(\phi - s\phi') - 2a_2\phi']\phi\phi''' + (2a_2 + a_1^2)[1 + (2a_2 - 3a_1^2)s^2]\phi(\phi'')^2 + [(2a_2 + a_1^2)(2a_2 - 3a_1^2)(\phi - s\phi')(3s\phi' - \phi) - 6a_2(\phi')^2]\phi''.$$

Then by (4.17), we get $f_0 = 0$, $f_2 = 0$ and $f_4 = 0$. To solve the system of ODEs $f_0 = 0$, $f_2 = 0$ and $f_4 = 0$ with $\phi(0) = 1$, we first express ϕ'' in terms of ϕ and ϕ' by eliminating ϕ''' from

$$s^{-2}f_0 + s^2 f_4 + f_2 = 0.$$

Then plug the expression of ϕ'' into f_0 and we can get the expression of ϕ'''. Now plugging the expressions of ϕ'' and ϕ''' into f_4, we obtain an ODE equivalent to

$$0 = 4(1 + k_1 s^2)(1 + k_2 s^2)^2\phi'^2 - 4s(1 + k_2 s^2)(k_1 + k_2 + 2k_1k_2s^2)\phi\phi' + [k_1 - k_2 + 4k_1k_2s^2(1 + k_2s^2)]\phi^2, \tag{4.18}$$

where we put

$$k_1 := 2a_2 - 3a_1^2, \quad k_2 := 2a_2 + a_1^2. \tag{4.19}$$

Then solving (4.18) with $\phi(0) = 1$ yields (1.3).

5 Proof of Theorem 1.1

By the result in [4], we only need to consider the case shown in Lemma 2.1, and only in this case it possibly occurs that $b \neq$ constant. Now suppose $\phi(s) \neq \sqrt{1 + \epsilon s^2} + ks$ for any constants ϵ and k, and $b \neq$ constant in any neighborhood. The discussions in Sections 3 and 4 imply that $\phi(s)$ is given by (1.3) and the dimension $n = 2$ (see Case 2B(III) in Section 4). Furthermore, plugging (4.11) and (4.16) and $n = 2$ into (4.10) yields

$$\delta = \frac{(3a_1^2 - 2a_2)[1 + (2a_2 + a_1^2)b^2]}{1 + 2a_2b^2}, \tag{5.1}$$

$$\lambda = \frac{(3a_1^2 - 2a_2)(2a_2 + a_1^2)b^2 + 2(a_1^2 - a_2)}{1 + 2a_2b^2}. \tag{5.2}$$

Since we have proved in Section 3 that $k = 0$ and $\epsilon = 0$, by (2.5) and (5.2) we obtain (1.2). At the end of Section 4, we have shown that $\phi(s)$ is given by (1.3) by solving (4.18) with $\phi(0) = 1$. Besides, the proof in Section 3 also shows $c = 0$, which implies $\boldsymbol{S} = 0$.

Remark 5.1. Plugging (5.1) and (5.2) into (2.9), we get

$$f(b) = \sqrt{1 + (2a_2 - 3a_1^2)b^2}. \tag{5.3}$$

One possibly wonders whether we can get (5.3) from (2.2) when we plug (1.3) and $n = 2$ into (2.2). This is true. One way to check it is to expand (2.2) and (5.3) into power series, respectively. One may try a direct verification.

6 Examples

In this section, we will construct some examples for the metric F given by (1.2)–(1.4).

Since every two-dimensional Riemann metric is locally conformally flat, we may put

$$\alpha = \mathrm{e}^{\sigma}\sqrt{(y^1)^2 + (y^2)^2}, \tag{6.1}$$

where $\sigma = \sigma(x)$ is a scalar function and $x = (x^1, x^2)$. Then β can be expressed as

$$\beta = \mathrm{e}^{\sigma}(\xi y^1 + \eta y^2). \tag{6.2}$$

Now we can show that (1.2) is equivalent to the following system of PDEs:

$$\sigma_1 = \frac{T_1}{T_0}, \quad \sigma_2 = \frac{T_2}{\xi T_0}, \quad \xi_1 = -\frac{\eta(\eta\eta_2 + \xi\xi_2 + \xi\eta_1)}{\xi^2}, \tag{6.3}$$

where

$$\begin{aligned}
T_0 &:= \xi[1 + k_2(\xi^2 + \eta^2)][1 + k_1(\xi^2 + \eta^2)], \\
T_1 &:= 2\xi\eta[(3k_1 - k_2)/4 + k_1k_2(\xi^2 + \eta^2)]\xi_2 \\
&\quad - [1 + (k_1 + k_2)\xi^2/2 + (k_2 - k_1)\eta^2/2 + k_1k_2(\xi^4 - \eta^4)]\eta_2, \\
T_2 &:= [(k_2 - k_1)\xi^2/2 + (k_1 + k_2)\eta^2 - k_1k_2(\xi^4 - \eta^4)](\xi\xi_2 + \eta\eta_2) \\
&\quad + \xi[1 + k_2(\xi^2 + \eta^2)][1 + k_1(\xi^2 + \eta^2)]\eta_1.
\end{aligned}$$

Proposition 6.1. *Let $F = \alpha\phi(s)$ and $s = \beta/\alpha$ be a two-dimensional (α, β)-metric on $\mathbb{R}^2$, where $b = \|\beta\|_\alpha \neq$ constant and $\phi(s)$ satisfies (1.3). Then F is of isotropic S-curvature if and only if α and β can be locally defined by (6.1) and (6.2), where ξ, η and σ are some scalar functions satisfying (6.3). In this case, $\boldsymbol{S} = 0$.*

If we take $\xi = x^2$ and $\eta = -x^1$, then σ determined by (6.3) is given by

$$\sigma = -\frac{1}{4}\{\ln[1 + k_2|x|^2] + 3\ln[1 + k_1|x|^2]\}, \tag{6.4}$$

where $|x|^2 := (x^1)^2 + (x^2)^2$. Thus we obtain the following example.

