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The Brezis-Nirenberg type critical problem for the nonlinear Choquard equation

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Abstract We establish some existence results for the Brezis-Nirenberg type problem of the nonlinear Choquard equation

$$-\Delta u = \left(\int_\Omega \frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu} dy\right)|u|^{2^*_\mu-2}u + \lambda u \quad \text{in} \quad \Omega,$$

where Ω is a bounded domain of $\mathbb{R}^N$ with Lipschitz boundary, λ is a real parameter, $N \geqslant 3$, $2^*_\mu = (2N-\mu)/(N-2)$ is the critical exponent in the sense of the Hardy-Littlewood-Sobolev inequality.

Keywords Brezis-Nirenberg problem, Choquard equation, Hardy-Littlewood-Sobolev inequality, critical exponent

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1 Introduction

In the recent decades, many people studied the elliptic equation

$$\begin{cases} -\Delta u = |u|^{2^*-2}u + \lambda u & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $2^* = \frac{2N}{N-2}$ is the critical exponent for the embedding of $H^1_0(\Omega)$ to $L^p(\Omega)$, and $\lambda \in (0, \lambda_1)$, where λ_1 is the first eigenvalue of $-\Delta$ set on the bounded domain. In a celebrated paper [8], Brezis and Nirenberg proved that: if $N \geqslant 4$ and $\lambda \in (0, \lambda_1)$, then (1.1) has a nontrivial solution; if $N = 3$ then there exists a constant $\lambda_* \in (0, \lambda_1)$ such that for any $\lambda \in (\lambda_*, \lambda_1)$ (1.1) has a positive solution. Furthermore, if Ω is a ball, then (1.1) has a positive solution if and only if $\lambda \in (\frac{\lambda_1}{4}, \lambda_1)$. Capozzi et al. [11] proved if $N \geqslant 4$ then (1.1) has a nontrivial solution for all $\lambda > 0$. In [13], for $N \geqslant 6$ and $\lambda \in (0, \lambda_1)$, Cerami et al. proved the existence of sign-changing solutions. While for the case Ω is a ball, $N \geqslant 7$ and $\lambda \in (0, \lambda_1)$, they also proved the existence of infinitely many radial solutions to (1.1). There is a great deal of work on elliptic equations with critical nonlinearity (see, for example, [10, 12, 16, 18, 19, 30, 32, 37] and the references therein).

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In the present paper, we consider the existence and nonexistence of solutions for the following nonlocal equation:

$$\begin{cases} -\Delta u = \left(\int_{\Omega} \frac{|u(y)|^{2_\mu^*}}{|x-y|^\mu} dy\right) |u|^{2_\mu^*-2}u + \lambda u \quad \text{in} \quad \Omega, \\ u \in H_0^1(\Omega), \end{cases} \tag{1.2}$$

where Ω is a bounded domain of $\mathbb{R}^N$ with Lipschitz boundary, λ is a real parameter, $N \geqslant 3$, $0 < \mu < N$ and $2_\mu^* = (2N-\mu)/(N-2)$. This nonlocal elliptic equation is closely related to the nonlinear Choquard equation

$$-\Delta u + V(x)u = \left(\frac{1}{|x|^\mu} * |u|^p\right)|u|^{p-2}u \quad \text{in} \quad \mathbb{R}^3. \tag{1.3}$$

Different from the fractional Laplacian where the pseudo-differential operator causes the nonlocal phenomena, for the Choquard equation the nonlocal term appears in the nonlinearity and influences the equation greatly. For $p = 2$ and $\mu = 1$, it goes back to the description of the quantum theory of a polaron at rest by Pekar [28] in 1954 and the modeling of an electron trapped in its own hole in 1976 in the work of Choquard, as a certain approximation to Hartree-Fock theory of one-component plasma (see [20]). In some particular cases, this equation is also known as the Schrödinger-Newton equation, which was introduced by Penrose [29] in his discussion on the selfgravitational collapse of a quantum mechanical wave function.

The existence and qualitative properties of solutions of (1.3) have been widely studied in the recent decades. In [20], Lieb proved the existence and uniqueness, up to translations, of the ground state. Later, in [22], Lions showed the existence of a sequence of radially symmetric solutions. Cingolani et al. [14], Ma and Zhao [23] and Moroz and Van Schaftingen [24] showed the regularity, positivity and radial symmetry of the ground states and derived decay property at infinity as well. Moreover, Moroz and Van Schaftingen [25] considered the existence of ground states under the assumptions of Berestycki-Lions type. For periodic potential V that changes sign and 0 lies in the gap of the spectrum of the Schrödinger operator $-\Delta + V$, the problem is strongly indefinite, and the existence of solution for $p = 2$ was considered in [9] by reduction arguments. In [3], Alves et al. studied the existence of multi-bump shaped solution for the nonlinear Choquard equation with deepening potential well. For a general case, Ackermann [1] proposed a new approach to prove the existence of infinitely many geometrically distinct weak solutions. For other related results, we refer the readers to [15,17] for the existence of sign-changing solutions, [4,5,26,33,36,39] for the existence and concentration behavior of the semiclassical solutions.

The starting point of the variational approach to (1.2) is the following well-known Hardy-Littlewood-Sobolev inequality.

Proposition 1.1 (Hardy-Littlewood-Sobolev inequality, see [21]). *Let $t, r > 1$ and $0 < \mu < N$ with $1/t + \mu/N + 1/r = 2$, $f \in L^t(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$. There exists a sharp constant $C(t, N, \mu, r)$, independent of f and h, such that*

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)h(y)}{|x-y|^\mu} dxdy \leqslant C(t, N, \mu, r)|f|_t |h|_r. \tag{1.4}$$

If $t = r = 2N/(2N-\mu)$, then

$$C(t, N, \mu, r) = C(N, \mu) = \pi^{\frac{\mu}{2}} \frac{\Gamma(\frac{N}{2} - \frac{\mu}{2})}{\Gamma(N - \frac{\mu}{2})} \left\{\frac{\Gamma(\frac{N}{2})}{\Gamma(N)}\right\}^{-1+\frac{\mu}{N}}.$$

In this case, the equality in (1.4) holds if and only if $f \equiv Ch$ and

$$h(x) = A(\gamma^2 + |x-a|^2)^{-(2N-\mu)/2}$$

for some $A \in \mathbb{C}$, $0 \neq \gamma \in \mathbb{R}$ and $a \in \mathbb{R}^N$.

Notice that, by the Hardy-Littlewood-Sobolev inequality, the integral

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^q |u(y)|^q}{|x-y|^\mu} dxdy$$

is well-defined if $|u|^q \in L^t(\mathbb{R}^N)$ for some $t > 1$ satisfying

$$\frac{2}{t} + \frac{\mu}{N} = 2.$$

Thus, for $u \in H^1(\mathbb{R}^N)$, by Sobolev embedding theorems, we know

$$2 \leqslant tq \leqslant \frac{2N}{N-2},$$

i.e.,

$$\frac{2N-\mu}{N} \leqslant q \leqslant \frac{2N-\mu}{N-2}.$$

Thus, $\frac{2N-\mu}{N}$ is called the lower critical exponent and $2_\mu^* = \frac{2N-\mu}{N-2}$ is the upper critical exponent in the sense of the Hardy-Littlewood-Sobolev inequality.

We need to point out that all the papers we mentioned above were about the nonlinear Choquard equation with superlinear subcritical nonlinearities. In a recent paper [27], Moroz and Van Schaftingen considered the nonlinear Choquard equation (1.3) in $\mathbb{R}^N$ with lower critical exponent $\frac{2N-\mu}{N}$. Moroz and Van Schaftingen [27] investigated the existence and nonexistence of solutions to the equation with nonconstant potential by minimizing arguments. However, as far as we know there seems no result for the nonlinear Choquard equation with upper critical exponent with respect to the Hardy-Littlewood-Sobolev inequality. In [2], Alves et al. studied the existence and concentrations of the solutions of a nonlocal Schrödinger with the critical exponential growth in $\mathbb{R}^2$, this problem is closely related to the Choquard equation. Recently, many people also studied the Brezis-Nirenberg problem for elliptic equation driven by the fractional Laplacian, this type of problem are nonlocal in nature and we may refer the readers to [6, 34, 35] and the references therein for a recent progress. In addition, it is quite natural to ask if the well-known results established by Brezis and Nirenberg [8] for the local elliptic equation still hold for the nonlocal Choquard equation. The main purpose of the present paper is to study the nonlinear Choquard equation with upper critical exponent $2_\mu^* = \frac{2N-\mu}{N-2}$ and give a confirm answer to the question of the existence and nonexistence of solutions. By the way, in a forthcoming paper, we will consider the existence of solutions for the following nonlinear Choquard equation with upper critical exponent 2_μ^* in the whole space:

$$-\Delta u + V(x)u = \left(\frac{1}{|x|^\mu} * G(u)\right) g(u) \quad \text{in} \quad \mathbb{R}^3, \tag{1.5}$$

where the nonlinearity g is of upper critical growth in the sense of the Hardy-Littlewood-Sobolev inequality.

From the Hardy-Littlewood-Sobolev inequality, for all $u \in D^{1,2}(\mathbb{R}^N)$ we know

$$\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*}}{|x-y|^\mu} dxdy\right)^{\frac{N-2}{2N-\mu}} \leqslant C(N,\mu)^{\frac{N-2}{2N-\mu}} |u|_{2^*}^2,$$

where $C(N,\mu)$ is defined as in Proposition 1.1. We use $S_{H,L}$ to denote best constant defined by

$$S_{H,L} := \inf_{u\in D^{1,2}(\mathbb{R}^N)\backslash\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*}}{|x-y|^\mu} dxdy)^{\frac{N-2}{2N-\mu}}}. \tag{1.6}$$

From the commentaries above, we can easily draw the following conclusion.

Lemma 1.2. *The constant* $S_{H,L}$ *defined in* (1.6) *is achieved if and only if*

$$u = C\left(\frac{b}{b^2 + |x-a|^2}\right)^{\frac{N-2}{2}},$$

where $C > 0$ *is a fixed constant,* $a \in \mathbb{R}^N$ *and* $b \in (0,\infty)$ *are parameters. Furthermore,*

$$S_{H,L} = \frac{S}{C(N,\mu)^{\frac{N-2}{2N-\mu}}},$$

where S *is the best Sobolev constant.*

Proof. On one hand, by the Hardy-Littlewood-Sobolev inequality, we can see

$$S_{H,L} \geqslant \frac{1}{C(N,\mu)^{\frac{N-2}{2N-\mu}}} \inf_{u\in D^{1,2}(\mathbb{R}^N)\backslash\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{|u|_{2^*}^2} = \frac{S}{C(N,\mu)^{\frac{N-2}{2N-\mu}}}.$$

On the other hand, notice that the equality in the Hardy-Littlewood-Sobolev inequality holds if and only if

$$h(x) = C\left(\frac{b}{b^2+|x-a|^2}\right)^{\frac{2N-\mu}{2}},$$

where $C>0$ is a fixed constant, $a\in\mathbb{R}^N$ and $b\in(0,\infty)$ are parameters. Thus

$$\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu} dxdy\right)^{\frac{N-2}{2N-\mu}} = C(N,\mu)^{\frac{N-2}{2N-\mu}} |u|_{2^*}^2,$$

if and only if

$$u = C\left(\frac{b}{b^2+|x-a|^2}\right)^{\frac{N-2}{2}}.$$

Then, by the definition of $S_{H,L}$, we know

$$S_{H,L} \leqslant \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu} dxdy)^{\frac{N-2}{2N-\mu}}} = \frac{1}{C(N,\mu)^{\frac{N-2}{2N-\mu}}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{|u|_{2^*}^2}$$

and thus we get

$$S_{H,L} \leqslant \frac{S}{C(N,\mu)^{\frac{N-2}{2N-\mu}}}.$$

From the above arguments, we know that $S_{H,L}$ is achieved if and only if $u = C(\frac{b}{b^2+|x-a|^2})^{\frac{N-2}{2}}$ and

$$S_{H,L} = \frac{S}{C(N,\mu)^{\frac{N-2}{2N-\mu}}}.$$

In particular, let $U(x) := \frac{[N(N-2)]^{\frac{N-2}{4}}}{(1+|x|^2)^{\frac{N-2}{2}}}$ be a minimizer for S. Then

$$\begin{aligned}\tilde{U}(x) &= S^{\frac{(N-\mu)(2-N)}{4(N-\mu+2)}} C(N,\mu)^{\frac{2-N}{2(N-\mu+2)}} U(x)\\ &= S^{\frac{(N-\mu)(2-N)}{4(N-\mu+2)}} C(N,\mu)^{\frac{2-N}{2(N-\mu+2)}} \frac{[N(N-2)]^{\frac{N-2}{4}}}{(1+|x|^2)^{\frac{N-2}{2}}}\end{aligned} \tag{1.7}$$

is the unique minimizer for $S_{H,L}$ and satisfies

$$-\Delta u = \left(\int_{\mathbb{R}^N} \frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu} dy\right)|u|^{2^*_\mu-2}u \quad \text{in} \quad \mathbb{R}^N.$$

Moreover,

$$\int_{\mathbb{R}^N} |\nabla\tilde{U}|^2 dx = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\tilde{U}(x)|^{2^*_\mu}|\tilde{U}(y)|^{2^*_\mu}}{|x-y|^\mu} dxdy = S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}.$$

This completes the proof. □

We have some more words about the best constant $S_{H,L}$.

