SCIENCE CHINA
Mathematics

• ARTICLES •

May 2018 Vol. 61 No. 5: 855–868
https://doi.org/10.1007/s11425-016-9077-8

Classification of certain qualitative properties of solutions for the quasilinear parabolic equations

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Received November 21, 2016; accepted March 28, 2017; published online November 10, 2017

Abstract In this paper, we mainly consider the initial boundary problem for a quasilinear parabolic equation

$$u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = -|u|^{\beta-1}u + \alpha|u|^{q-2}u,$$

where $p > 1, \beta > 0$, $q \geqslant 1$ and $\alpha > 0$. By using Gagliardo-Nirenberg type inequality, the energy method and comparison principle, the phenomena of blowup and extinction are classified completely in the different ranges of reaction exponents.

Keywords quasilinear parabolic equation, weak solution, blowup, extinction

MSC(2010) 35A01, 35B44, 35D30, 35K92

Citation: Li Y, Zhang Z C, Zhu L P. Classification of certain qualitative properties of solutions for the quasilinear parabolic equations. Sci China Math, 2018, 61: 855–868, https://doi.org/10.1007/s11425-016-9077-8

1 Introduction

In this paper, the following initial boundary problem is considered:

$$\begin{cases} u_t - \Delta_p u = -|u|^{\beta-1}u + \alpha|u|^{q-2}u, & x \in \Omega, \quad t > 0, \\ u = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geqslant 1)$ is a smoothly bounded domain and $p > 1, \beta > 0, q \geqslant 1, \alpha > 0$. The operator Δ_p is defined as follows:

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u).$$

We also suppose that $u_0(x) \geqslant 0, u_0(x) \not\equiv 0, u_0(x) \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Problem (1.1) arises in the theory of nonstationary filtration of non-Newtonian (or dilatant) fluids and combustion of solid fuels. The term $-|u|^{\beta-1}u$, which is negative as we can prove later that $u \geqslant 0$, is called a singular absorption term for $\beta < 0$ or a strong absorption one for $0 < \beta < 1$ or a weak absorption one for $\beta > 1$. $\alpha|u|^{q-2}u$ is an inner source term. It has been known for many years that the term $-|u|^{\beta-1}u$ with

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$\beta > 0$ may lead to finite time extinction, i.e., *there exists a* $T \in (0, +\infty)$ *such that* $u(x,t)$ *is nontrivial for* $t \in [0, T)$ *and* $u(x,t) \equiv 0$ *for* $t \in [T, +\infty)$ *a.e. in* $x \in \Omega$. On the other hand, $\alpha|u|^{q-2}u$ may lead to finite time blowup. However, if the two terms appear simultaneously in the first equation of (1.1), then the solutions will exhibit complicated properties which will be studied later. To be specific, both blowup and extinction can occur under some suitable conditions.

As the operator Δ_p is degenerate for $p > 2$ and is singular for $1 < p < 2$, it is impossible to consider the classical solutions of (1.1) generally. However, the concept of weak solutions is enough for our study. For the local existence of weak solutions of (1.1), various methods can be applied such as approximation by regular solution (see [4, 35]), the fixed point method (see [32]), the method of extension of semigroup (see [12]) and the developed Faedo-Galerkin method (see [2, 3, 13]).

As soon as the local existence is established, one may ask whether the weak solution is global or not. Moreover, we are eager to know when the solution is global in time and when it blows up in finite time. For the global solution, we also want to know whether it will become zero in finite time or not.

The phenomenon of finite time blowup was first considered by Fujita [10] in 1966. Since then, many people devoted themselves to this problem. The main equations they studied are the heat equations of the form $u_t - \Delta u = |u|^{p-1}u$ in bounded or unbounded smooth domains in $\mathbb{R}^N$. The theory of blowup for the heat equation is already developed; we refer the reader to [15, 18, 24, 25, 27] and the references therein. While for the p-Laplacian equations of the form $u_t - \Delta_p u = f(x, t, u, \nabla u)$, there are still many problems worth studying, such as the blowup rate, the blowup time estimate, the asymptotic behavior of blowup solutions, the blowup criteria and so on. Some related results can be found in [11, 13, 19, 23, 33–36] and the references therein. To be specific, in [13, 19, 23, 32, 35], criteria for the finite time blowup to occur were established in bounded domains for different kinds of source terms and values of p. Generally speaking, finite time blowup may occur if $f(x, t, s, \vec{r})$ grows faster than s^{p-1} $(p > 2)$ or s $(1 < p < 2)$ $(q = p - 1$ or $q = 1$ is called the critical blowup exponent) when $s \to \infty$ and the initial data is large enough. In [11], Galaktionov and Posashkov studied the blowup set for the equation $u_t - \operatorname{div}(|\nabla u|^{\sigma}\nabla u) = u^{\beta}$ with $\sigma > 0, \beta > 1$ and $x \in \mathbb{R}^N$. They proved that the radial solution will blow up at $|x| = 0$. For the blowup time estimate, Zhou and Yang [38] considered the equation $u_t - \operatorname{div}(|\nabla u|^{m-2}\nabla u) = |u|^{p(x)-1}u$ with Dirichlet boundary condition on bounded domains. They obtained an upper bound of the blowup time for some suitable conditions on $m, p(x)$ and initial data. Zhao and Liang [36] considered a Cauchy problem $u_t - \Delta_p u = u^q$ in the radial situation and obtained the blowup rate upper bound is of the order $(T - t)^{-1/(q-1)}$ for $q > p - 1$. In our latest papers [33, 34], we considered the equation $u_t - \Delta_p u = \lambda u^m + \mu|\nabla u|^q$ with $p > 2$ and $\lambda\mu < 0$, and proved that u will blow up in finite time in the L^{∞}-norm sense if $\lambda > 0, \mu < 0$ and $m > \max\{p - 1, q\}, q \leqslant p/2$. For the blowup of more general p-Laplacian equations, there are also some important results. In [29, 31], the Fujita exponent for equations with weighted source of the form

$$\frac{\partial u}{\partial t} = \operatorname{div}(|\nabla u|^{p-2}\nabla u) + k\frac{1}{|x|^2}|\nabla u|^{m-1}\nabla u \cdot x + |x|^{\lambda}u^p$$

was studied. In [22, 37], the global existence, blowup and the blowup point of solutions for the doubly degenerate equations, i.e., equations with $\operatorname{div}(|\nabla u^m|^{p-2}\nabla u^m)$ were carefully studied.

