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Stability of unduloid bridges with free boundary in a Euclidean slab

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Abstract We study the stability of unduloids with free boundary in the domain B between two parallel hyperplanes in $\mathbb{R}^{n+1}$. If the unduloid has one half of period in B and is sufficiently close to a cylinder, then for $2 \leqslant n \leqslant 10$, it is unstable; while for $n \geqslant 11$, it is stable. If the unduloid has two or more halves of period in B and is sufficiently close to a cylinder, then for all $n \geqslant 2$, it is unstable.

Keywords unduloid, free boundary, stability

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1 Introduction

Capillarity is an important physical phenomenon, which occurs when two different materials contact and do not mix. Given a container B with an incompressible liquid drop T in it, the interface of the liquid and the air is a capillary surface M. In absence of gravity, the interface M is of constant mean curvature and the contact angle of M to the boundary ∂B is constant. One should compare this setting with soap bubble (resp. soap film), where the surface has no boundary (resp. fixed boundary) and constant mean curvature.

The literature for the study of capillarity is extensive and we refer to the book of Finn [5], where the treatment of the theory is mainly in the nonparametric case and in the more general situation of presence of gravity. Also we mention [6] for a more recent survey about this topic.

In this paper, we are concerned with the special case that the container B is a Euclidean slab and no gravity is involved, i.e., B is assumed to be the domain between two parallel hyperplanes in $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. The (weak) stability for capillary hypersurfaces in a slab has been discussed by Vogel [18,19], Athanassenas [3], Zhou [21], Pedrosa and Ritoré [14] and Ainouz and Souam [1]. (See also [2, 4, 10–13, 20] for relevant works with planar boundaries and [8, 9, 15, 16] for those with other various boundaries.) Among these works, the results for free boundary case (the contact angle is $\pi/2$) are of special interest. More precisely, for $n = 2$, Vogel [18, 19] and Athanassenas [3] independently proved

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the unduloids in a slab with free boundary are unstable. Then Pedrosa and Ritoré [14] showed that for $2 \leqslant n \leqslant 7$, all unduloids with free boundary are unstable; while for $n \geqslant 9$ there exist unduloids with free boundary which are stable. Moreover, in the same paper [14], Pedrosa and Ritoré investigated the stability of unduloids close to spheres and cylinders with free boundary. They proved for any $n \geqslant 2$, unduloids sufficiently close to spheres are unstable, and for $2 \leqslant n \leqslant 8$, unduloids sufficiently close to cylinders are unstable. In the present paper, we continue their study by considering the remaining dimensions $n \geqslant 9$ for unduloids close enough to cylinders. In fact, we prove the following theorem.

Theorem 1.1. *Let $\mathcal{F}_m$ be the set of unduloids in the slab $B \subset \mathbb{R}^{n+1}$ with free boundary and with m halves of period in B. When $2 \leqslant n \leqslant 10$, the unduloids in $\mathcal{F}_1$ which are sufficiently close to cylinders are unstable; when $n \geqslant 11$, those in $\mathcal{F}_1$ sufficiently close to cylinders are stable. While for all $n \geqslant 2$, those in $\mathcal{F}_m$ $(m \geqslant 2)$ sufficiently close to cylinders are unstable.*

Here as mentioned above, "free boundary" means that the contact angle is $\pi/2$.

The outline of this paper is as follows. In Section 2, after fixing some notation and definitions, we introduce the Delaunay capillary hypersurface and recall Vogel's criterion for stability. Then in Section 3, we apply Vogel's criterion to discuss the stability of unduloids in the slab with free boundary which are close to cylinders, where we prove Theorem 1.1. In the proof, we utilize a perturbation property of Sturm-Liouville operators as a key ingredient, which may be of independent interest.

2 Preliminaries

2.1 Notation and definitions

Let $B = \{x = (x^1, \dots, x^n, x^{n+1}) \in \mathbb{R}^{n+1} : 0 \leqslant x^{n+1} \leqslant h\}$, $P_1 = \{x : x^{n+1} = 0\}$ and $P_2 = \{x : x^{n+1} = h\}$. Let $x : M^n \to \mathbb{R}^{n+1}$ be an orientable embedded hypersurface with $x(\mathrm{int}M) \subset \mathrm{int}B$ and $x(\partial M) \subset \partial B$. Suppose that $\Omega_i \subset P_i$ $(i = 1, 2)$ are both non-empty and that $\partial\Omega_1 \cup \partial\Omega_2 = x(\partial M)$. In addition, denote by $T \subset B$ the part satisfying $\partial T = x(M) \cup \Omega_1 \cup \Omega_2$.

Let N be the unit normal of M pointing inwards to T and $\bar{N}_i$ the unit outward normal of P_i $(i = 1, 2)$. Denote by ν_i and $\bar{\nu}_i$ the conormals of $\partial\Omega_i$ in M and Ω_i, where $i = 1, 2$, respectively. Let D (resp. ∇) be the Riemannian connection on $\mathbb{R}^{n+1}$ (resp. M). Then the second fundamental form of M in $\mathbb{R}^{n+1}$ is given by $\sigma(X_1, X_2) = -\langle X_2, D_{X_1} N\rangle$ for $\forall\, X_1, X_2 \in T_pM$. When taking an orthonormal basis $\{e_i\}_{i=1}^n$ on TM, we also denote by h_{ij} the components $\sigma(e_i, e_j)$. So the mean curvature H of M is $H = \frac{1}{n}\sum_{i=1}^n h_{ii}$. In addition, the second fundamental form of P_i in $\mathbb{R}^{n+1}$ is given by $\Pi_i(Y_1, Y_2) = \langle Y_2, D_{Y_1}\bar{N}_i\rangle$ for $\forall\, Y_1, Y_2 \in T_p(P_i)$. At last let $\theta_i \in (0, \pi)$ be the angle between ν_i and $\bar{\nu}_i$. See Figure 1 for an illustration.