Lemma 1.3. *Let $N\geqslant 3$. For every open subset Ω of $\mathbb{R}^N$,*

$$S_{H,L}(\Omega) := \inf_{u\in D_0^{1,2}(\Omega)\backslash\{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{(\int_\Omega\int_\Omega \frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu} dxdy)^{\frac{N-2}{2N-\mu}}} = S_{H,L}, \tag{1.8}$$

where $S_{H,L}(\Omega)$ is never achieved except when $\Omega=\mathbb{R}^N$.

Proof. It is clear that $S_{H,L} \leqslant S_{H,L}(\Omega)$ by $D_0^{1,2}(\Omega) \subset D^{1,2}(\mathbb{R}^N)$. Let $\{u_n\} \subset C_0^\infty(\mathbb{R}^N)$ be a minimizing sequence for $S_{H,L}$. We make translations and dilations for $\{u_n\}$ by choosing $y_n \in \mathbb{R}^N$ and $\tau_n > 0$ such that

$$u_n^{y_n,\tau_n}(x) := \tau_n^{\frac{N-2}{2}} u_n(\tau_n x + y_n) \in C_0^\infty(\Omega),$$

which satisfies

$$\int_{\mathbb{R}^N} |\nabla u_n^{y_n,\tau_n}|^2 dx = \int_{\mathbb{R}^N} |\nabla u_n|^2 dx$$

and

$$\int_\Omega \int_\Omega \frac{|u_n^{y_n,\tau_n}(x)|^{2_\mu^*} |u_n^{y_n,\tau_n}(y)|^{2_\mu^*}}{|x-y|^\mu} dxdy = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2_\mu^*} |u_n(y)|^{2_\mu^*}}{|x-y|^\mu} dxdy.$$

Hence we obtain $S_{H,L}(\Omega) \leqslant S_{H,L}$. Since $\tilde{U}(x)$ is the only class of functions such that the equality holds in the Hardy-Littlewood-Sobolev inequality, we know that $S_{H,L}(\Omega)$ is never achieved except when $\Omega = \mathbb{R}^N$. □

Next, we will denote the sequence of eigenvalues of the operator $-\Delta$ on Ω with homogeneous Dirichlet boundary data by

$$0 < \lambda_1 < \lambda_2 \leqslant \cdots \leqslant \lambda_j \leqslant \lambda_{j+1} \leqslant \cdots$$

and

$$\lambda_j \to +\infty$$

as $j \to +\infty$. Moreover, $\{e_j\}_{j\in\mathbb{N}} \subset L^\infty(\Omega)$ will be the sequence of eigenfunctions corresponding to $\{\lambda_j\}$. We recall that this sequence is an orthonormal basis of $L^2(\Omega)$ and an orthogonal basis of $H_0^1(\Omega)$. We denote

$$\mathbb{E}_{j+1} := \{u \in H_0^1(\Omega) : \langle u, e_i \rangle_{H_0^1} = 0, \forall\, i = 1, 2, \ldots, j\}, \tag{1.9}$$

while $\mathbb{Y}_j := \mathrm{span}\{e_1, \ldots, e_j\}$ will denote the linear subspace generated by the first j eigenfunctions of $-\Delta$ for any $j \in \mathbb{N}$. It is easily seen that $\mathbb{Y}_j$ is finite dimensional and $\mathbb{Y}_j \oplus \mathbb{E}_{j+1} = H_0^1(\Omega)$.

In order to study the problem by variational methods, we introduce the energy functional associated to (1.2) by

$$J_\lambda(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \frac{1}{2 \cdot 2_\mu^*} \int_\Omega \int_\Omega \frac{|u(x)|^{2_\mu^*} |u(y)|^{2_\mu^*}}{|x-y|^\mu} dxdy - \frac{\lambda}{2} \int_\Omega |u|^2 dx.$$

Then the Hardy-Littlewood-Sobolev inequality implies J_λ belongs to $C^1(H_0^1(\Omega), \mathbb{R})$. Moreover, u is a weak solution of (1.2) if and only if u is a critical point of functional J_λ.

The main results of this paper are stated in the following theorem.

Theorem 1.4. *Assume Ω is a bounded domain of $\mathbb{R}^N$, with Lipschitz boundary and $0 < \mu < N$. The following results hold*:

(i) *If $N \geqslant 4$, then* (1.2) *has a nontrivial solution for $\lambda > 0$, provided λ is not an eigenvalue of $-\Delta$ with homogeneous Dirichlet boundary data.*

(ii) *If $N = 3$, then there exists λ_* such that* (1.2) *has a nontrivial solution for $\lambda > \lambda_*$, provided λ is not an eigenvalue of $-\Delta$ with homogeneous Dirichlet boundary data.*

Throughout this paper, we denote the norm $\|u\| := (\int_\Omega |\nabla u|^2 dx)^{\frac{1}{2}}$ on $H_0^1(\Omega)$ and write $|\cdot|_q$ for the $L^q(\Omega)$-norm for $q \in [1, \infty]$ and always assume Ω is a bounded domain of $\mathbb{R}^N$ with Lipschitz boundary, λ is a real parameter. We denote positive constants by $C, C_1, C_2, C_3, \ldots$

Definition 1.5. Let I be a C^1 functional defined on Banach space X, we say that $\{u_n\}$ is a Palais-Smale sequence of I at c ($(\mathrm{PS})_c$ sequence, for short) if

$$I(u_n) \to c, \quad \text{and} \quad I'(u_n) \to 0, \quad \text{as} \quad n \to +\infty. \tag{1.10}$$

In addition, we say that I satisfies the Palais-Smale condition at the level c, if every Palais-Smale sequence at c has a convergent subsequence.

An outline of the paper is as follows: In Section 2, we give some preliminary results and prove the (PS) condition. In Section 3, we prove the existence of solutions for (1.2) when $N \geqslant 4$ and $0 < \lambda < \lambda_1$ by the mountain pass theorem. In Section 4, we prove the existence of solutions for (1.2) when $N \geqslant 4$ and $\lambda > \lambda_1$, provided λ is not an eigenvalue of $-\Delta$ with homogeneous Dirichlet boundary data, by the linking theorem. In Section 5, we investigate the existence of solutions for $\lambda > 0$ when $N = 3$. In Section 6, we prove a Pohožaev identity for (1.2) and use it to prove the nonexistence of solutions.

2 Preliminary results

To prove the (PS) condition, we need a key lemma which is inspired by the Brezis-Lieb convergence lemma (see [7]). The proof is analogous to that of [1, Lemma 3.5] or [24, Lemma 2.4], but we exhibit it here for completeness. First, we recall that pointwise convergence of a bounded sequence implies weak convergence (see [38, Proposition 5.4.7]).

Lemma 2.1. *Let $N \geqslant 3$, $q \in (1, +\infty)$ and $\{u_n\}$ be a bounded sequence in $L^q(\mathbb{R}^N)$. If $u_n \to u$ almost everywhere in $\mathbb{R}^N$ as $n \to \infty$, then $u_n \rightharpoonup u$ weakly in $L^q(\mathbb{R}^N)$.*

Lemma 2.2. *Let $N \geqslant 3$ and $0 < \mu < N$. If $\{u_n\}$ is a bounded sequence in $L^{\frac{2N}{N-2}}(\mathbb{R}^N)$ such that $u_n \to u$ almost everywhere in $\mathbb{R}^N$ as $n \to \infty$, then the following holds:*

$$\int_{\mathbb{R}^N} (|x|^{-\mu} * |u_n|^{2^*_\mu})|u_n|^{2^*_\mu} dx - \int_{\mathbb{R}^N} (|x|^{-\mu} * |u_n - u|^{2^*_\mu})|u_n - u|^{2^*_\mu} dx \to \int_{\mathbb{R}^N} (|x|^{-\mu} * |u|^{2^*_\mu})|u|^{2^*_\mu} dx$$

as $n \to \infty$.

Proof. First, similar to the proof of the Brezis-Lieb lemma (see [7]), we know that

$$|u_n - u|^{2^*_\mu} - |u_n|^{2^*_\mu} \to |u|^{2^*_\mu} \tag{2.1}$$

in $L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N)$ as $n \to \infty$. The Hardy-Littlewood-Sobolev inequality implies that

$$|x|^{-\mu} * (|u_n - u|^{2^*_\mu} - |u_n|^{2^*_\mu}) \to |x|^{-\mu} * |u|^{2^*_\mu} \tag{2.2}$$

in $L^{\frac{2N}{\mu}}(\mathbb{R}^N)$ as $n \to \infty$. On the other hand, we notice that

$$\begin{aligned} &\int_{\mathbb{R}^N} (|x|^{-\mu} * |u_n|^{2^*_\mu})|u_n|^{2^*_\mu} dx - \int_{\mathbb{R}^N} (|x|^{-\mu} * |u_n - u|^{2^*_\mu})|u_n - u|^{2^*_\mu} dx \\ &\quad = \int_{\mathbb{R}^N} (|x|^{-\mu} * (|u_n|^{2^*_\mu} - |u_n - u|^{2^*_\mu}))(|u_n|^{2^*_\mu} - |u_n - u|^{2^*_\mu}) dx \\ &\qquad + 2\int_{\mathbb{R}^N} (|x|^{-\mu} * (|u_n|^{2^*_\mu} - |u_n - u|^{2^*_\mu}))|u_n - u|^{2^*_\mu} dx. \end{aligned} \tag{2.3}$$

By Lemma 2.1, we have that

$$|u_n - u|^{2^*_\mu} \rightharpoonup 0 \tag{2.4}$$

in $L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N)$ as $n \to \infty$. From (2.1)–(2.4), we know that the result holds. □

Lemma 2.3. *Assume $N \geqslant 3$ and $0 < \mu < N$. Let*

$$\|\cdot\|_{NL} := \left(\int_\Omega \int_\Omega \frac{|\cdot|^{2^*_\mu} |\cdot|^{2^*_\mu}}{|x-y|^\mu} dx dy \right)^{\frac{1}{2 \cdot 2^*_\mu}}$$

and

$$X_{NL} := \{u : \Omega \to \mathbb{R}; \|u\|_{NL} < +\infty\}.$$

Then $\|\cdot\|_{NL}$ is a norm in X_{NL}. Moreover, under the norm $\|\cdot\|_{NL}$, X_{NL} is a Banach space.

Proof. By the semigroup property of the Riesz potential (see [31]), we obtain

$$\int_\Omega\int_\Omega \frac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy = \int_\Omega\left(\int_\Omega \frac{|u(y)|^{2_\mu^*}}{|x-y|^{\frac{N+\mu}{2}}}dy\right)^2 dx$$

for every $u\in H_0^1(\Omega)$. Then, by Minkowski's inequality, we know, for any $x\in\Omega$,

$$\begin{aligned}\left(\int_\Omega \frac{|u(y)+v(y)|^{2_\mu^*}}{|x-y|^{\frac{N+\mu}{2}}}dy\right)^2 &= \left(\int_\Omega \left|\frac{u(y)}{|x-y|^{\frac{N+\mu}{2}\cdot\frac{1}{2_\mu^*}}} + \frac{v(y)}{|x-y|^{\frac{N+\mu}{2}\cdot\frac{1}{2_\mu^*}}}\right|^{2_\mu^*}dy\right)^{\frac{1}{2_\mu^*}\cdot 2\cdot 2_\mu^*}\\ &\leqslant \left(\left(\int_\Omega \frac{|u(y)|^{2_\mu^*}}{|x-y|^{\frac{N+\mu}{2}}}dy\right)^{2\cdot\frac{1}{2\cdot 2_\mu^*}} + \left(\int_\Omega \frac{|v(y)|^{2_\mu^*}}{|x-y|^{\frac{N+\mu}{2}}}dy\right)^{2\cdot\frac{1}{2\cdot 2_\mu^*}}\right)^{2\cdot 2_\mu^*}.\end{aligned}$$

Notice that the integrals are non-negative and so, by Minkowski's inequality again, we have

$$\begin{aligned}&\left(\int_\Omega\left(\int_\Omega \frac{|u(y)+v(y)|^{2_\mu^*}}{|x-y|^{\frac{N+\mu}{2}}}dy\right)^2dx\right)^{\frac{1}{2\cdot 2_\mu^*}}\\ &\quad\leqslant \left(\int_\Omega\left(\int_\Omega \frac{|u(y)|^{2_\mu^*}}{|x-y|^{\frac{N+\mu}{2}}}dy\right)^2dx\right)^{\frac{1}{2\cdot 2_\mu^*}} + \left(\int_\Omega\left(\int_\Omega \frac{|v(y)|^{2_\mu^*}}{|x-y|^{\frac{N+\mu}{2}}}dy\right)^2dx\right)^{\frac{1}{2\cdot 2_\mu^*}},\end{aligned}$$

i.e.,

$$\|u+v\|_{NL}\leqslant \|u\|_{NL}+\|v\|_{NL}$$

for every $u,v\in L^{2^*}(\Omega)$. So, it is easy to verify that $\|\cdot\|_{NL}$ is a norm. The completeness of the space follows from a standard application of the monotone convergence theorem. □

Lemma 2.4. *Let $N\geqslant 3$, $0<\mu<N$ and $\lambda>0$. If $\{u_n\}$ is a $(\mathrm{PS})_c$ sequence of J_λ, then $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Let $u_0\in H_0^1(\Omega)$ be the weak limit of $\{u_n\}$. Then u_0 is a weak solution of* (1.2).

