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Determinant formula and a realization for the Lie algebra $W(2,2)$

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Abstract In this paper, an explicit determinant formula is given for the Verma modules over the Lie algebra $W(2,2)$. We construct a natural realization of a certain vaccum module for the algebra $W(2,2)$ via the Weyl vertex algebra. We also describe several results including the irreducibility, characters and the descending filtrations of submodules for the Verma module over the algebra $W(2,2)$.

Keywords Lie algebra, Verma module, highest weight representation, $W(2,2)$ algebra, vertex algebra

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1 Introduction

The Lie algebra $W(2,2)$ was introduced by Zhang and Dong [21] in their classification of moonshine type vertex operator algebras generated by two weight 2 vectors. Note that this algebra has also appeared in the framework of the non-relativistic conformal field theory (see [3]), the BMS/GCA correspondence (see [5, 6, 16]), and two-dimensional statistical systems (see [11]).

The representation theory of the Lie algebra $W(2,2)$ has been investigated from many algebraic perspectives (see [1, 3, 13, 14, 17, 19, 21]). Although it can be regarded as an extension of the Virasoro algebra, the representation theory of the algebra $W(2,2)$ is different from that for the Virasoro algebra significantly. It is important to note that the maximal submodule of a Verma module is not necessarily generated by some singular vectors. Instead, the submodule may be associated with some subsingular vectors, these being vectors which become singular in an appropriate factor module (see [14, 17]).

As is known, the structure of the Verma modules for a Virasoro algebra is partly encoded in the determinant of its Shapovalov form [10, 12, 15]. The purpose of the present paper is to give an explicit determinant formula of the Shapovalov form on the Verma module over the algebra $W(2,2)$. We determine the zeros of the determinant in terms of a proper total ordering on the basis and compute their exponents. As a byproduct, we also fix a subtle gap appeared in [21], in which the Gram matrix A or A_n in the proofs of Theorems 2.1 and 2.4 is actually not an upper triangular matrix if n is even and $n \geqslant 4$. We also discuss the characters and filtrations of the Verma modules over the algebra $W(2,2)$.

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The free field realizations play an important role in studying representation theory of the Virasoro algebra (see [7–9, 18]). An interesting free field realization for the algebra $W(2,2)$ has been recently constructed with the twisted Heisenberg-Virasoro algebra at level zero (see [1,2]). The second aim of this paper is to give a direct realization of a certain vacuum module over the algebra $W(2,2)$ via the Weyl vertex algebra.

The paper is organized as follows. In Section 2, we briefly review the relevant results on representations of the algebra $W(2,2)$. In Section 3, we define a contravariant form on the Verma module and derive an explicit determinant formula for this form. In Section 4, we construct a natural realization of a certain vacuum module over the algebra $W(2,2)$ via the Weyl vertex algebra. Finally in Section 5, we give a discussion containing the research background, the conclusion and the related research work. Throughout the paper, $\mathbb{Z}_+$ denotes the set of non-negative integers.

2 The Lie algebra $W(2,2)$

Definition 2.1 (See [14,21]). The Lie algebra $W(2,2)$ is equipped with basis $\{L_n, W_n \mid n \in \mathbb{Z}\} \cup \{\boldsymbol{c}, \boldsymbol{k}\}$ and the following commutation relations:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}\boldsymbol{c},$$
$$[L_m, W_n] = (m-n)W_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}\boldsymbol{k},$$
$$[W_m, W_n] = 0,$$

where $m, n \in \mathbb{Z}$ and $\boldsymbol{c}$ and $\boldsymbol{k}$ are central elements of the Lie algebra $W(2,2)$.

The Lie algebra $W(2,2)$ contains the Virasoro algebra

$$\mathrm{Vir} = \bigoplus_{m\in\mathbb{Z}} \mathbb{C}L_m \oplus \mathbb{C}\boldsymbol{c}$$

as a subalgebra. It is clear that the algebra $W(2,2)$ has the $\mathbb{Z}$-grading and the triangular decomposition

$$W(2,2) = W(2,2)_- \oplus W(2,2)_0 \oplus W(2,2)_+,$$

where

$$W(2,2)_\pm = \bigoplus_{n>0} \mathbb{C}L_{\pm n} \oplus \bigoplus_{n>0} \mathbb{C}W_{\pm n}, \quad W(2,2)_0 = \mathbb{C}L_0 \oplus \mathbb{C}W_0 \oplus \mathbb{C}\boldsymbol{c} \oplus \mathbb{C}\boldsymbol{k}.$$

From the definition, we know that there is an anti-involution σ on the algebra $W(2,2)$ given by

$$\sigma(L_n) = L_{-n}, \quad \sigma(W_n) = W_{-n}, \quad \sigma(\boldsymbol{c}) = \boldsymbol{c}, \quad \sigma(\boldsymbol{k}) = \boldsymbol{k}$$

for $n \in \mathbb{Z}$.