Finite time extinction is another important property of solutions of evolution equations. Since Kalashnikov first brought in the concept of extinction in 1974, it has attracted many mathematicians' interests and most of them focused on the fast diffusive equations (see [5–9, 14, 17, 30] for examples). Moreover, in [28], the homogeneous p-Laplacian equation $u_t = \Delta_p u$ with $p > 1, x \in \mathbb{R}^N$ was studied. It was shown that extinction can happen if and only if $1 < p \leqslant p_c = 2N/(N + 1)$. In [32], Yin and Jin considered the equation $u_t - \Delta_p u = \lambda u^q$ with $x \in \Omega$ and $1 < p < 2$. They proved that $q = p - 1$ is the critical extinction exponent. In [14], Gu considered the p-Laplacian equation $u_t - \Delta_p u = -|u|^{\beta-1}u$ with $p > 1$. In that paper, the conditions for extinction to occur were obtained for any $p > 1$ while the non-extinction condition was obtained only for $p \geqslant 2$. For the equation with absorption and source terms, i.e., $u_t - \Delta_p u = \lambda u^q - \beta u^k$ with $1 < p < 2$ and $0 < q, k < 1$, it was showed in [8] that the solution will exhibit extinction phenomenon under the assumptions that $u_0(x)$ or λ is small enough and that β is large enough. In [16, 23], the extinction phenomenon for p-Laplacian equations with Neumann boundary data and nonlocal absorption term was studied.

In this paper, we will deal with (1.1) for any $p > 1$. In Section 2, we will give some basic concepts and a weak comparison principle. Section 3 is devoted to the existence of the weak solution for (1.1) in a general case. The extinction phenomenon will be discussed in Section 4. We will give some blowup results under different conditions for $u_0(x)$ and p, β, q, α in Section 5. In Section 6, we will give some discussions.

2 Preliminaries

Before giving the definition of weak solutions, we bring in the following function space:

$$\mathbb{V} := \{v \in L^p(0,T; W_0^{1,p}(\Omega)) \,|\, \partial_t v \in L^{p'}(0,T; W^{-1,p'}(\Omega))\}. \tag{2.1}$$

Now, let us introduce the definition of the weak solution of (1.1).

Definition 2.1. Let $Q_T = \Omega \times (0,T), S_T = \partial\Omega \times (0,T), \partial Q_T = S_T \cup \{\overline{\Omega} \times \{0\}\}$. A function $u \in \mathbb{V} \cap C(0,T; L^2(\Omega))$ is called a weak solution of (1.1) if it satisfies the following:

(1) for every nonnegative test-function $\varphi \in \mathbb{V} \cap C(0,T; L^2(\Omega))$,

$$\iint_{Q_T} (\partial_t u\varphi + |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi) dxdt = -\iint_{Q_T} (|u|^{\beta-1}u - \alpha|u|^{q-2}u)\varphi dxdt. \tag{2.2}$$

(2) $u(x,0) = u_0(x)$ for a.e. $x \in \Omega$.

Moreover, if we replace "$=$" in (2.2) by "$\leqslant$" ("$\geqslant$") and assume that $u(x,0) \leqslant (\geqslant) u_0(x)$, $u(x,t)\,|_{x\in\partial\Omega} \leqslant (\geqslant) 0$, then the corresponding solution is called a sub-(sup-) solution.

For the weak solution of (1.1), we have the following weak comparison principle. Some similar results can be found in [4, 19, 32, 33].

Proposition 2.2. *Suppose that u and v are weak sub- and sup- solutions of* (1.1)*, respectively. If u and v are locally bounded, then $u \leqslant v$ a.e. in Q_T.*

Proof. Let $\varphi = \max\{u - v, 0\}$. Then $\varphi(x,0) = 0, \varphi(x,t)\,|_{x\in\partial\Omega} = 0$. By Definition 2.1, $\varphi(x,t)$ satisfies

$$\begin{aligned}
&\iint_{Q_T} \partial_t\varphi\varphi dxdt + \underbrace{\iint_{Q_T} (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)(\nabla u - \nabla v) dxdt}_{\mathcal{M}} \\
&\quad \leqslant -\iint_{Q_T} (|u|^{\beta-1}u - |v|^{\beta-1}v)\varphi dxdt + \alpha \iint_{Q_T} (|u|^{q-2}u - |v|^{q-2}v)\varphi dxdt \\
&\quad \leqslant -\underbrace{\iint_{Q_T} (|u|^{\beta-1}u - |v|^{\beta-1}v)\varphi dxdt}_{\mathcal{A}} + L\iint_{Q_T} \varphi^2 dxdt,
\end{aligned} \tag{2.3}$$

where L is a constant depending on the sup-norms of u and v.

Let us now estimate terms $\mathcal{M}$ and $\mathcal{A}$ appearing in (2.3). By the monotone inequality (see [20]), we have $\mathcal{M} \geqslant 0$ for any $p > 1$. For term $\mathcal{A}$, by the fact that

$$\begin{cases}
|u|^{\beta-1}u - |v|^{\beta-1}v = u^\beta - v^\beta > 0, & \text{if } u > v > 0, \\
|u|^{\beta-1}u - |v|^{\beta-1}v = u^\beta + |v|^\beta > 0, & \text{if } u > 0 > v, \\
|u|^{\beta-1}u - |v|^{\beta-1}v = -|u|^\beta + |v|^\beta > 0, & \text{if } 0 > u > v,
\end{cases} \tag{2.4}$$

we have $\mathcal{A} \geqslant 0$.

Following the above discussion, we have

$$\frac{1}{2}\int_\Omega \varphi^2 dx \leqslant L\iint_{Q_T} \varphi^2 dxdt. \tag{2.5}$$

By Gronwall's inequality, we have $\int_\Omega \varphi^2 \, dx = 0$. This implies that $\varphi = 0$ a.e. $x \in \Omega$, i.e. $u \leqslant v$ a.e. $(x,t) \in Q_T$. □

3 Existence of weak solutions

In this section, we will establish the local existence and global existence of weak solutions of (1.1). Analogous to the proofs in [2, 3, 13] and the compactness results in [26], we have the following local existence of the bounded weak solution for (1.1).

Theorem 3.1. *Suppose that $u_0 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), u_0 \geqslant 0, u_0 \not\equiv 0$ a.e. in Ω and that $q \geqslant 1$. Then there exists a $T^* = T^*(u_0) > 0$ such that for $0 < T < T^*$, (1.1) admits a solution*

$$u \in \mathbb{U} := \{u \in L^\infty(0,T; W_0^{1,p}(\Omega)) \cap L^\infty(Q_T) \,|\, \partial_t u \in L^2(Q_T)\}. \tag{3.1}$$

Moreover, $0 \leqslant u \leqslant M$ a.e. in Q_T for some M depending on $u_0(x)$.

Next, we will give some results focusing on the global existence of the weak solution for (1.1).

Denote by $\Lambda_1 > 0$ the first eigenvalue of the p-Laplacian operator with homogeneous Dirichlet boundary condition, i.e.,

$$\Lambda_1 := \inf\left\{ \int_\Omega |\nabla u|^p dx \,\middle|\, u \in W_0^{1,p}(\Omega), \int_\Omega |u|^p dx = 1 \right\}. \tag{3.2}$$

Theorem 3.2 (Global existence). *Let $u_0(x) \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), u_0(x) \geqslant 0$ and one of the following conditions is satisfied*:

(i) $q = p, \alpha < \Lambda_1$.

(ii) $q = p = \beta + 1, \alpha < \Lambda_1 + 1$.

(iii) $2 < p \leqslant q < \beta + 1$.