Example 6.2. Let F be a two-dimensional (α, β)-metric defined by (1.3). Define α and β by (6.1) and (6.2), where $\xi = x^2$ and $\eta = -x^1$, and σ is given by (6.4). Then F is of isotropic S-curvature $\boldsymbol{S} = 0$ by Theorem 1.1. Furthermore, we have $b^2 = \|\beta\|_\alpha^2 = |x|^2 \neq$ constant.

In Example 6.2, if we take $k_1 = 0$ and $k_2 = 4$, then by (1.3) and (1.4), we obtain

$$\phi(s) = (1 + 4s^2)^{\frac{1}{4}}\sqrt{2s + \sqrt{1 + 4s^2}},$$

and thus we get Example 1.2.

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Appendix A

Remark A.1. Let p_i be the coefficients of s^i in (3.1). We have

$$
\begin{aligned}
p_0 &= (-a_1 - 2b^2a_1a_2 - a_1n - 6b^2a_3 - 2a_1na_2b^2)k + 2(2a_2b^2+1)^2(n+1)c,\\
p_1 &= (-4a_2 - 6a_1na_3b^2 - 12b^2a_3a_1 - 20{a_2}^2b^2 - 24b^2a_4)k - 2(2a_2b^2+1)^2v\\
&\quad + 4(2a_2b^2+1)(2b^2a_1a_2 + 6b^2a_3 + a_1)(n+1)c + 4a_2b^2(2a_2b^2+1)\epsilon,\\
p_2 &= (-60b^2a_5 + 3a_3n - 3a_3 + 6a_1na_2 - 36b^2a_4a_1 - 114b^2a_3a_2 + 6a_2na_3b^2 + 6a_1n{a_2}^2b^2\\
&\quad + 2a_2a_1 - 14{a_2}^2a_1b^2 - 12a_1na_4b^2)k + 2(24b^2a_3a_1 - 4a_2 + {a_1}^2 - 4{a_2}^2b^2 + 8{a_2}^3b^4\\
&\quad + 36{a_3}^2b^4 + 48a_2b^4a_1a_3 + 24b^2a_4 + 4a_2b^2{a_1}^2 + 4{a_1}^2{a_2}^2b^4 + 48a_2b^4a_4)(n+1)c\\
&\quad + (a_1n + 8{a_2}^2b^4a_1 + a_1 + 18b^2a_3 + 6b^2a_1a_2 + 48a_2b^4a_3 + 2a_1na_2b^2)\epsilon\\
&\quad - 2(2a_2b^2+1)(2b^2a_1a_2 + 12b^2a_3 + a_1)v,\\
p_3 &= (-80b^2a_5a_1 + 10a_3a_1 + 16a_1na_3 + 8a_4 - 224b^2a_4a_2 + 34a_2a_1na_3b^2 + 18{a_3}^2nb^2\\
&\quad - 20a_1na_5b^2 + 24{a_2}^2 - 156b^2{a_3}^2 + 16a_2na_4b^2 + 8a_4n + 8{a_2}^3nb^2 + 4{a_2}^2n - 12{a_2}^3b^2\\
&\quad - 80a_2a_1b^2a_3 - 120b^2a_6)k + 4(40a_2b^4a_5 + 24b^2a_4a_1 - 18b^2a_3a_2 + 20b^2a_5 - 8{a_2}^2a_1b^2\\
&\quad + 6a_3b^2{a_1}^2 + 4a_1{a_2}^3b^4 + 72a_3b^4a_4 - 5a_2a_1 + 28{a_2}^2b^4a_3 + 36{a_3}^2b^4a_1 - 7a_3\\
&\quad + 12{a_1}^2a_2b^4a_3 + 48a_2b^4a_1a_4)(n+1)c + 2b^2(3a_1na_3 + 4{a_2}^3b^2 + 36b^2{a_3}^2 + 2{a_2}^2\\
&\quad + 24a_2a_1b^2a_3 + 48b^2a_4a_2 + 24a_4 + 12a_3a_1)\epsilon + (-48b^2a_4 - 24b^2a_3a_1 - 48a_2b^4a_1a_3\\
&\quad + 16{a_2}^2b^2 - 8{a_2}^3b^4 - 72{a_3}^2b^4 + 10a_2 - 96a_2b^4a_4)v,\\
p_4 &= (15\,{a_2}^2a_1 + 30\,a_4a_1 + 15\,a_5n + 130\,a_2a_3 + 30\,a_1na_4 + 30\,a_2na_5b^2 + 66\,a_1na_2b^2a_4\\
&\quad - 600\,a_3b^2a_4 + 84\,a_3nb^2a_4 + 6\,a_2na_3 - 210\,b^2a_7 + 54\,{a_2}^2nb^2a_3 - 120\,{a_3}^2a_1b^2\\
&\quad - 80\,{a_2}^2b^2a_3 - 30\,a_1na_6b^2 - 150\,b^2a_6a_1 - 9\,a_1n{a_2}^2 + 35\,a_5 + 48\,a_1n{a_3}^2b^2\\