Let $(c, h, \alpha, \beta) \in \mathbb{C}^4$. Consider $\mathbb{C}$ as a $W(2,2)_0$-module with

$$\boldsymbol{c}1 = c1, \quad L_0 1 = h1, \quad W_0 1 = \alpha 1, \quad \boldsymbol{k}1 = \beta 1.$$

Let $W(2,2)_+$ act trivially on $\mathbb{C}$, making $\mathbb{C}$ a $(W(2,2)_0 \oplus W(2,2)_-)$-module. The Verma module $M(c,h,\alpha,\beta)$ is defined by

$$M(c,h,\alpha,\beta) = U(W(2,2)) \otimes_{U(W(2,2)_0 \oplus W(2,2)_+)} \mathbb{C} \simeq U(W(2,2)_-)\mathbf{1},$$

where $\mathbf{1} = 1 \otimes 1$. It follows that $M(c,h,\alpha,\beta) = \bigoplus_{n\geqslant 0} M_n(c,h,\alpha,\beta)$, where

$$M_n(c,h,\alpha,\beta) = \{v \in M(c,h,\alpha,\beta) \mid L_0 v = (h+n)v\}.$$

It is clear that $M(c,h,\alpha,\beta)$ has a unique maximal submodule $J(c,h,\alpha,\beta)$ and the factor module

$$L(c,h,\alpha,\beta)=M(c,h,\alpha,\beta)/J(c,h,\alpha,\beta)$$

is an irreducible highest weight module.

Let $\langle\cdot,\cdot\rangle$ be a $\mathbb{C}$-valued bilinear form on $M(c,h,\alpha,\beta)$ defined by

$$\langle a\mathbf{1},b\mathbf{1}\rangle=\langle\mathbf{1},P(\sigma(a)b)\mathbf{1}\rangle,\quad \langle\mathbf{1},\mathbf{1}\rangle=1,\quad \text{for any}\quad a,b\in U(W(2,2)_{-}),$$

where $P: U(W(2,2))\to U(W(2,2)_0)$ is the Harish-Chandra projection, i.e., a projection along the decomposition

$$U(W(2,2))=U(W(2,2)_0)\oplus(W(2,2)_{-}U(W(2,2))+U(W(2,2))W(2,2)_{+}).$$

It follows that the form $\langle\cdot,\cdot\rangle$ is contragradient, in the sense that,

$$\langle L_m u,v\rangle=\langle u,L_{-m}v\rangle,\quad \langle W_m u,v\rangle=\langle u,W_{-m}v\rangle$$

for $m\in\mathbb{Z}$, $u,v\in M(c,h,\alpha,\beta)$. Since the distinct weight spaces of $M(c,h,\alpha,\beta)$ are orthogonal with respect to $\langle\cdot,\cdot\rangle$, it follows that the study of $\langle\cdot,\cdot\rangle$ on $M(c,h,\alpha,\beta)$ can be reduced to the study of the restrictions

$$\langle\cdot,\cdot\rangle_n: M_n(c,h,\alpha,\beta)\times M_n(c,h,\alpha,\beta)\to\mathbb{C}.$$

The Verma module $M(c,h,\alpha,\beta)$ is irreducible if and only if the forms $\langle\cdot,\cdot\rangle_n$ are nondegenerate for all $n\in\mathbb{Z}_+$.

3 Determinant formula

Recall that a partition of a positive integer n is a finite non-increasing sequence of positive integers $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_r)$ such that $n=\sum_{i=1}^r\lambda_i$. The λ_i is called the part of the partition λ. We call r the length of λ, denoted by $\ell(\lambda)$, and call the sum of λ_i's the weight of λ, denoted by $|\lambda|$. The number of partitions of n is given by the partition function $p(n)$. Denote by Λ the set of all partitions. Recall that the natural ordering on Λ is defined as follows:

$$\begin{aligned}&\lambda>\mu\Leftrightarrow\lambda_1=\mu_1,\ldots,\lambda_k=\mu_k,\quad \lambda_{k+1}>\mu_{k+1}\quad\text{for some}\quad k>0,\\&\lambda=\mu\Leftrightarrow\lambda_i=\mu_i\quad\text{for all}\quad i.\end{aligned}$$

For $n>0$, let

$$S_n=\{W_{-\lambda}L_{-\mu}\mid|\lambda|+|\mu|=n\text{ for any }\lambda,\mu\in\Lambda\},$$

where $W_{-\lambda}:=W_{-\lambda_1}\cdots W_{-\lambda_r}$ and $L_{-\mu}:=L_{-\mu_1}\cdots L_{-\mu_s}$.

