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Kreĭn space representation and Lorentz groups of analytic Hilbert modules

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Abstract This paper aims to introduce some new ideas into the study of submodules in Hilbert spaces of analytic functions. The effort is laid out in the Hardy space over the bidisk $H^2(\mathbb{D}^2)$. A closed subspace M in $H^2(\mathbb{D}^2)$ is called a submodule if $z_iM \subset M$ $(i = 1, 2)$. An associated integral operator (*defect operator*) C_M captures much information about M. Using a Kreĭn space indefinite metric on the range of C_M, this paper gives a representation of M. Then it studies the group (called Lorentz group) of isometric self-maps of M with respect to the indefinite metric, and in finite rank case shows that the Lorentz group is a complete invariant for congruence relation. Furthermore, the Lorentz group contains an interesting abelian subgroup (called little Lorentz group) which turns out to be a finer invariant for M.

Keywords submodules, Kreĭn spaces, reproducing kernels, defect operators, Lorentz group, little Lorentz group

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1 Introduction

The book [4] by Douglas and Paulsen formulated an analytic framework for studying commuting operator tuples. In this framework, the tuple of multiplcation by coordinates $(M_{z_1}, M_{z_2}, \ldots, M_{z_n})$, as well as their restrictions to joint invariant subspaces (called submodules), serves as a model for a large class of commuting operator tuples. In recent years research in this framework has been one of the most active fronts in multivariable operator theory, with encouraging developments in Hardy spaces, Bergman spaces, Dirichlet spaces and Duray-Aveson spaces. A very notable success of this study, which started even before [4] (see [15]), is the theory on the Hardy space over the bidisk $H^2(\mathbb{D}^2)$. In this setting, a closed subspace M of $H^2(\mathbb{D}^2)$ is a submodule if it is invariant under multiplications by both coordinate functions z_1 and z_2 (or z and w), or equivalently it is invariant under multiplcation by functions in the algebra $H^\infty(\mathbb{D}^2)$. Submodules are two-variable counterparts of shift invariant subspaces in the classical

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Hardy space $H^2(\mathbb{D})$, but they have much more complicated structure. One important tool in studying submodules is the core operator defined by

$$C_M f(z) = \int_{\mathbb{T}^2} G^M(z,\lambda) f(\lambda) dm(\lambda), \quad z \in \mathbb{D}^2, \quad f \in H^2(\mathbb{D}^2)$$

in [10], where $G^M(z,\lambda) = \frac{k_\lambda^M(z)}{k_\lambda(z)}$ is the quotient of the reproducing kernel for M over the Szegö kernel for $H^2(\mathbb{D}^2)$. A motivation behind this definition is the belief that the quotient will balance out singularities of the reproducing kernels on the distinguished boundary $\mathbb{T}^2$, hence C_M may be a nice operator. Indeed, as it turns out, C^M is Hilbert-Schmidt in all known examples. Moreover, it unifies some key elements in prior studies and gives rise to a classification of submodules (see [23]). This paper is a step further along this line.

If we let P_M denote the orthogonal projection from $H^2(\mathbb{D}^2)$ onto M, then it can be shown that the core operator of M is equal to

$$C_M = P_M - T_{z_1} P_M T_{z_1}^* P_M - T_{z_2} P_M T_{z_2}^* P_M + T_{z_1 z_2} P_M T_{z_1 z_2}^* P_M.$$

Since $C_M = 0$ on the complement $H^2(\mathbb{D}^2) \ominus M$, we may simply restrict C_M to M and write

$$C_M = I - R_{z_1} R_{z_1}^* - R_{z_2} R_{z_2}^* + R_{z_1 z_2} R_{z_1 z_2}^*,$$

where R_f stands for the compression of Toeplitz operator T_f to M. For this reason, it is also called the defect operator for the pair (R_{z_1}, R_{z_2}). Clearly, C_M is self-adjoint, and it is not hard to check that it is a contraction. Furthermore, it is shown to be Hilbert-Schmidt for almost all submodules (see [19, 20]). The following formula is important for this paper:

$$C_M k_\lambda = \frac{k_\lambda^M}{k_\lambda}. \tag{1.1}$$

This formula can be verified as follows: Since $P_M T_{z_1}^* k_\lambda^M = \overline{\lambda_1} k_\lambda^M$, we have that

$$\begin{aligned} C_M k_\lambda &= (P_M - T_{z_1} P_M T_{z_1}^* P_M - T_{z_2} P_M T_{z_2}^* P_M + T_{z_1 z_2} P_M T_{z_1 z_2}^* P_M) k_\lambda \\ &= (I_M - R_{z_1} R_{z_1}^* - R_{z_2} R_{z_2}^* + R_{z_1 z_2} R_{z_1 z_2}^*) k_\lambda^M \\ &= (1 - \overline{\lambda_1} z_1 - \overline{\lambda_2} z_2 + \overline{\lambda_1 \lambda_2} z_1 z_2) k_\lambda^M \\ &= \frac{k_\lambda^M}{k_\lambda}. \end{aligned}$$

Another important fact for this paper is the following decomposition of compact defect operator (see [23]) on the orthogonal complement of $\ker C_M$:

$$C_M = \begin{pmatrix} I_n & & & \\ & D & & \\ & & -I_{n-1} & \\ & & & -D \end{pmatrix}, \tag{1.2}$$

where I_k stands for the identity matrix of size $k \times k$, and D is an injective positive pure contraction. Clearly, if C_M is finite rank, then $\mathrm{rank} C_M$ is odd. This fact will be used in several places in the paper.

The paper is organized into two parts. Since, for a fixed λ, $1/k_\lambda$ is invertible in the algebra $H^\infty(\mathbb{D}^2)$, (1.1) implies that the range of C_M generates M. The first part (see Sections 2–5) of this paper takes a closer look at this fact and gives a representation of M through a Kreĭn space constructed from C_M, and we will show that for every submodule M with $\mathrm{rank}\, C_M = N < \infty$ there exist Kreĭn space operator D and N functions $\varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_{N-n}$ in M (these might be unbounded) such that

$$P_M = DD^\sharp = \sum_{1 \leqslant j \leqslant n} T_{\varphi_j} T_{\varphi_j}^* - \sum_{1 \leqslant k \leqslant N-n} T_{\psi_k} T_{\psi_k}^*$$

on span$\{k_\lambda(z) : \lambda \in \mathbb{D}^2\}$. We would like to emphasize that this equality may be viewed as a bidisk version of Beurling's theorem, since in $H^2(\mathbb{D})$, $M = \theta H^2(\mathbb{D})$ for some inner function θ by Beurling's theorem (see [3]) and one checks that $P_M = T_\theta T_\theta^*$.

In the second part (Sections 6–9) of this paper, we consider the group $\mathcal{G}(M)$ of invertible operators on M that preserve an indefinite metric for the Kreĭn space. To be precise, it is the collection of invertible operators T acting on M such that $T^* C_M T = C_M$. It is called Lorentz group of M because of its resemblence to the classical Lorentz group on Minkowski space-time. A classification of submodules is given in [23] based on *congruence*. It will be shown that, when the defect operators are finite rank, two submodules are congruent if and only if their associated Lorentz groups are isomorphic. In other words, submodules can in fact be classified by their Lorentz groups. Congruence (or the Lorentz group) is a coarse invariant for submodules. In an attempt to introduce a finer invariant to submodules, we define the little Lorentz group $\mathcal{G}_0(M)$, which is the set of invertible elements f in the algebra $H^\infty(\mathbb{D}^2)$ such that $R_f^* C_M R_f = C_M$. Since the map $f \to R_f$ is an embedding of $\mathcal{G}_0(M)$ into $\mathcal{G}(M)$, the former can be viewed as a subgroup of the latter. $\mathcal{G}_0(M)$ is apparently abelian, and it is nontrivial for every submodule. Moreover, it is invariant under unitary equivalence. In fact, if two submodules are unitarily equivalent then their little Lorentz groups are identical. The converse, however, is not true. Little Lorentz group is computed in some well-known examples. In Section 10, we give the concluding remarks.

Kreĭn spaces and Lorentz groups are new approaches in the study of submodules. In particular, it brings group theory into the vista. Although this paper is written in the setting of $H^2(\mathbb{D}^2)$, the ideas can be worked out as well in many other spaces of analytic functions, and it would be interesting to study how the Lorentz groups vary with respect to the change of settings.

2 Kreĭn space $\mathcal{K} \otimes H^2$ and operator D

Let C_M be the defect operator of a submodule M in H^2 (short for $H^2(\mathbb{D}^2)$). We consider the Jordan decomposition $C_M = C_+ - C_-$ of C_M. Now, we set $\mathcal{H}_\pm = \overline{\mathrm{ran}} C_\pm^{1/2}$ and $\mathcal{K} = \mathcal{H}_+ \oplus \mathcal{H}_-$. In the case, C_M has finite rank, and $\mathcal{K}$ is simply the range of C_M. We shall introduce an indefinite inner product on $\mathcal{K}$ defined as follows:

$$\left\langle \begin{pmatrix} u_+ \\ u_- \end{pmatrix}, \begin{pmatrix} v_+ \\ v_- \end{pmatrix} \right\rangle_{\mathcal{K}} := \langle u_+, v_+\rangle - \langle u_-, v_-\rangle, \quad \text{where} \quad \begin{pmatrix} u_+ \\ u_- \end{pmatrix}, \begin{pmatrix} v_+ \\ v_- \end{pmatrix} \in \mathcal{H}_+ \oplus \mathcal{H}_-.$$

Kreĭn space $\mathcal{K} \otimes H^2$ will play an important role in our study. One of standard references on the theory of Kreĭn spaces will be [7].