&\quad - 150\,a_2a_1b^2a_4 - 370\,a_2b^2a_5)k + 4\,(144\,a_3b^4a_1a_4 + 80\,a_2b^4a_1a_5 + 28\,a_1{a_2}^2b^4a_3\\
&\quad + 24\,{a_1}^2a_2b^4a_4 - 52\,a_2a_1b^2a_3 - 38\,a_2b^2a_4 + 52\,{a_2}^2b^4a_4 + 120\,a_3b^4a_5 + 60\,{a_3}^2b^4a_2\\
&\quad + 60\,a_2b^4a_6 - 6\,{a_1}^2{a_2}^2b^2 + 18\,{a_1}^2{a_3}^2b^4 + 12\,a_4b^2{a_1}^2 + 40\,b^2a_5a_1 - {a_2}^2 - 14\,a_4\\
&\quad - 36\,{a_3}^2b^2 - 15\,a_3a_1 - 10\,{a_2}^3b^2 + 72\,{a_4}^2b^4 + 30\,b^2a_6 - 3\,a_2{a_1}^2 + 2\,{a_2}^4b^4)(n+1)c\\
&\quad + (100\,b^2a_5 + 72\,{a_3}^2b^4a_1 + 60\,b^2a_4a_1 - 6\,a_2a_1 - 6\,a_2na_3b^2 + 14\,a_2b^2a_3 - 6\,{a_2}^2a_1b^2\\
&\quad + 160\,a_2b^4a_5 + 96\,a_2b^4a_1a_4 + 56\,{a_2}^2b^4a_3 + 12\,a_1na_4b^2 - 6\,a_1n{a_2}^2b^2 - 3\,a_3n\\
&\quad + 288\,a_3b^4a_4 - 6\,a_1na_2 - 9\,a_3)\epsilon + (-56\,{a_2}^2b^4a_3 - 72\,{a_3}^2b^4a_1 + 30\,a_3 - 288\,a_3b^4a_4\\
&\quad - 160\,a_2b^4a_5 + 12\,a_2a_1 - 80\,b^2a_5 + 24\,{a_2}^2a_1b^2 + 104\,a_2b^2a_3 - 96\,a_2b^4a_1a_4 - 48\,b^2a_4a_1)v,\\
p_5 &= (114\,a_2nb^2{a_3}^2 + 96\,{a_2}^2nb^2a_4 - 468\,a_3a_1b^2a_4 - 48\,a_1na_2a_3 - 42\,a_1na_7b^2\\
&\quad - 232\,a_2a_1b^2a_5 + 150\,a_3na_5b^2 + 48\,a_2na_6b^2 + 174\,{a_3}^2 + 24\,a_6n + 68\,a_5a_1 - 12\,{a_2}^3n\\
&\quad - 12\,{a_3}^2n - 576\,{a_4}^2b^2 - 336\,b^2a_8 + 84\,a_6 + 108\,a_2a_1na_5b^2 + 186\,a_1na_3b^2a_4 + 248\,a_2a_4\\
&\quad - 972\,a_3b^2a_5 - 552\,a_2b^2a_6 + 48\,a_1na_5 + 96\,{a_4}^2nb^2 + 86\,a_2a_1a_3 - 252\,b^2a_7a_1\\
&\quad - 168\,{a_3}^2a_2b^2 - 140\,{a_2}^2b^2a_4 + 12\,{a_2}^3)k + 4\,(52\,a_1{a_2}^2b^4a_4 + 240\,a_3b^4a_1a_5\\
&\quad + 216\,a_3b^4a_2a_4 - 150\,a_3b^2a_4 + 42\,b^2a_7 - 64\,a_2b^2a_5 - 70\,{a_2}^2b^2a_3 - 84\,{a_3}^2a_1b^2\\
&\quad + 60\,b^2a_6a_1 + 3\,a_2a_3 + 3\,{a_2}^2a_1 - 29\,a_4a_1 + 84\,a_2b^4a_7 + 20\,a_5b^2{a_1}^2 - 12\,a_1{a_2}^3b^2\\
&\quad + 144\,{a_4}^2b^4a_1 + 240\,a_4b^4a_5 + 180\,a_3b^4a_6 + 84\,{a_2}^2b^4a_5 + 16\,{a_2}^3b^4a_3 + 72\,{a_1}^2a_3b^4a_4\\
&\quad - 104\,a_2a_1b^2a_4 - 8\,a_3{a_1}^2 + 36\,{a_3}^3b^4 - 23\,a_5 - 34\,{a_1}^2a_2b^2a_3 + 40\,{a_1}^2a_2b^4a_5\\
&\quad + 120\,a_2b^4a_1a_6 + 60\,a_1a_2b^4{a_3}^2)(n+1)c + (-18\,{a_3}^2nb^2 - 16\,a_1na_3 - 4\,{a_2}^2n - 32\,a_4\\
&\quad + 104\,{a_2}^2b^4a_4 - 8\,{a_2}^3nb^2 + 24\,a_2b^2a_4 - 22\,a_3a_1 - 8\,a_4n - 16\,a_2na_4b^2 + 288\,a_3b^4a_1a_4\\
&\quad + 120\,b^2a_5a_1 + 240\,a_2b^4a_6 + 120\,{a_3}^2b^4a_2 + 180\,b^2a_6 + 20\,a_1na_5b^2 - 16\,{a_2}^2 - 8\,{a_2}^3b^2\\
&\quad + 480\,a_3b^4a_5 - 36\,a_2a_1b^2a_3 + 288\,{a_4}^2b^4 - 34\,a_1na_2b^2a_3 + 160\,a_2b^4a_1a_5)\epsilon
\end{aligned}
$$