(iv) $q < p$.

Then the solution of (1.1) *is globally in time bounded, i.e., there exists a constant M depending only on $p, q, \beta, \Lambda_1, \alpha, u_0$ and Ω such that for every $T > 0, 0 \leqslant u \leqslant M$.*

Proof. (i) Let $\widetilde{\Omega} \subset \mathbb{R}^N$ be a smooth domain which satisfies $\Omega \subset\subset \widetilde{\Omega}$. Denote by ϕ and $\Lambda_1(\widetilde{\Omega})$ the first eigenfunction and the first eigenvalue related to the following Dirichlet problem:

$$-\Delta_p \phi = \Lambda_1(\widetilde{\Omega})|\phi|^{p-2}\phi \quad \text{in} \quad \widetilde{\Omega}, \quad \phi = 0 \quad \text{on} \quad \partial\widetilde{\Omega}, \quad \int_{\widetilde{\Omega}} |\phi|^p dx = 1. \tag{3.3}$$

Then by [19, Lemma 1.1], we know that $\phi > 0$ in $\widetilde{\Omega}$ and that $\Lambda_1(\widetilde{\Omega}) < \Lambda_1(\Omega)$. Moreover, by [21, Theorem 3.2], $\Lambda_1(\widetilde{\Omega})$ continuously depends on $\widetilde{\Omega}$ and $\Lambda_1(\widetilde{\Omega}) \to \Lambda_1(\Omega)$ as $\widetilde{\Omega} \to \Omega$ in the Hausdorff complementary topology. Thus, we can choose a suitable $\widetilde{\Omega}$ and $\theta > 0$ such that $\alpha \leqslant \Lambda_1(\widetilde{\Omega}) \leqslant \Lambda_1(\Omega)$. Let $\Phi = K\phi \geqslant K\mu \geqslant \|u_0\|_{L^\infty(\Omega)}$ with $\mu = \inf_\Omega \phi > 0$. Then a simple calculation shows that for every nonnegative test-function $\varphi \in \mathbb{V} \cap (0,T; L^2(\Omega))$,

$$\begin{aligned} \iint_{Q_T} \partial_t \Phi \varphi + |\nabla \Phi|^{p-2} \nabla \Phi \cdot \nabla \varphi dx dt &\geqslant \Lambda_1(\widetilde{\Omega}) \iint_{Q_T} \Phi^{p-1} \varphi dx dt \\ &\geqslant \alpha \iint_{Q_T} \Phi^{p-1} \varphi dx dt. \end{aligned} \tag{3.4}$$

This implies that Φ is a sup-solution of (1.1). Then by Proposition 2.2, we have $0 \leqslant u \leqslant \Phi$ a.e. in Q_T. We can also see from the construction of Φ that it is independent of t which enables us to continue the procedure above on any time interval $[T, T']$. Then, we can assert that the solution of (1.1) is globally in time bounded.

The proof of (ii) is the same as the one of (i).

(iii) Without loss of generality, we assume $\alpha = 1$ and the method below is still valid for the general case with a little modification. Denote by $\rho(\Omega)$ the diameter of Ω. Then we can easily know that $\rho(\Omega) < \infty$

as Ω is bounded. Let $\varepsilon \in (0,1)$ satisfy: there exists a ball of radius ε belonging to $B(\cdot,\rho(\Omega)+1)\cap\Omega^c$. For any $a\in\Omega$, let x_a satisfy

$$B(x_a,\varepsilon)\subset B(x_a,\rho(\Omega)+1)\cap\Omega^c,\quad |x_a-a|<\rho(\Omega)+1. \tag{3.5}$$

Let

$$V(x,t)=L\mathrm{e}^{\sigma r},\quad r=|x-x_a|,\quad x\in\Omega. \tag{3.6}$$

Define $\mathcal{L}_p v := v_t-\Delta_p v - v^{q-1}+v^\beta$. Then $V(x,t)$ satisfies

$$\mathcal{L}_p V = -(p-1)(L\sigma)^{p-1}\mathrm{e}^{(p-1)\sigma r}-\frac{N-1}{r}(L\sigma)^{p-1}\mathrm{e}^{(p-1)\sigma r}-L^{q-1}\mathrm{e}^{(q-1)\sigma r}+L^\beta \mathrm{e}^{\beta\sigma r}. \tag{3.7}$$

In order to derive that $\mathcal{L}_pV\geqslant 0$, we need to choose suitable σ and L such that

$$(p-1)\sigma^p+\frac{N-1}{r}\sigma^{p-1}\leqslant L^{\beta+1-p}\mathrm{e}^{(\beta+1-p)\sigma r}-L^{q-p}\mathrm{e}^{(q-p)\sigma r}. \tag{3.8}$$

By (3.5) and (3.6), we know that $\varepsilon\leqslant r<\rho(\Omega)+1$. Then if we want (3.8) to be satisfied, it is sufficient that

$$(p-1)\sigma^p+\frac{N-1}{\varepsilon}\sigma^{p-1}+L^{q-p}\mathrm{e}^{(q-p)\sigma(\rho(\Omega)+1)}\leqslant L^{\beta+1-p}. \tag{3.9}$$

If $q>p$, let σ and L satisfy

$$\sigma=\frac{1}{(q-p)(\rho(\Omega)+1)},\quad L=\max\left\{(2\mathrm{e})^{\frac{1}{\beta+1-q}},\left(2\left((p-1)\sigma^p+\frac{N-1}{\varepsilon}\sigma^{p-1}\right)\right)^{\frac{1}{\beta+1-p}}\right\}. \tag{3.10}$$

If $q=p$, let σ and L satisfy

$$\sigma=1,\quad L=\max\left\{2^{\frac{1}{\beta+1-q}},\left(2\left(p-1+\frac{N-1}{\varepsilon}\right)\right)^{\frac{1}{\beta+1-p}}\right\}. \tag{3.11}$$

Then it holds that $\mathcal{L}_pV\geqslant 0$. If we assume furthermore that $L\geqslant\|u_0\|_{L^\infty(\Omega)}$, then $V(x,0)\geqslant u_0(x)$. Thus, we have proved that $V(x,t)$ is a sup-solution of (1.1). By Proposition 2.2, we have

$$u(x,t)\leqslant L\mathrm{e}^{\sigma(\rho(\Omega)+1)}<\infty. \tag{3.12}$$

Notice that the right-hand side of (3.12) is in fact independent of t, which enables us to continue the procedure above in any time interval $[T,T']$. Hence, we can conclude that $u(x,t)$ is globally in time bounded.

In the case $q<p$, by Young's inequality, there exists a small $\gamma>0$ such that $\alpha|s|^{q-2}s\leqslant\alpha|s|^{q-1}+C_\alpha\leqslant\gamma|s|^{p-1}+C(\gamma)$. Then the conclusion follows from the same procedure as above. □

4 Finite time extinction and decay

Before proving our main results, we first introduce the following Gagliardo-Nirenberg type inequality which can be found in [6,13] and the references therein.