Lemma 2.1. *For any F in $\mathcal{K} \otimes H^2$, let*

$$F = \sum_{i,j \geqslant 0} \begin{pmatrix} u_{ij} \\ v_{ij} \end{pmatrix} \otimes z_1^i z_2^j$$

be the Taylor expansion of F. Then $\sum_{i,j\geqslant 0}((C_+^{1/2} u_{ij})(\lambda) - (C_-^{1/2} v_{ij})(\lambda))\lambda_1^i \lambda_2^j$ $(\lambda = (\lambda_1, \lambda_2) \in \mathbb{D}^2)$ converges uniformly on any compact subset in $\mathbb{D}^2$.

Proof. Since $\{\|u_{ij}\|\}_{i,j}$ is bounded and

$$\begin{aligned} \sum_{i,j} |(C_+^{1/2} u_{ij})(\lambda)\lambda_1^i \lambda_2^j| &= \sum_{i,j} |\langle C_+^{1/2} u_{ij}, k_\lambda^M\rangle \lambda_1^i \lambda_2^j| \\ &\leqslant \|k_\lambda^M\| \sum_{i,j} \|u_{ij}\| |\lambda_1|^i |\lambda_2|^j \\ &< \infty, \end{aligned}$$

$\sum_{i,j\geqslant 0}(C_+^{1/2} u_{ij})(\lambda)\lambda_1^i \lambda_2^j$ converges uniformly on any compact subset in $\mathbb{D}^2$. This concludes the proof. □

The following mapping is the key of our discussion: Define

$$D : F = \sum_{i,j\geqslant 0} \begin{pmatrix} u_{ij} \\ v_{ij} \end{pmatrix} \otimes z_1^i z_2^j \mapsto \sum_{i,j\geqslant 0} (C_+^{1/2} u_{ij} - C_-^{1/2} v_{ij}) z_1^i z_2^j.$$

By Lemma 2.1, D is well-defined as a linear mapping from $\mathcal{K} \otimes H^2$ to $\mathrm{Hol}(\mathbb{D}^2)$, the set of all holomorphic functions on $\mathbb{D}^2$. The following formula is useful.

Lemma 2.2. *It holds that*

$$D \begin{pmatrix} u \\ v \end{pmatrix} \otimes f = (C_+^{1/2} u - C_-^{1/2} v) f.$$

Proof. Trivially $(C_+^{1/2} u - C_-^{1/2} v) f$ belongs to $\mathrm{Hol}(\mathbb{D}^2)$. Writing $f = \sum_{i,j\geqslant 0} c_{ij} z_1^i z_2^j$, by Lemma 2.1 we have that

$$\begin{aligned} D \begin{pmatrix} u \\ v \end{pmatrix} \otimes f &= D \sum_{i,j\geqslant 0} \begin{pmatrix} u \\ v \end{pmatrix} \otimes c_{ij} z_1^i z_2^j \\ &= \sum_{i,j\geqslant 0} (C_+^{1/2} u - C_-^{1/2} v) c_{ij} z_1^i z_2^j \\ &= (C_+^{1/2} u - C_-^{1/2} v) \sum_{i,j\geqslant 0} c_{ij} z_1^i z_2^j \\ &= (C_+^{1/2} u - C_-^{1/2} v) f. \end{aligned}$$

Thus we have the desired identity. □

Before we state results, let us take a look at the idea of D in the classical setting $H^2(\mathbb{D})$. This shall explain why it is of importance. By Beurling's theorem, every shift invariant subspace M is of the form $\theta H^2(\mathbb{D})$. The corresponding defect operator $C_M = P_M - T_z P_M T_z^* P_M = \theta \otimes \theta$. Hence, $C_M = C_+$ and $\mathcal{H}_+ = \theta$. So the map $D : \theta \otimes H^2(\mathbb{D}) \to \theta H^2(\mathbb{D})$ is simply the multiplication $D(\theta \otimes f) = \theta f$.

However, in the setting of $H^2(\mathbb{D}^2)$, D may not be bounded, so we will deal with it as an unbounded operator with domain $\mathrm{dom}\, D = \{F \in \mathcal{K} \otimes H^2 : DF \in M\}$. Observe that $\mathrm{dom} D$ is a module over $H^\infty(\mathbb{D}^2)$ with module action defined by $p(u \otimes h) = u \otimes ph$ for $p \in H^\infty(\mathbb{D}^2)$.

3 Basic properties of D

We shall give some basic properties of D as a linear operator from $\mathrm{dom}\, D$ to M.

Theorem 3.1. *Let M be a submodule of H^2. Then D is a densely defined closed module map with dense range.*

Proof. First it is easy to check that $D(pF) = p(DF)$ for every $p \in [z_1, z_2]$. Indeed, it suffices to show the statement in the case where p is a monomial. Let

$$F = \sum_{i,j\geqslant 0} \begin{pmatrix} u_{ij} \\ v_{ij} \end{pmatrix} \otimes z_1^i z_2^j$$

be the Taylor expansion of F in $\mathcal{K} \otimes H^2$. Then $z_1^k z_2^l DF$ belongs to M and

$$\begin{aligned} (z_1^k z_2^l DF)(\lambda) &= \lambda_1^k \lambda_2^l \sum_{i,j\geqslant 0} (C_+^{1/2} u_{ij}(\lambda) - C_-^{1/2} v_{ij}(\lambda)) \lambda_1^i \lambda_2^j \\ &= \sum_{i,j\geqslant 0} (C_+^{1/2} u_{ij}(\lambda) - C_-^{1/2} v_{ij}(\lambda)) \lambda_1^{i+k} \lambda_2^{j+l} \\ &= \left(D \sum_{i,j\geqslant 0} \begin{pmatrix} u_{ij} \\ v_{ij} \end{pmatrix} \otimes z_1^{i+k} z_2^{j+l} \right)(\lambda), \end{aligned}$$

which implies that $D(z_1^k z_2^l F)$ is defined as a function in M. Therefore, $z_1^k z_2^l F$ belongs to $\operatorname{dom} D$ and $z_1^k z_2^l DF = D(z_1^k z_2^l F)$. Hence D is a module map.

Now we shall show that D is densely defined. For any $(aC_+^{1/2}k_\lambda, bC_-^{1/2}k_\mu)$ in $\mathcal{K}$ where a and b are in $\mathbb{C}$, λ and μ are in $\mathbb{D}^2$, and any f in H^∞, by Lemma 2.2,

$$D\begin{pmatrix} aC_+^{1/2}k_\lambda \\ bC_-^{1/2}k_\mu \end{pmatrix} \otimes f = (aC_+k_\lambda - bC_-k_\mu)f$$

is a function in M. Hence D is densely defined. Furthermore, considering the case where $f = k_\lambda$ and $\lambda = \mu$, by Lemma 2.2 and (1.1), we have that

$$D\begin{pmatrix} C_+^{1/2}k_\lambda \\ C_-^{1/2}k_\lambda \end{pmatrix} \otimes k_\lambda = (C_+k_\lambda - C_-k_\lambda)k_\lambda = (C_M k_\lambda)k_\lambda = k_\lambda^M.$$

Hence the range of D is dense in M.

Next, we shall show that D is closed. Suppose that F_n is in $\operatorname{dom} D$, $F_n \to F$ in $\mathcal{K} \otimes H^2$ and $DF_n \to g$ in M. Setting

$$F_n = \sum \begin{pmatrix} u_{ij}^{(n)} \\ v_{ij}^{(n)} \end{pmatrix} \otimes z_1^i z_2^j \quad \text{and} \quad F = \sum \begin{pmatrix} u_{ij} \\ v_{ij} \end{pmatrix} \otimes z_1^i z_2^j,$$

we have that $u_{ij}^{(n)} \to u_{ij}$ in $\mathcal{H}_+$ and $v_{ij}^{(n)} \to v_{ij}$ in $\mathcal{H}_-$ as $n \to \infty$. Hence we have that

$$\begin{aligned}(DF_n)(\lambda) &= \sum((C_+^{1/2}u_{ij}^{(n)})(\lambda) - (C_-^{1/2}v_{ij}^{(n)})(\lambda))\lambda_1^i\lambda_2^j \\ &\to \sum((C_+^{1/2}u_{ij})(\lambda) - (C_-^{1/2}v_{ij})(\lambda))\lambda_1^i\lambda_2^j \\ &= (DF)(\lambda)\end{aligned}$$

by Lemma 2.1 and the Lebesgue dominated convergence theorem. Furthermore, since $(DF_n)(\lambda) \to g(\lambda)$, we have that $(DF)(\lambda) = g(\lambda)$ for any λ in $\mathbb{D}^2$. Hence F belongs to $\operatorname{dom} D$ and $DF = g$, i.e., D is closed. This concludes the proof. □

Let $\{e_{+,j}\}_j$ (resp. $\{e_{-,j}\}_j$) be an orthonormal basis of $\mathcal{H}_+$ (resp. $\mathcal{H}_-$). Furthermore, we set

$$\varphi_j = D\begin{pmatrix} e_{+,j} \\ 0 \end{pmatrix} \otimes 1 \quad \text{and} \quad \psi_j = D\begin{pmatrix} 0 \\ e_{-,j} \end{pmatrix} \otimes 1,$$

i.e.,

$$\varphi_j = C_+^{1/2}e_{+,j} \quad \text{and} \quad \psi_j = -C_-^{1/2}e_{-,j}.$$

In particular, if C is compact, we will choose eigenvectors of C_+ (resp. C_-) as $\{e_{+,j}\}_j$ (resp. $\{e_{-,j}\}_j$). In this case, we have the following:

$$\varphi_j = \lambda_{+,j}^{1/2}e_{+,j} \quad \text{and} \quad \psi_j = -\lambda_{-,j}^{1/2}e_{-,j},$$

where $\lambda_{+,j}$ (resp. $\lambda_{-,j}$) denotes an eigenvalue of C_+ (resp. C_-).