From the Poincaré-Birkhoff-Witt (PBW) theorem, S_n is a basis of $U(W(2,2)_{-})$. Let

$$p_2(n)=\dim U(W(2,2)_{-})_n=|S_n|.$$

Then $p_2(n)$ is finite and can be counted by the generating series

$$\sum_{n=0}^{\infty}p_2(n)q^n=\prod_{k\geqslant1}(1-q^k)^{-2}.\tag{3.1}$$

In particular,

$$p_2(n)=\sum_{i=0}^{n}p(i)p(n-i).\tag{3.2}$$

From [21], we have the following total ordering $\succ$ on S_n:

$$W_{-\lambda}L_{-\mu} \succ W_{-\lambda'}L_{-\mu'} \Leftrightarrow |\lambda'| > |\lambda|, \quad \text{or} \quad |\lambda'| = |\lambda| \quad \text{and} \quad \lambda > \lambda', \quad \text{or} \quad \lambda = \lambda', \quad \text{and} \quad \mu' > \mu.$$

Let

$$S_n = \{b_i \mid b_i \succ b_j \text{ for } i < j\}, \quad \text{where} \quad b_i = W_{-\lambda^{(i)}}L_{-\mu^{(i)}},$$

with $\lambda^{(i)}, \mu^{(i)} \in \Lambda$ and $|\lambda^{(i)}| + |\mu^{(i)}| = n$ for $i = 1, 2, \ldots, p_2(n)$. Let

$$B_n = \{b_1\mathbf{1}, \ldots, b_{p_2(n)}\mathbf{1}\}.$$

Then B_n is a basis of $M_n(c, h, \alpha, \beta)$. Let $G_n = (g_{ij})$ be the Gram matrix of the form $\langle \cdot, \cdot \rangle$ defined by

$$g_{ij} = \langle b_i\mathbf{1}, b_j\mathbf{1} \rangle \quad \text{for} \quad i, j = 1, \ldots, p_2(n).$$

In order to compute the determinant of the Gram matrix G_n, we introduce a dual basis B_n^* of $M_n(c, h, \alpha, \beta)$ as follows: For $b_i = W_{-\lambda^{(i)}}L_{-\mu^{(i)}} \in S_n$, we define

$$b_i^* = W_{-\mu^{(i)}}L_{-\lambda^{(i)}}, \quad B_n^* = \{b_1^*\mathbf{1}, \ldots, b_{p_2(n)}^*\mathbf{1}\},$$

where $\lambda^{(i)}, \mu^{(i)} \in \Lambda$ and $|\lambda^{(i)}| + |\mu^{(i)}| = n$ for $i = 1, 2, \ldots, p_2(n)$.

Lemma 3.1. *Let $D_n = (d_{ij})$, where $d_{ij} = \langle b_i\mathbf{1}, b_j^*\mathbf{1} \rangle$ for $i, j = 1, \ldots, p_2(n)$. Then the matrix D_n is upper triangular.*

Proof. Let

$$b_p = W_{-\lambda^{(p)}}L_{-\mu^{(p)}}, \quad b_q = W_{-\lambda^{(q)}}L_{-\mu^{(q)}}, \quad b_p^* = W_{-\mu^{(p)}}L_{-\lambda^{(p)}}, \quad b_q^* = W_{-\mu^{(q)}}L_{-\lambda^{(q)}}.$$

Since $b_q \succ b_p$ for $p > q$ in S_n, we can discuss it into three different cases.

(1) If $|\lambda^{(p)}| > |\lambda^{(q)}|$, then $\langle W_{-\lambda^{(p)}}\mathbf{1}, L_{-\lambda^{(q)}}\mathbf{1} \rangle = 0$. It follows that

$$d_{pq} = \langle b_p\mathbf{1}, b_q^*\mathbf{1} \rangle = L_{\underline{\mu^{(p)}}}W_{\underline{\lambda^{(p)}}}W_{-\mu^{(q)}}L_{-\lambda^{(q)}}\mathbf{1} = L_{\underline{\mu^{(p)}}}W_{-\mu^{(q)}}W_{\underline{\lambda^{(p)}}}L_{-\lambda^{(q)}}\mathbf{1} = 0,$$

where $L_{\underline{\mu}} = L_{\mu_r}L_{\mu_{r-1}} \cdots L_{\mu_1}$ for $\mu = (\mu_1, \ldots, \mu_r) \in \Lambda$.

(2) If $|\lambda^{(p)}| = |\lambda^{(q)}|$ and $\lambda^{(p)} \succ \lambda^{(q)}$, then

$$d_{pq} = \langle b_p\mathbf{1}, b_q^*\mathbf{1} \rangle = L_{\underline{\mu^{(p)}}}W_{-\mu^{(q)}}W_{\lambda^{(p)}}L_{-\lambda^{(q)}}\mathbf{1} = 0.$$

(3) If $|\lambda^{(p)}| = |\lambda^{(q)}|$ and $\lambda^{(p)} = \lambda^{(q)}$, and $\mu^{(p)} \succ \mu^{(q)}$, then $W_{\mu^{(q)}}L_{-\mu^{(p)}}\mathbf{1} = 0$. It follows that

$$d_{pq} = \langle b_p\mathbf{1}, b_q^*\mathbf{1} \rangle = L_{\underline{\lambda^{(q)}}}W_{\underline{\mu^{(q)}}}W_{-\lambda^{(p)}}L_{-\mu^{(p)}}\mathbf{1} = L_{\underline{\lambda^{(q)}}}W_{-\lambda^{(p)}}W_{\mu^{(q)}}L_{-\mu^{(p)}}\mathbf{1} = 0.$$