Proof. It is easy to see that there exists $C_1>0$ such that

$$|J_\lambda(u_n)|\leqslant C_1,\quad \left|\left\langle J_\lambda'(u_n),\frac{u_n}{\|u_n\|}\right\rangle\right|\leqslant C_1.$$

In order to prove $\{u_n\}$ is bounded in $H_0^1(\Omega)$, we consider the two cases: $0<\lambda<\lambda_1$ and $\lambda\in[\lambda_j,\lambda_{j+1})$ for some $j\in\mathbb{N}$ separately.

Case 1. $0<\lambda<\lambda_1$.

For n large enough, we have

$$\begin{aligned}C_1(1+\|u_n\|) &\geqslant J_\lambda(u_n)-\frac{1}{2\cdot 2_\mu^*}\langle J_\lambda'(u_n),u_n\rangle\\ &= \left(\frac12-\frac{1}{2\cdot 2_\mu^*}\right)(\|u_n\|^2-\lambda|u_n|_2^2)\\ &\geqslant \left(\frac12-\frac{1}{2\cdot 2_\mu^*}\right)\delta_1\|u_n\|^2\end{aligned}$$

for some $\delta_1>0$. Thus $\{u_n\}$ is bounded in $H_0^1(\Omega)$.

Case 2. $\lambda\in[\lambda_j,\lambda_{j+1})$ for some $j\in\mathbb{N}$.

Let $\beta\in(\frac{1}{2\cdot 2_\mu^*},\frac12)$. For n large enough, we have

$$\begin{aligned}C_1(1+\|u_n\|) &\geqslant J_\lambda(u_n)-\beta\langle J_\lambda'(u_n),u_n\rangle\\ &= \left(\frac12-\beta\right)(\|u_n\|^2-\lambda|u_n|_2^2)+\left(\beta-\frac{1}{2\cdot 2_\mu^*}\right)\|u_n\|_{NL}^{2\cdot 2_\mu^*}\\ &= \left(\frac12-\beta\right)(\|z_n\|^2+\|y_n\|^2-\lambda|z_n|_2^2-\lambda|y_n|_2^2)+\left(\beta-\frac{1}{2\cdot 2_\mu^*}\right)\|u_n\|_{NL}^{2\cdot 2_\mu^*}\\ &\geqslant \left(\frac12-\beta\right)(\delta_2\|z_n\|^2+(\lambda_1-\lambda)|y_n|_2^2)+\left(\beta-\frac{1}{2\cdot 2_\mu^*}\right)\|u_n\|_{NL}^{2\cdot 2_\mu^*}\end{aligned}$$

for some $\delta_2 > 0$, where $u_n = z_n + y_n$, $z_n \in \mathbb{E}_{j+1}$, $y_n \in \mathbb{Y}_j$, where $\mathbb{E}_{j+1}$ is defined in (1.9). It is then easy to verify that $\{u_n\}$ is bounded in $H_0^1(\Omega)$ using the fact that $\mathbb{Y}_j$ is finite dimensional and Lemma 2.3.

Since $H_0^1(\Omega)$ is reflexive, up to a subsequence, still denoted by u_n, there exists $u_0 \in H_0^1(\Omega)$ such that $u_n \rightharpoonup u_0$ in $H_0^1(\Omega)$ and $u_n \rightharpoonup u_0$ in $L^{2^*}(\Omega)$ as $n \to +\infty$. Then

$$|u_n|^{2_\mu^*} \rightharpoonup |u_0|^{2_\mu^*} \quad \text{in} \quad L^{\frac{2N}{2N-\mu}}(\Omega)$$

as $n \to +\infty$. By the Hardy-Littlewood-Sobolev inequality, the Riesz potential defines a linear continuous map from $L^{\frac{2N}{2N-\mu}}(\Omega)$ to $L^{\frac{2N}{\mu}}(\Omega)$, we know that

$$|x|^{-\mu} * |u_n|^{2_\mu^*} \rightharpoonup |x|^{-\mu} * |u_0|^{2_\mu^*} \quad \text{in} \quad L^{\frac{2N}{\mu}}(\Omega)$$

as $n \to +\infty$. Combining with the fact that

$$|u_n|^{2_\mu^*-2}u_n \rightharpoonup |u_0|^{2_\mu^*-2}u_0 \quad \text{in} \quad L^{\frac{2N}{N-\mu+2}}(\Omega)$$

as $n \to +\infty$, we have

$$(|x|^{-\mu} * |u_n|^{2_\mu^*})|u_n|^{2_\mu^*-2}u_n \rightharpoonup (|x|^{-\mu} * |u_0|^{2_\mu^*})|u_0|^{2_\mu^*-2}u_0 \quad \text{in} \quad L^{\frac{2N}{N+2}}(\Omega)$$

as $n \to +\infty$. Since, for any $\varphi \in H_0^1(\Omega)$,

$$0 \leftarrow \langle J_\lambda'(u_n), \varphi\rangle = \int_\Omega \nabla u_n \nabla \varphi dx - \lambda \int_\Omega u_n \varphi dx - \int_\Omega \int_\Omega \frac{|u_n(x)|^{2_\mu^*}|u_n(y)|^{2_\mu^*-2}u_n(y)\varphi(y)}{|x-y|^\mu} dxdy,$$

passing to the limit as $n \to +\infty$, we obtain

$$\int_\Omega \nabla u_0 \nabla \varphi dx - \lambda \int_\Omega u_0 \varphi dx - \int_\Omega \int_\Omega \frac{|u_0(x)|^{2_\mu^*}|u_0(y)|^{2_\mu^*-2}u_0(y)\varphi(y)}{|x-y|^\mu} dxdy = 0$$

for any $\varphi \in H_0^1(\Omega)$, which means u_0 is a weak solution of (1.2).

Finally, taking $\varphi = u_0 \in H_0^1(\Omega)$ as a test function in (1.2), we have

$$\int_\Omega |\nabla u_0|^2 dx = \lambda \int_\Omega u_0^2 dx + \int_\Omega \int_\Omega \frac{|u_0(x)|^{2_\mu^*}|u_0(y)|^{2_\mu^*}}{|x-y|^\mu} dxdy,$$

and so

$$J_\lambda(u_0) = \frac{N+2-\mu}{4N-2\mu} \int_\Omega \int_\Omega \frac{|u_0(x)|^{2_\mu^*}|u_0(y)|^{2_\mu^*}}{|x-y|^\mu} dxdy \geqslant 0.$$

This completes the proof. □

Lemma 2.5. *Let $N \geqslant 3$, $0 < \mu < N$ and $\lambda > 0$. If $\{u_n\}$ is a* $(\mathrm{PS})_c$ *sequence of J_λ with*

$$c < \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}, \tag{2.5}$$

then $\{u_n\}$ has a convergent subsequence.

Proof. Let u_0 be the weak limit of $\{u_n\}$ obtained in Lemma 2.4 and define $v_n := u_n - u_0$. Then we know $v_n \rightharpoonup 0$ in $H_0^1(\Omega)$ and $v_n \to 0$ a.e. in Ω. Moreover, by [7, the Brezis-Lieb lemma] and Lemma 2.2, we know

$$\int_\Omega |\nabla u_n|^2 dx = \int_\Omega |\nabla v_n|^2 dx + \int_\Omega |\nabla u_0|^2 dx + o_n(1)$$

and

$$\int_\Omega \int_\Omega \frac{|u_n(x)|^{2_\mu^*}|u_n(y)|^{2_\mu^*}}{|x-y|^\mu} dxdy = \int_\Omega \int_\Omega \frac{|v_n(x)|^{2_\mu^*}|v_n(y)|^{2_\mu^*}}{|x-y|^\mu} dxdy + \int_\Omega \int_\Omega \frac{|u_0(x)|^{2_\mu^*}|u_0(y)|^{2_\mu^*}}{|x-y|^\mu} dxdy + o_n(1).$$

Then, we have

$$
\begin{aligned}
c \leftarrow J_\lambda(u_n) &= \frac{1}{2}\int_\Omega |\nabla u_n|^2 dx - \frac{\lambda}{2}\int_\Omega u_n^2 dx - \frac{1}{2\cdot 2_\mu^*}\int_\Omega\int_\Omega \frac{|u_n(x)|^{2_\mu^*}|u_n(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy \\
&= \frac{1}{2}\int_\Omega |\nabla v_n|^2 dx + \frac{1}{2}\int_\Omega |\nabla u_0|^2 dx - \frac{\lambda}{2}\int_\Omega u_0^2 dx \\
&\quad - \frac{1}{2\cdot 2_\mu^*}\int_\Omega\int_\Omega \frac{|v_n(x)|^{2_\mu^*}|v_n(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy - \frac{1}{2\cdot 2_\mu^*}\int_\Omega\int_\Omega \frac{|u_0(x)|^{2_\mu^*}|u_0(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy + o_n(1) \\
&= J_\lambda(u_0) + \frac{1}{2}\int_\Omega |\nabla v_n|^2 dx - \frac{1}{2\cdot 2_\mu^*}\int_\Omega\int_\Omega \frac{|v_n(x)|^{2_\mu^*}|v_n(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy + o_n(1) \\
&\geqslant \frac{1}{2}\int_\Omega |\nabla v_n|^2 dx - \frac{1}{2\cdot 2_\mu^*}\int_\Omega\int_\Omega \frac{|v_n(x)|^{2_\mu^*}|v_n(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy + o_n(1),
\end{aligned} \tag{2.6}
$$

since $J_\lambda(u_0) \geqslant 0$ and $\int_\Omega u_n^2 dx \to \int_\Omega u_0^2 dx$, as $n \to +\infty$. Similarly, since $\langle J'_\lambda(u_0), u_0\rangle = 0$, we have

$$
\begin{aligned}
o_n(1) &= \langle J'_\lambda(u_n), u_n\rangle \\
&= \int_\Omega |\nabla u_n|^2 dx - \lambda\int_\Omega u_n^2 dx - \int_\Omega\int_\Omega \frac{|u_n(x)|^{2_\mu^*}|u_n(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy \\
&= \int_\Omega |\nabla v_n|^2 dx + \int_\Omega |\nabla u_0|^2 dx - \lambda\int_\Omega u_0^2 dx \\
&\quad - \int_\Omega\int_\Omega \frac{|v_n(x)|^{2_\mu^*}|v_n(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy - \int_\Omega\int_\Omega \frac{|u_0(x)|^{2_\mu^*}|u_0(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy + o_n(1) \\
&= \langle J'_\lambda(u_0), u_0\rangle + \int_\Omega |\nabla v_n|^2 dx - \int_\Omega\int_\Omega \frac{|v_n(x)|^{2_\mu^*}|v_n(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy + o_n(1) \\
&= \int_\Omega |\nabla v_n|^2 dx - \int_\Omega\int_\Omega \frac{|v_n(x)|^{2_\mu^*}|v_n(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy + o_n(1).
\end{aligned} \tag{2.7}
$$

From (2.7), we know there exists a non-negative constant b such that

$$\int_\Omega |\nabla v_n|^2 dx \to b$$

and

$$\int_\Omega\int_\Omega \frac{|v_n(x)|^{2_\mu^*}|v_n(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy \to b,$$

as $n \to +\infty$. From (2.6) and (2.7), we obtain

$$c \geqslant \frac{N+2-\mu}{4N-2\mu}b. \tag{2.8}$$

By the definition of the best constant $S_{H,L}$ in (1.6), we have

$$S_{H,L}\left(\int_\Omega\int_\Omega \frac{|v_n(x)|^{2_\mu^*}|v_n(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy\right)^{\frac{N-2}{2N-\mu}} \leqslant \int_\Omega |\nabla v_n|^2 dx,$$

which yields $b \geqslant S_{H,L} b^{\frac{N-2}{2N-\mu}}$. Thus we have either $b = 0$ or $b \geqslant S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}$. If $b = 0$, the proof is completed. Otherwise $b \geqslant S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}$, then we obtain from (2.8),

$$\frac{N+2-\mu}{4N-2\mu}S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} \leqslant \frac{N+2-\mu}{4N-2\mu}b \leqslant c,$$

which contradicts with the fact that

$$c < \frac{N+2-\mu}{4N-2\mu}S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}.$$

Thus $b=0$, and

$$\|u_n-u_0\|\to 0$$

as $n\to+\infty$. This ends the proof of Lemma 2.5. □

3 The case $N\geqslant 4$, $0<\lambda<\lambda_1$

We devote this section to proving Theorem 1.4 for the case $N\geqslant 4$ and $0<\lambda<\lambda_1$.