Following [15], we discuss the variation of M.

Definition 2.1. An admissible variation of $x : M^n \to \mathbb{R}^{n+1}$ is a differentiable map $X : (-\varepsilon, \varepsilon) \times M \to \mathbb{R}^{n+1}$ so that $X_t : M^n \to \mathbb{R}^{n+1}, t \in (-\varepsilon, \varepsilon)$ given by $X_t(p) = X(t, p), p \in M$ is an immersion satisfying $X_t(\mathrm{int}M) \subset \mathrm{int}B$ and $X_t(\partial M) \subset \partial B$ for all t, and $X_0 = x$.

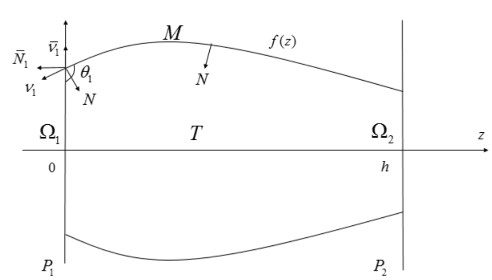

Figure 1 A typical illustration

Now for given $\theta_i \in (0,\pi)$ $(i=1,2)$, we define an energy functional

$$E(t) = |M(t)| - \sum_{i=1}^{2} \cos\theta_i |\Omega_i(t)|, \tag{2.1}$$

where $M(t)$ is the image of M under X_t and $\Omega_i(t)$ has the same definition as Ω_i but with x replaced by X_t. The symbol $|\cdot|$ denotes the area function. In addition, the volume functional can be defined as

$$V(t) = \int_{[0,t]\times M} X^* dv,$$

where dv is the standard volume element of $\mathbb{R}^{n+1}$.

Under these constraints, we have the following definition.

Definition 2.2. An immersed hypersurface $x: M^n \to \mathbb{R}^{n+1}$ is called capillary if $E'(0)=0$ for any admissible volume-preserving variation of x. By volume-preserving, we mean $V(t)$ is always equal to $V(0)=0$.

Note that we have the following formulas (see, e.g., [15, 16, 18]):

$$E'(0) = -n\int_M H\varphi da + \sum_{i=1}^{2}\int_{\partial\Omega_i} \langle Y, \nu_i - \cos\theta_i \bar{\nu}_i\rangle ds, \tag{2.2}$$

$$V'(0) = \int_M \varphi da, \tag{2.3}$$

where Y is the variational vector field $Y(p) = \frac{\partial X}{\partial t}(p)\,|_{t=0}$, φ is its normal component $\varphi = \langle Y, N\rangle$, and da and ds are the corresponding area elements.

From these formulas we see that M is capillary if and only if it has constant mean curvature and makes constant contact angle θ_i with P_i.

By [14, Lemma 5.1], the embedding M is a rotationally symmetric hypersurface, with the rotational axis orthogonal to P_1 and P_2. So in the next subsection we will recall rotational hypersurfaces with constant mean curvature in $\mathbb{R}^{n+1}$.

2.2 Delaunay hypersurfaces in Euclidean space

In this subsection, following [7] we review some facts about Delaunay hypersurfaces, which are rotational and of constant mean curvature H.

Let $M^n \subset \mathbb{R}^{n+1}$ be a hypersurface which is invariant under the action of the orthogonal group $O(n)$ fixing the x^{n+1}-axis. Assume M is generated by a curve Γ contained in the $x^{n+1}x^n$-plane. Then it suffices to determine the curve Γ.

Parametrize the curve $\Gamma = (x^{n+1}, x^n)$ by arc-length s. Denote by α the angle between the tangent to Γ and the positive x^{n+1}-direction and choose the normal vector $N = (\sin\alpha, -\cos\alpha)$. Then $(x^{n+1}, x^n; \alpha)$ satisfies the following system of ordinary differential equations:

$$\begin{cases} (x^{n+1})' = \cos\alpha, \\ (x^n)' = \sin\alpha, \\ \alpha' = -nH + (n-1)\dfrac{\cos\alpha}{x^n}. \end{cases}$$

The first integral of this system is given by $(x^n)^{n-1}\cos\alpha - H(x^n)^n = J$, where the constant J is called the force of the curve Γ and it together with H will determine the curve as follows. (See [7, Proposition 4.3].)

Proposition 2.3. *The curve Γ and the hypersurface M generated by Γ have the following several possible types*:

(1) *If $JH > 0$ then Γ is a periodic graph over the x^{n+1}-axis. It generates a periodic embedded unduloid, or a cylinder.*

(2) *If $JH < 0$, then Γ is a locally convex curve and M is a nodoid, which has self-intersections.*

(3) *If $J = 0$ and $H \neq 0$, then M is a sphere.*

(4) *If $H = 0$ and $J \neq 0$, we obtain a catenary which generates an embedded catenoid M with $J > 0$ if the normal points down and $J < 0$ if the normal points up.*

(5) *If $H = 0$ and $J = 0$, then Γ is a straight line orthogonal to the x^{n+1}-axis which generates a hyperplane.*

(6) *If M touches the x^{n+1}-axis, then it must be a sphere or a hyperplane.*

(7) *The curve Γ is determined, up to translation along the x^{n+1}-axis, by the pair (H, J).*