Hence, $d_{pq} = 0$ for $p > q$, which implies that D_n is upper triangular. □

Corollary 3.2. *The following holds*:

$$\det G_n = \det R_n \prod_{p=1}^{p_2(n)} d_{pp}, \tag{3.3}$$

where

$$d_{pp} = \langle b_p\mathbf{1}, b_p^*\mathbf{1} \rangle = \langle W_{-\lambda^{(p)}}\mathbf{1}, L_{-\lambda^{(p)}}\mathbf{1} \rangle \langle L_{-\mu^{(p)}}\mathbf{1}, W_{-\mu^{(p)}}\mathbf{1} \rangle$$

for $p = 1, \ldots, p_2(n)$, and R_n is the basis transformation matrix from B_n^* to B_n given by

$$(b_1\mathbf{1}, \ldots, b_{p_2(n)}\mathbf{1}) = (b_1^*\mathbf{1}, \ldots, b_{p_2(n)}^*\mathbf{1})R_n.$$

Proof. It follows directly from $G_n = D_n R_n$ and Lemma 3.1. $\square$

Lemma 3.3. *The determinant* $\det G_n$ *is a polynomial in* α *of degree*

$$\sum_{\substack{r,s\in\mathbb{Z}_+\\ 1\leqslant rs\leqslant n}} p_2(n-rs). \tag{3.4}$$

Proof. The degree of α equals

$$\begin{aligned}\sum_{p=1}^{p_2(n)}(\ell(\lambda^{(p)})+\ell(\mu^{(p)})) &= \sum_{i=0}^{n} p(i) \sum_{\nu\in\Lambda,|\nu|=n-i}\ell(\nu)\\ &= \sum_{i=0}^{n} p(i) \sum_{\substack{r,s\in\mathbb{Z}_+\\ 1\leqslant rs\leqslant n-i}} p(n-i-rs)\\ &= \sum_{\substack{r,s\in\mathbb{Z}_+\\ 1\leqslant rs\leqslant n}} \sum_{i=0}^{n-rs} p(i)p(n-i-rs)\\ &= \sum_{\substack{r,s\in\mathbb{Z}_+\\ 1\leqslant rs\leqslant n}} p_2(n-rs),\end{aligned}$$

where we used the following identity:

$$\sum_{\lambda\in\Lambda_n,|\lambda|=n}\ell(\lambda) = \sum_{\substack{r,s\in\mathbb{Z}_+\\ 1\leqslant rs\leqslant n}} p(n-rs).$$

The proof is complete. $\square$

Recall a basic result from linear algebra.

Lemma 3.4 (See [12]). *Let* V *be a linear space of dimension* n*, and let* $\mathcal{A}(t) \in \mathrm{End}(V)[t]$*. Then* $\det \mathcal{A}(t)$ *is divisible by* t^k*, where* k *is the dimension of* $\ker \mathcal{A}(0)$*.*

Lemma 3.5. *The determinant* $\det G_n$ *is divisible by* $\phi_{r,s}^{p_2(n-rs)}$*, where*

$$\phi_{r,s} = \begin{cases}\phi_r\phi_s, & r\neq s,\\ \phi_r, & r=s,\end{cases} \qquad \phi_r = \alpha + \frac{r^2-1}{24}\beta, \tag{3.5}$$

for any $r,s\in\mathbb{Z}_+$ *satisfying* $1\leqslant rs\leqslant n$*.*

Proof. Let $J_n(c,h,\alpha,\beta) = J(c,h,\alpha,\beta)\cap M_n(c,h,\alpha,\beta)$. Then

$$J_n(c,h,\alpha,\beta) = \ker G_n = \ker\langle\cdot,\cdot\rangle_n.$$

It follows that $\det G_n(c,h,\alpha,\beta)=0$ if and only if $J_n(c,h,\alpha,\beta)\neq 0$.

Suppose that k is the smallest positive integer for which the determinant $\det G_k$ vanishes at $\alpha = h_0$. It follows that there exists $0\neq u\in J_k(c_L,\beta,h,h_0)$ such that $L_m u = 0$, $W_m u = 0$ for all $m>0$. Otherwise, we assume that $L_i u\neq 0$ or $W_i u\neq 0$ for some $i>0$. Then $\langle v, L_i u\rangle = \langle L_{-i}v,u\rangle = 0$ or $\langle v, W_i u\rangle = \langle W_{-i}v,u\rangle = 0$ for any $v\in M(c,h,\alpha,\beta)$ and $L_i u, W_i u\in M_{k-i}(c,h,\alpha,\beta)$; this contradicts the minimality of k. Then $\langle u\rangle$ is a submodule of $J(c,h,\alpha,\beta)$. The subspace $\langle u\rangle\cap M_n(c,h,\alpha,\beta)$ is spanned by the elements $W_{-\lambda}L_{-\mu}u$ where $\lambda,\mu\in\Lambda, |\lambda|+|\mu| = n-k$. These vectors are linearly independent, since $U(W(2,2))$ has no divisors of zero. Therefore, $J_n(c,h,\alpha,\beta)$ has a subspace of dimension $p_2(n-k)$. Then G_n has a kernel of at least dimension $p_2(n-k)$.