By Lemma 1.2, we know that $U(x)=\frac{[N(N-2)]^{\frac{N-2}{4}}}{(1+|x|^2)^{\frac{N-2}{2}}}$ is a minimizer for both S and $S_{H,L}$. Without loss of generality, we may assume that $0\in\Omega$ and $B_\delta\subset\Omega\subset B_{\kappa_0\delta}$ for some positive κ_0. Let $\psi\in C_0^\infty(\Omega)$ such that

$$\begin{cases}\psi(x)=\begin{cases}1, & \text{if} \quad x\in B_\delta,\\ 0, & \text{if} \quad x\in\mathbb{R}^N\setminus\Omega,\end{cases}\\ 0\leqslant\psi(x)\leqslant 1, \qquad \forall\, x\in\mathbb{R}^N,\\ |D\psi(x)|\leqslant C=\text{const}, \quad \forall\, x\in\mathbb{R}^N.\end{cases}$$

We define, for $\varepsilon>0$,

$$U_\varepsilon(x):=\varepsilon^{\frac{2-N}{2}}U\left(\frac{x}{\varepsilon}\right),$$
$$u_\varepsilon(x):=\psi(x)U_\varepsilon(x).$$

From [37, Lemma 1.46] and Lemma 1.2, we know

$$|\nabla U_\varepsilon|_2^2=|U_\varepsilon|_{2^*}^{2^*}=S^{\frac{N}{2}}, \tag{3.1}$$

and as $\varepsilon\to 0^+$,

$$\int_\Omega|\nabla u_\varepsilon|^2dx=S^{\frac{N}{2}}+O(\varepsilon^{N-2})=C(N,\mu)^{\frac{N-2}{2N-\mu}\cdot\frac{N}{2}}S_{H,L}^{\frac{N}{2}}+O(\varepsilon^{N-2}), \tag{3.2}$$

$$\int_\Omega|u_\varepsilon|^{2^*}dx=S^{\frac{N}{2}}+O(\varepsilon^N) \tag{3.3}$$

and

$$\int_\Omega|u_\varepsilon|^2dx\geqslant\begin{cases}d\varepsilon^2|\ln\varepsilon|+O(\varepsilon^2), & \text{if} \quad N=4,\\ d\varepsilon^2+O(\varepsilon^{N-2}), & \text{if} \quad N\geqslant 5,\end{cases} \tag{3.4}$$

where d is a positive constant.

Using the Hardy-Littlewood-Sobolev inequality, on one hand, we get

$$\begin{aligned}\left(\int_\Omega\int_\Omega\frac{|u_\varepsilon(x)|^{2^*_\mu}|u_\varepsilon(y)|^{2^*_\mu}}{|x-y|^\mu}dxdy\right)^{\frac{N-2}{2N-\mu}}&\leqslant C(N,\mu)^{\frac{N-2}{2N-\mu}}|u_\varepsilon|_{2^*}^2\\ &=C(N,\mu)^{\frac{N-2}{2N-\mu}}(S^{\frac{N}{2}}+O(\varepsilon^N))^{\frac{N-2}{N}}\\ &=C(N,\mu)^{\frac{N-2}{2N-\mu}}(C(N,\mu)^{\frac{N-2}{2N-\mu}\cdot\frac{N}{2}}S_{H,L}^{\frac{N}{2}}+O(\varepsilon^N))^{\frac{N-2}{N}}\\ &=C(N,\mu)^{\frac{N-2}{2N-\mu}\cdot\frac{N}{2}}S_{H,L}^{\frac{N-2}{2}}+O(\varepsilon^{N-2}).\end{aligned} \tag{3.5}$$

While on the other hand,

$$\begin{aligned}\int_\Omega\int_\Omega\frac{|u_\varepsilon(x)|^{2^*_\mu}|u_\varepsilon(y)|^{2^*_\mu}}{|x-y|^\mu}dxdy&\geqslant\int_{B_\delta}\int_{B_\delta}\frac{|u_\varepsilon(x)|^{2^*_\mu}|u_\varepsilon(y)|^{2^*_\mu}}{|x-y|^\mu}dxdy\\ &=\int_{B_\delta}\int_{B_\delta}\frac{|U_\varepsilon(x)|^{2^*_\mu}|U_\varepsilon(y)|^{2^*_\mu}}{|x-y|^\mu}dxdy\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|U_\varepsilon(x)|^{2_\mu^*}|U_\varepsilon(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy \\
&\quad - 2\int_{\mathbb{R}^N\setminus B_\delta}\int_{B_\delta} \frac{|U_\varepsilon(x)|^{2_\mu^*}|U_\varepsilon(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy \\
&\quad - \int_{\mathbb{R}^N\setminus B_\delta}\int_{\mathbb{R}^N\setminus B_\delta} \frac{|U_\varepsilon(x)|^{2_\mu^*}|U_\varepsilon(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy \\
&=: C(N,\mu)^{\frac{N}{2}} S_{H,L}^{\frac{2N-\mu}{2}} - 2\mathbb{D} - \mathbb{E},
\end{aligned} \tag{3.6}
$$

where

$$
\mathbb{D} = \int_{\mathbb{R}^N\setminus B_\delta}\int_{B_\delta} \frac{|U_\varepsilon(x)|^{2_\mu^*}|U_\varepsilon(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy
$$

and

$$
\mathbb{E} = \int_{\mathbb{R}^N\setminus B_\delta}\int_{\mathbb{R}^N\setminus B_\delta} \frac{|U_\varepsilon(x)|^{2_\mu^*}|U_\varepsilon(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy.
$$

By a direct computation, we know

$$
\begin{aligned}
\mathbb{D} &= \int_{\mathbb{R}^N\setminus B_\delta}\int_{B_\delta} \frac{\varepsilon^{\mu-2N}[N(N-2)]^{\frac{2N-\mu}{2}}}{(1+|\frac{x}{\varepsilon}|^2)^{\frac{2N-\mu}{2}}|x-y|^\mu(1+|\frac{y}{\varepsilon}|^2)^{\frac{2N-\mu}{2}}}dxdy \\
&= \varepsilon^{2N-\mu}[N(N-2)]^{\frac{2N-\mu}{2}} \int_{\mathbb{R}^N\setminus B_\delta}\int_{B_\delta} \frac{1}{(\varepsilon^2+|x|^2)^{\frac{2N-\mu}{2}}|x-y|^\mu(\varepsilon^2+|y|^2)^{\frac{2N-\mu}{2}}}dxdy \\
&\leqslant O(\varepsilon^{2N-\mu})\bigg(\int_{\mathbb{R}^N\setminus B_\delta} \frac{1}{(\varepsilon^2+|x|^2)^N}dx\bigg)^{\frac{2N-\mu}{2N}}\bigg(\int_{B_\delta}\frac{1}{(\varepsilon^2+|y|^2)^N}dy\bigg)^{\frac{2N-\mu}{2N}} \\
&\leqslant O(\varepsilon^{2N-\mu})\bigg(\int_{\mathbb{R}^N\setminus B_\delta} \frac{1}{|x|^{2N}}dx\bigg)^{\frac{2N-\mu}{2N}}\bigg(\int_0^\delta \frac{r^{N-1}}{(\varepsilon^2+r^2)^N}dr\bigg)^{\frac{2N-\mu}{2N}} \\
&= O(\varepsilon^{\frac{2N-\mu}{2}})\bigg(\int_0^{\frac{\delta}{\varepsilon}} \frac{z^{N-1}}{(1+z^2)^N}dz\bigg)^{\frac{2N-\mu}{2N}} \\
&\leqslant O(\varepsilon^{\frac{2N-\mu}{2}})\bigg(\int_0^{+\infty} \frac{z^{N-1}}{(1+z^2)^N}dz\bigg)^{\frac{2N-\mu}{2N}} \\
&= O(\varepsilon^{\frac{2N-\mu}{2}})
\end{aligned} \tag{3.7}
$$

and

$$
\begin{aligned}
\mathbb{E} &= \int_{\mathbb{R}^N\setminus B_\delta}\int_{\mathbb{R}^N\setminus B_\delta} \frac{\varepsilon^{\mu-2N}[N(N-2)]^{\frac{2N-\mu}{2}}}{(1+|\frac{x}{\varepsilon}|^2)^{\frac{2N-\mu}{2}}|x-y|^\mu(1+|\frac{y}{\varepsilon}|^2)^{\frac{2N-\mu}{2}}}dxdy \\
&= \varepsilon^{2N-\mu}[N(N-2)]^{\frac{2N-\mu}{2}} \int_{\mathbb{R}^N\setminus B_\delta}\int_{\mathbb{R}^N\setminus B_\delta} \frac{1}{(\varepsilon^2+|x|^2)^{\frac{2N-\mu}{2}}|x-y|^\mu(\varepsilon^2+|y|^2)^{\frac{2N-\mu}{2}}}dxdy \\
&\leqslant \varepsilon^{2N-\mu}[N(N-2)]^{\frac{2N-\mu}{2}} \int_{\mathbb{R}^N\setminus B_\delta}\int_{\mathbb{R}^N\setminus B_\delta} \frac{1}{|x|^{2N-\mu}|x-y|^\mu|y|^{2N-\mu}}dxdy \\
&= O(\varepsilon^{2N-\mu}).
\end{aligned} \tag{3.8}
$$

It follows from (3.6) to (3.8) that

$$
\begin{aligned}
\bigg(\int_\Omega\int_\Omega \frac{|u_\varepsilon(x)|^{2_\mu^*}|u_\varepsilon(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy\bigg)^{\frac{N-2}{2N-\mu}} &\geqslant \big(C(N,\mu)^{\frac{N}{2}} S_{H,L}^{\frac{2N-\mu}{2}} - O(\varepsilon^{\frac{2N-\mu}{2}}) - O(\varepsilon^{2N-\mu})\big)^{\frac{N-2}{2N-\mu}} \\
&= \big(C(N,\mu)^{\frac{N}{2}} S_{H,L}^{\frac{2N-\mu}{2}} - O(\varepsilon^{\frac{2N-\mu}{2}})\big)^{\frac{N-2}{2N-\mu}}.
\end{aligned} \tag{3.9}
$$

When $N=3$, (3.2) and (3.9) also hold.

Lemma 3.1. *If $N \geqslant 4$ and $\lambda > 0$, then, there exists $v \in H_0^1(\Omega)\backslash\{0\}$ such that*

$$\frac{|\nabla v|_2^2 - \lambda |v|_2^2}{\|v\|_{NL}^2} < S_{H,L}.$$

Proof. If $N = 4$, from (3.4), (3.2) and (3.9), we can obtain

$$\begin{aligned}
\frac{|\nabla u_\varepsilon|_2^2 - \lambda |u_\varepsilon|_2^2}{\|u_\varepsilon\|_{NL}^2} &\leqslant \frac{C(4,\mu)^{\frac{4}{8-\mu}} S_{H,L}^2 - \lambda d\varepsilon^2 |\ln \varepsilon| + O(\varepsilon^2)}{(C(4,\mu)^2 S_{H,L}^{\frac{8-\mu}{2}} - O(\varepsilon^{4-\frac{\mu}{2}}))^{\frac{2}{8-\mu}}} \\
&= S_{H,L} - \frac{\lambda d\varepsilon^2 |\ln \varepsilon|}{(C(4,\mu)^2 S_{H,L}^{\frac{8-\mu}{2}} - O(\varepsilon^{4-\frac{\mu}{2}}))^{\frac{2}{8-\mu}}} + O(\varepsilon^2) \\
&\leqslant S_{H,L} - \lambda d\varepsilon^2 |\ln \varepsilon| + O(\varepsilon^2) \\
&< S_{H,L}
\end{aligned} \tag{3.10}$$

for $\varepsilon > 0$ sufficiently small. Analogously, if $N \geqslant 5$, we have

$$\frac{|\nabla u_\varepsilon|_2^2 - \lambda |u_\varepsilon|_2^2}{\|u_\varepsilon\|_{NL}^2} < S_{H,L} \tag{3.11}$$

for $\varepsilon > 0$ sufficiently small. From the above arguments, we may take $v := u_\varepsilon$ with ε small enough and then the conclusion follows immediately. □

Lemma 3.2. *If $N \geqslant 3$ and $\lambda \in (0, \lambda_1)$, then, the functional J_λ satisfies the following properties*:

(i) *There exist $\alpha, \rho > 0$ such that $J_\lambda(u) \geqslant \alpha$ for $\|u\| = \rho$.*

(ii) *There exists $e \in H_0^1(\Omega)$ with $\|e\| > \rho$ such that $J_\lambda(e) < 0$.*

Proof. (i) By $\lambda \in (0, \lambda_1)$, the Sobolev embedding and the Hardy-Littlewood-Sobolev inequality, for all $u \in H_0^1(\Omega)\backslash \{0\}$ we have

$$\begin{aligned}
J_\lambda(u) &\geqslant \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{\lambda}{2\lambda_1}\int_\Omega |\nabla u|^2 dx - \frac{1}{22_\mu^*} C_1 |u|_{2^*}^{2(\frac{2N-\mu}{N-2})} \\
&\geqslant \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_1}\right)\|u\|^2 - \frac{1}{22_\mu^*} C_2 \|u\|^{2(\frac{2N-\mu}{N-2})}.
\end{aligned}$$

Since $2 < 2(\frac{2N-\mu}{N-2})$, we can choose some $\alpha, \rho > 0$ such that $J_\lambda(u) \geqslant \alpha$ for $\|u\| = \rho$.