Lemma 4.1. *Let $1<p<+\infty$ and $r\in[\beta+1,+\infty)$ if $p\geqslant N$, and $r\in[\beta+1,\frac{Np}{N-p}]$ if $p<N$. Then there exists a constant $C>0$, depending only on p,r,N,β and $|\Omega|$, such that for every $u\in W_0^{1,p}(\Omega)$,*

$$\|u\|_{L^r(\Omega)}\leqslant C\|\nabla u\|^{\theta}_{L^p(\Omega)}\|u\|^{1-\theta}_{L^{\beta+1}(\Omega)}\quad \textit{with}\quad \theta=\frac{\frac{1}{\beta+1}-\frac{1}{r}}{\frac{1}{N}-\frac{1}{p}+\frac{1}{\beta+1}}\in[0,1]. \tag{4.1}$$

Remark 4.2. We can see from the expression of θ with $r>\beta+1$ that

$$\theta<\frac{\frac{1}{\beta+1}-\frac{1}{r}}{-\frac{1}{p}+\frac{1}{\beta+1}} \tag{4.2}$$

and that

$$r\left(\frac{\theta}{p}+\frac{1-\theta}{\beta+1}\right)>1, \tag{4.3}$$

which will play an important role in establishing a desired ordinary differential inequality later.

4.1 Finite time extinction

The following theorem deals with the finite time extinction.

Theorem 4.3. *Let $\beta+1 \leqslant q \leqslant p$ and $\beta < \min\{1, p-1\}$. Assume additionally that $\alpha < \min\{1, \Lambda_1\}$. Then there exists a finite time $T^* > 0$, such that $u = 0$ a.e. in Ω for $t \geqslant T^*$.*

Proof. By Theorem 3.2, u exists globally in time. Let $y(t) = \|u\|^2_{L^2(\Omega)}$. Then it satisfies

$$\frac{1}{2}y'(t) + \int_\Omega |\nabla u|^p dx = \alpha \int_\Omega u^q dx - \int_\Omega u^{\beta+1} dx. \tag{4.4}$$

By the assumption that $\beta+1 \leqslant q \leqslant p$, we have

$$\begin{aligned}\int_\Omega u^q dx &= \int_{\Omega\cap\{u\geqslant 1\}} u^q dx + \int_{\Omega\cap\{u\leqslant 1\}} u^q dx \leqslant \int_{\Omega\cap\{u\geqslant 1\}} u^p dx + \int_{\Omega\cap\{u\leqslant 1\}} u^{\beta+1} dx \\ &\leqslant \int_\Omega (u^p + u^{\beta+1}) dx \leqslant \frac{1}{\Lambda_1}\int_\Omega |\nabla u|^p dx + \int_\Omega u^{\beta+1} dx,\end{aligned} \tag{4.5}$$

where we used Poincaré's inequality $\Lambda_1\|u\|^p_{L^p(\Omega)} \leqslant \|\nabla u\|^p_{L^p(\Omega)}$. Combining (4.4) with (4.5), we find that for

$$D = \begin{cases} 1 - \alpha\max\left\{\dfrac{1}{\Lambda_1}, 1\right\}, & \text{if } \ \beta+1 < q < p, \\ 1-\alpha, & \text{if } \ \beta+1 = q < p, \\ 1 - \dfrac{\alpha}{\Lambda_1}, & \text{if } \ \beta+1 < q = p, \end{cases} \tag{4.6}$$

it holds that

$$\frac{1}{2}y'(t) + D\int_\Omega (|\nabla u|^p + u^{\beta+1}) dx \leqslant 0. \tag{4.7}$$

Our next goal is to obtain the following differential inequality from (4.7):

$$y'(t) + Ky^\gamma(t) \leqslant 0, \quad \text{with} \quad K > 0, \quad 0 < \gamma < 1. \tag{4.8}$$

Integrating (4.8) with respect to t, we have

$$y(t) \leqslant (y^{1-\gamma}(0) - K(1-\gamma)t)^{\frac{1}{1-\gamma}}, \tag{4.9}$$

which implies

$$y(t) \to 0 \quad \text{as} \quad t \to T^* := \frac{y^{1-\gamma}(0)}{K(1-\gamma)}. \tag{4.10}$$

Thus, the finite time extinction for the solution of (1.1) is proved.

To obtain (4.8), we divide our proof into two parts: $p > \frac{2N}{N+2}$ and $1 < p < \frac{2N}{N+2}$.

(i) If $p \geqslant \frac{2N}{N+2}$, then $\frac{Np}{N-p} \geqslant 2$ for $p < N$ which implies that we can choose $r = 2$ in (4.1). If $p \geqslant N$, then $r \in [\beta+1, +\infty)$ which enables us to set $r = 2$ in (4.1). In both cases, we can obtain

$$\begin{aligned}\|u\|_{L^2(\Omega)} &\leqslant C\|\nabla u\|^\theta_{L^p(\Omega)}\|u\|^{1-\theta}_{L^{\beta+1}(\Omega)} = C\biggl(\int_\Omega |\nabla u|^p dx\biggr)^{\frac{\theta}{p}}\biggl(\int_\Omega u^{\beta+1} dx\biggr)^{\frac{1-\theta}{\beta+1}} \\ &\leqslant C\biggl(\int_\Omega (|\nabla u|^p + u^{\beta+1}) dx\biggr)^{\frac{\theta}{p}+\frac{1-\theta}{\beta+1}}\end{aligned} \tag{4.11}$$

from (4.1) with $r = 2$. Then

$$C^{-2}y(t) \leqslant \biggl(\int_\Omega (|\nabla u|^p + u^{\beta+1}) dx\biggr)^{2(\frac{\theta}{p}+\frac{1-\theta}{\beta+1})}. \tag{4.12}$$

Combining (4.12) with (4.7), we can obtain (4.8) with

$$\frac{1}{\gamma}=2\bigg(\frac{\theta}{p}+\frac{1-\theta}{\beta+1}\bigg)>1,\quad K=2DC^{-2\gamma}. \tag{4.13}$$

(ii) If $1<p<\frac{2N}{N+2}$, let $2>r\in(\beta+1,\frac{Np}{N-p}]$ and $M=\|u\|_{L^\infty(Q_T)}$. Then we have

$$y(t)=\|u\|_{L^2(\Omega)}=\int_\Omega u^{2-r}u^r dx\leqslant M^{2-r}\|u\|^r_{L^r(\Omega)}. \tag{4.14}$$

By (4.1) with $r\in(\beta+1,2)$, it holds that

$$\begin{aligned} y(t)&\leqslant M^{2-r}(C\|\nabla u\|^\theta_{L^p(\Omega)}\|u\|^{1-\theta}_{L^{\beta+1}(\Omega)})^r\\ &\leqslant M^{2-r}(D^{\frac{\theta}{p}+\frac{1-\theta}{\beta+1}})^{-r}C^r\bigg(D\int_\Omega(|\nabla u|^p+u^{\beta+1})dx\bigg)^{r(\frac{\theta}{p}+\frac{1-\theta}{\beta+1})}. \end{aligned} \tag{4.15}$$

Combining (4.15) with (4.7), we can derive (4.8) with

$$\frac{1}{\gamma}=r\bigg(\frac{\theta}{p}+\frac{1-\theta}{\beta+1}\bigg)>1,\quad K=2DM^{\gamma(r-2)}C^{-r\gamma}. \tag{4.16}$$

This completes the proof. □

Remark 4.4. In the case $1<p<2$, Fang et al. [8] obtained some similar extinction results. The results there needed stronger conditions for the coefficients of absorption and source terms. Moreover, the initial data was also chosen small enough. However, our results hold for any nontrivial initial data and some α which does not need to be sufficiently small. Besides, our proof is more efficient.