It follows from Lemma 3.4 that $\det G_n$ is divisible by $(\alpha-h_0)^{p_2(n-k)}$. Since $\det G_n$ has a zero at $\alpha = -\frac{r^2-1}{24}\beta$ for $r\in\mathbb{Z}_+$ satisfying $1\leqslant r\leqslant n$, the determinant $\det G_n$ is divisible by $\phi_r^{p_2(n-r)}$. From

$$p_2(n-rs)\leqslant \min\{p_2(n-r), p_2(n-s)\}$$

for any $r, s \in \mathbb{Z}_+$ satisfying $1 \leqslant rs \leqslant n$, we have $\det G_n$ is divisible by $\phi_{r,s}^{p_2(n-rs)}$ for any $r, s \in \mathbb{Z}_+$ satisfying $1 \leqslant rs \leqslant n$. Finally, with the degree and the coefficient of the highest power of α, we have

$$\det G_n = k_n \prod_{\substack{r,s\in\mathbb{Z}_+\\ 1\leqslant rs\leqslant n}} \phi_{r,s}^{p_2(n-rs)},$$

where k_n is a nonzero constant independent of c, h, α and β. □

Theorem 3.6. *For $n \in \mathbb{Z}_+$, the determinant of the Gram matrix of $\langle \cdot, \cdot \rangle_n$ has the form*

$$\det G_n = k_n \prod_{\substack{r,s\in\mathbb{Z}_+\\ 1\leqslant rs\leqslant n}} \phi_{r,s}^{p_2(n-rs)}, \tag{3.6}$$

where k_n is a nonzero constant independent of c, h, α, β and

$$\phi_{r,s} = \begin{cases} \phi_r\phi_s, & r \neq s, \\ \phi_r, & r = s, \end{cases} \qquad \phi_r = \alpha + \frac{r^2-1}{24}\beta.$$

As a corollary, we have the following corollary.

Corollary 3.7. *$M(c, h, \alpha, \beta)$ is irreducible if and only if*

$$2\alpha + \frac{m^2-1}{12}\beta \neq 0 \quad \text{for all} \quad m \in \mathbb{Z}_+.$$

Next, we recall and summarize the structure of the Verma module $M(c, h, \alpha, \beta)$.

Assume that $\beta \neq 0$ and $p = \sqrt{1 - \frac{24\alpha}{\beta}} \in \mathbb{Z}_+$. It follows from [14, Theorem 2.7] that the Verma module $M(c, h, \alpha, \beta)$ possesses a singular vector

$$u = S\mathbf{1} \in M_p(c, h, \alpha, \beta), \tag{3.7}$$

where $S = W_{-p} + Q(W)$ and $Q(W)$ is a polynomial of W_{-i} with $0 < i < p$. Let

$$h(r) = \frac{\alpha}{\beta}c + \frac{(13p+1)(p-1)}{12} + \frac{(1-r)p}{2} \tag{3.8}$$

for any $r \in \mathbb{Z}_+$. Using [14, Corollary 3.20] (see also [17, Theorem 3.1]), we have the following theorem.

Theorem 3.8. (1) *If $h \neq h(r)$ for all $r \in \mathbb{Z}_+$, then*

$$L(c, h, \alpha, \beta) = M(c, h, \alpha, \beta)/\langle u \rangle$$

and its character is given by

$$q^h(1-q^p)\prod_{k\geqslant 1}(1-q^k)^{-2}. \tag{3.9}$$

(2) *Assume $h = h(r)$ for some $r \in \mathbb{Z}_+$. Let*

$$T = L_{-p} + Q_1(W)L_{1-p} + \cdots + Q_{p-1}(W)L_{-1} + Q_p(W) \in U_p(W(2,2)_+).$$

Then

$$L(c, h, \alpha, \beta) = M(c, h, \alpha, \beta)/\langle v_r \rangle,$$

where

$$v_r = (T^r + Q_1(W)T^{r-1} + \cdots + Q_{r-1}(W)T + Q_r(W))\mathbf{1}, \tag{3.10}$$

where $Q_i(W) \in U(W_+)_{ip}$ and W_{-p} does not occur in $Q_i(W)$ for $i = 1, \ldots, r$. Here, this vector v_r is called the subsingular vector in $M(c, h, \alpha, \beta)$. Furthermore, the character of the irreducible module $L(c, h, \alpha, \beta)$ is given by

$$q^h(1-q^p)(1-q^{rp})\prod_{k\geqslant 1}(1-q^k)^{-2}. \tag{3.11}$$

Corollary 3.9. *If $2\alpha+\frac{p^2-1}{12}\beta=0$ for some $p\in\mathbb{Z}_+$, there is a singular vector u_n at level np in $M(c,h,\alpha,\beta)$ for non-negative integer n and the singular vector is*

$$u_n=S^n\mathbf{1}\in M_{np}(c,h,\alpha,\beta).$$

The submodule $\langle u_n\rangle$ generated by u_n is isomorphic to $M(c,h+np,\alpha,\beta)$, and there exits the following descending filtration:

$$M(c,h,\alpha,\beta)=\langle u_0\rangle\supset\langle u_1\rangle\supset\langle u_2\rangle\supset\langle u_3\rangle\supset\cdots \tag{3.12}$$

for the submodules of Verma module $M(c,h,\alpha,\beta)$.