Corollary 3.2. *Let M be a submodule of H^2. If C_M is of finite rank and $\operatorname{ran} C_M$ is contained in H^∞, then we have the following*:

(1) *φ_j and ψ_j are bounded*;

(2) *D is bounded.*

Proof. Since $\varphi_j = C_+^{1/2}e_{+,j} \in \operatorname{ran} C_M$, (1) is trivial. We shall show (2). For any F in $\mathcal{K} \otimes H^2$, let

$$\sum_{i,j=0}^{\infty} \begin{pmatrix} u_{ij} \\ v_{ij} \end{pmatrix} \otimes z_1^i z_2^j$$

be the Taylor expansion of F. Then F can be rewritten as follows:

$$\begin{aligned}F &= \sum_{i,j=0}^{\infty}\left\{\sum_{k=1}^{m} c_{ij}^{(k)}\begin{pmatrix}e_{+,k}\\0\end{pmatrix}\otimes z_1^i z_2^j + \sum_{l=1}^{n} d_{ij}^{(l)}\begin{pmatrix}0\\e_{-,l}\end{pmatrix}\otimes z_1^i z_2^j\right\}\\ &= \sum_{k=1}^{m}\sum_{i,j=0}^{\infty} c_{ij}^{(k)}\begin{pmatrix}e_{+,k}\\0\end{pmatrix}\otimes z_1^i z_2^j + \sum_{l=1}^{n}\sum_{i,j=0}^{\infty} d_{ij}^{(l)}\begin{pmatrix}0\\e_{-,l}\end{pmatrix}\otimes z_1^i z_2^j\\ &= \sum_{k=1}^{m}\begin{pmatrix}e_{+,k}\\0\end{pmatrix}\otimes f_k + \sum_{l=1}^{n}\begin{pmatrix}0\\e_{-,l}\end{pmatrix}\otimes g_l,\end{aligned}$$

where we should note that f_k and g_l are in H^2. Hence we have that

$$DF = \sum_{k=1}^{m}\varphi_k f_k + \sum_{l=1}^{n}\psi_l g_l \in M$$

by Lemma 2.2. Therefore, by Theorem 3.1 and the closed graph theorem for Kreĭn spaces (see [7, p. 147]), D is bounded. □

Remark 3.3. Two assumptions of Corollary 3.2, that C_M is of finite rank and $\operatorname{ran} C$ is contained in H^∞, are satisfied in many concrete examples.

4 A representation of P_M

Let $D^\sharp$ denote the Kreĭn space adjoint of D. Then $D^\sharp k_\lambda^M$ is calculated as follows.

Lemma 4.1. *If M is a submodule, then k_λ^M belongs to $\operatorname{dom} D^\sharp$ and*

$$D^\sharp k_\lambda^M = \begin{pmatrix}C_+^{1/2}k_\lambda\\C_-^{1/2}k_\lambda\end{pmatrix}\otimes k_\lambda.$$

Proof. Since

$$\begin{aligned}\left\langle k_\lambda^M, D\begin{pmatrix}C_+^{1/2}u\\C_-^{1/2}v\end{pmatrix}\otimes z_1^i z_2^j\right\rangle &= \langle k_\lambda^M, (C_+u - C_-v)z_1^i z_2^j\rangle\\ &= \langle k_\lambda^M, z_1^i z_2^j C_+u\rangle - \langle k_\lambda^M, z_1^i z_2^j C_-v\rangle\\ &= \langle C_+^{1/2}\overline{\lambda_1}^i\overline{\lambda_2}^j k_\lambda^M, C_+^{1/2}u\rangle - \langle C_-^{1/2}\overline{\lambda_1}^i\overline{\lambda_2}^j k_\lambda^M, C_-^{1/2}u\rangle\\ &= \left\langle\begin{pmatrix}C_+^{1/2}\overline{\lambda_1}^i\overline{\lambda_2}^j k_\lambda^M\\C_-^{1/2}\overline{\lambda_1}^i\overline{\lambda_2}^j k_\lambda^M\end{pmatrix}\otimes z_1^i z_2^j, \begin{pmatrix}C_+^{1/2}u\\C_-^{1/2}v\end{pmatrix}\otimes z_1^i z_2^j\right\rangle_{\mathcal{K}\otimes H^2},\end{aligned}$$

by the orthogonality of $\{z_1^i z_2^j : i,\, j \geqslant 0\}$, we have that

$$\begin{aligned}D^\sharp k_\lambda^M &= \sum_{i,j\geqslant 0}\begin{pmatrix}C_+^{1/2}\overline{\lambda_1}^i\overline{\lambda_2}^j k_\lambda\\C_-^{1/2}\overline{\lambda_1}^i\overline{\lambda_2}^j k_\lambda\end{pmatrix}\otimes z_1^i z_2^j\\ &= \sum_{i,j\geqslant 0}\begin{pmatrix}C_+^{1/2}k_\lambda\\C_-^{1/2}k_\lambda\end{pmatrix}\otimes \overline{\lambda_1}^i\overline{\lambda_2}^j z_1^i z_2^j\\ &= \begin{pmatrix}C_+^{1/2}k_\lambda\\C_-^{1/2}k_\lambda\end{pmatrix}\otimes k_\lambda.\end{aligned}$$

This completes the proof. □

In the following argument, we shall identify $D^\sharp$ with the following operator matrix:

$$\begin{pmatrix} D^\sharp & 0 \\ 0 & 0 \end{pmatrix} \quad \text{on} \quad \operatorname{dom} D^\sharp \oplus M^\perp \subset H^2.$$

Next, we shall show that P_M is factorized with D.

Lemma 4.2. *Let M be a submodule of H^2. Then $P_M = DD^\sharp$ on the linear space generated by reproducing kernels.*

Proof. By Lemma 4.1 and (1.1), we have that

$$DD^\sharp k_\lambda = D \begin{pmatrix} C_+^{1/2} k_\lambda \\ C_-^{1/2} k_\lambda \end{pmatrix} \otimes k_\lambda = (C_M k_\lambda) k_\lambda = k_\lambda^M = P_M k_\lambda.$$

Thus we have the conclusion. □

Let T_{φ_j} denote the Toeplitz operator with symbol φ_j. We note that T_{φ_j} might be unbounded. Indeed, it is known that there exist submodules which contain no bounded functions other than 0 (see [15, p. 71]).

In [23], it was shown that the rank of C_M is odd if it is finite. The next theorem shows us that Kreĭn space operators appear naturally in our problems.

Theorem 4.3. *Let M be a submodule of H^2 with* $\operatorname{rank} C_M = 2N+1 < \infty$. *Then there exist $2N+1$ functions $\varphi_1, \dots, \varphi_{N+1}, \psi_1, \dots, \psi_N$ in M such that*

(1) $P_M = DD^\sharp = \sum_{j=1}^{N+1} T_{\varphi_j} T_{\varphi_j}^* - \sum_{k=1}^{N} T_{\psi_k} T_{\psi_k}^*$ *on the linear space generated by reproducing kernels*;

(2) $\sum_{j=1}^{N+1} |\varphi_j(\lambda)|^2 - \sum_{k=1}^{N} |\psi_k(\lambda)|^2 \to 1$ *as λ tends radially to $\mathbb{T}^2$ a.e., where $\mathbb{T}$ denotes the unit circle* $\{e^{i\theta} : \theta \in [0, 2\pi)\}$.

Proof. Let $\otimes$ denote the Schatten form. Then C_M is represented as follows:

$$C_M = \sum_{j=1}^{N+1} \varphi_j \otimes \varphi_j - \sum_{k=1}^{N} \psi_k \otimes \psi_k.$$

Hence, we have that

$$\begin{aligned} k_\lambda^M &= (C_M k_\lambda) k_\lambda \\ &= \bigg(\sum_{j=1}^{N+1} \overline{\varphi_j(\lambda)} \varphi_j - \sum_{k=1}^{N} \overline{\psi_k(\lambda)} \psi_k \bigg) k_\lambda \\ &= \bigg(\sum_{j=1}^{N+1} T_{\varphi_j} T_{\varphi_j}^* - \sum_{k=1}^{N} T_{\psi_k} T_{\psi_k}^* \bigg) k_\lambda. \end{aligned}$$

Hence, by Lemma 4.1, we have (1). By (1) and [10, Theorem 2.1], we have (2). □

Corollary 4.4. *Let M be a submodule of H^2. If* $\operatorname{rank} C_M = 2N+1$ *and* $\operatorname{ran} C_M$ *is contained in H^∞, then $\sum_{j=1}^{N+1} \varphi_j H^2 + \sum_{k=1}^{N} \psi_k H^2$ is a dense subspace of M.*