In this paper, we are concerned with embedded hypersurfaces with free boundary, i.e., we will only consider unduloids with cylinders being a limit case. In this case, we can write $(x^{n+1}, x^n) = (z, y)$ and use $y = f(z), z \in [0, h]$ to represent the generating curve. Then we can compute the principal curvatures of the hypersurface to get

$$\kappa_1 = -\frac{f''(z)}{(1+(f'(z))^2)^{3/2}}, \quad \kappa_2 = \cdots = \kappa_n = \frac{1}{f(1+(f'(z))^2)^{1/2}}.$$

So we have the equation for capillary hypersurfaces

$$-\frac{f''(z)}{(1+(f'(z))^2)^{3/2}} + \frac{n-1}{f(1+(f'(z))^2)^{1/2}} = nH, \tag{2.4}$$

$$-\frac{f'(0)}{\sqrt{1+(f'(0))^2}} = \cos\theta_1, \quad \frac{f'(h)}{\sqrt{1+(f'(h))^2}} = \cos\theta_2. \tag{2.5}$$

2.3 Stability and its criterion

Furthermore, for a capillary hypersurface, one can compute the second derivative of $E(t)$ at $t = 0$ with respect to an admissible volume-preserving variation to get (see, e.g., [15, Appendix])

$$E''(0) = -\int_M (\Delta\varphi + (|\sigma|^2 + \widetilde{\mathrm{Ric}}(N))\varphi)\varphi da + \sum_{i=1}^{2}\int_{\partial\Omega_i}\left(\frac{\partial\varphi}{\partial\nu_i} - q_i\varphi\right)\varphi ds, \tag{2.6}$$

where $\varphi \in \mathcal{F} := \{\varphi \in H^1(M), \int_M \varphi da = 0\}$, $\widetilde{\mathrm{Ric}}(N)$ is the Ricci curvature of the ambient space and

$$q_i = \frac{1}{\sin\theta_i}\Pi_i(\bar\nu_i, \bar\nu_i) + \cot\theta_i\sigma(\nu_i, \nu_i). \tag{2.7}$$

Here, $H^1(M)$ is the first Sobolev space. In our setting, $\widetilde{\mathrm{Ric}}(N) = 0$ and $\Pi_i(\bar\nu_i, \bar\nu_i) = 0$.

Definition 2.4. A capillary hypersurface M is called (weakly) stable if $E''(0) \geqslant 0$ for all $\varphi \in \mathcal{F}$.

In our context the second variation formula (2.6) can be rewritten as

$$E''(0) = \int_M (|\nabla\varphi|^2 - |\sigma|^2\varphi^2)da - \sum_{i=1}^{2}\int_{\partial\Omega_i}\cot\theta_i\sigma(\nu_i, \nu_i)\varphi^2 ds. \tag{2.8}$$

Denote by μ the volume element of the sphere $\mathbb{S}^{n-1}$ and by ω_{n-1} the volume of $\mathbb{S}^{n-1}$, so that $da = f^{n-1}\sqrt{1+f'^2}\mu \wedge dz$ and $ds = f^{n-1}\mu$. Then we can get

$$\begin{aligned}\frac{1}{\omega_{n-1}}E''(0) = &\int_0^h \left[\frac{\varphi'^2}{1+f'^2} - \left(\frac{f''^2}{(1+f'^2)^3} + \frac{n-1}{f^2(1+f'^2)}\right)\varphi^2\right] f^{n-1}\sqrt{1+f'^2}dz \\ &- \frac{f'(z)f''(z)f(z)^{n-1}}{(1+f'(z)^2)^{3/2}}\varphi(z)^2\Big|_{z=0} + \frac{f'(z)f''(z)f(z)^{n-1}}{(1+f'(z)^2)^{3/2}}\varphi(z)^2\Big|_{z=h},\end{aligned}$$

with the volume-preserving constraint $\int_M \varphi da = 0$ becoming $\int_0^h \varphi f^{n-1}\sqrt{1+f'^2}dz = 0$. Now let $\psi = \varphi\sqrt{1+f'^2}$. Then integrating by parts and using the following equality:

$$\left(-\frac{f''(z)}{(1+f'(z)^2)^{3/2}} + \frac{n-1}{f(1+f'(z)^2)^{1/2}}\right)' = (nH)' = 0,$$

we can get

$$\frac{1}{\omega_{n-1}}E''(0) = \int_0^h \left[\frac{f^{n-1}\psi'^2}{(1+f'^2)^{3/2}} - (n-1)\frac{f^{n-3}\psi^2}{(1+f'^2)^{1/2}}\right]dz$$

with the volume-preserving constraint $\int_0^h \psi f^{n-1}dz = 0$. In summary we arrive at the following stability condition.

Theorem 2.5. *Let $f \in C^2[0,h]$ be the solution of* (2.4) *and* (2.5). *Then the hypersurface M generated by $y = f(z)$ is stable, if*

$$\beta(\psi) = \int_0^h \left[\frac{f^{n-1}\psi'^2}{(1+f'^2)^{3/2}} - (n-1)\frac{f^{n-3}\psi^2}{(1+f'^2)^{1/2}}\right]dz$$

is positive definite on the space

$$(f^{n-1})^{\perp} := \left\{\psi \in C^2[0,h] \,\middle|\, \int_0^h \psi f^{n-1}dz = 0\right\}.$$

Remark 2.6. Here our proof of Theorem 2.5 is different from that in [18] for the surface case ($n = 2$). Our proof seems more transparent.