$$+ (-240\, a_2 b^4 a_6 - 480\, a_3 b^4 a_5 - 80\, b^2 a_5 a_1 + 168\, {a_3}^2 b^2 + 58\, a_4 - 120\, b^2 a_6 - 288\, a_3 b^4 a_1 a_4 - 120\, {a_3}^2 b^4 a_2 - 104\, {a_2}^2 b^4 a_4 + 136\, a_2 a_1 b^2 a_3 + 32\, a_3 a_1 - 288\, {a_4}^2 b^4 - 6\, {a_2}^2 - 160\, a_2 b^4 a_1 a_5 + 24\, {a_2}^3 b^2 + 208\, a_2 b^2 a_4) v.$$

Remark A.2. In (3.2), A_1, A_2 and A_3 are given by

$$\begin{aligned}
A_1 = {} & 432\, {a_3}^3 {a_1}^2 + 224\, {a_2}^5 a_1 - 1440\, {a_2}^3 a_5 + 288\, {a_2}^4 a_3 - 48\, {a_1}^3 {a_2}^4 - 4320\, {a_3}^2 a_5 \\
& - 2880\, a_4 a_2 a_5 + 2160\, a_6 a_2 a_3 + 5328\, {a_2}^2 a_3 a_4 + 864\, a_1 {a_3}^2 a_4 - 960\, a_2 a_5 n a_1 a_3 \\
& + 80\, {a_2}^2 {a_1}^2 a_5 - 16\, {a_2}^5 n^2 a_1 + 240\, a_1 {a_2}^2 a_6 - 48\, {a_1}^3 n {a_2}^4 - 24\, {a_1}^4 {a_2}^2 a_3 + 432\, {a_3}^3 {a_1}^2 n \\
& + 72\, {a_3}^2 {a_1}^3 a_2 - 108\, a_2 n {a_3}^3 - 96\, {a_1}^3 {a_2}^2 a_4 + 688\, {a_2}^3 a_4 a_1 - 32\, {a_2}^3 n^2 a_4 a_1 \\
& + 40\, {a_2}^2 a_5 n {a_1}^2 + 240\, {a_2}^2 a_1 n a_6 - 40\, {a_2}^2 {a_1}^2 n^2 a_5 + 108\, {a_3}^2 {a_1}^3 n a_2 + 12\, {a_1}^4 {a_2}^2 n^2 a_3 \\
& - 12\, {a_1}^4 n {a_2}^2 a_3 - 96\, {a_1}^3 n {a_2}^2 a_4 - 52\, {a_1}^2 n^2 a_3 {a_2}^3 + 36\, {a_1}^3 n^2 {a_3}^2 a_2 - 1008\, {a_2}^2 n {a_3}^2 a_1 \\
& - 432\, a_4 {a_1}^2 a_2 a_3 + 864\, a_1 n a_4 {a_3}^2 + 656\, a_1 n a_4 {a_2}^3 - 92\, {a_1}^2 n {a_2}^3 a_3 - 960\, a_1 a_3 a_2 a_5 \\
& + 5184\, {a_4}^2 a_3 - 4536\, a_2 {a_3}^3 + 72\, {a_1}^2 n^2 a_3 a_2 a_4 - 1008\, {a_3}^2 {a_2}^2 a_1 - 40\, {a_1}^2 {a_2}^3 a_3 \\
& + 208\, a_1 n {a_2}^5 - 360\, {a_1}^2 n a_4 a_2 a_3, \\
A_2 = {} & 2052\, {a_3}^3 + 80\, {a_1}^2 a_2 a_5 - 216\, {a_1}^2 a_3 a_4 + 192\, a_1 n {a_2}^4 + 54\, {a_3}^2 {a_1}^3 n + 76\, a_3 {a_1}^2 {a_2}^2 \\
& + 240\, a_6 a_1 a_2 - 216\, a_2 a_4 a_3 - 480\, a_1 {a_3}^2 a_2 - 480\, a_5 a_1 a_3 - 48\, {a_1}^3 n {a_2}^3 - 96\, {a_1}^3 a_2 a_4 \\
& + 36\, {a_3}^2 {a_1}^3 + 672\, a_1 {a_2}^2 a_4 - 24\, {a_2}^4 n^2 a_1 - 24\, {a_1}^4 a_2 a_3 - 720\, {a_2}^2 a_5 + 18\, {a_1}^3 n^2 {a_3}^2 \\
& + 40\, a_2 a_5 n {a_1}^2 - 74\, {a_1}^2 n^2 {a_2}^2 a_3 + 36\, {a_1}^2 n^2 a_4 a_3 + 240\, a_6 a_1 n a_2 - 48\, {a_2}^2 n^2 a_4 a_1 \\
& - 40\, {a_1}^2 n^2 a_5 a_2 - 480\, a_1 n a_3 a_5 + 12\, {a_1}^4 a_2 n^2 a_3 - 12\, {a_1}^4 n a_2 a_3 - 96\, {a_1}^3 n a_2 a_4 \\
& + 624\, a_1 n a_4 {a_2}^2 - 180\, {a_1}^2 n a_4 a_3 + 2\, {a_2}^2 {a_1}^2 n a_3 - 480\, a_2 a_1 n {a_3}^2 - 48\, {a_1}^3 {a_2}^3 \\
& - 1440\, a_5 a_4 + 1080\, a_6 a_3 + 216\, {a_2}^4 a_1 - 648\, {a_2}^3 a_3 - 54\, {a_3}^3 n, \\
A_3 = {} & -26\, {a_1}^2 n^2 a_3 a_2 + 120\, a_1 n {a_3}^2 - 24\, {a_1}^3 n a_4 + 10\, {a_1}^2 n a_5 + 156\, a_1 a_2 a_4 + 60\, a_6 a_1 n \\
& + 52\, {a_1}^2 a_2 a_3 - 12\, {a_1}^3 n {a_2}^2 + 3\, {a_1}^4 n^2 a_3 - 3\, {a_1}^4 n a_3 - 72\, {a_2}^2 a_3 + 120\, a_1 {a_3}^2 \\
& - 10\, {a_1}^2 n^2 a_5 - 12\, {a_1}^3 {a_2}^2 - 12\, {a_2}^3 n^2 a_1 + 36\, a_1 n {a_2}^3 + 48\, a_1 {a_2}^3 - 6\, {a_1}^4 a_3 + 60\, a_6 a_1 \\
& - 24\, {a_1}^3 a_4 + 20\, {a_1}^2 a_5 - 144\, a_4 a_3 + 26\, a_2 {a_1}^2 n a_3 + 132\, a_1 n a_2 a_4 - 24\, a_2 n^2 a_4 a_1.
\end{aligned}$$