Proof. By Corollary 3.2 and Theorem 4.3, we have the conclusion. □

Corollary 4.5. *Let M be a submodule of H^2. If* $\operatorname{rank} C_M = 2N+1$ *and* $\operatorname{ran} C_M$ *is contained in H^∞, then the following two identities hold*:

$$P_M = \sum_{j=1}^{N+1} T_{\varphi_j} T_{\varphi_j}^* - \sum_{k=1}^{N} T_{\psi_k} T_{\psi_k}^*,$$

$$I_{H^2} = \sum_{j=1}^{N+1} T_{\varphi_j}^* T_{\varphi_j} - \sum_{k=1}^{N} T_{\psi_k}^* T_{\psi_k}.$$

Proof. By Corollary 3.2 and Theorem 4.3, we have the first identity. Let f be any function in H^2, and m denote the normalized Lebesgue measure on $\mathbb{T}^2$. Then, by Theorem 4.3(ii), we have that

$$\begin{aligned}\left\langle\left(\sum_{j=0}^N T_{\varphi_j}^* T_{\varphi_j} - \sum_{k=1}^N T_{\psi_k}^* T_{\psi_k}\right) f, f\right\rangle &= \sum_{j=0}^N \|T_{\varphi_j} f\|^2 - \sum_{k=1}^N \|T_{\psi_k} f\|^2 \\ &= \sum_{j=0}^N \int_{\mathbb{T}^2} |\varphi_j f|^2 dm - \sum_{k=1}^N \int_{\mathbb{T}^2} |\psi_k f|^2 dm \\ &= \int_{\mathbb{T}^2} \left(\sum_{j=0}^N |\varphi_j|^2 - \sum_{k=1}^N |\psi_k|^2\right) |f|^2 dm \\ &= \int_{\mathbb{T}^2} |f|^2 dm \\ &= \langle f, f\rangle.\end{aligned}$$

By the polarization identity we have the second identity. □

Corollary 4.6. *Let M be a submodule of H^2 with* $\operatorname{rank} C_M = 2N+1 < \infty$. *Then λ is a zero of M if and only if $\sum_{j=1}^{N+1} |\varphi_j(\lambda)|^2 - \sum_{k=1}^N |\psi_k(\lambda)|^2 = 0$.*

Proof. By Theorem 4.3, the Berezin transform of P_M is $\sum_{j=1}^{N+1} |\varphi_j(\lambda)|^2 - \sum_{k=1}^N |\psi_k(\lambda)|^2$. This concludes the proof. □

5 Examples

We shall compute some examples. Throughout this section, $\otimes$ will denote the Schatten form.

Example 5.1. If $M = z_1 H^2 + z_2 H^2$, then it is easy to see that

$$\begin{aligned}&C_M = z_1 \otimes z_1 + z_2 \otimes z_2 - z_1 z_2 \otimes z_1 z_2, \\ &C_+ = z_1 \otimes z_1 + z_2 \otimes z_2 \quad \text{and} \quad C_- = z_1 z_2 \otimes z_1 z_2.\end{aligned}$$

By Theorem 4.3, we have that

$$P_M = T_{z_1} T_{z_1}^* + T_{z_2} T_{z_2}^* - T_{z_1 z_2} T_{z_1 z_2}^*.$$

More generally, the submodule $M = q_1 H^2 + q_2 H^2$, where $q_1 = q_1(z_1)$ and $q_2 = q_2(z_2)$ are one variable inner functions, is well-studied (see [12,13]). Then the defect operator of M is calculated as follows (see [21]):

$$C_M = q_1 \otimes q_1 + q_2 \otimes q_2 - q_1 q_2 \otimes q_1 q_2,$$

and

$$\sigma(C_M) = \{0, 1, \pm\sqrt{(1-|q_1(0)|^2)(1-|q_2(0)|^2)}\}.$$

Eigenfunctions can be also described, however they are complicated.

Example 5.2. This example computes a rank 3 defect operator of another type. Fix the following notation:

$$K_{rw}(z) = \frac{\sqrt{1-r^2}}{1-r\overline{w}z}, \quad 0 \leqslant r < 1.$$

Then it is easy to check that $\{z^j \overline{w} K_{rw}(z)\}_{j\in\mathbb{Z}}$ is an orthonormal system in $L^2(\mathbb{T})$. Now, we set

$$\mathcal{L} = H^2 \oplus \bigoplus_{j=0}^{\infty} \mathbb{C} z^j \overline{w} K_{rw}(z).$$

Then $\mathcal{L}$ is invariant under the multiplication of z and w in L^2. Indeed, trivially $\mathcal{L}$ is invariant under the multiplication of z, and for any $j \geqslant 0$ we have that

$$\begin{aligned} wz^j\overline{w}K_{rw}(z) &= z^jK_{rw}(z) \\ &= z^j(\sqrt{1-r^2} + rz\overline{w}K_{rw}(z)) \\ &= \sqrt{1-r^2}z^j + rz^{j+1}\overline{w}K_{rw}(z), \end{aligned}$$

i.e., $\mathcal{L}$ is invariant under the multiplication of w. This type of invariant subspace was discovered by Izuchi and Ohno [14]. Furthermore, there exists an inner function θ such that

$$M = \theta\bigg(H^2 \oplus \bigoplus_{j=0}^{\infty} \mathbb{C}z^j\overline{w}K_{rw}(z)\bigg)$$

is a submodule in H^2. This type of submodule was discussed by Izuch [11] in detail. We shall calculate C_M. Setting $L_z = P_{\mathcal{L}}M_z|_{\mathcal{L}}$, trivially we have that $R_z = M_\varphi L_z M_\varphi^*$. Hence, it suffices to deal with L_z and L_w on $\mathcal{L}$. First observe that

$$\begin{aligned} C_{\mathcal{L}} &= I_{\mathcal{L}} - L_zL_z^* - L_wL_w^* + L_zL_wL_z^*L_w^* \\ &= I_{\mathcal{L}} - L_zL_z^* - L_w(I_{\mathcal{L}} - L_zL_z^*)L_w^*. \end{aligned}$$

Now, it is easy to see that $I_{\mathcal{L}} - L_zL_z^*$ is the orthogonal projection onto $H^2(w) \oplus \mathbb{C}\overline{w}K_{rw}(z)$. Moreover, $L_w(I_{\mathcal{L}} - L_zL_z^*)L_w^*$ is the orthogonal projection onto $wH^2(w) \oplus \mathbb{C}K_{rw}(z)$. Hence, we have that

$$C_{\mathcal{L}} = 1 \otimes 1 + \overline{w}K_{rw} \otimes \overline{w}K_{rw} - K_{rw} \otimes K_{rw}.$$

This concludes that

$$C_M = \varphi \otimes \varphi + \varphi\overline{w}K_{rw} \otimes \varphi\overline{w}K_{rw} - \varphi K_{rw} \otimes \varphi K_{rw}.$$

Example 5.3. $H^2(z_1)$ denotes the Hardy space over $\mathbb{D}$ with variable z_1. Let $\{q_j\}_{j\geqslant 0}$ be an inner sequence, which is a sequence of inner functions in $H^2(z_1)$ such that every q_j/q_{j+1} is also inner. Then

$$M = \sum_{j\geqslant 0} \oplus q_jH^2(z_1)z_2^j$$

is a submodule in $H^2(\mathbb{D}^2)$ (see [16, 17] for details). The defect operator of M is calculated as follows:

$$C_M = q_0 \otimes q_0 \oplus \sum_{j=1}^{\infty} \oplus (q_jz_2^j \otimes q_jz_2^j - q_{j-1}z_2^j \otimes q_{j-1}z_2^j).$$

Lemma 5.4. *If* $(q_{j-1}/q_j)(0) \neq 0$, *then*

$$q_j \otimes q_j - q_{j-1} \otimes q_{j-1} = \alpha_je_{+,j} \otimes e_{+,j} - \alpha_je_{-,j} \otimes e_{-,j}$$

gives the spectral resolution, where we set $\alpha_j = \sqrt{1 - |(q_{j-1}/q_j)(0)|^2}$,

$$e_{+,j} = \frac{q_{j-1} - \frac{1}{\alpha_j}T_{z_1q_j}T_{z_1q_j}^*q_{j-1}}{\sqrt{2(1-\alpha_j)}} \quad \textit{and} \quad e_{-,j} = \frac{q_{j-1} + \frac{1}{\alpha_j}T_{z_1q_j}T_{z_1q_j}^*q_{j-1}}{\sqrt{2(1+\alpha_j)}}.$$

Proof. Since

$$q_j \otimes q_j - q_{j-1} \otimes q_{j-1} = T_{q_j}\bigg(1 \otimes 1 - \frac{q_{j-1}}{q_j} \otimes \frac{q_{j-1}}{q_j}\bigg)T_{q_j}^*,$$

it suffices to see the spectral resolution of $A = 1 \otimes 1 - q \otimes q$, where q is an inner function with $q(0) \neq 0$ in $H^2(\mathbb{D})$. A has the following matrix representation with respect to orthonormal system $\{1, T_{z_1}T_{z_1}^*q/\|T_{z_1}T_{z_1}^*q\|\}$:

$$\begin{pmatrix} \alpha^2 & -\alpha q(0) \\ -\alpha\overline{q(0)} & -\alpha^2 \end{pmatrix} \quad \text{where} \quad \alpha = \sqrt{1-|q(0)|^2}.$$

By elementary linear algebra, we have the conclusion. □

For simplicity, we assume that $(q_{j-1}/q_j)(0) \neq 0$ for every $j \geqslant 1$. By Lemma 5.4, we have that

$$C_M = q_0 \otimes q_0 + \sum_{j=1}^{\infty} (\alpha_j e_{+,j} z_2^j \otimes e_{+,j} z_2^j - \alpha_j e_{-,j} z_2^j \otimes e_{-,j} z_2^j).$$