Next, we recall the useful criterion for stability due to Vogel [18]. To that end we consider, more generally, quadratic forms of the form $\beta(\psi) = \int_0^h (P(\psi')^2 + Q(\psi)^2)dz$, where P and Q are functions of z, and $P > 0$ on $[0,h]$. For a given nonzero $g \in C^1[0,h]$, Vogel [18] determined the necessary and sufficient conditions for $\beta(\psi)$ to be positive definite on the space

$$g^{\perp} = \left\{\psi \in C^1[0,h] : \int_0^h \psi \cdot g dz = 0\right\},$$

and the precise formulation for these conditions involves a careful examination of the eigenvalue distribution of the Sturm-Liouville operator associated with β, i.e., $L(\psi) = -(P\psi')' + Q\psi = \lambda\psi$, under boundary conditions $\psi'(0) = \psi'(h) = 0$.

Specifically, recall that according to the theory of Sturm-Liouville, the sequence of eigenvalues of L, which we denote by $\{\lambda_i\}$, $i = 0, 1, 2, \ldots$, is strictly increasing. Moreover, following Vogel [18], we divide the analysis into four cases and examine them individually as follows:

(1) $\lambda_0 > 0$. Then evidently $\beta(\psi)$ is positive definite on $g^{\perp}$.

(2) $\lambda_0 = 0$. $\beta(\psi)$ must be positive semidefinite on $g^{\perp}$. Moreover, it is positive definite on $g^{\perp}$ if and only if the first eigenfunction ϕ_0 of L does not lie in $g^{\perp}$.

(3) $\lambda_0 < \lambda_1 \leqslant 0$. In this case, $\beta(\psi)$ cannot be positive definite on $g^{\perp}$.

(4) $\lambda_0 < 0 < \lambda_1$. This case is the most involved, and Vogel [18] proved the following theorem.

Theorem 2.7 (See [18])**.** *Assume that the first two eigenvalues of the Sturm-Liouville problem satisfy $\lambda_0 < 0 < \lambda_1$. Solve $\phi(z)$ from $L(\phi) = g$ with $\phi'(0) = \phi'(h) = 0$. Then $\beta(\phi)$ is positive definite on $g^{\perp}$, if and only if $\int_0^h g\phi dz < 0$.*

This criterion is not easy to be applied. So Vogel [18] further proved the following theorem, which embeds the given $f(z)$ into a family of constant mean curvature hypersurfaces, so that the function ϕ needed in Theorem 2.7 is easily constructed.

Theorem 2.8 (See [18])**.** *Let $f(z)$ solve* (2.4) *and* (2.5). *Then the hypersurface M generated by $f(z)$ is stable, if the following hold*:

(1) *The eigenvalues of the Sturm-Liouville problem*

$$\begin{cases} L(\psi) = -\left(\dfrac{f^{n-1}\psi'}{(1+(f')^2)^{3/2}}\right)' - (n-1)\dfrac{f^{n-3}\psi}{(1+(f')^2)^{1/2}} = \lambda\psi, \\ \psi'(0) = \psi'(h) = 0, \end{cases}$$

satisfy $\lambda_0 < 0 < \lambda_1$.

(2) *Assume that* $f(z,\varepsilon)$ *solves* (2.4) *and* (2.5) *with* $H=H(\varepsilon)$, *and* $f(z)=f(z,\varepsilon_0)$. *Denote by* $V(\varepsilon)$ *the volume* (*in* B) *bounded by the hypersurface generated by* $f(z,\varepsilon)$. *Then* $H'(\varepsilon_0)V'(\varepsilon_0)<0$.
Moreover, if $\lambda_1<0$, *then* M *is unstable*; *if* $H'(\varepsilon_0)V'(\varepsilon_0)>0$, M *is also unstable.*

3 Stability of unduloids close to cylinders

3.1 Stability of cylinders

First, we will illustrate how to use Theorem 2.8 to analyse the stability of cylinders. The solution for the cylinder is given by $y=c$. Its mean curvature is $H(c)=\frac{n-1}{nc}$, and volume is $V(c)=\frac{\omega_{n-1}hc^n}{n}$. Thus we have $H'(c)V'(c)<0$, which is the required condition (2) in Theorem 2.8. To check the condition (1), we solve the Sturm-Liouville problem

$$\begin{cases}\psi''=-\left(\dfrac{\lambda}{c^{n-1}}+\dfrac{n-1}{c^2}\right)\psi,\\ \psi'(0)=\psi'(h)=0.\end{cases}$$

The eigenfunctions are

$$\psi_k=\cos\left(\frac{k\pi z}{h}\right),\quad k=0,1,2,\dots,$$

and corresponding eigenvalues are

$$\lambda_k=c^{n-1}\left(\left(\frac{k\pi}{h}\right)^2-\frac{n-1}{c^2}\right).$$

So the condition (1) in Theorem 2.8 means that $c>\frac{h}{\pi}\sqrt{n-1}$. Therefore, we obtain the following.

Proposition 3.1 (See [14, 18]). *The cylinder of* $y=c$ *is stable if* $c>\frac{h}{\pi}\sqrt{n-1}$, *and unstable if* $c<\frac{h}{\pi}\sqrt{n-1}$.

3.2 The asymptotic expansion of unduloids close to cylinders

In this paper, we are concerned with the unduloids in $\mathcal{F}_m$ which are close to cylinders. So first we intend to get the expression of their generating function $f(z)$. The following lemma shows that to get $f(z)$ in $\mathcal{F}_m$ which solves an ODE with boundary conditions, we only need to get $y(z,A)$ which solves an ODE with initial conditions.