(ii) For some $u_1 \in H_0^1(\Omega)\backslash \{0\}$, we have

$$J_\lambda(tu_1) = \frac{t^2}{2}\int_\Omega |\nabla u_1|^2 dx - \frac{\lambda t^2}{2}\int_\Omega u_1^2 dx - \frac{t^{2\cdot 2_\mu^*}}{2\cdot 2_\mu^*}\int_\Omega\int_\Omega \frac{|u_1(x)|^{2_\mu^*}|u_1(y)|^{2_\mu^*}}{|x-y|^\mu} dxdy < 0$$

for $t > 0$ large enough. Hence, we can take an $e := t_1 u_1$ for some $t_1 > 0$ and (ii) follows. □

Proposition 3.3. *By Lemma* 3.2 *and the mountain pass theorem without* (PS) *condition* (*see* [37]), *there exists a* (PS) *sequence $\{u_n\}$ such that $J_\lambda(u_n) \to c$ and $J'_\lambda(u_n) \to 0$ in $H_0^1(\Omega)^{-1}$ at the minimax level*

$$c^* = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_\lambda(\gamma(t)) > 0, \tag{3.12}$$

where

$$\Gamma := \{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = 0, J_\lambda(\gamma(1)) < 0\}.$$

Proof of Theorem 1.4. Case $N \geqslant 4$, $0 < \lambda < \lambda_1$. From Lemma 3.1, we know there exists $v \in H_0^1(\Omega)\backslash\{0\}$ such that

$$\frac{|\nabla v|_2^2 - \lambda |v|_2^2}{\|v\|_{NL}^2} < S_{H,L}.$$

Therefore,

$$
\begin{aligned}
0 < \max_{t\geqslant 0} J_\lambda(tv) &= \max_{t\geqslant 0}\left\{\frac{t^2}{2}\int_\Omega |\nabla v|^2dx - \frac{\lambda t^2}{2}\int_\Omega v^2dx - \frac{t^{2\cdot 2_\mu^*}}{2\cdot 2_\mu^*}\int_\Omega\int_\Omega \frac{|v(x)|^{2_\mu^*}|v(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy\right\} \\
&= \frac{N+2-\mu}{4N-2\mu}\left(\frac{|\nabla v|_2^2 - \lambda|v|_2^2}{\|v\|_{NL}^2}\right)^{\frac{2N-\mu}{N+2-\mu}} \\
&< \frac{N+2-\mu}{4N-2\mu}S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}.
\end{aligned}
$$

By the definition of c^*, we know $c^* < \frac{N+2-\mu}{4N-2\mu}S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}$. Let $\{u_n\}$ be the (PS) sequence obtained in Proposition 3.3. Applying Lemma 2.5, we know $\{u_n\}$ contains a convergent subsequence. In addition, we have J_λ has a critical value $c^* \in (0, \frac{N+2-\mu}{4N-2\mu}S_{H,L}^{\frac{2N-\mu}{N+2-\mu}})$ and (1.2) has a nontrivial solution. □

4 The case $N \geqslant 4$, $\lambda \geqslant \lambda_1$

We may suppose that $\lambda \in [\lambda_j, \lambda_{j+1})$ for some $j \in \mathbb{N}$, where λ_j is the j-th eigenvalue of $-\Delta$ on Ω with boundary condition $u = 0$. e_j is the j-th eigenfunctions corresponding to the eigenvalue λ_j.

Lemma 4.1. *If $N \geqslant 3$ and $\lambda \in [\lambda_j, \lambda_{j+1})$ for some $j \in \mathbb{N}$, then, the functional J_λ satisfies the following properties*:

(i) *There exist $\alpha, \rho > 0$ such that for any $u \in \mathbb{E}_{j+1}$ with $\|u\| = \rho$ it results that $J_\lambda(u) \geqslant \alpha$.*

(ii) *$J_\lambda(u) < 0$ for any $u \in \mathbb{Y}_j$.*

(iii) *Let $\mathbb{F}$ be a finite dimensional subspace of $H_0^1(\Omega)$. There exists $R > \rho$ such that for any $u \in \mathbb{F}$ with $\|u\| \geqslant R$ it results that $J_\lambda(u) \leqslant 0$.*

Proof. (i) Since $\lambda \in [\lambda_j, \lambda_{j+1})$, by the Sobolev embedding and the Hardy-Littlewood-Sobolev inequality, for all $u \in \mathbb{E}_{j+1} \setminus \{0\}$ we have

$$
\begin{aligned}
J_\lambda(u) &\geqslant \frac{1}{2}\int_\Omega |\nabla u|^2dx - \frac{\lambda}{2\lambda_{j+1}}\int_\Omega |\nabla u|^2dx - \frac{1}{22_\mu^*}C_1|u|_{2^*}^{2(\frac{2N-\mu}{N-2})} \\
&\geqslant \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_{j+1}}\right)\|u\|^2 - \frac{1}{22_\mu^*}C_2\|u\|^{2(\frac{2N-\mu}{N-2})}.
\end{aligned}
$$

Since $2 < 2(\frac{2N-\mu}{N-2})$, we can choose some $\alpha, \rho > 0$ such that $J_\lambda(u) \geqslant \alpha$ for $u \in \mathbb{E}_{j+1}$ with $\|u\| = \rho$.

(ii) Let $u \in \mathbb{Y}_j$, i.e., $u = \sum_{i=1}^j l_ie_i$, where $l_i \in \mathbb{R}, i = 1, \dots, j$. Since $\{e_i\}_{i\in\mathbb{N}}$ is an orthonormal basis of $L^2(\Omega)$ and $H_0^1(\Omega)$, we have

$$
\int_\Omega u^2dx = \sum_{i=1}^j l_i^2 \quad \text{and} \quad \int_\Omega |\nabla u|^2dx = \sum_{i=1}^j l_i^2|\nabla e_i|_2^2.
$$

Then, we get

$$
\begin{aligned}
J_\lambda(u) &= \frac{1}{2}\sum_{i=1}^j l_i^2(|\nabla e_i|_2^2 - \lambda) - \frac{1}{22_\mu^*}\int_\Omega\int_\Omega \frac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy \\
&< \frac{1}{2}\sum_{i=1}^j l_i^2(\lambda_i - \lambda) \\
&\leqslant 0,
\end{aligned}
$$

thanks to $\lambda_i \leqslant \lambda_j \leqslant \lambda$.

(iii) For $u \in \mathbb{F} \setminus \{0\}$, the non-negativity of λ gives

$$
\begin{aligned}
J_\lambda(u) &= \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}|u|_2^2 - \frac{1}{2\cdot 2_\mu^*}\|u\|_{NL}^{2\cdot 2_\mu^*} \\
&\leqslant \frac{1}{2}\|u\|^2 - \frac{1}{2\cdot 2_\mu^*}\|u\|_{NL}^{2\cdot 2_\mu^*} \\
&\leqslant \frac{1}{2}\|u\|^2 - \frac{C_1}{2\cdot 2_\mu^*}\|u\|^{2\cdot 2_\mu^*}
\end{aligned}
$$

for some positive constant C_1, since all norms on finite dimensional space are equivalent. So, $J_\lambda(u) \to -\infty$ as $\|u\| \to +\infty$. Hence, there exists $R > \rho$ such that for any $u \in \mathbb{F}$ with $\|u\| \geqslant R$ it results that $J_\lambda(u) \leqslant 0$ and (iii) follows. □

From Lemma 3.1, if $N \geqslant 4$ and $\lambda > 0$, then for ε small enough,

$$
\frac{|\nabla u_\varepsilon|_2^2 - \lambda |u_\varepsilon|_2^2}{\|u_\varepsilon\|_{NL}^2} < S_{H,L}.
$$

For any $j \in \mathbb{N}$, we define the linear space

$$
\mathbb{G}_{j,\varepsilon} := \operatorname{span}\{e_1, \ldots, e_j, u_\varepsilon\}
$$

and set

$$
m_{j,\varepsilon} := \max_{u \in \mathbb{G}_{j,\varepsilon}, \|u\|_{NL}=1} \left(\int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega |u|^2 dx \right),
$$

where $\|\cdot\|_{NL}$ is defined in Lemma 2.3.

Lemma 4.2. *If $N \geqslant 4$ and $\lambda \in [\lambda_j, \lambda_{j+1})$ for some $j \in \mathbb{N}$, then*

(i) *$m_{j,\varepsilon}$ is achieved at some $u_m \in \mathbb{G}_{j,\varepsilon}$ and u_m can be written as follows*:

$$
u_m = v + t u_\varepsilon
$$

with $v \in \mathbb{Y}_j$ and $t \geqslant 0$.

(ii) *The following estimate holds true*:

$$
m_{j,\varepsilon} \leqslant \begin{cases} (\lambda_j - \lambda)|v|_2^2, & \text{if} \quad t = 0, \\ (\lambda_j - \lambda)|v|_2^2 + A_\varepsilon(1 + |v|_2 O(\varepsilon^{\frac{N-2}{2}})) + O(\varepsilon^{\frac{N-2}{2}})|v|_2, & \text{if} \quad t > 0, \end{cases} \tag{4.1}
$$

as $\varepsilon \to 0$, where v is given in (i), *u_ε is given in Section* 3 *and*

$$
A_\varepsilon = \frac{|\nabla u_\varepsilon|_2^2 - \lambda |u_\varepsilon|_2^2}{\|u_\varepsilon\|_{NL}^2}. \tag{4.2}
$$

Proof. (i) Since $\mathbb{G}_{j,\varepsilon}$ is a finite dimensional space, m_ε is achieved at some $u_m \in \mathbb{G}_{j,\varepsilon}$, i.e.,

$$
m_{j,\varepsilon} = |\nabla u_m|_2^2 - \lambda |u_m|_2^2 \quad \text{and} \quad \|u_m\|_{NL} = 1.
$$

Obviously, $u_m \not\equiv 0$. From the definition of $\mathbb{G}_{j,\varepsilon}$ we have that

$$
u_m = v + t u_\varepsilon
$$

for some $v \in \mathbb{Y}_j$ and $t \in \mathbb{R}$. We can suppose that $t \geqslant 0$, otherwise, if $t < 0$ we can replace u_m with $-u_m$. The result follows.