Hence, if C is of finite rank, then we have that

$$C_+ = q_0 \otimes q_0 \oplus \sum_{j=1}^{n} \oplus \alpha_j e_{+,j} z_2^j \otimes e_{+,j} z_2^j \quad \text{and} \quad C_- = \sum_{j=1}^{n} \oplus \alpha_j e_{-,j} z_2^j \otimes e_{-,j} z_2^j.$$

By Theorem 4.3, we have that

$$P_M = T_{q_0} T_{q_0}^* + \sum_{j=1}^{n} \alpha_j T_{e_{+,j} z_2^j} T_{e_{+,j} z_2^j}^* - \sum_{j=1}^{n} \alpha_j T_{e_{-,j} z_2^j} T_{e_{-,j} z_2^j}^*.$$

6 Lorentz group

A prototype of Kreĭn space is the four-dimensional space-time $\mathbb{R}^4$, where the indefinite metric is the Minkowski metric given by the indefinite inner product

$$(u,\ v) = u_1 v_1 + u_2 v_2 + u_3 v_3 - u_4 v_4.$$

Here, the metric matrix is

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The Lorentz transform is the collection of invertible 4×4 matrices T that preserves the Minkowski metric, e.g., $(Tu, Tv) = (u, v)$, or equivalently $T^* C T = C$. This notion can be generalized to the Kreĭn spaces with more general metric matrices. For example, every non-singular $n \times n$ Hermitian matrix C induces an indefinite inner product on $\mathbb{C}^n$ defined by $(u,\ v)_C = \langle Cu,\ v \rangle$. The associated Lorentz group $\mathcal{G}(\mathbb{C}^n)$ is thus the collection of non-singular matrices T such that $T^* C T = C$. Since C is congruent to its signature matrix $I_p \oplus (-I_q)$, where p is the number of positive eigenvalues and q is the number of negative ones, $\mathcal{G}(\mathbb{C}^n)$ is easily verified to be isomorphic to the so-called pseudo-unitary group $U(p, q)$ (see [2]). Hence up to isomorphism $\mathcal{G}(\mathbb{C}^n)$ is completely determined by C's signature. Things are more complicated if C is an Hermitian operator on an infinite dimensional Hilbert space, for example the case here when C is the defect operator for a submodule. Nonetheless the notion of Lorentz group still makes good sense. For an algebra $\mathcal{B}$, $\mathcal{B}^{-1}$ will denote the set of invertible elements in $\mathcal{B}$.

Definition 6.1. Let M be a submodule of $H^2(\mathbb{D}^2)$ and denote $B(M)$ the set of all bounded linear operators on M. Then we call

$$\mathcal{G}(M) = \{g \in B^{-1}(M) : g^* C_M g = C_M\},$$

the Lorentz group of M.

We first verify that $\mathcal{G}(M)$ is indeed a group. For $g \in \mathcal{G}(M)$, it is easy to check $(g^*)^{-1} C_M (g)^{-1} = C_M$, so g^{-1} is in $\mathcal{G}(M)$. For g_1, $g_2 \in \mathcal{G}(M)$, we have

$$\begin{aligned}(g_1 g_2)^* C_M (g_1 g_2) &= {g_2}^* ({g_1}^* C_M g_1) g_2 \\ &= {g_2}^* C_M g_2 \\ &= C_M,\end{aligned}$$

and it follows that $\mathcal{G}(M)$ is a multiplicative group.

Before we make a study of $\mathcal{G}(M)$, let us take a closer look at the operators $T \in B(M)$ that satisfies the equation $T^*C_MT = C_M$. Assume C_M is compact and $C_M = \sum_j \eta_j \phi_j \otimes \phi_j$ is its spectral decomposition, where $\{\phi_j\}$ is an orthonormal basis for $\mathcal{K}$ formed by eigenvectors with corresponding eigenvalues η_j. Then C_M's integral kernel is

$$G^M(\lambda, z) = \sum_j \eta_j \overline{\phi_j(\lambda)} \phi_j(z).$$

The following proposition provides a description of T.

Proposition 6.2. *Assume C_M is compact. Then $T^*C_MT = C_M$ if and only if*

$$\sum_j \eta_j \overline{T^*\phi_j(\lambda)} T^*\phi_j(z) = G^M(\lambda, z).$$

Proof. If

$$\sum_j \eta_j \overline{T^*\phi_j(\lambda)} T^*\phi_j(z) = G^M(\lambda, z),$$

then

$$\begin{aligned} C_M f(z) &= \int_{\mathbb{T}^2} G^M(z, \lambda) f(\lambda) dm(\lambda) \\ &= \sum_j \eta_j \int_{\mathbb{T}^2} \overline{T^*\phi_j(\lambda)} T^*\phi_j(z) f(\lambda) dm(\lambda) \\ &= \sum_j \eta_j \langle f, T^*\phi_j \rangle T^*\phi_j(z) \\ &= T^* \left(\sum_j \eta_j \langle Tf, \phi_j \rangle \phi_j(z) \right) \\ &= T^* C_M T f(z). \end{aligned}$$

Tracing the above arguments from the bottom up and use the fact that G^M is uniquely determined by C_M, we have

$$\sum_j \eta_j \overline{T^*\phi_j(\lambda)} T^*\phi_j(z) = G^M(\lambda, z).$$

This completes the proof. □

Recall that two submodules M_1 and M_2 are said to be congruent if there is a bounded invertible operator $J: M_2 \to M_1$ such that the defect operators satisfy

$$C_{M_1} = J C_{M_2} J^*.$$

Congruence relation was introduced in [20] in an attempt to classify submodules. Lorentz group is invariant under congruence relation.

Proposition 6.3. *If two submodules M_1 and M_2 are congruent, then their Lorentz groups $\mathcal{G}(M_1)$ and $\mathcal{G}(M_2)$ are isomorphic.*

Proof. Suppose that M_1 and M_2 are congruent, and we let C_{M_1} and C_{M_2} be their defect operators. Then there is a bounded invertible operator $J : M_2 \to M_1$ such that

$$C_{M_1} = J C_{M_2} J^*.$$

Let us define

$$\begin{aligned} &\varphi : \mathcal{G}(M_1) \mapsto \mathcal{G}(M_2), \\ &g \mapsto J^* g (J^*)^{-1}. \end{aligned}$$

For any $g \in \mathcal{G}(M_1)$, we have

$$\begin{aligned}\varphi^*(g)C_{M_2}\varphi(g) &= J^{-1}g^*JC_{M_2}J^*g(J^*)^{-1}\\ &= J^{-1}g^*C_{M_1}g(J^*)^{-1}\\ &= J^{-1}C_{M_1}(J^*)^{-1}\\ &= C_{M_2}.\end{aligned}$$

So it is well-defined. It is clear that φ is invertible and

$$\begin{aligned}\varphi(g_1g_2) &= J^*g_1g_2(J^*)^{-1}\\ &= J^*g_1(J^*)^{-1}J^*g_2(J^*)^{-1}\\ &= \varphi(g_1)\varphi(g_2).\end{aligned}$$

Hence, $\mathcal{G}(M_1)$ and $\mathcal{G}(M_2)$ are isomorphic. □

If C_M is of finite rank, then $\mathcal{G}(M)$ can be determined. First, consider general block matrices

$$T = \begin{pmatrix} T_0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where T_0 is an invertible Hermitian matrix and A is an operator. Then one computes easily that $A^*TA = T$ if and only if

$$A_{11}^*T_0A_{11} = T_0, \quad A_{11}^*T_0A_{12} = 0, \quad A_{12}^*T_0A_{12} = 0. \tag{6.1}$$

Since A_{11} and T_0 are finite matrices and T_0 is invertible, A_{11} is invertible by the first equation, which by the second identity implies $A_{12} = 0$. If A is invertible then A_{22} is invertible, but there is no restriction on A_{21}.

If C_M is finite rank with rank $2p+1$, we can decompose M as $M = \mathcal{K} \oplus \ker C_M$, where $\mathcal{K}$ is now equal to the range of C_M. In addition, C_M is then congruent to

$$T = \begin{pmatrix} I_{p+1} \oplus -I_p & 0 \\ 0 & 0 \end{pmatrix}.$$

So by (6.1), $A^*TA = T$ if and only if $A_{11} \in U(p+1, p)$ and $A_{22} \in \mathcal{B}^{-1}(\ker C_M)$. It is not hard to check that the space $zwM \subset \ker C_M$, hence $\mathcal{B}^{-1}(\ker C_M)$ is isomorphic to $\mathcal{B}^{-1}(\mathcal{H})$, where $\mathcal{H}$ is any separable complex Hilbert space. In conclusion we have the following.

Proposition 6.4. *If C_M has rank $2p+1$, then $\mathcal{G}(M)$ is isomorphic to the group*

$$\begin{pmatrix} U(p+1,p) & 0 \\ \mathcal{B}(\mathbb{C}^{2p+1}, \mathcal{H}) & \mathcal{B}^{-1}(\mathcal{H}) \end{pmatrix}.$$

The following corollary is immediate.

Corollary 6.5. *If C_{M_1} and C_{M_2} are finite rank, then $\mathcal{G}(M_1)$ and $\mathcal{G}(M_2)$ are isomorphic if and only if* $\mathrm{rank}C_{M_1} = \mathrm{rank}C_{M_2}$.