Lemma 3.2. *Let* $\mathcal{G}:=\{y(z,A)\mid 0<A<\frac{n}{n-1}\}$, *where* $y(z,A)$ *is the solution to the ODE initial value problem*

$$\begin{cases}-\dfrac{y''}{(1+(y')^2)^{3/2}}+\dfrac{n-1}{y(1+(y')^2)^{1/2}}=n-1,\\ y(0)=A,\quad y'(0)=0.\end{cases}\tag{3.1}$$

Then there is a one-to-one correspondence between $\mathcal{G}$ *and* $\mathcal{F}_m$ *for any given* $m\geqslant 1$. *In other words,* A *can be taken as a parameter of* $\mathcal{F}_m$.

Proof. We only prove it for $m=1$. The cases for $m\geqslant 2$ are similar. Let $y(z,A)\in\mathcal{G}$ and $P(A)$ be its period. Then $f(z)=\frac{2h}{P(A)}y(\frac{zP(A)}{2h},A)$ solves

$$\begin{cases}-\dfrac{f''}{(1+(f')^2)^{3/2}}+\dfrac{n-1}{f(1+(f')^2)^{1/2}}=(n-1)\dfrac{P(A)}{2h},\\ f'(0)=f'(h)=0.\end{cases}$$

So $f(z)\in\mathcal{F}_1$.

On the other hand, let $f(z)\in\mathcal{F}_1$ with constant mean curvature H. Let $y(z)=\frac{nH}{n-1}f(\frac{n-1}{nH}z)$. Then $y(z)\in\mathcal{G}$. □

We comment that this lemma is in essence a scale transformation, and the surface in $\mathcal{F}_1$ which corresponds to $y(z,A)$ in $\mathcal{G}$ has mean curvature $\frac{n-1}{n}\frac{P(A)}{2h}$. With the above lemma at hand, we can only consider the solutions in $\mathcal{G}$, which are in general easier to obtain than those in $\mathcal{F}_m$.

Denote $\lambda=\sqrt{n-1}$. Then Equation (3.1) is equivalent to

$$\begin{cases} y'=\tan\sigma,\\ \sigma'=\lambda^2\Big(-\dfrac{1}{\cos\sigma}+\dfrac{1}{y}\Big),\\ y(0)=A,\quad \sigma(0)=0.\end{cases}$$

Let $y=y(z,A)$ and $\sigma=\sigma(z,A)$ be its solution. Then $y(z,1)=1$ and $\sigma(z,1)=0$ corresponds to the cylinder. Let $\delta=A-1$. Then we have the following Taylor expansion:

$$\begin{aligned} y(z,A)&=1+y_1(z)\delta+y_2(z)\delta^2+y_3(z)\delta^3+O(\delta^4),\\ \sigma(z,A)&=1+\sigma_1(z)\delta+\sigma_2(z)\delta^2+\sigma_3(z)\delta^3+O(\delta^4),\end{aligned}$$

where (see [14] for details)

$$\begin{aligned} y_1(z)&=\cos(\lambda z),\quad \sigma_1(z)=-\lambda\sin(\lambda z),\\ y_2(z)&=\frac{1}{12}[(-3\lambda^2+6)+(4\lambda^2-4)\cos(\lambda z)-(\lambda^2+2)\cos(2\lambda z)],\\ \sigma_2(z)&=\frac{1}{12}[(-2\lambda^3+2\lambda)\sin(\lambda z)+(\lambda^3+2\lambda)\sin(2\lambda z)],\\ y_3(z)&=\frac{1}{288}[-(48\lambda^4-144\lambda^2+96)+(12\lambda^5-96\lambda^3+12\lambda)z\sin(\lambda z)\\ &\quad+(61\lambda^4-128\lambda^2+49)\cos(\lambda z)-(16\lambda^4+16\lambda^2-32)\cos(2\lambda z)\\ &\quad+(3\lambda^4+15)\cos(3\lambda z)],\\ \sigma_3(z)&=\frac{1}{288}[(12\lambda^6-96\lambda^4+12\lambda^2)z\cos(\lambda z)+(-49\lambda^5+104\lambda^3-37\lambda)\sin(\lambda z)\\ &\quad+(32\lambda^5+32\lambda^3-64\lambda)\sin(2\lambda z)-(9\lambda^5+24\lambda^3+45\lambda)\sin(3\lambda z)].\end{aligned}$$

Then as in [14], we can derive the following lemma.

Lemma 3.3 (See [14])**.** *The half period of an unduloid generated by $y(z,A)$ is*

$$\frac{P(A)}{2}=\frac{\pi}{\lambda}+\frac{\pi}{24\lambda}(n^2-10n+10)(A-1)^2+O((A-1)^3),$$

for A close to 1.

Remark 3.4. Note that in [14, p. 1375], there is a minor mistake for $P(A)/2$. Thus we include the proof here.

Proof. Denote $A=1+\delta$. Let $p(\delta)$ be half the period of $y(z,1+\delta)$. Then $p(\delta)$ satisfies $\sigma(p(\delta),1+\delta)=0$. Define

$$F(z,\delta)=\frac{1}{\delta}\sigma(z,1+\delta)=\sigma_1(z)+\sigma_2(z)\delta+\sigma_3(z)\delta^2+O(\delta^3).$$

Then $F(p(\delta),\delta)=0$. Since $F(\pi/\lambda,0)=0$ and $F_z(\pi/\lambda,0)=\sigma_1'(\pi/\lambda)=\lambda^2<0$, by the implicit function theorem, the equation $F(p(\delta),\delta)=0$ defines a function $p=p(\delta)$ around $\delta=0, p=\pi/\lambda$, and we have

$$p'(\delta)=-\frac{F_\delta}{F_z}\bigg|_{(\pi/\lambda,0)}=-\frac{\sigma_2}{\sigma_1'}\bigg|_{\pi/\lambda}=0.$$

Moreover, taking the second derivative on $F(p(\delta),\delta)=0$ with respect to δ, we get

$$F_{zz}(p_\delta)^2+F_zp_{\delta\delta}+2F_{z\delta}p_\delta+F_{\delta\delta}=0.$$

Let $\delta = 0$. We have

$$p_{\delta\delta} = -\frac{F_{\delta\delta}}{F_z}\bigg|_{(\pi/\lambda,0)} = -\frac{2\sigma_3(\pi/\lambda)}{\lambda^2}.$$

Now by a direct computation we have

$$\sigma_3(\pi/\lambda) = -\frac{\pi\lambda}{24}(\lambda^4 - 8\lambda^2 + 1).$$

Note $\lambda^2 = n - 1$. We get the conclusion. □

3.3 Stability of unduloids close to cylinders in $\mathcal{F}_m$ with $m \geqslant 2$

We shall prove the following proposition.