Remark A.3. In (3.17), f_0, f_2 and f_4 are given by (define $\phi_1 := \phi', \phi_2 := \phi''$ and $\phi_3 := \phi'''$)

$$\begin{aligned}
f_0 = {} & -\phi\, s^2 (s\phi_1 - \phi)(2\, a_2 s^2 - s^2 {a_1}^2 + 1)\phi_3 - s^3 \phi\, (-s\phi\, a_1 - \phi\, s n a_1 + n + 4\, s^2 {a_1}^2 \\
& + 2\, s^2 n a_2 + s^2 {a_1}^2 n - 4 - 8\, a_2 s^2){\phi_2}^2 + s(s\phi_1 - \phi)(-2\, s^3 \phi_1 a_2 + s^3 \phi_1 {a_1}^2 + s^3 \phi_1 n {a_1}^2 \\
& - 2\, s^3 \phi_1 n a_2 + 6\, s^2 \phi\, a_2 - 3\, s^2 \phi\, {a_1}^2 n - 3\, s^2 \phi\, {a_1}^2 - 2\, s^2 \phi\, n a_2 - s\phi_1 n - s\phi_1 + 2\, s\phi^2 n a_1 \\
& + 2\, s\phi^2 a_1 + \phi - \phi\, n)\phi_2 - (s\phi_1 - \phi)^2 (2\, s^2 \phi_1 a_2 n - s^2 \phi_1 {a_1}^2 + 2\, s^2 \phi_1 a_2 - s^2 \phi_1 n {a_1}^2 \\
& - 4\, s\phi\, a_2 + 2\, s\phi\, {a_1}^2 n + 2\, s\phi\, {a_1}^2 + \phi_1 n + \phi_1 - \phi^2 n a_1 - \phi^2 a_1), \\
f_2 = {} & -\phi\, (-1 + 2\, a_2 s^2)(2\, a_2 s^2 - s^2 {a_1}^2 + 1)(s\phi_1 - \phi)\phi_3 + s\phi\, (-2\, s\phi\, a_1 - 2\, \phi\, s n a_1 \\
& - 8\, s^4 a_2 {a_1}^2 + 16\, s^4 {a_2}^2 + n - 4\, s^4 n {a_2}^2 - 4 + 4\, s^2 {a_1}^2 + 2\, s^4 n a_2 {a_1}^2 + 3\, s^2 {a_1}^2 n){\phi_2}^2 \\
& + (s\phi_1 - \phi)(-4\, s^4 \phi_1 {a_2}^2 - 4\, s^4 \phi_1 n {a_2}^2 + 2\, s^4 \phi_1 n a_2 {a_1}^2 + 2\, s^4 \phi_1 a_2 {a_1}^2 + 12\, s^3 \phi\, {a_2}^2 \\
& - 6\, s^3 \phi\, a_2 {a_1}^2 - 4\, s^3 \phi\, n {a_2}^2 + 2\, s^3 \phi\, n a_2 {a_1}^2 - s^2 \phi_1 {a_1}^2 - s^2 \phi_1 n {a_1}^2 + 2\, s\phi\, {a_1}^2 \\
& - 2\, s\phi\, a_2 - 2\, s\phi\, n a_2 + 4\, s\phi\, {a_1}^2 n + \phi_1 n + \phi_1 - 2\, \phi^2 a_1 - 2\, \phi^2 n a_1)\phi_2 \\
& - 2\, a_2 (s\phi_1 - \phi)^2 (2\, s^2 \phi_1 a_2 + 2\, s^2 \phi_1 a_2 n - s^2 \phi_1 n {a_1}^2 - s^2 \phi_1 {a_1}^2 - 4\, s\phi\, a_2 \\
& + 2\, s\phi\, {a_1}^2 + \phi_1 n + \phi_1), \\
f_4 = {} & 2\, \phi\, a_2 (2\, a_2 s^2 - s^2 {a_1}^2 + 1)(s\phi_1 - \phi)\phi_3 + \phi\, (\phi\, a_1 + \phi\, n a_1 - 16\, s^3 {a_2}^2 \\
& - 8\, s a_2 - 2\, s {a_1}^2 n + 2\, s n a_2 + 8\, s^3 a_2 {a_1}^2 + 4\, s^3 n {a_2}^2 - 2\, s^3 n a_2 {a_1}^2){\phi_2}^2 \\
& - 2\, (s\phi_1 - \phi) a_2 (-2\, s^2 \phi_1 a_2 - 2\, s^2 \phi_1 a_2 n + s^2 \phi_1 n {a_1}^2 + s^2 \phi_1 {a_1}^2
\end{aligned}$$