The situation when C_M is of infinite rank seems rather complicated. It was known that C_M is Hilbert-Schmidt in almost all examples (see [20]). So the decreasing speed of C_M's eigenvalues is of importance here. Indeed it is shown in [18,20] that if λ_j are the eigenvalues of C_{M_1} and η_j are the eigenvalues of C_{M_1}, both arranged such that $|\lambda_j|$ and $|\eta_j|$ are decreasing, then C_{M_1} and C_{M_2} are congruent if and only if the ratio $\frac{|\lambda_j|}{|\eta_j|}$ are bounded above and below by positive constants. So if we let $\sigma_1(t) = \sum_j |\lambda_j|t^j$ and $\sigma_2(t) = \sum_j |\eta_j|t^j$, then σ_1 and σ_2 have the same radius of convergence. Now we are in position to state a conjecture.

Conjecture. If $\mathcal{G}(M_1)$ and $\mathcal{G}(M_2)$ are isomorphic, then σ_1 and σ_2 have the same radius of convergence.

Observe that the conjecture is trivial when C_M is of finite rank because f is a polynomial in this case.

In the remaining part of this section, we show that when C_M is finite rank there is an element in $\mathcal{G}(M)$ of order 2. This fact will be used later. It is sufficient to construct such a matrix on $\mathcal{K}$. Then we extend it to M by simply adding the identity operator on $\ker C_M$. On $\mathcal{K}$, we have by (1.2)

$$C_M = \begin{pmatrix} I_n & & & \\ & D & & \\ & & -I_{n-1} & \\ & & & -D \end{pmatrix}.$$

Let

$$T = \begin{pmatrix} I_n & & & \\ & D & & \\ & & I_{n-1} & \\ & & & D \end{pmatrix}, \quad J = \begin{pmatrix} J_{2n} & \\ & J_{2n-1} \end{pmatrix}, \quad I' = \begin{pmatrix} I_{2n} & \\ & -I_{2n-1} \end{pmatrix},$$

where

$$J_k = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}_{k\times k}.$$

It is easy to see $J^2 = I$. Since C_M is finite rank, it is invertible on $\mathcal{K}$, and hence so is T. We set $K = T^{-1/2}JT^{1/2}$, and verify that

$$\begin{aligned} K^*CK &= T^{\frac{1}{2}}JT^{-\frac{1}{2}}CT^{\frac{-1}{2}}JT^{\frac{1}{2}} \\ &= T^{\frac{1}{2}}JI'JT^{\frac{1}{2}} \\ &= T^{\frac{1}{2}}I'T^{\frac{1}{2}} = C. \end{aligned}$$

It is easy to see that $K^2 = I$ and $K \neq I$ when $n \geqslant 1$. Hence $K \oplus I_{\ker C_M}$ is an element in $\mathcal{G}(M)$ of order 2. The existence of such K in $\mathcal{G}(M)$ will be used to show the difference between the Lorentz group and the little Lorentz group (to be defined later). The map $\varphi(g) = KgK$ defines an inner automorphism of $\mathcal{G}(M)$. We give an interesting example below.

Example 6.6. Let $M = q_1(z_1)H^2(\mathbb{D}^2) + q_2(z_2)H^2(\mathbb{D}^2)$ be as in Example 5.1. It is computed in [21] that on $\mathcal{K}$,

$$C_M = \begin{pmatrix} 1 & & \\ & \eta & \\ & & -\eta \end{pmatrix},$$

where $\eta = (1-|q_1(0)|^2)^{1/2}(1-|q_2(0)|^2)^{1/2}$. By the construction of K and φ, one checks that $\varphi : \mathcal{G} \mapsto \mathcal{G}$ has the form

$$\varphi\begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} = \begin{pmatrix} g_{22} & \eta g_{21} & \sqrt{\eta} g_{23} \\ \frac{g_{12}}{\eta} & g_{11} & \frac{g_{13}}{\sqrt{\eta}} \\ \frac{g_{32}}{\sqrt{\eta}} & \sqrt{\eta} g_{31} & g_{33} \end{pmatrix}.$$

In particular, if both q_1 and q_2 vanish at 0, then $\eta = 1$ and

$$\varphi\begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} = \begin{pmatrix} g_{22} & g_{21} & g_{23} \\ g_{12} & g_{11} & g_{13} \\ g_{32} & g_{31} & g_{33} \end{pmatrix}.$$

7 Subgroups in $(H^\infty)^{-1}$

In preparation for the definition of little Lorentz group in the next section, we look into some subgroups in $(H^\infty(\mathbb{D}))^{-1}$ and $(H^\infty(\mathbb{D}^2))^{-1}$ in this section. We believed it was a well-studied subject because both are classical commutative groups of analytic functions. But to our surprise, there is not much in the literature. However, this section makes no attempt to do a general study. Instead, it looks into some subgroups that are relevant to the discussion in later sections.

First of all, the set of nonzero complex numbers $\mathbb{C}_\times$ is a trivial subgroup in $(H^\infty)^{-1}$, so we only look at subgroups in $(H^\infty(\mathbb{D}^i))^{-1}/\mathbb{C}_\times$, $i=1,2$. In addition, we observe that if J is an ideal in H^∞, then $\mathcal{G}(J) := (1+J)\cap(H^\infty)^{-1}$ is a group. To see it, we let $1+f$ and $1+g$ be in the set. Clearly, $(1+f)(1+g)\in\mathcal{G}(J)$. Furthermore, $(1+f)^{-1} = 1-f(1+f)^{-1}$, which is in $\mathcal{G}(J)$.

Now we consider the ideal

$$J^n = \{f\in H^\infty(\mathbb{D}) : f^{(k)}(0)=0,\, 0\leqslant k\leqslant n\},$$

where $f^{(k)}$ stands for the k-th derivative of f, and set $\mathcal{G}^n = \mathcal{G}(J^n)$. On $\mathbb{D}^2$, similar subgroups can be defined. For non-negative integers n_1 and n_2, we let

$$J^{n_1,n_2} = \left\{ f\in H^\infty(\mathbb{D}^2) : \frac{\partial^i f}{\partial z^i}\bigg|_{(0,0)} = 0, \frac{\partial^j f}{\partial w^j}\bigg|_{(0,0)} = 0,\, 0\leqslant i\leqslant n_1,\, 0\leqslant j\leqslant n_2 \right\},$$

and set $\mathcal{G}^{n_1,n_2} = (1+J^{n_1,n_2})\cap(H^\infty(\mathbb{D}^2))^{-1}$.

Clearly, $\mathcal{G}^n$ is a subgroup in $\mathcal{G}^{n-1}$ for each $n\geqslant 1$. However, they are all isomorphic to each other.

Theorem 7.1. *$\mathcal{G}^0$ is isomorphic to $\mathcal{G}^n$ for each $n\geqslant 1$.*

Proof. We prove the case $n=1$. The other cases are similar. For $\varphi\in\mathcal{G}^0$, define $\rho(\varphi) = \varphi(z)^z = \mathrm{e}^{z\log\varphi(z)}$. First of all, since φ is nonvanishing and $\mathbb{D}$ is contractable to a point, $\log\varphi$ is well-defined and analytic. It is clear that $\rho(1) = 1^z = 1$. In addition, ρ is a homomorphism, since

$$\begin{aligned}\rho(\varphi_1\varphi_2) &= (\varphi_1(z)\varphi_2(z))^z\\ &= \varphi_1(z)^z\varphi_2(z)^z\\ &= \rho(\varphi_1)\rho(\varphi_2).\end{aligned}$$

To check $\rho(\varphi)$ is in $(H^\infty)^{-1}$, we verify that it is bounded above and below by positive numbers. Let $z = x+\mathrm{i}y$. Then

$$\begin{aligned}|\rho(\varphi)| &= \mathrm{e}^{\mathrm{Re}[z\log\varphi(z)]}\\ &= \mathrm{e}^{\mathrm{Re}\{(x+\mathrm{i}y)[\log|\varphi(z)|+\mathrm{i}\mathrm{Arg}\varphi(z)]\}}\\ &= \mathrm{e}^{x\log|\varphi(z)|-y\mathrm{Arg}\varphi(z)}.\end{aligned}$$

Since $\varphi(z)$ is in $(H^\infty(\mathbb{D}))^{-1}$, there exist constants $0<m<1$ and $M>1$, such that $m\leqslant|\varphi(z)|\leqslant M$ on $\mathbb{D}$. Since $|x|<1$, we have

$$\begin{gathered}\mathrm{Min}\{\log m, -\log M\}\leqslant x\log|\varphi(z)|\leqslant\mathrm{Max}\{\log M, -\log m\},\\ \mathrm{Min}\left\{m,\frac{1}{M}\right\}\leqslant\mathrm{e}^{x\log|\varphi(z)|}\leqslant\mathrm{Max}\left\{M,\frac{1}{m}\right\}.\end{gathered}$$

Since $|y|<1$, $|\mathrm{Arg}\varphi(z)|\leqslant 2\pi$, $-2\pi\leqslant y\mathrm{Arg}\varphi(z)\leqslant 2\pi$ and $\mathrm{e}^{-2\pi}\leqslant\mathrm{e}^{-y\mathrm{Arg}\varphi(z)}\leqslant\mathrm{e}^{2\pi}$, we have

$$\mathrm{Min}\left\{m,\frac{1}{M}\right\}\cdot\mathrm{e}^{-2\pi}\leqslant|\rho(\varphi)|\leqslant\mathrm{Max}\left\{M,\frac{1}{m}\right\}\cdot\mathrm{e}^{2\pi}.$$

It follows that $\rho(\varphi)\in H^\infty(\mathbb{D})$ and $\rho^{-1}(\varphi) = \rho(\frac{1}{\varphi})\in H^\infty(\mathbb{D})$, e.g., $\rho(\varphi)\in(H^\infty)^{-1}$.