Proposition 3.5. *For any $n \geqslant 2$, the unduloids in $\mathcal{F}_m$ with $m \geqslant 2$ which are sufficiently close to cylinders are unstable.*

Proof. First, we have $\lim_{A\to 1} \frac{P(A)}{2} = \frac{\pi}{\lambda}$. Then by Lemma 3.2, $f(x) = \frac{\lambda h}{m\pi}$ is the cylinder solution in $\mathcal{F}_m$.

As in the proof of Proposition 3.1, the first two eigenvalues of the Sturm-Liouville problem associated with $f(z)$ satisfy $\lambda_0 < \lambda_1 < 0$. Then by continuity, the eigenvalues for $f(z, A)$ as $A \to 1$ satisfy the same relation $\lambda_0(A) < \lambda_1(A) < 0$. Thus by Vogel's criterion (see Theorem 2.8) they generate unstable unduloids. □

3.4 Stability of unduloids close to cylinders in $\mathcal{F}_1$

This case is more involved, since in $\mathcal{F}_1$, $\lambda_1(A)\,|_{A=1} = 0$. But we still need to use Vogel's criterion, Theorem 2.8. To that end we need to study the behavior of λ_1 near $A = 1$. As a first step, we get the expression of unduloids close to cylinders in $\mathcal{F}_1$. For simplicity, set $h = \pi/\sqrt{n-1}$.

Proposition 3.6. *The unduloids close to cylinders in $\mathcal{F}_1$ can be represented by*

$$f(z,\delta) = 1 + f_1(z)\delta + f_2(z)\delta^2 + O(\delta^3), \quad \delta \to 0, \tag{3.2}$$

where

$$f_1(z) = \cos\lambda z, \quad f_2(z) = \frac{-n^2 + 4n + 8}{24} + \frac{n-2}{3}\cos\lambda z - \frac{n+1}{12}\cos 2\lambda z.$$

Proof. Recall that in $\mathcal{G}$ the functions close to cylinders are given by

$$y(z, 1+\delta) = 1 + y_1(z)\delta + y_2(z)\delta^2 + O(\delta^3),$$

where

$$y_1(z) = \cos(\lambda z),$$
$$y_2(z) = \frac{1}{12}[(-3\lambda^2 + 6) + (4\lambda^2 - 4)\cos(\lambda z) - (\lambda^2 + 2)\cos(2\lambda z)].$$

Meanwhile we have

$$\frac{P(\delta)}{2} = \frac{\pi}{\lambda} + \frac{\pi}{24\lambda}(n^2 - 10n + 10)\delta^2 + O(\delta^3).$$

Plugging them into

$$f(z,\delta) = \frac{2h}{P}y\left(\frac{zP}{2h}, 1+\delta\right)$$

as in Lemma 3.2, we get the desired result. □

Now we can check the condition (2) in Theorem 2.8.

Proposition 3.7. *For $f(z,\delta) \in \mathcal{F}_1$ with δ sufficiently small, when $2 \leqslant n \leqslant 8$ or $n \geqslant 11$, we have $H'(\delta)V'(\delta) < 0$; when $n = 9, 10$, we have $H'(\delta)V'(\delta) > 0$.*

Proof. The mean curvature of surfaces generated by functions in $\mathcal{G}$ is $\frac{n-1}{n}$. So by Lemma 3.2, the mean curvature of the surface generated by $f(z,\delta)$ in $\mathcal{F}_1$ is equal to

$$H(\delta)=\frac{n-1}{n}\frac{P(A)}{2h}=\frac{n-1}{n}\bigg(1+\frac{1}{24}(n^2-10n+10)\delta^2+O(\delta^3)\bigg),$$

which implies

$$H'(\delta)=\frac{n-1}{12n}(n^2-10n+10)\delta+O(\delta^2).$$

On the other hand, we can compute the volume of the domain enclosed by $f(z,\delta)$ as follows:

$$\begin{aligned}V(\delta)&=\frac{\omega_{n-1}}{n}\int_0^h(1+f_1(z)\delta+f_2(z)\delta^2+O(\delta^3))^n dz\\&=\frac{\omega_{n-1}}{n}\int_0^h\bigg(1+nf_1(z)\delta+nf_2(z)\delta^2+\frac{n(n-1)}{2}f_1(z)^2\delta^2+O(\delta^3)\bigg)dz,\end{aligned}$$

which yields

$$\begin{aligned}V'(\delta)&=\frac{\omega_{n-1}}{n}\int_0^h\bigg(nf_1(z)+2\bigg(nf_2(z)+\frac{n(n-1)}{2}f_1(z)^2\bigg)\delta+O(\delta^2)\bigg)dz\\&=\frac{\omega_{n-1}\pi}{12\lambda}(-n^2+10n+2)\delta+O(\delta^2).\end{aligned}$$

Therefore, we can conclude that

$$H'(\delta)V'(\delta)=-\frac{\omega_{n-1}\pi\lambda}{144n}(n^2-10n+10)(n^2-10n-2)\delta^2+O(\delta^3).$$

Then by elementary analysis we get our result. □

On the other hand, we get the following result concerning the condition (1) in Theorem 2.8.