(1) *If $h\neq h(r)$ for all $r\in\mathbb{Z}_+$, then the subquotient $\langle u_r\rangle/\langle u_{r+1}\rangle$ is the irreducible module $L(c,h+rp,\alpha,\beta)$ and $\langle u_1\rangle$ is the maximal submodule of $M(c,h,\alpha,\beta)$.*

(2) *If $h=h(2m)$ for some $m\in\mathbb{Z}_+$, there are subsingular vectors and the following descending filtration:*

$$\begin{aligned}M_0=\langle u_0\rangle\supset M_1=\langle v_{2m}\rangle\supset M_2=\langle u_1\rangle\supset M_3=\langle v_{2m-2}\rangle\supset M_4=\langle u_2\rangle\supset\cdots\\ \supset M_{2m-1}=\langle v_2\rangle\supset M_{2m}=\langle u_m\rangle\supset M_{2m+1}=\langle u_{m+1}\rangle\supset\cdots\end{aligned} \tag{3.13}$$

such that the subquotient M_i/M_{i+1} is the irreducible Verma module over the algebra $W(2,2)$ for any $i\in\mathbb{Z}_+$ and $M_1=\langle v_{2m}\rangle$ is the maximal submodule of $M(c,h,\alpha,\beta)$.

(3) *If $h=h(2m-1)$ for some $m\in\mathbb{Z}_+$, there are subsingular vectors and the following descending filtration:*

$$\begin{aligned}M_0=\langle u_0\rangle\supset M_1=\langle v_{2m-1}\rangle\supset M_2=\langle u_1\rangle\supset M_3=\langle v_{2m-3}\rangle\supset M_4=\langle u_2\rangle\supset\cdots\\ \supset M_{2m-1}=\langle v_1\rangle\supset M_{2m}=\langle u_m\rangle\supset M_{2m+1}=\langle u_{m+1}\rangle\supset\cdots\end{aligned} \tag{3.14}$$

such that the subquotient M_i/M_{i+1} is the irreducible Verma module over the algebra $W(2,2)$ for any $i\in\mathbb{Z}_+$ and $M_1=\langle v_{2m-1}\rangle$ is the maximal submodule of $M(c,h,\alpha,\beta)$.

Proof. The proof of (1) follows from the proof of Remark 3.22 in [17]. It suffices to give the proof of (2). The filtration in (2) is due to (3.12) and

$$h+ip=h(2m-2i)$$

for $i=0,\dots,m-1$. Clearly, by (1), these subquotients $\langle u_i\rangle/\langle v_{2m-2i}\rangle$ are the irreducible modules $L(c,h+ip,\alpha,\beta)$ and the subquotients $\langle u_k\rangle/\langle u_{k+1}\rangle$ are the irreducible modules $L(c,h+kp,\alpha,\beta)$ for $k\geqslant m$. By the proof of Corollary 3.19 in [17], we obtain that the subquotients $\langle v_{2m+2-2i}\rangle/\langle u_i\rangle$ are isomorphic to the irreducible modules $L(c,h+(2m+2-2i)p,\alpha,\beta)$, respectively. □

4 A realization of the vaccum module

For $(c,\beta)\in\mathbb{C}^2$ with $\beta\neq0$, it follows from Corollary 3.7 that the Verma module $M(c,0,0,\beta)$ is reducible. In particular, $(U(W(2,2))L_{-1}\mathbf{1}+U(W(2,2))W_{-1}\mathbf{1})$ is a proper submodule of $M(c,0,0,\beta)$. We call the factor module

$$V(c,\beta)=M(c,0,0,\beta)/(U(W(2,2))L_{-1}\mathbf{1}+U(W(2,2))W_{-1}\mathbf{1})$$

the vaccum module over the algebra $W(2,2)$. Furthermore, $V(c,\beta)$ is an irreducible $W(2,2)$-module.

Let the symbols x,x_1 and x_2 denote mutually commuting independent formal variables. We denote by $\mathrm{End}V(c,\beta)[[x,x^{-1}]]$ the vector space of (doubly infinite) formal Laurent series in x with coefficients in $\mathrm{End}V(c,\beta)$. We have the following generating functions:

$$L(x)=\sum_{n\in\mathbb{Z}}L_nx^{-n-2},\quad W(x)=\sum_{n\in\mathbb{Z}}W_nx^{-n-2}\in\mathrm{End}V(c,\beta)[[x,x^{-1}]].$$

Set $\bar{\mathbf{1}} = \mathbf{1} + (U(W(2,2))L_{-1}\mathbf{1} + U(W(2,2))W_{-1}\mathbf{1})$ and let

$$L = L_{-2}\bar{\mathbf{1}}, \quad W = W_{-2}\bar{\mathbf{1}} \in V(c,\beta).$$

It follows from [21] that there exists a vertex algebra structure on $V(c,\beta)$, which is uniquely determined by the condition that $\bar{\mathbf{1}}$ is the vacuum vector and

$$Y(L,x) = L(x), \quad Y(W,x) = W(x).$$

As a vertex algebra, $V(c,\beta)$ is generated by the subset $\{L, W\}$.