Moreover, $\rho(\varphi)(0)=\varphi(0)^0=1^0=1$, and

$$\rho(\varphi)'(z)=\mathrm{e}^{z\log\varphi(z)}\left\{\log\varphi(z)+\frac{z\varphi'(z)}{\varphi(z)}\right\},$$

so $\rho(\varphi)'(0)=0$. Hence $\rho(\varphi)\in\mathcal{G}^1$, and ρ is well-defined.

Furthermore, since $\rho(\varphi)=1$ if and only if $\log\varphi(z)=0$, or equivalently, $\varphi(z)=1$ on $\mathbb{D}$. Hence ρ is injective.

Lastly, we will show that ρ is a surjective. For any $\widetilde{\varphi}\in\mathcal{G}^1$, let us consider $\varphi=\mathrm{e}^{\frac{\log\widetilde{\varphi}}{z}}$. Since $\widetilde{\varphi}=1+z^2h$ for some $h\in H^\infty(\mathbb{D})$, $\log\widetilde{\varphi}=z^2h+o(z^2)$. Hence $\varphi(z)=\mathrm{e}^{zh+o(z)}$, in particular $\varphi(0)=1$. So there exists $0<\varepsilon<1$ such that when $|z|<\varepsilon$, $|\varphi(z)|>\frac{1}{2}$. When $\varepsilon\leqslant|z|<1$,

$$\begin{aligned}|\varphi(z)|&=\mathrm{e}^{\mathrm{Re}\{\frac{\log\widetilde{\varphi}}{z}\}}\\&=\mathrm{e}^{\mathrm{Re}\{\frac{\log|\widetilde{\varphi}|+\mathrm{i}\mathrm{Arg}\log\widetilde{\varphi}}{x+\mathrm{i}y}\}}\\&=\mathrm{e}^{\mathrm{Re}\{\frac{(\log|\widetilde{\varphi}|+\mathrm{i}\mathrm{Arg}\log\widetilde{\varphi})(x-\mathrm{i}y)}{x^2+y^2}\}}\\&=\mathrm{e}^{\frac{x\log|\widetilde{\varphi}|+y\mathrm{Arg}\log\widetilde{\varphi}}{x^2+y^2}}.\end{aligned}$$

Since $\widetilde{\varphi}\in\mathcal{G}^1$, there exist positive constants k and K, such that $k\leqslant|\widetilde{\varphi}|\leqslant K$. So

$$\frac{\mathrm{Min}\{\log k,-\log K\}}{\varepsilon^2}\leqslant\frac{x\log|\widetilde{\varphi}|}{x^2+y^2}\leqslant\frac{\mathrm{Max}\{\log K,-\log k\}}{\varepsilon^2},$$

and

$$\frac{-2\pi}{\varepsilon^2}\leqslant\frac{y\mathrm{Arg}\log\widetilde{\varphi}}{x^2+y^2}\leqslant\frac{2\pi}{\varepsilon^2}.$$

Hence $|\varphi(z)|$ is bounded above and below by positive constants, e.g., $\varphi\in(H^\infty(\mathbb{D}))^{-1}$, and hence $\varphi\in\mathcal{G}^0$. This shows ρ is a surjective, and it concludes that ρ is an isomorphism from $\mathcal{G}^0$ to $\mathcal{G}^1$.

For $\varphi\in\mathcal{G}^0$, if we let $\rho^n(\varphi)=\varphi(z)^{z^n}=\mathrm{e}^{z^n\log\varphi(z)}$, then using polar coordinate and similar arguments, we can show that ρ^n is an isomorphism from $\mathcal{G}^0$ to $\mathcal{G}^n$. □

The same map ρ can be defined from $\mathcal{G}^{0,0}$ to $\mathcal{G}^{1,0}$, and its well-definedness and injectivity still hold. However, the map fails to be onto. For example, $1+0.5w$ is in $\{\mathcal{G}^{1,0}\}$ but not in the range of ρ, as is easily seen from the power series of $\rho(\varphi)$.

Since the groups $\mathcal{G}^n$ and $\mathcal{G}^{m,n}$ are multiplicative, it is sometimes more informative to see their additive counterparts. Let $L^\infty_{\mathbb{R}}(\mathbb{T})$ be the set of real-valued functions in $L^\infty(\mathbb{T})$. Clearly, it is an additive abelian group. The following lemma from [8] gives a group homomorphism from $\mathcal{G}_0$ to $L^\infty_{\mathbb{R}}(\mathbb{T})$.

Lemma 7.2. *There is a surjective group homomorphism ρ from $(H^\infty(\mathbb{D}))^{-1}$ to $L^\infty_{\mathbb{R}}(\mathbb{T})$ with* $\ker\rho=\mathbb{T}$.

Proof. The main idea of its proof is to consider the map ρ defined by $\rho(f(z))=\log|f^*(\theta)|$, where $|f^*(\theta)|$ is the radial limit. It is not hard to check ρ is a group homomorphism with $\ker\rho=\mathbb{T}$. For surjectivity, for each $g(\theta)\in L^\infty_{\mathbb{R}}(\mathbb{T})$, let

$$f(z)=\exp\left(\frac{1}{2\pi}\int_{\mathbb{T}}\frac{\mathrm{e}^{\mathrm{i}\theta}+z}{\mathrm{e}^{\mathrm{i}\theta}-z}g(\theta)d\theta\right).$$

For simplicity, we set $c=\frac{1}{2\pi}$ in the sequel. Then verify that $f\in(H^\infty(\mathbb{D}))^{-1}$, and

$$\begin{aligned}\log|f(z)|&=\log\left[\exp\left(c\int_{\mathbb{T}}P_r(\theta)g(\theta)d\theta\right)\right]\\&=c\int_{\mathbb{T}}P_r(\theta)g(\theta)d\theta,\end{aligned}$$

so $\rho(f(z))=\log|f^*(\theta)|=g(\theta)$. This idea of proof will be generalized to $\mathbb{D}^2$ later. □

It follows from Lemma 7.2 that $(H^\infty(\mathbb{D}))^{-1}/\mathbb{T}$ is isomorphic to $L^\infty_{\mathbb{R}}(\mathbb{T})$. Now consider the following chain of subgroups of $L^\infty_{\mathbb{R}}(\mathbb{T})$:

$$\mathcal{N}^n = \left\{ g(\theta) \in L^\infty_{\mathbb{R}}(\mathbb{T}) : \int_{\mathbb{T}} g(\theta) \mathrm{e}^{-\mathrm{i}k\theta} d\theta = 0,\ k = 0, 1, 2, \dots, n \right\}.$$

Clearly, $\mathcal{N}^{n+1} \subseteq \mathcal{N}^n$ for any $n \geqslant 0$.

Proposition 7.3. *The restriction of ρ to $\mathcal{G}^n$ is an isomorphism from $\mathcal{G}^n$ to $\mathcal{N}^n$.*

Proof. It is clear that $\ker(\rho|_{\mathcal{G}^n}) = 1$, since $\ker \rho = \mathbb{T}$ and $f(0) = 1$ in $\mathcal{G}^n$.

For each $g(\theta) \in \mathcal{N}^n$, as in the proof of Lemma 7.2, there exists some $f(z) \in (H^\infty(\mathbb{D}))^{-1}$ satisfying

$$\begin{aligned} f(z) &= \exp\left(c \int_{\mathbb{T}} \frac{\mathrm{e}^{\mathrm{i}\theta} + z}{\mathrm{e}^{\mathrm{i}\theta} - z} g(\theta) d\theta \right), \\ f(0) &= \exp\left(c \int_{\mathbb{T}} g(\theta) d\theta \right). \end{aligned}$$

Then

$$\begin{aligned} f'(z) &= \exp\left(c \int_{\mathbb{T}} \frac{\mathrm{e}^{\mathrm{i}\theta} + z}{\mathrm{e}^{\mathrm{i}\theta} - z} g(\theta) d\theta \right) \cdot c \int_{\mathbb{T}} \frac{2\mathrm{e}^{\mathrm{i}\theta}}{(\mathrm{e}^{\mathrm{i}\theta} - z)^2} g(\theta) d\theta, \\ f'(0) &= \exp\left(c \int_{\mathbb{T}} g(\theta) d\theta \right) \cdot c \int_{\mathbb{T}} 2\mathrm{e}^{-\mathrm{i}\theta} g(\theta) d\theta, \\ f''(z) &= \exp\left(c \int_{\mathbb{T}} \frac{\mathrm{e}^{\mathrm{i}\theta} + z}{\mathrm{e}^{\mathrm{i}\theta} - z} g(\theta) d\theta \right) \cdot \left\{ \left[c \int_{\mathbb{T}} \frac{2\mathrm{e}^{\mathrm{i}\theta}}{(\mathrm{e}^{\mathrm{i}\theta} - z)^2} g(\theta) d\theta \right]^2 + c \int_{\mathbb{T}} \frac{4\mathrm{e}^{\mathrm{i}\theta}}{(\mathrm{e}^{\mathrm{i}\theta} - z)^3} g(\theta) d\theta \right\}, \\ f''(0) &= \exp\left(c \int_{\mathbb{T}} g(\theta) d\theta \right) \cdot \left\{ \left[c \int_{\mathbb{T}} 2\mathrm{e}^{-\mathrm{i}\theta} g(\theta) d\theta \right]^2 + c \int_{\mathbb{T}} 4\mathrm{e}^{-2\mathrm{i}\theta} \cdot g(\theta) d\theta \right\}, \end{aligned}$$