Proposition 3.8. *For $f(z,\delta)\in\mathcal{F}_1$ with δ sufficiently small, when $2\leqslant n\leqslant 8$, the first two eigenvalues of its Sturm-Liouville problem satisfy $\lambda_0(\delta)<\lambda_1(\delta)<0$; when $n\geqslant 9$, they satisfy $\lambda_0(\delta)<0<\lambda_1(\delta)$.*

Remark 3.9. The result for $2\leqslant n\leqslant 8$ in Proposition 3.8 can be obtained by combining Proposition 3.1 and [14, Corollary 2.8]. Here, we will give a unified proof for $n\geqslant 2$, which is different from that in [14].

Combining Propositions 3.7 and 3.8, we obtain the following theorem.

Theorem 3.10. *When $2\leqslant n\leqslant 10$, the unduloids sufficiently close to cylinders in $\mathcal{F}_1$ are unstable. When $n\geqslant 11$, they are stable.*

It remains to prove Proposition 3.8. For that purpose, we need the following proposition, which indicates how the Taylor expansion of Sturm-Liouville operator determines that of its eigenvalues.

Proposition 3.11 (See [17, Chapter 17])**.** *Let $\{L_\varepsilon\}$ with parameter ε be a family of Sturm-Liouville operators on*

$$\{\phi\in C^2[0,h]:\phi'(0)=\phi'(h)=0\}.$$

Suppose $L_\varepsilon=L+L_1\varepsilon+L_2\varepsilon^2+O(\varepsilon^3)$. Let $\psi_{\varepsilon i}$ and $\lambda_{\varepsilon i}$ be the eigenfunctions and eigenvalues of L_ε, respectively, with the expansion

$$\psi_{\varepsilon i}=\psi_i+\psi_i^{(1)}\varepsilon+\psi_i^{(2)}\varepsilon^2+O(\varepsilon^3),\quad \lambda_{\varepsilon i}=\lambda_i+\lambda_i^{(1)}\varepsilon+\lambda_i^{(2)}\varepsilon^2+O(\varepsilon^3),$$

where ψ_i and λ_i are the eigenfunctions and eigenvalues of L, respectively, and $\{\psi_i\}$ is orthonormal. Then we have

$$\lambda_i^{(1)}=(\psi_i,L_1\psi_i),\quad \psi_i^{(1)}=\sum_{j\neq i}\frac{(\psi_j,L_1\psi_i)}{\lambda_i-\lambda_j}\psi_j.$$

Moreover, if $\lambda_i=0$, then $\lambda_i^{(2)}=(\psi_i,L_2\psi_i)+(\psi_i^{(1)},L_1\psi_i)$.

Proof. Plugging those expansions in the proposition into $L_\varepsilon\psi_{\varepsilon i}=\lambda_{\varepsilon i}\psi_{\varepsilon i}$ and comparing the coefficients, we get

$$L\psi_i=\lambda_i\psi_i, \tag{3.3}$$

$$L_1\psi_i+L\psi_i^{(1)}=\lambda_i^{(1)}\psi_i+\lambda_i\psi_i^{(1)}, \tag{3.4}$$

$$L_2\psi_i+L_1\psi_i^{(1)}+L\psi_i^{(2)}=\lambda_i^{(2)}\psi_i+\lambda_i^{(1)}\psi_i^{(1)}+\lambda_i\psi_i^{(2)}. \tag{3.5}$$

For fixed i, let $\psi_i^{(1)}=\sum_{j=0}^{+\infty}\alpha_j\psi_j$. Without loss of generality, assume $\alpha_i=0$. (Otherwise replace $\psi_{\varepsilon i}$ with $\psi_{\varepsilon i}/(1+\varepsilon\alpha_i)$.) Then taking the inner product of (3.4) with ψ_k we obtain

$$(\psi_k,L_1\psi_i)+\sum_{j\neq i}\lambda_j\alpha_j(\psi_k,\psi_j)=\lambda_i^{(1)}(\psi_k,\psi_i)+\lambda_i\sum_{j\neq i}\alpha_j(\psi_k,\psi_j). \tag{3.6}$$

So letting $k=i$ gives $\lambda_i^{(1)}=(\psi_i,L_1\psi_i)$; while letting $k\neq i$ yields α_k and further gives

$$\psi_i^{(1)}=\sum_{k\neq i}\frac{(\psi_k,L_1\psi_i)}{\lambda_i-\lambda_k}\psi_k.$$

Now if $\lambda_i=0$, taking the inner product of (3.5) with ψ_i and noting that $(\psi_i,L\psi_i^{(2)})=(\psi_i^{(2)},L\psi_i)=0$, we finally get $\lambda_i^{(2)}=(\psi_i,L_2\psi_i)+(\psi_i^{(1)},L_1\psi_i)$. □

Now we are in a position to prove Proposition 3.8.

Proof of Proposition 3.8. Recall that $h=\pi/\sqrt{n-1}$ and $\lambda=\sqrt{n-1}$. The cylinder solution in $\mathcal{F}_1$ is $f(z,0)=1$. Its corresponding Sturm-Liouville operator is $L(\psi)=-\psi''-(n-1)\psi$, with the eigenvalues and eigenfunctions given by

$$\lambda_k=(k^2-1)(n-1),\quad k=0,1,2,\dots,$$
$$\psi_k=\sqrt{\frac{2\lambda}{\pi}}\cos(k\lambda z),\quad k=1,2,\dots,\quad \psi_0=\sqrt{\frac{\lambda}{\pi}},$$

respectively. Here $\{\psi_k\}$ furnishes an orthonormal basis for $L^2([0,h])$.