Different from Theorem 4.3, the following theorem shows that finite time extinction can also occur for $q>p$ and $1<p<2$ with small initial data.

Theorem 4.5. *Assume that $q>p, 1<p<2$. Then the solution of* (1.1) *will vanish at finite time provided the initial data is small enough.*

Proof. The proof here is same as the one in [32, Theorem 4.1], so we omit it. □

4.2 Decay

Let us now consider the decay of the solution.

Theorem 4.6. *Assume that $\beta\geqslant1$ and $p\geqslant2$. Then the solution of* (1.1) *will not extinguish in finite time. Assume additionally $\beta\leqslant q-1$. Then there exists a constant $\epsilon>0$, such that if $u_0\geqslant0$ and $\|u_0\|_{L^\infty(\Omega)}<\epsilon$, then the solution will decay to zero as $t\to+\infty$. Moreover, we have the following estimates*:

$$\begin{cases} 0\leqslant u\leqslant C_1(t+C_2)^{-\gamma},\quad \gamma=\dfrac{1}{\beta-1},\quad \textit{for}\quad 1<\beta\leqslant q-1,\\ 0\leqslant u\leqslant C_3\mathrm{e}^{-C_4t},\quad \textit{for}\quad \beta=1,\quad q\geqslant2. \end{cases} \tag{4.17}$$

The constants $C_i, i=1,2,3,4$ appearing above depend on q,β and α.

Proof. By [14, Theorem 3.3], we know that the solution of

$$\begin{cases} v_t-\Delta_p v=-|v|^{\beta-1}v, & x\in\Omega,\quad t>0,\\ v=0, & x\in\partial\Omega,\quad t>0,\\ v(x,0)=u_0(x), & x\in\Omega \end{cases} \tag{4.18}$$

will not extinguish in finite time if $p\geqslant2,\beta\geqslant1,u_0(x)\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega),u_0(x)\not\equiv0$. As was shown in Theorem 3.1, $u\geqslant0$. Thus, v is a sub-solution of (1.1). By comparison principle, u will not extinguish in finite time.

Let us now consider the decay of the solution of (1.1). For convenience, we define $\mathcal{L}_p$ as

$$\mathcal{L}_p\varphi = \varphi_t - \Delta_p\varphi - \alpha|\varphi|^{q-2}\varphi + |\varphi|^{\beta-1}\varphi.$$

If $1 < \beta \leqslant q-1$, let

$$w(x,t) = C_1(t+C_2)^{-\gamma}, \quad \gamma = \frac{1}{\beta-1}, \tag{4.19}$$

where $C_1, C_2 > 0$ are constants to be decided later. By a direct computation, we have

$$\mathcal{L}_p w = (t+C_2)^{-\gamma-1}(-\gamma C_1 + C_1^{\beta} - \alpha C_1^{q-1}(t+C_2)^{-(q-1-\beta)\gamma}). \tag{4.20}$$

If $\beta < q-1$, let $C_1, C_2 > 0$ satisfy $(2\gamma)^{\gamma} \leqslant C_1 \leqslant (2\alpha)^{\frac{1}{\beta-q+1}} C_2^{\gamma}$. Then we have $\mathcal{L}_p w \geqslant 0$. Assume additionally that $\|u_0\|_{L^\infty(\Omega)} \leqslant C_1 C_2^{-\gamma}$. Then we have $w(x,0) \geqslant u_0(x)$. Thus, we have shown that C_1 and C_2 satisfy

$$\max\{\|u_0\|_{L^\infty(\Omega)} C_2^{\gamma}, (2\gamma)^{\gamma}\} \leqslant (2\alpha)^{\frac{1}{\beta-q+1}} C_2^{\gamma}. \tag{4.21}$$

To make (4.21) satisfied, we need

$$\|u_0\|_{L^\infty(\Omega)} \leqslant \epsilon := (2\alpha)^{\frac{1}{\beta-q+1}} \tag{4.22}$$

and

$$C_2 \geqslant 2^{\frac{1}{\gamma(q-\beta-1)}+1}\gamma. \tag{4.23}$$

For C_1 and C_2 satisfying (4.21) and (4.23), we know that w is a sup-solution, which implies that

$$0 \leqslant u \leqslant C_1(t+C_2)^{-\gamma}, \quad \gamma = \frac{1}{\beta-1}, \quad \text{for} \quad 1 < \beta < q-1 \tag{4.24}$$

provided u_0 satisfies (4.22).

If $1 < \beta = q-1$, assume additionally that $\alpha < 1$, and we can still obtain the first estimate in (4.17) for C_1, C_2 and u_0 satisfying

$$C_1 \geqslant \max\left\{C_2^{\gamma}, \left(\frac{\gamma}{1-\alpha}\right)^{\gamma}\right\}. \tag{4.25}$$

If $\beta = 1, q \geqslant 2$, let

$$w = C_1 \mathrm{e}^{-C_2 t} \tag{4.26}$$

with

$$\begin{cases} \alpha C_1^{q-2} + C_2 \leqslant 1, \quad \|u_0\|_{L^\infty(\Omega)} \leqslant C_1, \quad \text{for} \quad q > 2, \\ 0 < C_2 \leqslant 1-\alpha, \quad \|u_0\|_{L^\infty(\Omega)} \leqslant C_1, \quad \text{for} \quad q = 2. \end{cases} \tag{4.27}$$

We can still verify that w is a sup-solution of (1.1). Then we obtain the desired result by the comparison principle. Thus, the proof is completed. □

5 Finite time blowup

In this section, we will use two different methods to show that the solution of (1.1) will blow up in finite time. We first introduce the following blowup result which is based on the construction of a self-similar sub-solution and comparison principle.