(ii) If $t = 0$, then $u_m = v \in \mathbb{Y}_j$ and

$$
m_{j,\varepsilon} = |\nabla u_m|_2^2 - \lambda |u_m|_2^2 = |\nabla v|_2^2 - \lambda |v|_2^2 \leqslant (\lambda_j - \lambda)|v|_2^2.
$$

We consider the case $t > 0$. Since $e_1, \ldots, e_j \in L^{\infty}(\Omega)$, we also have $v \in L^{\infty}(\Omega)$. By a direct computation, we have

$$\begin{aligned}
&\int_{B_{\kappa_0\delta}}\int_{B_{\kappa_0\delta}}\frac{|u_\varepsilon(x)|^{2^*_\mu}|u_\varepsilon(y)|^{2^*_\mu-1}}{|x-y|^\mu}dxdy\\
&=\int_{B_{\kappa_0\delta}}\int_{B_{\kappa_0\delta}}\frac{|U_\varepsilon(x)|^{2^*_\mu}|U_\varepsilon(y)|^{2^*_\mu-1}}{|x-y|^\mu}dxdy\\
&=\varepsilon^{\frac{2\mu-3N-2}{2}}[N(N-2)]^{\frac{3N-2\mu+2}{4}}\int_{B_{\kappa_0\delta}}\int_{B_{\kappa_0\delta}}\frac{1}{(1+|\frac{x}{\varepsilon}|^2)^{\frac{2N-\mu}{2}}|x-y|^\mu(1+|\frac{y}{\varepsilon}|^2)^{\frac{N-\mu+2}{2}}}dxdy\\
&=\varepsilon^{\frac{2\mu-3N-2}{2}}[N(N-2)]^{\frac{3N-2\mu+2}{4}}\varepsilon^{2N-\mu}\int_{B_{\frac{\kappa_0\delta}{\varepsilon}}}\int_{B_{\frac{\kappa_0\delta}{\varepsilon}}}\frac{1}{(1+|x|^2)^{\frac{2N-\mu}{2}}|x-y|^\mu(1+|y|^2)^{\frac{N-\mu+2}{2}}}dxdy\\
&\leqslant O(\varepsilon^{\frac{N-2}{2}})\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{1}{(1+|x|^2)^{\frac{2N-\mu}{2}}|x-y|^\mu(1+|y|^2)^{\frac{N-\mu+2}{2}}}dxdy,
\end{aligned}$$

where κ_0 is given in Section 3. If $\mu > 1$, by the Hardy-Littlewood-Sobolev inequality, we have

$$\begin{aligned}
&\int_{B_{\kappa_0\delta}}\int_{B_{\kappa_0\delta}}\frac{|u_\varepsilon(x)|^{2^*_\mu}|u_\varepsilon(y)|^{2^*_\mu-1}}{|x-y|^\mu}dxdy\\
&\leqslant O(\varepsilon^{\frac{N-2}{2}})\bigg(\int_{\mathbb{R}^N}\bigg(\frac{1}{(1+|x|^2)^{\frac{2N-\mu}{2}}}\bigg)^{\frac{N}{N-1}}dx\bigg)^{\frac{N-1}{N}}\bigg(\int_{\mathbb{R}^N}\bigg(\frac{1}{(1+|x|^2)^{\frac{N-\mu+2}{2}}}\bigg)^{\frac{N}{N-\mu+1}}dx\bigg)^{\frac{N-\mu+1}{N}}\\
&=O(\varepsilon^{\frac{N-2}{2}}).
\end{aligned}$$

If $\mu \leqslant 1$, by the Hardy-Littlewood-Sobolev inequality again, we have

$$\begin{aligned}
&\int_{B_{\kappa_0\delta}}\int_{B_{\kappa_0\delta}}\frac{|u_\varepsilon(x)|^{2^*_\mu}|u_\varepsilon(y)|^{2^*_\mu-1}}{|x-y|^\mu}dxdy\\
&\leqslant O(\varepsilon^{\frac{N-2}{2}})\bigg(\int_{\mathbb{R}^N}\bigg(\frac{1}{(1+|x|^2)^{\frac{2N-\mu}{2}}}\bigg)^{\frac{2N}{2N-\mu}}dx\bigg)^{\frac{2N-\mu}{2N}}\bigg(\int_{\mathbb{R}^N}\bigg(\frac{1}{(1+|x|^2)^{\frac{N-\mu+2}{2}}}\bigg)^{\frac{2N}{2N-\mu}}dx\bigg)^{\frac{2N-\mu}{2N}}\\
&=O(\varepsilon^{\frac{N-2}{2}}).
\end{aligned}$$

Thus, we obtain

$$\int_\Omega\int_\Omega\frac{|u_\varepsilon(x)|^{2^*_\mu}|u_\varepsilon(y)|^{2^*_\mu-1}}{|x-y|^\mu}dxdy\leqslant O(\varepsilon^{\frac{N-2}{2}}).$$

On the other hand, by a direct computation, we have

$$\begin{aligned}
&\int_{B_\delta}\int_{B_\delta}\frac{|u_\varepsilon(x)|^{2^*_\mu}|u_\varepsilon(y)|^{2^*_\mu-1}}{|x-y|^\mu}dxdy\\
&=\int_{B_\delta}\int_{B_\delta}\frac{|U_\varepsilon(x)|^{2^*_\mu}|U_\varepsilon(y)|^{2^*_\mu-1}}{|x-y|^\mu}dxdy\\
&=\varepsilon^{\frac{2\mu-3N-2}{2}}[N(N-2)]^{\frac{3N-2\mu+2}{4}}\int_{B_\delta}\int_{B_\delta}\frac{1}{(1+|\frac{x}{\varepsilon}|^2)^{\frac{2N-\mu}{2}}|x-y|^\mu(1+|\frac{y}{\varepsilon}|^2)^{\frac{N-\mu+2}{2}}}dxdy\\
&=\varepsilon^{\frac{2\mu-3N-2}{2}}[N(N-2)]^{\frac{3N-2\mu+2}{4}}\varepsilon^{2N-\mu}\int_{B_{\frac{\delta}{\varepsilon}}}\int_{B_{\frac{\delta}{\varepsilon}}}\frac{1}{(1+|x|^2)^{\frac{2N-\mu}{2}}|x-y|^\mu(1+|y|^2)^{\frac{N-\mu+2}{2}}}dxdy\\
&\geqslant O(\varepsilon^{\frac{N-2}{2}})\int_{B_\delta}\int_{B_\delta}\frac{1}{(1+|x|^2)^{\frac{2N-\mu}{2}}|x-y|^\mu(1+|y|^2)^{\frac{N-\mu+2}{2}}}dxdy\\
&=O(\varepsilon^{\frac{N-2}{2}})
\end{aligned}$$

provided $\varepsilon < 1$ and so

$$\int_\Omega\int_\Omega\frac{|u_\varepsilon(x)|^{2^*_\mu}|u_\varepsilon(y)|^{2^*_\mu-1}}{|x-y|^\mu}dxdy\geqslant O(\varepsilon^{\frac{N-2}{2}}).$$

Then we can get

$$\int_\Omega\int_\Omega \frac{|u_\varepsilon(x)|^{2_\mu^*}|u_\varepsilon(y)|^{2_\mu^*-1}}{|x-y|^\mu}dxdy = O(\varepsilon^{\frac{N-2}{2}}).$$

By convexity, we obtain

$$\begin{aligned}
1 &= \int_\Omega\int_\Omega \frac{|u_m(x)|^{2_\mu^*}|u_m(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy \\
&= \int_\Omega\int_\Omega \frac{|v(x)+tu_\varepsilon(x)|^{2_\mu^*}|v(y)+tu_\varepsilon(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy \\
&\geqslant \int_\Omega\int_\Omega \frac{|tu_\varepsilon(x)|^{2_\mu^*}|tu_\varepsilon(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy + 2\cdot 2_\mu^*\int_\Omega\int_\Omega \frac{|tu_\varepsilon(x)|^{2_\mu^*-1}v(x)|tu_\varepsilon(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy \\
&\geqslant t^{2\cdot 2_\mu^*}\int_\Omega\int_\Omega \frac{|u_\varepsilon(x)|^{2_\mu^*}|u_\varepsilon(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy - 2\cdot 2_\mu^* t^{2\cdot 2_\mu^*-1}|v|_\infty \int_\Omega\int_\Omega \frac{|u_\varepsilon(x)|^{2_\mu^*-1}|u_\varepsilon(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy \\
&\geqslant t^{2\cdot 2_\mu^*}\int_\Omega\int_\Omega \frac{|u_\varepsilon(x)|^{2_\mu^*}|u_\varepsilon(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy - C_2 t^{2\cdot 2_\mu^*-1}|v|_2 O(\varepsilon^{\frac{N-2}{2}}),
\end{aligned} \tag{4.3}$$

where we used the fact that $\mathbb{Y}_j$ is a finite dimensional space and all norms on $\mathbb{Y}_j$ are equivalent. This implies that $t < C_3$ for some constant $C_3 > 0$. Taking (4.3) into account, we have

$$\int_\Omega\int_\Omega \frac{|tu_\varepsilon(x)|^{2_\mu^*}|tu_\varepsilon(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy \leqslant 1 + O(\varepsilon^{\frac{N-2}{2}})|v|_2.$$

By (4.2), one can see that

$$\begin{aligned}
m_{j,\varepsilon} &= \int_\Omega |\nabla(v+tu_\varepsilon)|^2dx - \lambda\int_\Omega |v+tu_\varepsilon|^2dx \\
&\leqslant (\lambda_j-\lambda)|v|_2^2 + A_\varepsilon\left(\int_\Omega\int_\Omega \frac{|tu_\varepsilon(x)|^{2_\mu^*}|tu_\varepsilon(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy\right)^{\frac{N-2}{2N-\mu}} + C_4|u_\varepsilon|_1|v|_2 \\
&\leqslant (\lambda_j-\lambda)|v|_2^2 + A_\varepsilon(1+|v|_2 O(\varepsilon^{\frac{N-2}{2}}))^{\frac{N-2}{2N-\mu}} + C_4|u_\varepsilon|_1|v|_2 \\
&\leqslant (\lambda_j-\lambda)|v|_2^2 + A_\varepsilon(1+|v|_2 O(\varepsilon^{\frac{N-2}{2}})) + O(\varepsilon^{\frac{N-2}{2}})|v|_2,
\end{aligned}$$

where we had used the estimate in [37, Lemma 2.25] that $|u_\varepsilon|_1 = O(\varepsilon^{\frac{N-2}{2}})$. □

Lemma 4.3. *If $N \geqslant 4$ and $\lambda \in (\lambda_j, \lambda_{j+1})$ for some $j \in \mathbb{N}$, then,*

$$\frac{|\nabla u|_2^2 - \lambda|u|_2^2}{\|u\|_{NL}^2} < S_{H,L}$$

for any $u \in \mathbb{G}_{j,\varepsilon}$.

Proof. We only need to check that

$$m_{j,\varepsilon} = \max_{u\in\mathbb{G}_{j,\varepsilon},\|u\|_{NL}=1}\left(\int_\Omega |\nabla u|^2dx - \lambda\int_\Omega |u|^2dx\right) < S_{H,L}.$$

If $t = 0$ in (4.1), by the choice of $\lambda \in (\lambda_j, \lambda_{j+1})$, we get that

$$m_{j,\varepsilon} \leqslant (\lambda_j - \lambda)|v|_2^2 < 0 < S_{H,L}.$$

Now we suppose that $t > 0$ and discuss the cases $N \geqslant 5$ and $N = 4$, separately.

If $N \geqslant 5$, we have

$$m_{j,\varepsilon} \leqslant (\lambda_j-\lambda)|v|_2^2 + \frac{|\nabla u_\varepsilon|_2^2 - \lambda|u_\varepsilon|_2^2}{\|u_\varepsilon\|_{NL}^2}(1+|v|_2 O(\varepsilon^{\frac{N-2}{2}})) + O(\varepsilon^{\frac{N-2}{2}})|v|_2$$

$$
\begin{aligned}
&\leqslant (\lambda_j-\lambda)|v|_2^2+\frac{C(N,\mu)^{\frac{N-2}{2N-\mu}\cdot\frac{N}{2}}S_{H,L}^{\frac{N}{2}}-\lambda d\varepsilon^2+O(\varepsilon^{N-2})}{(C(N,\mu)^{\frac{N}{2}}S_{H,L}^{\frac{2N-\mu}{2}}-O(\varepsilon^{N-\frac{\mu}{2}}))^{\frac{N-2}{2N-\mu}}}(1+|v|_2O(\varepsilon^{\frac{N-2}{2}}))+O(\varepsilon^{\frac{N-2}{2}})|v|_2\\
&\leqslant \left(S_{H,L}-\frac{\lambda d\varepsilon^2}{(C(N,\mu)^{\frac{N}{2}}S_{H,L}^{\frac{2N-\mu}{2}}-O(\varepsilon^{N-\frac{\mu}{2}}))^{\frac{N-2}{2N-\mu}}}+O(\varepsilon^{\frac{N}{2}})\right)\\
&\quad\times(1+|v|_2O(\varepsilon^{\frac{N-2}{2}}))+(\lambda_j-\lambda)|v|_2^2+O(\varepsilon^{\frac{N-2}{2}})|v|_2\\
&\leqslant S_{H,L}-\frac{\lambda d\varepsilon^2}{(C(N,\mu)^{\frac{N}{2}}S_{H,L}^{\frac{2N-\mu}{2}}-O(\varepsilon^{N-\frac{\mu}{2}}))^{\frac{N-2}{2N-\mu}}}+O(\varepsilon^{\frac{N}{2}})+(\lambda_j-\lambda)|v|_2^2+O(\varepsilon^{\frac{N-2}{2}})|v|_2
\end{aligned}
$$

for $\varepsilon>0$ sufficiently small. Since $\lambda\in(\lambda_j,\lambda_{j+1})$, we know

$$
(\lambda_j-\lambda)|v|_2^2+O(\varepsilon^{\frac{N-2}{2}})|v|_2\leqslant\frac{1}{4(\lambda_j-\lambda)}O(\varepsilon^{N-2})=O(\varepsilon^{N-2}). \tag{4.4}
$$

Therefore

$$
m_{j,\varepsilon}\leqslant S_{H,L}-\lambda d\varepsilon^2+O(\varepsilon^{\frac{N}{2}})<S_{H,L}
$$

for $\varepsilon>0$ sufficiently small.