$$+ 4\,s\phi\,a_2 - 2\,s\phi\,{a_1}^2 - \phi_1 n - \phi_1)\phi_2.$$

Remark A.4. Let p_i be the coefficients of s^i in (4.1). We have

$$\begin{aligned}
p_0 &= 2\,a_2b^2(1+2\,a_2b^2)\lambda - (1+2\,a_2b^2)^2\delta + 6\,a_1b^2a_3 - 2\,a_2 + {a_1}^2n + 2\,b^2{a_1}^2a_2\\
&\quad - 4\,{a_2}^2b^2 + 2\,{a_1}^2na_2b^2 + {a_1}^2,\\
p_1 &= (1+2\,a_2b^2)(2\,a_2b^2a_1 + 12\,b^2a_3 + a_1n + a_1)\lambda - (1+2\,a_2b^2)(2\,a_2b^2a_1\\
&\quad + 12\,b^2a_3 + a_1)\delta + 12\,b^2{a_1}^2a_3 - 6\,a_3 + 4\,a_1n{a_2}^2b^2 - 12\,a_2b^2a_3 + 2\,a_1na_2\\
&\quad + 24\,a_1b^2a_4 + 6\,{a_1}^2na_3b^2 + 12\,{a_2}^2a_1b^2,\\
p_2 &= 6\,b^2(3\,a_3a_1 + 4\,a_2b^2a_3a_1 + 8\,b^2a_2a_4 + {a_2}^2 + 6\,a_4 + 6\,{a_3}^2b^2 + a_1na_3)\lambda\\
&\quad + (12\,{a_2}^2b^2 - 24\,b^2a_4 - 48\,a_2b^4a_4 - 36\,{a_3}^2b^4 - 12\,a_1b^2a_3 + 6\,a_2 - 24\,a_2b^4a_3a_1)\delta\\
&\quad + 72\,a_2b^2a_3a_1 - 6\,{a_1}^2n{a_2}^2b^2 - 6\,a_2{a_1}^2 + 36\,b^2{a_1}^2a_4 - 12\,a_4 + 6\,{a_2}^2 + 60\,a_1b^2a_5\\
&\quad - 6\,{a_1}^2na_2 - 12\,a_3a_1 + 6\,{a_2}^2{a_1}^2b^2 - 18\,{a_3}^2b^2 + 24\,{a_2}^3b^2 + 12\,{a_1}^2na_4b^2 + 12\,a_1na_2b^2a_3,\\
p_3 &= (24\,a_2b^2a_3 + 80\,b^2a_5 - 7\,a_2a_1 - 3\,a_3n - 7\,a_1na_2 + 48\,a_1b^2a_4 - 4\,{a_2}^2a_1b^2\\
&\quad + 144\,a_3b^4a_4 + 36\,{a_3}^2b^4a_1 - 4\,{a_2}^2b^4a_3 + 80\,a_2b^4a_5 - 4\,{a_2}^3a_1b^4 - 9\,a_3 + 48\,a_2b^4a_4a_1\\
&\quad - 6\,a_2nb^2a_3 - 8\,a_1n{a_2}^2b^2 + 12\,a_1na_4b^2)\lambda + (-40\,b^2a_5 - 24\,a_1b^2a_4 + 7\,a_2a_1\\
&\quad + 4\,{a_2}^2b^4a_3 + 17\,a_3 - 36\,{a_3}^2b^4a_1 - 80\,a_2b^4a_5 - 144\,a_3b^4a_4 - 48\,a_2b^4a_4a_1\\
&\quad + 16\,{a_2}^2a_1b^2 + 72\,a_2b^2a_3 + 4\,{a_2}^3a_1b^4)\delta - 20\,a_5 - 16\,a_1n{a_2}^2 - 4\,a_1a_4n + 80\,b^2{a_1}^2a_5\\
&\quad - 16\,{a_1}^2na_3 - 48\,a_3b^2a_4 + 120\,a_1b^2a_6 - 40\,a_4a_1 + 18\,a_3a_2 - 20\,{a_2}^2a_1 - 22\,a_3{a_1}^2\\
&\quad - 20\,a_1n{a_2}^3b^2 + 32\,a_2{a_1}^2b^2a_3 + 160\,a_2b^2a_4a_1 - 12\,{a_2}^2nb^2a_3 + 84\,{a_3}^2b^2a_1\\
&\quad - 34\,{a_1}^2na_2b^2a_3 + 20\,{a_1}^2na_5b^2 + 20\,{a_2}^3a_1b^2 + 40\,a_2b^2a_5 + 172\,{a_2}^2b^2a_3\\
&\quad - 6\,a_2na_3 + 16\,a_1na_2b^2a_4.
\end{aligned}$$