Theorem 5.1. *Suppose that $q > \max\{p, 2, \beta+1\}$. Then the solution of (1.1) will blow up in finite time for some large $u_0(x)$ satisfying $u_0(x) > 0$ in $\Omega' \subset \Omega$.*

Proof. Without loss of generality, we assume that $0 \in \Omega$. Define $v(x,t)$ as

$$v(x,t) = \frac{1}{(1-\varepsilon t)^k} V\left(\frac{|x|}{(1-\varepsilon t)^m}\right), \quad t_0 \leqslant t < \frac{1}{\varepsilon}, \tag{5.1}$$

where

$$V(y)=1+\frac{A}{\sigma}-\frac{y^{\sigma}}{\sigma A^{\sigma-1}},\quad y\geqslant 0, \tag{5.2}$$

and

$$\sigma=\frac{p}{p-1},\quad k=\frac{1}{q-2},\quad 1<m<\frac{q-p}{p(q-2)},\quad A>\frac{2k}{m},\quad 0<\varepsilon<\frac{\alpha}{k(1+\frac{A}{\sigma})}. \tag{5.3}$$

Let

$$R=(A^{\sigma-1}(\sigma+A))^{\frac{1}{\sigma}},\quad D:=\left\{(x,t)\,\middle|\, t_0\leqslant t<\frac{1}{\varepsilon},|x|<R(1-\varepsilon t)^m\right\}. \tag{5.4}$$

Then $V(y)\geqslant 0$ is smooth in D and $v(y)<0$ if $y>R$. Moreover, $V(y)$ satisfies

$$\begin{cases}1\leqslant V(y)\leqslant 1+\dfrac{A}{\sigma},\quad -1\leqslant V'(y)\leqslant 0, & \text{if } 0\leqslant y\leqslant A,\\ 0\leqslant V(y)\leqslant 1,\quad -\dfrac{R^{\sigma-1}}{A^{\sigma-1}}\leqslant V'(y)\leqslant -1, & \text{if } A\leqslant y\leqslant R,\\ (|V'|^{p-2}V')'+\dfrac{N-1}{y}|V'|^{p-2}V'=-\dfrac{N}{A}.\end{cases} \tag{5.5}$$

Define

$$\mathcal{L}_p v=v_t-\Delta_p v-\alpha|v|^{q-2}v+|v|^{\beta-1}v. \tag{5.6}$$

Then

$$\mathcal{L}_p v=\frac{\varepsilon(kV+myV')}{(1-\varepsilon t)^{k+1}}-\frac{(|V'|^{p-2}V')'+\frac{N-1}{y}|V'|^{p-2}V'}{(1-\varepsilon t)^{(k+m)(p-1)+m}}-\frac{\alpha V^{q-1}}{(1-\varepsilon t)^{k(q-1)}}+\frac{V^{\beta}}{(1-\varepsilon t)^{k\beta}}. \tag{5.7}$$

By (5.3), we can easily see that $k+1=k(q-1),k\beta<k+1,(k+m)(p-1)+m<k+1$. Then, for $0\leqslant\frac{1}{\varepsilon}-t_0\ll 1$ and $t_0\leqslant t<\frac{1}{\varepsilon}$, if $y\in[0,A]$,

$$\begin{aligned}\mathcal{L}_p v&=\frac{1}{(1-\varepsilon t)^{k+1}}\bigg\{\varepsilon(kV+myV')+\frac{N}{A}(1-\varepsilon t)^{k+1-m-(k+m)(p-1)}-\alpha V^{q-1}\\&\quad+V^{\beta}(1-\varepsilon t)^{k+1-k\beta}\bigg\}\\&\leqslant\frac{1}{(1-\varepsilon t)^{k+1}}\bigg\{\varepsilon k\left(1+\frac{A}{\sigma}\right)+\frac{N}{A}(1-\varepsilon t)^{k+1-m-(k+m)(p-1)}-\alpha\\&\quad+V^{\beta}(1-\varepsilon t)^{k+1-k\beta}\bigg\}\\&\leqslant 0,\quad\text{for}\quad \varepsilon\ll\frac{\alpha}{k(1+\frac{A}{\sigma})}.\end{aligned} \tag{5.8}$$

Similarly, if $y\in[A,R]$,

$$\begin{aligned}\mathcal{L}_p v&\leqslant\frac{1}{(1-\varepsilon t)^{k+1}}\bigg\{\varepsilon(k-mA)+\frac{N}{A}(1-\varepsilon t)^{k+1-m-(k+m)(p-1)}+(1-\varepsilon t)^{k+1-k\beta}\bigg\}\\&\leqslant 0.\end{aligned} \tag{5.9}$$

Thus, we have proved that $\mathcal{L}_p v\leqslant 0$ in D. In order for $v(x,t)$ to be a sub-solution, we also need to choose suitable initial data and boundary value. Let t_0 be such that $u_0(x)>0$ in $B(0,R(1-\varepsilon t_0)^m)\subset\Omega$ and $u_0(x)\geqslant v(\cdot,t_0)$ in $B(0,R(1-\varepsilon t_0)^m)$. According to Theorem 3.1 and the definition of $v,u(x,t)\geqslant 0=v(x,t)$ in $\partial B(0,R(1-\varepsilon t)^m)\times(t_0,\frac{1}{\varepsilon})$. Thus, we have shown that $v(x,t+t_0)$ is a sub-solution for (1.1) in $D(t_0):=\{(x,t)\,|\,0\leqslant t\leqslant\frac{1}{\varepsilon}-t_0,|x|<R(1-\varepsilon(t+t_0))^m\}$. By Proposition 2.2, we have

$$u(x,t)\geqslant v(x,t+t_0),\quad (x,t)\in D(t_0). \tag{5.10}$$

Noticing that $\lim_{t\to 1/\varepsilon}v(0,t)\to+\infty$, we have u must blow up at a finite time $T\leqslant\frac{1}{\varepsilon}-t_0<\infty$. □

Remark 5.2. If $1 < p < 2$ we can also choose m such that $0 < m < \frac{2-p}{p(q-2)}$ in (5.3).

Remark 5.3. The method we used above is first introduced by Souplet and Weissler [27] for $p = 2$. Li and Xie [19] developed this method for $p > 2$. In our latest papers [33,34], we used this method to study the blowup results of the initial boundary problem for a p-Laplacian parabolic equation with a nonlinear gradient term.

Next, we will introduce some blowup results whose proofs are based on the energy method and concavity method which were also used in [1,19,32,35] and the references therein. In the proof of our desired results, the following lemma concerning the so-called "energy" is useful.

Lemma 5.4. *Let*

$$E(t) = \int_\Omega \left(\frac{1}{p}|\nabla u|^p + \frac{1}{\beta+1}u^{\beta+1} - \frac{\alpha}{q}u^q\right)dx. \tag{5.11}$$

If $E(0) < 0$, then $E(t) < 0$ for all $t > 0$.

Proof. By a direct computation, we can see that

$$\begin{aligned} E'(t) &= \int_\Omega (|\nabla u|^{p-2}\nabla u \cdot \nabla u_t + u^\beta u_t - \alpha u^{q-1}u_t)dx \\ &= \int_\Omega (-\Delta_p u + u^\beta - \alpha u^{q-1})u_t dx = -\int_\Omega u_t^2 dx \leqslant 0. \end{aligned} \tag{5.12}$$

Hence, $E(t) \leqslant E(0) < 0$ for all $t > 0$. □

The following theorem is the main result of this section.