If $N=4$, by (4.4), we have

$$
\begin{aligned}
m_{j,\varepsilon}&\leqslant(\lambda_j-\lambda)|v|_2^2+\frac{|\nabla u_\varepsilon|_2^2-\lambda|u_\varepsilon|_2^2}{\|u_\varepsilon\|_{NL}^2}(1+|v|_2O(\varepsilon))+O(\varepsilon)|v|_2\\
&\leqslant(\lambda_j-\lambda)|v|_2^2+\frac{C(4,\mu)^{\frac{4}{8-\mu}}S_{H,L}^2-\lambda d\varepsilon^2|\ln\varepsilon|+O(\varepsilon^2)}{(C(4,\mu)^2S_{H,L}^{\frac{8-\mu}{2}}-O(\varepsilon^{4-\frac{\mu}{2}}))^{\frac{2}{8-\mu}}}(1+|v|_2O(\varepsilon))+O(\varepsilon)|v|_2\\
&\leqslant\left(S_{H,L}-\frac{\lambda d\varepsilon^2|\ln\varepsilon|}{(C(4,\mu)^2S_{H,L}^{\frac{8-\mu}{2}}-O(\varepsilon^{4-\frac{\mu}{2}}))^{\frac{2}{8-\mu}}}+O(\varepsilon^2)\right)(1+|v|_2O(\varepsilon))+(\lambda_j-\lambda)|v|_2^2+O(\varepsilon)|v|_2\\
&\leqslant S_{H,L}-\frac{\lambda d\varepsilon^2|\ln\varepsilon|}{(C(4,\mu)^2S_{H,L}^{\frac{8-\mu}{2}}-O(\varepsilon^{4-\frac{\mu}{2}}))^{\frac{2}{8-\mu}}}+O(\varepsilon^2)+(\lambda_j-\lambda)|v|_2^2+O(\varepsilon)|v|_2\\
&\leqslant S_{H,L}-\lambda d\varepsilon^2|\ln\varepsilon|+O(\varepsilon^2)\\
&<S_{H,L}
\end{aligned}
$$

for $\varepsilon>0$ sufficiently small. The result follows. □

Proof of Theorem 1.4. Case $N\geqslant4$, $\lambda>\lambda_1$. From the definition of $\mathbb{G}_{j,\varepsilon}$, we know

$$
u_m=\overline{v}+tz_\varepsilon,
$$

where

$$
\overline{v}=v+t\sum_{i=1}^{j}\left(\int_\Omega u_\varepsilon e_i dx\right)e_i\in\mathbb{Y}_j
$$

and

$$
z_\varepsilon=u_\varepsilon-\sum_{i=1}^{j}\left(\int_\Omega u_\varepsilon e_i dx\right)e_i,
$$

so that $\overline{v}$ and z_ε are orthogonal in $L^2(\Omega)$. This implies that

$$
|u_m|_2^2=|\overline{v}|_2^2+t^2|z_\varepsilon|_2^2.
$$

Then,

$$
\mathbb{G}_{j,\varepsilon}=\mathbb{Y}_j\oplus\mathbb{R}z_\varepsilon.
$$

Applying Lemma 4.1, we know that J_λ satisfies the geometric structure of the linking theorem (see [30, Theorem 5.3]), i.e.,

$$\inf_{u\in\mathbb{E}_{j+1},\|u\|=\rho} J_\lambda(u) \geqslant \alpha > 0, \quad \sup_{u\in\mathbb{Y}_j} J_\lambda(u) < 0$$

and

$$\sup_{u\in\mathbb{G}_{j,\varepsilon},\|u\|\geqslant R} J_\lambda(u) \leqslant 0,$$

where α and R are as in Lemma 4.1. Define the linking critical level of J_λ, i.e.,

$$c^\star = \inf_{\gamma\in\Gamma}\max_{u\in V} J_\lambda(\gamma(u)) > 0, \tag{4.5}$$

where

$$\Gamma := \{\gamma \in C(\overline{V}, H_0^1(\Omega)) : \gamma = \mathrm{id} \ \text{on} \ \partial V\}$$

and

$$V := (\overline{B_R} \cap \mathbb{Y}_j) \oplus \{rz_\varepsilon : r \in (0,R)\}.$$

For any $\gamma \in \Gamma$, we have

$$c^\star \leqslant \max_{u\in V} J_\lambda(\gamma(u))$$

and in particular, if we take $\gamma = \mathrm{id}$ on $\overline{V}$, then

$$c^\star \leqslant \max_{u\in V} J_\lambda(u) \leqslant \max_{u\in\mathbb{G}_{j,\varepsilon}} J_\lambda(u).$$

Note that for any $u \in H_0^1(\Omega)\backslash\{0\}$,

$$\max_{t\geqslant 0} J_\lambda(tu) = \frac{N+2-\mu}{4N-2\mu}\left(\frac{|\nabla u|_2^2 - \lambda|u|_2^2}{\|u\|_{NL}^2}\right)^{\frac{2N-\mu}{N+2-\mu}}.$$

From the face that $\mathbb{G}_{j,\varepsilon}$ is a linear space we have

$$\max_{u\in\mathbb{G}_{j,\varepsilon}} J_\lambda(u) = \max_{u\in\mathbb{G}_{j,\varepsilon},t\neq 0} J_\lambda\left(|t|\frac{u}{|t|}\right) = \max_{u\in\mathbb{G}_{j,\varepsilon},t>0} J_\lambda(tu) \leqslant \max_{u\in\mathbb{G}_{j,\varepsilon},t\geqslant 0} J_\lambda(tu).$$

Thus, by Lemma 4.3, we have

$$\begin{aligned} c^\star &\leqslant \max_{u\in\mathbb{G}_{j,\varepsilon},t\geqslant 0} J_\lambda(tu) \\ &= \max_{u\in\mathbb{G}_{j,\varepsilon}} \frac{N+2-\mu}{4N-2\mu}\left(\frac{|\nabla u|_2^2 - \lambda|u|_2^2}{\|u\|_{NL}^2}\right)^{\frac{2N-\mu}{N+2-\mu}} \\ &< \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}. \end{aligned}$$

Therefore, the linking theorem and Lemma 2.5 yield that (1.2) admits a nontrivial solution $u \in H_0^1(\Omega)$ with critical value $c^\star \geqslant \alpha$. □

5 The case $N = 3$

In this section, we prove Theorem 1.4 for the case $N = 3$ by using the mountain pass theorem and the linking theorem. We still denote $\mathbb{F}$ to be a finite dimensional subspace of $H_0^1(\Omega)$ and

$$\mathbb{G}_{j,\varepsilon} := \mathrm{span}\{e_1, \ldots, e_j, u_\varepsilon\},$$

for any $j \in \mathbb{N}$.

Lemma 5.1. *Let $N=3$ and u_ε be as in Section 3. Then, there exists λ_* such that for any $\lambda>\lambda_*$,*

$$\frac{|\nabla u_\varepsilon|_2^2-\lambda|u_\varepsilon|_2^2}{\|u_\varepsilon\|_{NL}^2}<S_{H,L}$$

provided $\varepsilon>0$ is sufficiently small.

Proof. By the definition of u_ε, we can get

$$\int_\Omega |u_\varepsilon|^2dx \geqslant \int_{B_\delta}|U_\varepsilon|^2dx \geqslant C_0\varepsilon \tag{5.1}$$

for $\varepsilon>0$ sufficiently small. By (3.2), (3.9) and (5.1), we have

$$\begin{aligned}\frac{|\nabla u_\varepsilon|_2^2-\lambda|u_\varepsilon|_2^2}{\|u_\varepsilon\|_{NL}^2} &\leqslant \frac{C(3,\mu)^{\frac{1}{6-\mu}\cdot\frac{3}{2}}S_{H,L}^{\frac{3}{2}}-\lambda C_0\varepsilon+O(\varepsilon)}{(C(3,\mu)^{\frac{3}{2}}S_{H,L}^{\frac{6-\mu}{2}}-O(\varepsilon^{3-\frac{\mu}{2}}))^{\frac{1}{6-\mu}}}\\ &= S_{H,L}-\frac{(\lambda C_0-O(1))\varepsilon}{(C(3,\mu)^{\frac{3}{2}}S_{H,L}^{\frac{6-\mu}{2}}-O(\varepsilon^{3-\frac{\mu}{2}}))^{\frac{1}{6-\mu}}}\\ &< S_{H,L}\end{aligned}$$

if λ is large enough, say $\lambda>\lambda_*>0$, while $\varepsilon>0$ is sufficiently small. □

We show that J_λ has the geometric structure of the mountain pass theorem when $\lambda\in(0,\lambda_1)$ and the geometric structure of the linking theorem when $\lambda\in[\lambda_j,\lambda_{j+1})$ for some $j\in\mathbb{N}$.

We set

$$m_{j,\varepsilon}:=\max_{u\in\mathbb{G}_{j,\varepsilon},\|u\|_{NL}=1}\left(\int_\Omega|\nabla u|^2dx-\lambda\int_\Omega|u|^2dx\right).$$

Related to Lemma 4.2, we also have the corresponding result for $N=3$, so, we have the following lemma.

Lemma 5.2. *If $N=3$ and $\lambda\in[\lambda_j,\lambda_{j+1})$ for some $j\in\mathbb{N}$, then,*

(i) *m_ε is achieved in $u_m\in\mathbb{G}_{j,\varepsilon}$ and u_m can be written as follows*:

$$u_m=v+tu_\varepsilon$$

with $v\in\mathbb{Y}_j$ and $t\geqslant0$.

(ii) *The following estimate holds true*:

$$m_{j,\varepsilon}\leqslant\begin{cases}(\lambda_j-\lambda)|v|_2^2, & \textit{if}\quad t=0,\\ (\lambda_j-\lambda)|v|_2^2+A_\varepsilon(1+|v|_2O(\varepsilon^{\frac{1}{2}}))+O(\varepsilon^{\frac{1}{2}})|v|_2, & \textit{if}\quad t>0,\end{cases}\tag{5.2}$$

as $\varepsilon\to0$, where v is given in (i), *u_ε is given in Section 3 and*

$$A_\varepsilon=\frac{|\nabla u_\varepsilon|_2^2-\lambda|u_\varepsilon|_2^2}{\|u_\varepsilon\|_{NL}^2}.$$

Lemma 5.3. *If $N=3$, $\lambda\in(\lambda_j,\lambda_{j+1})$ for some $j\in\mathbb{N}$ and $\lambda>\lambda_*$, then,*

$$\frac{|\nabla u|_2^2-\lambda|u|_2^2}{\|u\|_{NL}^2}<S_{H,L}$$

for any $u\in\mathbb{G}_{j,\varepsilon}$.

Proof. If $t=0$ in (5.2), by the choice of $\lambda\in(\lambda_j,\lambda_{j+1})$, we get that

$$m_\varepsilon\leqslant(\lambda_j-\lambda)|v|_2^2\leqslant0<S_{H,L}.$$

When $t>0$, by (3.2), (3.9), (5.1) and Lemma 5.2, using the similar estimates to that in (4.4), we have

$$\begin{aligned}
m_{j,\varepsilon} &\leqslant (\lambda_j-\lambda)|v|_2^2+\frac{|\nabla u_\varepsilon|_2^2-\lambda|u_\varepsilon|_2^2}{\|u_\varepsilon\|_{NL}^2}(1+|v|_2O(\varepsilon^{\frac{1}{2}}))+O(\varepsilon^{\frac{1}{2}})|v|_2\\
&\leqslant (\lambda_j-\lambda)|v|_2^2+\frac{C(3,\mu)^{\frac{1}{6-\mu}\cdot\frac{3}{2}}S_{H,L}^{\frac{3}{2}}-\lambda C_0\varepsilon+O(\varepsilon)}{(C(3,\mu)^{\frac{3}{2}}S_{H,L}^{\frac{6-\mu}{2}}-O(\varepsilon^{3-\frac{\mu}{2}}))^{\frac{1}{6-\mu}}}(1+|v|_2O(\varepsilon^{\frac{1}{2}}))+O(\varepsilon^{\frac{1}{2}})|v|_2\\
&\leqslant \left(S_{H,L}-\frac{(\lambda C_0-O(1))\varepsilon}{(C(3,\mu)^{\frac{3}{2}}S_{H,L}^{\frac{6-\mu}{2}}-O(\varepsilon^{3-\frac{\mu}{2}}))^{\frac{1}{6-\mu}}}\right)(1+|v|_2O(\varepsilon^{\frac{1}{2}}))+(\lambda_j-\lambda)|v|_2^2+O(\varepsilon^{\frac{1}{2}})|v|_2\\
&\leqslant S_{H,L}-\frac{(\lambda C_0-O(1))\varepsilon}{(C(3,\mu)^{\frac{3}{2}}S_{H,L}^{\frac{6-\mu}{2}}-O(\varepsilon^{3-\frac{\mu}{2}}))^{\frac{1}{6-\mu}}}+(\lambda_j-\lambda)|v|_2^2+O(\varepsilon^{\frac{1}{2}})|v|_2\\
&\leqslant S_{H,L}-\lambda C_0\varepsilon+O(\varepsilon)\\
&< S_{H,L}
\end{aligned}$$

for $\varepsilon>0$ sufficiently small, since $\lambda>\lambda_*$ and $\lambda\in(\lambda_j,\lambda_{j+1})$. The result follows. □

Proof of Theorem 1.4. Case $N=3$. We consider the two cases: $\lambda_1>\lambda_*$ and $\lambda_1\leqslant\lambda_*$, separately.