Remark A.5. In (4.9), f_0, f_2 and f_4 are given by (define $\phi_1 := \phi', \phi_2 := \phi''$ and $\phi_3 := \phi'''$)

$$\begin{aligned}
f_0 &= s(\phi - s\phi_1)(\phi\,s^2(\phi - s\phi_1)\phi_3 - \phi\,s^3(-2+n){\phi_2}^2 + s(\phi - s\phi_1)(\phi + s\phi_1)(n+1)\phi_2\\
&\quad - \phi_1(\phi - s\phi_1)^2(n+1)),\\
f_2 &= (-\phi\,s^3(k + 2\,s^2a_2k - 4\,a_2s^2){\phi_1}^2 + \phi^2s^2(1 - 8\,a_2s^2 + k + 4\,s^2a_2k)\phi_1\\
&\quad - s\phi^3(-4\,a_2s^2 + 1 + 2\,s^2a_2k))\phi_3 + (-\phi\,s^3(12\,a_2s^2 + 2 - 3\,k - 6\,s^2a_2k + 2\,s^2na_2k\\
&\quad - 4\,s^2na_2 + nk)\phi_1 + \phi^2s^2(-4\,s^2na_2 - k + 2\,s^2na_2k + 12\,a_2s^2 - 6\,s^2a_2k + n)){\phi_2}^2\\
&\quad + (-s^3(k + 2\,s^2a_2k - 4\,a_2s^2)(n+1){\phi_1}^3 + \phi\,s^2(2\,s^2na_2k + 2\,n + k - 4\,s^2na_2\\
&\quad - 12\,a_2s^2 + 6\,s^2a_2k){\phi_1}^2 + \phi^2s(-2+k)(n - 6\,a_2s^2 - 1 + 2\,s^2na_2)\phi_1\\
&\quad - \phi^3(-2+k)(-2\,a_2s^2 + 2\,s^2na_2 - 1))\phi_2 - s^2(-1 - 4\,a_2s^2 + k + 2\,s^2a_2k)(n+1){\phi_1}^4\\
&\quad + 2\,\phi\,s(k - 1 + nk - n + 3\,s^2na_2k + 4\,s^2a_2k - 6\,s^2na_2 - 8\,a_2s^2){\phi_1}^3 - \phi^2(-12\,s^2na_2\\
&\quad + k + 12\,s^2a_2k - 1 + nk - 24\,a_2s^2 - n + 6\,s^2na_2k){\phi_1}^2 + 2\,s\phi^3a_2(-2+k)(n+4)\phi_1\\
&\quad - 2\,\phi^4a_2(-2+k),\\
f_4 &= (\phi\,s(-1 - 4\,a_2s^2 + k + 2\,s^2a_2k){\phi_1}^2 - \phi^2(4\,s^2a_2k - 1 + k - 8\,a_2s^2)\phi_1\\
&\quad + 2\,s\phi^3a_2(-2+k))\phi_3 + (\phi\,s(2\,s^2na_2k + 4 - 3\,k + 12\,a_2s^2 + nk - 6\,s^2a_2k\\
&\quad - n - 4\,s^2na_2)\phi_1 - \phi^2(-2+k)(2\,s^2na_2 - 6\,a_2s^2 - 1)){\phi_2}^2 + (s(-1 - 4\,a_2s^2\\
&\quad + k + 2\,s^2a_2k)(n+1){\phi_1}^3 - \phi\,(nk - n + k + 4\,s^2na_2k - 1 + 6\,s^2a_2k\\
&\quad - 12\,a_2s^2 - 8\,s^2na_2){\phi_1}^2 + 2\,\phi^2sa_2(-2+k)(3+n)\phi_1 - 2\,\phi^3a_2(-2+k))\phi_2.
\end{aligned}$$

Remark A.6. In (4.15), f_0, f_2 and f_4 are given by (define $\phi_1 := \phi', \phi_2 := \phi''$ and $\phi_3 := \phi'''$)

$$f_0 = -s^2\phi\,(-\phi + s\phi_1)(-\phi_1s^2{a_1}^2 + 2\,\phi_1s^2a_2 + s\phi\,{a_1}^2 - 2\,s\phi\,a_2 + \phi_1)\phi_3$$

$$
\begin{aligned}
&- s^2\phi\,(3\,s^3\phi_1{a_1}^2 + 2\,s^3\phi_1 n a_2 - 6\,s^3\phi_1 a_2 - 3\,s^2\phi\,{a_1}^2 + 6\,s^2 a_2\phi - 2\,s^2\phi\,n a_2\\
&- 3\,s\phi_1 + \phi_1 s n + \phi){\phi_2}^2 - (-\phi + s\phi_1)(-s^4{\phi_1}^2 n{a_1}^2 - s^4{\phi_1}^2{a_1}^2 + 2\,s^4{\phi_1}^2 a_2\\
&+ 2\,s^4{\phi_1}^2 n a_2 + 2\,s^3\phi_1\phi\,{a_1}^2 n + 2\,s^3\phi_1\phi\,{a_1}^2 - 4\,s^3\phi\,\phi_1 a_2 + s^2{\phi_1}^2 n + s^2{\phi_1}^2\\
&+ 2\,s^2\phi^2 a_2 - s^2\phi^2 n{a_1}^2 - 2\,s^2\phi^2 n a_2 - s^2\phi^2{a_1}^2 + \phi\,n\phi_1 s + \phi^2)\phi_2\\
&- (-\phi + s\phi_1)^2(2\,s^2{\phi_1}^2 a_2 - s^2{\phi_1}^2{a_1}^2 - s^2{\phi_1}^2 n{a_1}^2 + 2\,s^2 a_2{\phi_1}^2 n\\
&- 4\,s\phi\,\phi_1 a_2 + 2\,s\phi_1\phi\,{a_1}^2 + 2\,s\phi_1\phi\,{a_1}^2 n - 2\,s\phi_1\phi\,n a_2 + {\phi_1}^2 n + {\phi_1}^2\\
&+ 2\,\phi^2 a_2 - \phi^2{a_1}^2 - \phi^2{a_1}^2 n),
\end{aligned}
$$