Theorem 5.5. *Suppose $u_0(x)$ satisfies*

$$\int_\Omega \left(\frac{1}{p}|\nabla u_0|^p + \frac{1}{\beta+1}u_0^{\beta+1} - \frac{\alpha}{q}u_0^q\right)dx < 0. \tag{5.13}$$

Then the solution of (1.1) *will blow up in finite time provided that one of the following cases occurs*:

(a) $0 < \beta < \min\{1, p-1\}, q > \max\{p, 2\}$;
(b) $q = p, 1 < \beta < p-1$;
(c) $\beta = p-1, q > \max\{p, 2\}$;
(d) $1 < \beta < p-1, q > p > 2$;
(e) $\beta + 1 = q = p > 2$;
(f) $q > \beta + 1 > p > 2$, *and* $\|u_0\|^2_{L^2(\Omega)}$ *is large enough.*

Proof. Let $y(t) = \|u\|^2_{L^2(\Omega)}$. Then it satisfies

$$\frac{1}{2}y'(t) = \int_\Omega uu_t dx = \int_\Omega (u\Delta_p u - u^{\beta+1} + \alpha u^q)dx = \int_\Omega (-|\nabla u|^p - u^{\beta+1} + \alpha u^q)dx. \tag{5.14}$$

By Lemma 5.4, we can get

$$\begin{aligned} \frac{1}{2p}y'(t) &= \int_\Omega \left(-\frac{1}{p}|\nabla u|^p - \frac{1}{p}u^{\beta+1} - \frac{\alpha}{p}u^q\right)dx \\ &= -E(t) + \left(\frac{1}{\beta+1} - \frac{1}{p}\right)\int_\Omega u^{\beta+1}dx + \alpha\left(\frac{1}{p} - \frac{1}{q}\right)\int_\Omega u^q dx \\ &> \left(\frac{1}{\beta+1} - \frac{1}{p}\right)\int_\Omega u^{\beta+1}dx + \alpha\left(\frac{1}{p} - \frac{1}{q}\right)\int_\Omega u^q dx. \end{aligned} \tag{5.15}$$

Let us now estimate (5.15) furthermore in different cases.

(a) $0 < \beta < \min\{1, p-1\}, q > \max\{p, 2\}$. In this case, by Hölder's inequality, (5.15) can be rewritten as

$$\frac{1}{2p}y'(t) \geqslant \alpha\left(\frac{1}{p} - \frac{1}{q}\right)\int_\Omega u^q dx \geqslant \alpha\left(\frac{1}{p} - \frac{1}{q}\right)|\Omega|^{\frac{2-q}{2}}y^{\frac{q}{2}}, \tag{5.16}$$

i.e.,

$$y'(t) \geqslant 2p\alpha\left(\frac{1}{p}-\frac{1}{q}\right)|\Omega|^{\frac{2-q}{2}} y^{\frac{q}{2}}(t). \tag{5.17}$$

Integrating (5.17) with respect to t, we have

$$y(t) \geqslant \left(y^{\frac{2-q}{2}}(0) - p\alpha(q-2)\left(\frac{1}{p}-\frac{1}{p}\right)|\Omega|^{\frac{2-q}{2}} t\right)^{\frac{2}{2-q}}, \tag{5.18}$$

which implies that

$$y(t) \to +\infty, \quad \text{as} \quad t \to T_1^* := \frac{y^{\frac{2-q}{2}}(0)|\Omega|^{\frac{2}{2-q}}}{p\alpha(q-2)(\frac{1}{p}-\frac{1}{q})}. \tag{5.19}$$

(b) $q = p, 1 < \beta < p-1$. In this case, it holds that

$$y'(t) > 2p\left(\frac{1}{\beta+1}-\frac{1}{p}\right)\int_\Omega u^{\beta+1}dx \geqslant 2p\left(\frac{1}{\beta+1}-\frac{1}{p}\right)|\Omega|^{\frac{1-\beta}{2}} y^{\frac{\beta+1}{2}}(t). \tag{5.20}$$

Then

$$y(t) \geqslant \left(y^{\frac{1-\beta}{2}}(0) - p(\beta-1)\left(\frac{1}{\beta+1}-\frac{1}{p}\right)|\Omega|^{\frac{1-\beta}{2}} t\right)^{\frac{2}{1-\beta}}. \tag{5.21}$$

Thus

$$y(t) \to +\infty, \quad \text{as} \quad t \to T_2^* := \frac{y^{\frac{1-\beta}{2}}(0)|\Omega|^{\frac{2}{1-\beta}}}{p(\beta-1)(\frac{1}{\beta+1}-\frac{1}{p})}. \tag{5.22}$$

(c) $\beta = p-1, q > \max\{p, 2\}$. Similarly to (a), we can derive that $y(t) \to +\infty$, as $t \to T_3^* = T_1^*$.

(d) $1 < \beta < p-1, q > p > 2$. We can rewrite (5.15) as

$$\begin{aligned} y'(t) &\geqslant 2p\left(\frac{1}{\beta+1}-\frac{1}{p}\right)|\Omega|^{\frac{1-\beta}{2}} y^{\frac{\beta+1}{2}}(t) + 2p\alpha\left(\frac{1}{p}-\frac{1}{q}\right)|\Omega|^{\frac{2-q}{2}} y^{\frac{q}{2}}(t) \\ &\geqslant 4p\sqrt{\alpha\left(\frac{1}{\beta+1}-\frac{1}{p}\right)\left(\frac{1}{p}-\frac{1}{q}\right)}|\Omega|^{\frac{3-q-\beta}{4}} y^{\frac{q+\beta+1}{4}}(t). \end{aligned} \tag{5.23}$$

Then $y(t) \to +\infty$, as $t \to T_4^* \leqslant \min\{T_1^*, T_2^*, T'\}$ with

$$T' := \frac{y^{\frac{3-q-\beta}{4}}(0)|\Omega|^{\frac{4}{3-q-\beta}}}{p(q+\beta-3)\sqrt{\alpha(\frac{1}{\beta+1}-\frac{1}{p})(\frac{1}{p}-\frac{1}{q})}}. \tag{5.24}$$

(e) $\beta+1 = q = p$. If this happens, then we can only derive from (5.15) that $y'(t) > 0$ which cannot be used to show that $y(t) \to +\infty$ as $t \to \widetilde{T} < \infty$. However, if $p > 2$, we can still obtain the desired result by the concavity method. The proof here is same as the one of [19, Lemma 3.4]. Here, we just provided the final ordinary inequality

$$y'(t) \geqslant \frac{y'(0)}{y^{\frac{p}{2}}(0)} y^{\frac{p}{2}}(t). \tag{5.25}$$

(f) $q > \beta+1 > p > 2$. As $\beta+1 > p$, the first term of the right-hand side in (5.15) is negative, we cannot use the above procedure directly. However, by the fact that $q > \beta+1$, we can still obtain the desired result. Indeed, by Young's inequality, we have for small $\epsilon > 0$,

$$\int_\Omega u^{\beta+1}dx \leqslant \frac{\epsilon(\beta+1)}{q}\int_\Omega u^q dx + C(\epsilon)\frac{q-\beta-1}{q}|\Omega|. \tag{5.26}$$