The Weyl algebra $\mathcal{W}$ has generators $a(n), a^*(n)$ $(n \in \mathbb{Z})$, and relations

$$[a(m), a^*(n)] = \delta_{m+n,0}, \quad [a^*(m), a^*(n)] = [a(m), a(n)] = 0$$

for $m, n \in \mathbb{Z}$. The simple $\mathcal{W}$-module $V_{\mathcal{W}}$ is generated by a vector $\mathbf{1}$ which satisfies

$$a(n)\mathbf{1} = 0, \quad n \geqslant 2, \quad a^*(n)\mathbf{1} = 0, \quad n > -2.$$

Set

$$a(x) = \sum_{n\in\mathbb{Z}} a(n)x^{-n+1}, \quad a^*(x) = \sum_{n\in\mathbb{Z}} a^*(n)x^{-n-2}.$$

Then

$$[a(x_1), a^*(x_2)] = x_1^{-1}\delta\left(\frac{x_2}{x_1}\right).$$

There exists a unique vertex algebra structure $(V_{\mathcal{W}}, Y, \mathbf{1})$ on $V_{\mathcal{W}}$ such that $\mathbf{1}$ is the vacuum vector and the vertex operator map for this vertex algebra structure is given by

$$Y(a(1)\mathbf{1}, x) = a(x), \quad Y(a^*(-2)\mathbf{1}, x) = a^*(x).$$

For $u, v \in V_{\mathcal{W}}$, we define the normal order of vertex operators as follows:

$$\circ\!\!\!\circ\, Y(u,x)Y(v,x) \,\circ\!\!\!\circ = Y^+(u,x)Y(v,x) + Y(v,x)Y^+(u,x),$$

where

$$Y(u,x) = \sum_{n\in\mathbb{Z}} u_n x^{-n-1} = Y^+(u,x) + Y^-(u,x) = \sum_{n\geqslant 0} u_n x^{-n-1} + \sum_{n<0} u_n x^{-n-1}.$$

Let

$$T(x) = -\,\circ\!\!\!\circ\, a(x)\partial_x a^*(x) \,\circ\!\!\!\circ - 2 \,\circ\!\!\!\circ\, a^*(x)\partial_x a(x) \,\circ\!\!\!\circ\, .$$

It follows from the well-known Feigh-fuchs construction (see [8, 9]) of the Virasoro algebra that

$$\begin{aligned}
[T(x_1), T(x_2)] &= \partial_{x_2}T(x_2)x_1^{-1}\delta\left(\frac{x_2}{x_1}\right) + 2T(x_2)\frac{\partial}{\partial x_2}x_1^{-1}\delta\left(\frac{x_2}{x_1}\right) + \frac{13}{6}\left(\frac{\partial}{\partial x_2}\right)^3 x_1^{-1}\delta\left(\frac{x_2}{x_1}\right),\\
[T(x_1), a(x_2)] &= \partial_{x_2}a(x_2)x_1^{-1}\delta\left(\frac{x_2}{x_1}\right) - a(x_2)\frac{\partial}{\partial x_2}x_1^{-1}\delta\left(\frac{x_2}{x_1}\right),\\
[T(x_1), a^*(x_2)] &= \partial_{x_2}a^*(x_2)x_1^{-1}\delta\left(\frac{x_2}{x_1}\right) + 2a^*(x_2)\frac{\partial}{\partial x_2}x_1^{-1}\delta\left(\frac{x_2}{x_1}\right).
\end{aligned}$$

Let

$$T(x) = \sum_{n\in\mathbb{Z}} T_n x^{-n-2}.$$

It follows that

$$\begin{aligned}
[T_m, T_n] &= (m-n)T_{m+n} + \frac{13}{6}(m^3 - m)\delta_{m+n,0},\\
[T_m, a_n] &= -(2m+n)a_{m+n},\\
[T_m, a_n^*] &= (m-n)a^*_{m+n}
\end{aligned}$$

for $m, n \in \mathbb{Z}$.