$$
\begin{aligned}
f_2 =\ &\phi\,(-1 + 2\,s^2 a_2)(-\phi + s\phi_1)(\phi_1 s^2{a_1}^2 - 2\,\phi_1 s^2 a_2 - s\phi\,{a_1}^2 + 2\,s\phi\,a_2 - \phi_1)\phi_3\\
&- \phi\,(-4\,\phi\,s^4 n{a_2}^2 + 2\,s^2\phi\,n a_2 - 4\,s^2 a_2\phi + 12\,\phi\,s^4{a_2}^2 + 2\,\phi\,s^4 n{a_1}^2 a_2 - \phi\\
&+ \phi\,s^2 n{a_1}^2 + 3\,s^2\phi\,{a_1}^2 - 6\,\phi\,s^4{a_1}^2 a_2 - 3\,s^3\phi_1{a_1}^2 + 3\,s\phi_1 - 2\,s^5\phi_1 n{a_1}^2 a_2\\
&- 12\,s^5\phi_1{a_2}^2 - \phi_1 s n - s^3\phi_1{a_1}^2 n + 6\,s^5\phi_1{a_1}^2 a_2 + 4\,s^5\phi_1 n{a_2}^2){\phi_2}^2\\
&+ (-\phi + s\phi_1)(2\,s^4{\phi_1}^2 n{a_1}^2 a_2 - 4\,s^4{\phi_1}^2{a_2}^2 + 2\,s^4{\phi_1}^2{a_1}^2 a_2 - 4\,s^4{\phi_1}^2 n{a_2}^2\\
&- 4\,s^3\phi_1\phi\,{a_1}^2 a_2 + 8\,s^3\phi\,\phi_1{a_2}^2 - s^2{\phi_1}^2{a_1}^2 - s^2{\phi_1}^2 n{a_1}^2 - 2\,s^2\phi^2 n{a_1}^2 a_2\\
&+ 4\,s^2\phi^2 n{a_2}^2 - 4\,s^2\phi^2{a_2}^2 + 2\,s^2\phi^2{a_1}^2 a_2 + 2\,s\phi_1\phi\,{a_1}^2 + 3\,s\phi_1\phi\,{a_1}^2 n\\
&- 4\,s\phi\,\phi_1 a_2 - 4\,s\phi_1\phi\,n a_2 + {\phi_1}^2 n + {\phi_1}^2 - 2\,\phi^2{a_1}^2 n - \phi^2{a_1}^2)\phi_2\\
&- 2\,a_2(-\phi + s\phi_1)^2(2\,s^2 a_2{\phi_1}^2 n + 2\,s^2{\phi_1}^2 a_2 - s^2{\phi_1}^2{a_1}^2 - s^2{\phi_1}^2 n{a_1}^2 - 2\,s\phi_1\phi\,n a_2\\
&- 4\,s\phi\,\phi_1 a_2 + s\phi_1\phi\,{a_1}^2 n + 2\,s\phi_1\phi\,{a_1}^2 + 2\,\phi^2 a_2 + {\phi_1}^2 + {\phi_1}^2 n - \phi^2{a_1}^2),
\end{aligned}
$$

$$
\begin{aligned}
f_4 =\ &2\,\phi\,a_2(-\phi + s\phi_1)(-\phi_1 s^2{a_1}^2 + 2\,\phi_1 s^2 a_2 + s\phi\,{a_1}^2 - 2\,s\phi\,a_2 + \phi_1)\phi_3\\
&+ \phi\,(2\,\phi\,a_2 - 4\,\phi\,s^2 n{a_2}^2 + 2\,\phi\,s^2 n{a_1}^2 a_2 + \phi\,{a_1}^2 n + 12\,s^2\phi\,{a_2}^2 - 6\,s^2\phi\,{a_1}^2 a_2\\
&+ 6\,s^3\phi_1{a_1}^2 a_2 + 4\,s^3\phi_1 n{a_2}^2 - 2\,s^3\phi_1 n{a_1}^2 a_2 - s\phi_1{a_1}^2 n + 2\,\phi_1 s n a_2 - 12\,s^3\phi_1{a_2}^2\\
&- 6\,s\phi_1 a_2){\phi_2}^2 - 2\,(-\phi + s\phi_1)a_2(-2\,s^2 a_2{\phi_1}^2 n - 2\,s^2{\phi_1}^2 a_2 + s^2{\phi_1}^2 n{a_1}^2\\
&+ s^2{\phi_1}^2{a_1}^2 + 4\,s\phi\,\phi_1 a_2 + 2\,s\phi_1\phi\,n a_2 - s\phi_1\phi\,{a_1}^2 n - 2\,s\phi_1\phi\,{a_1}^2\\
&- 2\,\phi^2 a_2 - {\phi_1}^2 - {\phi_1}^2 n + \phi^2{a_1}^2)\phi_2.
\end{aligned}
$$