Choose a suitable ϵ such that

$$\left(\frac{1}{\beta+1}-\frac{1}{p}\right)\frac{\epsilon(\beta+1)}{q} \geqslant -\frac{\alpha}{2}\left(\frac{1}{p}-\frac{1}{q}\right). \tag{5.27}$$

Then we have

$$\frac{1}{2p}y'(t) \geqslant \frac{\alpha}{2}\left(\frac{1}{p}-\frac{1}{q}\right)\int_\Omega u^q dx + C(\epsilon)\left(\frac{1}{\beta+1}-\frac{1}{p}\right)\frac{q-\beta-1}{q}|\Omega|. \tag{5.28}$$

If we assume additionally that $\|u_0\|^2_{L^2(\Omega)}$ is large enough, then we can derive

$$y'(t) \geqslant \frac{p\alpha}{2}\left(\frac{1}{p}-\frac{1}{q}\right)|\Omega|^{\frac{2-q}{2}} y^{\frac{q}{2}}(t), \tag{5.29}$$

which implies that

$$y(t) \to +\infty, \quad \text{as} \quad t \to T_5^* := 4T_1^*. \tag{5.30}$$

The proof of Theorem 5.5 is now completed. □

Remark 5.6. Following the same manner as in [19, Theorem 3.5], we can still obtain the desired blowup results in Theorem 5.5(e) if we assume that $\alpha > \Lambda_1 + 1$ instead of (5.13).

Remark 5.7. During the proof of Theorem 5.5, we also obtain an upper bound of the blowup time in each case.

6 Discussions

As was shown in the previous sections, the relation of p, q and β plays an important role in determining the properties of the weak solution of (1.1). To be specific, we will state it for $1 < p < 2$ and $p > 2$, respectively. Moreover, we will use two figures to state the results of blowup, extinction and global existence intuitionally. For simplicity, we will not point out which domain the boundary lines and the coordinate axis belong to.

We first discuss the case $1 < p < 2$ (see Figure 1). In this case, if $q > \max\{2, \beta+1\}$ or $0 < \beta \leqslant p-1$, $q > 2$, then finite time blowup will occur for some suitably large initial data (see Theorems 5.1, 5.5(a) and 5.5(c)). If $q \in (\beta+1, p)$, or $q = \beta+1$, or $q = p$, then finite time extinction will happen with suitable α and any nontrivial initial data (see Theorem 4.3). If $q > p, \beta > 0$, then small initial data can lead to finite time extinction (see Theorem 4.5). Noticing that if $q > 2$, then large initial data can lead to finite time blowup while small initial data implies finite time extinction which is interesting.

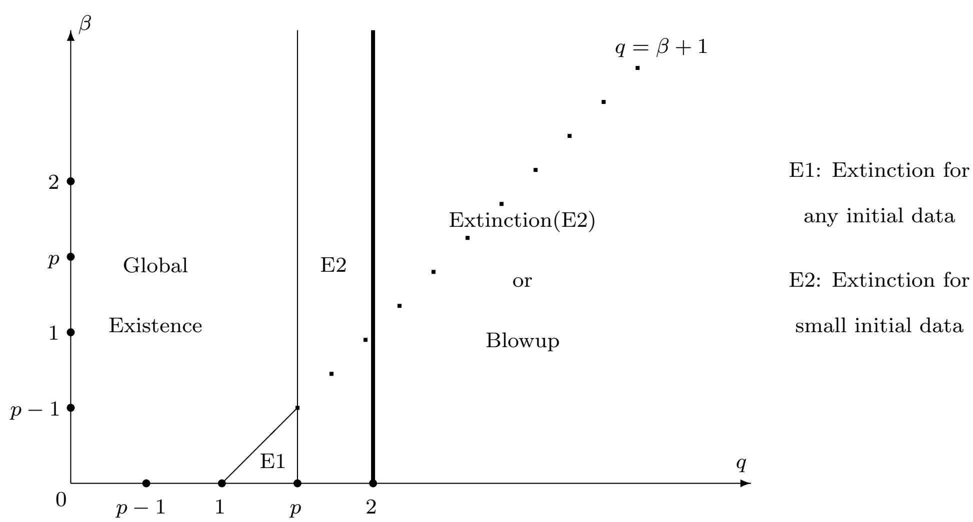

Figure 1 $1 < p < 2$

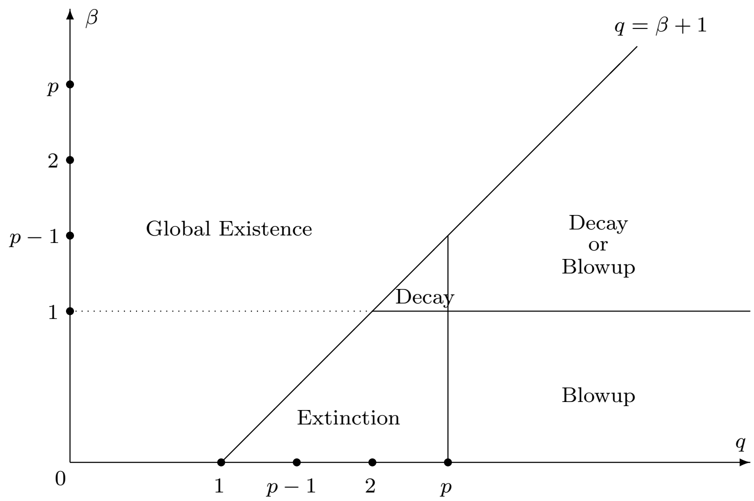

Figure 2 $p > 2$

Next, let us consider the case $p > 2$ (see Figure 2). In this case, if $q > \max\{p, \beta+1\}$, or $q > p,\ 0 < \beta \leqslant q-1$, or $q = p \geqslant \beta+1 > 2$, then for some suitably large initial data, the solution of (1.1) will blow up in finite time (see Theorems 5.1, 5.5(b) and 5.5(d)–5.5(f)). If $q \in (\beta+1, p),\ \beta < 1$, or $q = \beta+1 < 2$, or $q = p$, then finite time extinction will happen with suitable α and any nontrivial initial data (see Theorem 4.3). Besides, if $1 \leqslant \beta \leqslant q-1$, then as was shown in Theorem 4.6, the solution of (1.1) cannot extinguish in finite time, while it will decay to zero as $t \to +\infty$ for some suitably small u_0.

We also need to point out that finite time extinction is not a singularity property for solution of (1.1) as β and $q-1$ are positive. If finite time extinction happens, we have in fact shown that the solution of (1.1) is global in time bounded which is also an important property of the solution of (1.1). For the global existence of the weak solution, we can see from Theorem 3.2 that the critical value for q is p if $1 < p < 2$. While in the degenerate case, the critical value is p and $\beta+1$. Moreover, if $q \leqslant p$ or $q < \beta+1$, then we can obtain the global existence.

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant Nos. 11371286 and 11401458), the Special Fund of Education Department (Grant No. 2013JK0586) and the Youth Natural Science Grant of Shaanxi Province of China (Grant No. 2013JQ1015).

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