Theorem 4.1. *For $\beta \in \mathbb{C}$, there exists a homomorphism of vertex algebras*

$$\Phi : V(26,\beta) \to V_{\mathcal{W}}$$

uniquely determined by

$$L(x) \mapsto -\,{}^{\circ}_{\circ}\, a(x)\partial_x a^*(x)\, {}^{\circ}_{\circ} - 2\, {}^{\circ}_{\circ}\, a^*(x)\partial_x a(x)\, {}^{\circ}_{\circ} - \frac{\beta}{12}\partial_x^3 a(x),$$
$$W(x) \mapsto a^*(x).$$

Proof. Let

$$\begin{aligned}\tilde{L}(x) &= -\,{}^{\circ}_{\circ}\, a(x)\partial_x a^*(x)\, {}^{\circ}_{\circ} - 2\, {}^{\circ}_{\circ}\, a^*(x)\partial_x a(x)\, {}^{\circ}_{\circ} - \frac{\beta}{12}\partial_x^3 a(x)\\ &= T(x) - \frac{\beta}{12}\partial_x^3 a(x) = \sum_{n\in\mathbb{Z}} \tilde{L}_n x^{-n-2}.\end{aligned}$$

It follows that

$$\begin{aligned}[\tilde{L}_m, \tilde{L}_n] &= \left[T_m + \frac{1}{12}\beta(m^3-m)a_m, T_n + \frac{1}{12}\beta(n^3-n)a_n\right]\\ &= (m-n)T_{m+n} + \frac{13}{6}(m^3-m)\delta_{m+n,0} + \frac{1}{12}\beta(n^3-n)[T_m, a_n]\\ &\quad - \frac{1}{12}\beta(m^3-m)[L_n, a_m]\\ &= (m-n)T_{m+n} + \frac{13}{6}(m^3-m)\delta_{m+n,0} - \frac{1}{12}\beta(n^3-n)(2m+n)\\ &\quad + \frac{1}{12}\beta(m^3-m)(2n+m)\\ &= (m-n)\left(T_{m+n} + \frac{1}{12}\beta((m+n)^3-m-n)a_{m+n}\right) + \frac{13}{6}(m^3-m)\delta_{m+n,0},\end{aligned}$$

and

$$[\tilde{L}_m, a_n^*] = \left[T_m + \frac{1}{12}\beta(m^3-m)a_m, a_n^*\right] = (m-n)a_{m+n}^* + \frac{1}{12}\beta(m^3-m)\delta_{m+n,0}.$$

We have

$$\begin{aligned}[\tilde{L}(x_1), \tilde{L}(x_2)] &= \partial_{x_2}\tilde{L}(x_2)x_1^{-1}\delta\left(\frac{x_2}{x_1}\right) + \tilde{L}(x_2)\frac{\partial}{\partial x_2}x_1^{-1}\delta\left(\frac{x_2}{x_1}\right) + \frac{26}{12}\left(\frac{\partial}{\partial x_2}\right)^3 x_1^{-1}\delta\left(\frac{x_2}{x_1}\right),\\ [\tilde{L}(x_1), a^*(x_2)] &= \partial_{x_2}a^*(x_2)x_1^{-1}\delta\left(\frac{x_2}{x_1}\right) + a^*(x_2)\frac{\partial}{\partial x_2}x_1^{-1}\delta\left(\frac{x_2}{x_1}\right) + \frac{\beta}{12}\left(\frac{\partial}{\partial x_2}\right)^3 x_1^{-1}\delta\left(\frac{x_2}{x_1}\right),\end{aligned}$$

which implies that Φ is a $W(2,2)$-module homomorphism. Since $L(x)$ and $W(x)$ generate $V(26,\beta)$ as a vertex algebra, Φ is a homomorphism of vertex algebras. □

5 Discussion

In the recent years, great attention has been paid to the study of the representations of infinite-dimensional Virasoro type Lie algebras. The Lie algebra $W(2,2)$, which is also called the centrally extended BMS_3 algebra in physics, can be regarded as an extension of the Virasoro algebra. The induced representations, coadjont representations, and the characters associated with certain induced representations over the centrally extended BMS_3 have recently been studied in [5,6,16]. The associated $\mathfrak{bms}_3$ algebra is isomorphic to the infinite-dimensional extension of the Galilean conformal algebra in two dimensions, which is closely related to non-relativistic conformal symmetries (see [4]). A class of non-unitary representations of central extended GCA in 2D have been studied in some details (see [3]). The $\mathfrak{bms}_3$ algebra has also appeared in the framework of two-dimensional statistical systems in the form of an infinite-dimensional extension of $\mathfrak{alt}_1$ (see [11]) and turned out to be related to the classifications of vertex operator algebras in the form of the algebra $W(2,2)$. The representations of the algebra $W(2,2)$ have been investigated from a purely algebraic point of view (see [2, 13, 14, 17, 19–21]).

In this article, we give an explicit determinant formula of the contravariant form on the Verma module of the algebra $W(2,2)$. We also discuss the structure of the Verma module such as irreducibility, non-unitarity, singular vectors, characters and filtrations. Finally, we give a direct realization of certain vacuum module over the algebra $W(2,2)$ via the Weyl vertex algebra. Especially, the result related to the maximal submodule of the Verma module over the algebra $W(2,2)$ is partly different from that given by Radobolja [17]. We believe that these results provide valuable insights into the nature of $W(2,2)$-modules, free field realizations and some algebraic properties for the other Virasoro type Lie algebras.

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