Case 1. $\lambda_1>\lambda_*$.

For this case, we use the mountain pass theorem if $\lambda\in(\lambda_*,\lambda_1)$ while the linking theorem if $\lambda\in(\lambda_j,\lambda_{j+1})$ for some $j\in\mathbb{N}$.

If $\lambda\in(\lambda_*,\lambda_1)$, by Lemma 3.2 and the mountain pass theorem without the (PS) condition (see [37]), there exists a (PS) sequence $\{u_n\}$ such that $J_\lambda(u_n)\to c^*$ and $J'_\lambda(u_n)\to 0$ in $H_0^1(\Omega)^{-1}$ at the mountain pass level c^*. From Lemma 5.1, we have there exists $v\in H_0^1(\Omega)\backslash\{0\}$ such that

$$\frac{|\nabla v|_2^2-\lambda|v|_2^2}{\|v\|_{NL}^2}<S_{H,L}.$$

Thus,

$$\begin{aligned}
0<\max_{t\geqslant 0}J_\lambda(tv)&=\max_{t\geqslant 0}\left\{\frac{t^2}{2}\int_\Omega|\nabla v|^2dx-\frac{\lambda t^2}{2}\int_\Omega v^2dx-\frac{t^{2\cdot 2_\mu^*}}{2\cdot 2_\mu^*}\int_\Omega\int_\Omega\frac{|v(x)|^{2_\mu^*}|v(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy\right\}\\
&=\frac{5-\mu}{12-2\mu}\left(\frac{|\nabla v|_2^2-\lambda|v|_2^2}{\|v\|_{NL}^2}\right)^{\frac{6-\mu}{5-\mu}}\\
&<\frac{5-\mu}{12-2\mu}S_{H,L}^{\frac{6-\mu}{5-\mu}}.
\end{aligned}$$

By the definition of c, we know $c<\frac{5-\mu}{12-2\mu}S_{H,L}^{\frac{6-\mu}{5-\mu}}$.

From Lemma 2.5, we obtain $\{u_n\}$ contains a convergent subsequence. So, we have J_λ has a critical value $c^*\in(0,\frac{5-\mu}{12-2\mu}S_{H,L}^{\frac{6-\mu}{5-\mu}})$ and (1.2) has a nontrivial solution.

If $\lambda\in(\lambda_j,\lambda_{j+1})$ for some $j\in\mathbb{N}$, we define

$$z_\varepsilon=u_\varepsilon-\sum_{i=1}^n\left(\int_\Omega u_\varepsilon e_i dx\right)e_i,$$

then,

$$\mathbb{G}_{j,\varepsilon}=\mathbb{Y}_j\oplus\mathbb{R}u_\varepsilon=\mathbb{Y}_j\oplus\mathbb{R}z_\varepsilon.$$

By Lemma 4.1, we get that J_λ has the geometric structure required by the linking theorem (see [30, Theorem 5.3]). Thus we can define the linking critical level c_L of J_λ as in (4.5) and

$$c_L\leqslant\max_{u\in V}J_\lambda(u)\leqslant\max_{u\in\mathbb{G}_{j,\varepsilon}}J_\lambda(u).$$

On the other hand, we note that for any $u \in H_0^1(\Omega)\backslash\{0\}$,

$$\max_{t\geqslant 0} J_\lambda(tu) = \frac{5-\mu}{12-2\mu}\left(\frac{|\nabla u|_2^2 - \lambda|u|_2^2}{\|u\|_{NL}^2}\right)^{\frac{6-\mu}{5-\mu}}.$$

As the same arguments in Section 4, we have

$$\begin{aligned} c_L &\leqslant \max_{u\in\mathbb{G}_{j,\varepsilon}, t\geqslant 0} J_\lambda(tu) \\ &= \max_{u\in\mathbb{G}_{j,\varepsilon}} \frac{5-\mu}{12-2\mu}\left(\frac{|\nabla u|_2^2 - \lambda|u|_2^2}{\|u\|_{NL}^2}\right)^{\frac{6-\mu}{5-\mu}} \\ &< \frac{5-\mu}{12-2\mu} S_{H,L}^{\frac{6-\mu}{5-\mu}}. \end{aligned}$$

Therefore, the linking theorem and Lemma 2.5 yield that (1.2) admits a solution $u \in H_0^1(\Omega)$ with critical value $c_L \geqslant \alpha$. Since $\alpha > 0 = J_\lambda(0)$, we deduce that u is not identically zero.

Case 2. $\lambda_1 \leqslant \lambda_*$.

We only consider $\lambda \in (\lambda_j, \lambda_{j+1})$ for some $j \in \mathbb{N}$ and $\lambda > \lambda_*$. We can argue as in the last part of Case 1. In this way, we get that for any $\lambda > \lambda_*$ different from an eigenvalue of $-\Delta$, (1.2) admits a solution $u \in H_0^1(\Omega)$ with critical value $c_L \geqslant \alpha$ and u is not identically zero. □

6 Nonexistence

In this section, we discuss the nonexistence of solutions for (1.2) by using the Pohožaev identity. Firstly, we are going to show that the solutions for (1.2) possess some regularity which will be used to prove the Pohožaev identity.

Lemma 6.1. *If $N \geqslant 3$, $\lambda < 0$ and $u \in H^1(\Omega)$ solves* (1.2), *then $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$ for any $p \geqslant 1$.*

Proof. Denote by $H = K = |u|^{2_\mu^*-1} = |u|^{\frac{N-\mu+2}{N-2}}$. Then $H, K \in L^{\frac{2N}{N-\mu+2}}(\Omega)$. Using [25, Proposition 3.2], we know $u \in L^p(\Omega)$ for every $p \in [2, \frac{2N^2}{(N-\mu)(N-2)})$. Moreover, there exists a constant C_p independent of u such that

$$\left(\int_\Omega |u|^p dx\right)^{\frac{1}{p}} \leqslant C_p\left(\int_\Omega |u|^2 dx\right)^{\frac{1}{2}}.$$

Thus, $|u|^{2_\mu^*} \in L^q(\Omega)$ for every $q \in [\frac{2(N-2)}{2N-\mu}, \frac{2N^2}{(N-\mu)(2N-\mu)})$. Since $\frac{2(N-2)}{2N-\mu} < \frac{N}{N-\mu} < \frac{2N^2}{(N-\mu)(2N-\mu)}$, we have

$$\int_\Omega \frac{|u|^{2_\mu^*}}{|x-y|^\mu} dy \in L^\infty(\Omega),$$

and so

$$|-\Delta u - \lambda u| \leqslant C|u|^{\frac{N-\mu+2}{N-2}}.$$

By the classical bootstrap method for subcritical local problems in bounded domains, we deduce that $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$ for any $p \geqslant 1$. □

Proposition 6.2. *If $N \geqslant 3$, $\lambda < 0$ and $u \in H^1(\Omega)$ solves* (1.2), *then the following equality holds*:

$$\frac{1}{2}\int_{\partial\Omega}(x\cdot\nu)|\nabla u|^2 ds + \frac{N-2}{2}\int_\Omega |\nabla u|^2 dx = \frac{2N-\mu}{2\cdot 2_\mu^*}\int_\Omega\int_\Omega \frac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*}}{|x-y|^\mu} dxdy + \frac{\lambda N}{2}\int_\Omega |u|^2 dx,$$

where ν denotes the unit outward normal to $\partial\Omega$.

Proof. Since u is a solution of (1.2) and Lemma 6.1, u satisfies

$$-\Delta u = \left(\int_\Omega \frac{|u(y)|^{2_\mu^*}}{|x-y|^\mu} dy\right)|u|^{2_\mu^*-2}u + \lambda u. \tag{6.1}$$

Then

$$-\int_\Omega (x\cdot\nabla u)\Delta u dx=\int_\Omega (x\cdot\nabla u)\left(\int_\Omega \frac{|u(y)|^{2_\mu^*}}{|x-y|^\mu}dy\right)|u|^{2_\mu^*-1}dx+\lambda\int_\Omega (x\cdot\nabla u)udx. \tag{6.2}$$

Calculating the first term on the right-hand side, we know

$$\begin{aligned}
&\int_\Omega (x\cdot\nabla u(x))\left(\int_\Omega \frac{|u(y)|^{2_\mu^*}}{|x-y|^\mu}dy\right)|u(x)|^{2_\mu^*-1}dx\\
&=-\int_\Omega u(x)\nabla\left(x\int_\Omega \frac{|u(y)|^{2_\mu^*}}{|x-y|^\mu}dy|u(x)|^{2_\mu^*-1}\right)dx\\
&=-\int_\Omega u(x)\left(N\int_\Omega \frac{|u(y)|^{2_\mu^*}}{|x-y|^\mu}dy|u(x)|^{2_\mu^*-1}+(2_\mu^*-1)|u(x)|^{2_\mu^*-2}x\cdot\nabla u(x)\int_\Omega \frac{|u(y)|^{2_\mu^*}}{|x-y|^\mu}dy\right.\\
&\quad\left.+|u(x)|^{2_\mu^*-1}\int_\Omega (-\mu)x\cdot(x-y)\frac{|u(y)|^{2_\mu^*}}{|x-y|^{\mu+2}}dy\right)dx\\
&=-N\int_\Omega\int_\Omega \frac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy-(2_\mu^*-1)\int_\Omega x\cdot\nabla u(x)\int_\Omega \frac{|u(y)|^{2_\mu^*}}{|x-y|^\mu}dy|u(x)|^{2_\mu^*-1}dx\\
&\quad+\mu\int_\Omega\int_\Omega x\cdot(x-y)\frac{|u(y)|^{2_\mu^*}}{|x-y|^{\mu+2}}|u(x)|^{2_\mu^*}dydx.
\end{aligned}\tag{6.3}$$

This implies that

$$\begin{aligned}
&2_\mu^*\int_\Omega (x\cdot\nabla u(x))\left(\int_\Omega \frac{|u(y)|^{2_\mu^*}}{|x-y|^\mu}dy\right)|u(x)|^{2_\mu^*-1}dx\\
&\quad=-N\int_\Omega\int_\Omega \frac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy+\mu\int_\Omega\int_\Omega x\cdot(x-y)\frac{|u(y)|^{2_\mu^*}}{|x-y|^{\mu+2}}|u(x)|^{2_\mu^*}dydx,
\end{aligned}$$

similarly,

$$\begin{aligned}
&2_\mu^*\int_\Omega (y\cdot\nabla u(y))\left(\int_\Omega \frac{|u(x)|^{2_\mu^*}}{|x-y|^\mu}dx\right)|u(y)|^{2_\mu^*-1}dy\\
&\quad=-N\int_\Omega\int_\Omega \frac{|u(y)|^{2_\mu^*}|u(x)|^{2_\mu^*}}{|x-y|^\mu}dydx+\mu\int_\Omega\int_\Omega y\cdot(y-x)\frac{|u(x)|^{2_\mu^*}}{|x-y|^{\mu+2}}|u(y)|^{2_\mu^*}dxdy
\end{aligned}$$

and consequently, we get

$$\int_\Omega (x\cdot\nabla u(x))\left(\int_\Omega \frac{|u(y)|^{2_\mu^*}}{|x-y|^\mu}dy\right)|u(x)|^{2_\mu^*-1}dx=\frac{\mu-2N}{22_\mu^*}\int_\Omega\int_\Omega \frac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy. \tag{6.4}$$

Moreover, we already know that

$$\int_\Omega (x\cdot\nabla u)udx=-\frac{N}{2}\int_\Omega u^2dx \tag{6.5}$$

and

$$\int_{\partial\Omega}(x\cdot\nu)|\nabla u|^2ds=(2-N)\int_\Omega |\nabla u|^2dx+2\int_\Omega (x\cdot\nabla u)\Delta udx. \tag{6.6}$$

From the above equalities, we know the result holds. □

Using the Pohožaev identity obtained above, we can easily draw the following conclusion, the proof is standard and we omit it here.

Theorem 6.3. *If $N\geqslant 3$, $\lambda<0$ and $\Omega\neq\mathbb{R}^N$ is a smooth (possibly unbounded) domain in $\mathbb{R}^N$, which is strictly star-shaped with respect to the origin in $\mathbb{R}^N$, then any solution $u\in H_0^1(\Omega)$ of (1.2) is trivial.*

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