Denote $f_\delta(z)=f(z,\delta)\in\mathcal{F}_1$. The Sturm-Liouville operator corresponding to $f_\delta(z)$ is

$$L_\delta(\psi)=-\left[\frac{f_\delta^{n-1}\psi'}{(1+f_\delta'^2)^{3/2}}\right]'-(n-1)\frac{f_\delta^{n-3}\psi}{(1+f_\delta'^2)^{1/2}}.$$

Plugging (3.2) into it, we get

$$L_\delta(\psi)=L(\psi)+L_1(\psi)\delta+L_2(\psi)\delta^2+O(\delta^3), \tag{3.7}$$

with

$$L_1(\psi)=-(A\psi')'-(n-1)C\psi,\quad L_2(\psi)=-(B\psi')'-(n-1)D\psi.$$

Here, the coefficients A,B,C,D are given by

$$A=(n-1)\cos\lambda z,$$
$$B=\frac{(n-1)(n-2)}{2}\cos^2\lambda z-\frac{3}{2}\lambda^2\sin^2\lambda z$$
$$+(n-1)\left[\frac{-n^2+4n+8}{24}+\frac{n-2}{3}\cos\lambda z-\frac{n+1}{12}\cos 2\lambda z\right],$$
$$C=(n-3)\cos\lambda z,$$
$$D=\frac{(n-3)(n-4)}{2}\cos^2\lambda z-\frac{1}{2}\lambda^2\sin^2\lambda z$$

$$+ (n-3)\left[\frac{-n^2+4n+8}{24} + \frac{n-2}{3}\cos\lambda z - \frac{n+1}{12}\cos 2\lambda z\right].$$

Using the notation in Proposition 3.11, we assume that

$$L_\delta \psi_{\delta k} = \lambda_{\delta k}\psi_{\delta k}, \quad k = 0, 1, 2, \ldots,$$

where

$$\psi_{\delta k} = \psi_k + \psi_k^{(1)}\delta + \psi_k^{(2)}\delta^2 + O(\delta^3), \quad \lambda_{\delta k} = \lambda_k + \lambda_k^{(1)}\delta + \lambda_k^{(2)}\delta^2 + O(\delta^3).$$

Note that $\lambda_0 < 0 = \lambda_1 < \lambda_2$. So for small enough δ we have $\lambda_{\delta 0} < 0 < \lambda_{\delta 2}$. Then we only need to consider the expression for $\lambda_{\delta 1}$,

$$\lambda_{\delta 1} = \lambda_1 + \lambda_1^{(1)}\delta + \lambda_1^{(2)}\delta^2 + O(\delta^3) = \lambda_1^{(1)}\delta + \lambda_1^{(2)}\delta^2 + O(\delta^3).$$

Note that

$$\begin{aligned}
L_1\psi_1 &= \sqrt{\frac{2\lambda}{\pi}}((-(n-1)\cos\lambda z(-\lambda\sin\lambda z))' - (n-1)(n-3)\cos^2\lambda z) \\
&= \sqrt{\frac{2\lambda}{\pi}}\left[\frac{n^2-1}{2}\cos 2\lambda z - \frac{(n-1)(n-3)}{2}\right] \\
&= \frac{n^2-1}{2}\psi_2 - \frac{(n-1)(n-3)}{\sqrt{2}}\psi_0,
\end{aligned}$$

which implies

$$(\psi_j, L_1\psi_1) = \begin{cases} \dfrac{n^2-1}{2}, & j = 2, \\ -\dfrac{(n-1)(n-3)}{\sqrt{2}}, & j = 0, \\ 0, & \text{otherwise}. \end{cases}$$

Therefore by Proposition 3.11 we have $\lambda_1^{(1)} = (\psi_1, L_1\psi_1) = 0$. Moreover, $\lambda_1^{(2)} = (\psi_1, L_2\psi_1) + (\psi_1^{(1)}, L_1\psi_1)$. The first term on the right hand side can be computed as follows:

$$\begin{aligned}
(\psi_1, L_2\psi_1) &= \int_0^h (B(\psi_1')^2 - (n-1)D\psi_1^2)dz \\
&= \frac{2\lambda}{\pi}(n-1)\int_0^h (B\sin^2\lambda z - D\cos^2\lambda z)dz \\
&= \frac{n-1}{12}(-3n^2+18n+33),
\end{aligned}$$

while the second term is

$$\begin{aligned}
(\psi_1^{(1)}, L_1\psi_1) &= \left(\sum_{j\neq 1}\frac{(\psi_j, L_1\psi_1)}{\lambda_1-\lambda_j}\psi_j, L_1\psi_1\right) \\
&= \sum_{j\neq 1}\frac{(\psi_j, L_1\psi_1)^2}{\lambda_1-\lambda_j} \\
&= \frac{(\psi_2, L_1\psi_1)^2}{\lambda_1-\lambda_2} + \frac{(\psi_0, L_1\psi_1)^2}{\lambda_1-\lambda_0} \\
&= \frac{n-1}{12}(5n^2-38n+53).
\end{aligned}$$

Therefore we obtain

$$\lambda_1^{(2)} = \frac{n-1}{6}(n^2-10n+10).$$

As a result, when $2 \leqslant n \leqslant 8$, $\lambda_1^{(2)} < 0$; when $n \geqslant 9$, $\lambda_1^{(2)} > 0$. So we complete the proof. □

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