and in general,

$$\begin{aligned} f^{(n)}(z) &= \exp\left(c \int_{\mathbb{T}} \frac{\mathrm{e}^{\mathrm{i}\theta} + z}{\mathrm{e}^{\mathrm{i}\theta} - z} g(\theta) d\theta \right) \cdot \left\{ \left[c \int_{\mathbb{T}} \frac{2\mathrm{e}^{\mathrm{i}\theta}}{(\mathrm{e}^{\mathrm{i}\theta} - z)^2} g(\theta) d\theta \right]^n \right. \\ &\quad \left. + \cdots + c \int_{\mathbb{T}} \frac{2n!\mathrm{e}^{\mathrm{i}\theta}}{(\mathrm{e}^{\mathrm{i}\theta} - z)^{(n+1)}} g(\theta) d\theta \right\}, \\ f^{(n)}(0) &= \exp\left(c \int_{\mathbb{T}} g(\theta) d\theta \right) \cdot \left\{ \left[c \int_{\mathbb{T}} 2\mathrm{e}^{-\mathrm{i}\theta} g(\theta) d\theta \right]^n + \cdots + c \int_{\mathbb{T}} 2n!\mathrm{e}^{-\mathrm{i}n\theta} g(\theta) d\theta \right\}. \end{aligned}$$

Inductively, it shows that $f(0) = 1, f^{(k)}(0) = 0$ for all $k \leqslant n$ is equivalent to $\int g(\theta)\mathrm{e}^{-\mathrm{i}k\theta} d\theta = 0$ for all $k \leqslant n$. □

Things become more complicated in two variables. It is well known that every real harmonic function in $\mathbb{D}$ is the real part of a holomorphic function. However, a distinction arises in bidisk. For example, $z\bar{w}$ is harmonic in each variable, but it is not the real part of any holomorphic function. We use $\mathbb{R}P(\mathbb{D}^2)$ to denote the class of all functions in $\mathbb{D}^2$, which are real parts of holomorphic functions. For $f \in H^2(\mathbb{D}^2)$, define

$$f^*(t) = \lim_{r_1, r_2 \to 1^-} f(r_1 t_1, r_2 t_2)$$

at every $t \in \mathbb{T}^2$ where this radial limit exists, and we let $P(\log |f^*|)$ be the Poisson integral of $\log |f^*|$ (which is in $L^1(\mathbb{T}^2)$). If f does not vanish on $\mathbb{D}^2$, then we let $u[f]$ be the least 2-harmonic majorant of $\log |f|$. We refer the readers to [15] for details. Now we consider $P(L^\infty_{\mathbb{R}}(\mathbb{T}^2)) \cap \mathbb{R}P(\mathbb{D}^2)$. The following fact is analogous to Lemma 7.2.

Lemma 7.4. *$(H^\infty(\mathbb{D}^2))^{-1}/\mathbb{T}$ is isomorphic to $P(L^\infty_{\mathbb{R}}(\mathbb{T}^2)) \cap \mathbb{R}P(\mathbb{D}^2)$.*

Proof. For $f \in (H^\infty(\mathbb{D}^2))^{-1}$, we define $\Psi(f) = P(\log|f^*|)$. Clearly, $P(\log|f^*|) \in P(L^\infty_{\mathbb{R}}(\mathbb{T}^2))$. We need to show that $P(\log|f^*|) \in \mathbb{R}P(\mathbb{D}^2)$. By [15, p. 46, Theorem 3.3.5],

$$u[f] = P(\log|f^*| + d\sigma_f),$$

for some real singular measure $d\sigma_f \leqslant 0$. Since $f \in (H^\infty(\mathbb{D}^2))^{-1}$, $|f(z)|$ is bounded away from 0. Hence $\log f$ is holomorphic, which in particular implies $u[f] = \log|f| \in \mathbb{R}P(\mathbb{D}^2)$. It then only remains to show that $d\sigma_f = 0$. To this end, one observes that

$$\begin{aligned} 0 &= \log|f| + \log|f^{-1}| \\ &= u[f] + u[f^{-1}] \\ &= P(\log|f^*| + d\sigma_f + \log|(f^{-1})^*| + d\sigma_{f^{-1}}) \\ &= P(d\sigma_f + d\sigma_{f^{-1}}). \end{aligned}$$

Since both $d\sigma_f$ and $d\sigma_{f^{-1}}$ are non-positive, this implies $d\sigma_f = d\sigma_{f^{-1}} = 0$. Therefore, $P(\log|f^*|) = u[f] = \log|f| \in \mathbb{R}P(\mathbb{D}^2)$, e.g., Ψ is well-defined. It is easy to see that Ψ is a homomorphism. Moreover, $P(\log|f^*|) = 0$ if and only if $|f| \equiv 1$ on $\mathbb{D}^2$, which means f is a constant of modulus 1, so $\ker\Psi = \mathbb{T}$.

To see Ψ is surjective, we let $u = P[g]$ for some function $g \in L^\infty_{\mathbb{R}}(\mathbb{T}^2)$. Then u is bounded and 2-harmonic. And by maximum principle $\|u\|_{\infty,\mathbb{D}} \leqslant \|g\|_\infty$. If in addition, $u \in \mathbb{R}P(\mathbb{D}^2)$, then there exists a unique real valued function v (called u's harmonic conjugate) with $v(0) = 0$, such that $u + \mathrm{i}v$ is holomorphic. Let $f = \mathrm{e}^{u+\mathrm{i}v}$. Then $\|f\|_\infty = \|\mathrm{e}^{u+\mathrm{i}v}\|_\infty = \mathrm{e}^{\|u\|_\infty}$, and $|f(z)| = \mathrm{e}^{u(z)} \geqslant \mathrm{e}^{-\|u\|_\infty}$. Hence $f \in (H^\infty(\mathbb{D}^2))^{-1}$, so $|f^*|$ exists almost everywhere on $\mathbb{T}^2$ and $\log|f^*| = u^* = g$ a.e. on $\mathbb{T}^2$. So $\Psi(f) = u$, and it concludes that Ψ is surjective. □

Now let us consider the following chain of additive subgroups:

$$\mathcal{N}^{n_1,n_2} = \left\{ g \in L^\infty_{\mathbb{R}}(\mathbb{T}^2) : P[g] \in \mathbb{R}P \text{ and } \int_{\mathbb{T}^2} g(\xi)\bar{\xi}_i^{k_i} dm(\xi) = 0,\ 0 \leqslant k_i \leqslant n_i,\ i = 1,2 \right\},$$

where $dm(\xi)$ stands for the normalized Lebesgue measure on $\mathbb{T}^2$.

Proposition 7.5. *The restriction of Ψ to $\mathcal{G}^{n_1,n_2}$ is an isomorphism from $\mathcal{G}^{n_1,n_2}$ to $\mathcal{N}^{n_1,n_2}$.*

Proof. Assume $f \in \mathcal{G}^{n_1,n_2}$. First, by direct computation, we see that $f(0,0) = 1$ and $\frac{\partial^k f}{\partial z^k}|_{(0,0)} = 0$, $1 \leqslant k \leqslant n_1$ if and only if $\frac{\partial^k \log f}{\partial z^k}|_{(0,0)} = 0$, $0 \leqslant k \leqslant n_1$.

We write $\log f = \log|f| + \mathrm{i}\mathrm{Arg}f := u + \mathrm{i}v$. Note that $v(0,0) = 0$ and $u = \Psi(f)$ by the proof of Lemma 7.4. Since $\log f$ is holomorphic, $\frac{\partial u}{\partial \bar{z}} + \mathrm{i}\frac{\partial v}{\partial \bar{z}} = 0$, by taking conjugate we have $\frac{\partial u}{\partial z} - \mathrm{i}\frac{\partial v}{\partial z} = 0$ on $\mathbb{D}^2$. Hence,

$$\frac{\partial^k \log f}{\partial z^k} = \frac{\partial^k u}{\partial z^k} + \mathrm{i}\frac{\partial^k v}{\partial z^k} = 2\frac{\partial^k u}{\partial z^k}.$$

So in particular $\frac{\partial^k \log f}{\partial z^k}|_{(0,0)} = 0$ if and only if $\frac{\partial^k u}{\partial z^k}|_{(0,0)} = 0$. Since

$$u(z,w) = P[\log|f^*|] = \int_{\mathbb{T}^2} \frac{(1-|z|^2)(1-|w|^2)}{|1-z\bar{\xi_1}|^2|1-w\bar{\xi_2}|^2} \log|f^*(\xi)| dm(\xi),$$

we have

$$\frac{\partial^k u}{\partial z^k}(z,w) = k! \int_{\mathbb{T}^2} \frac{(1-|z|^2)(1-|w|^2)}{(1-z\bar{\xi_1})^{k+1}(1-\bar{z}\xi_1)|1-w\bar{\xi_2}|^2} (\log|f^*(\xi)|)\bar{\xi_1}^k dm(\xi),$$

which implies

$$\frac{\partial^k u}{\partial z^k}(0,0) = k! \int_{\mathbb{T}^2} (\log|f^*(\xi)|)\bar{\xi_1}^k dm(\xi).$$

Therefore, $f(0,0) = 1$ and $\frac{\partial^k f}{\partial z^k}|_{(0,0)} = 0$, $1 \leqslant k \leqslant n_1$ if and only if

$$\int_{\mathbb{T}^2} (\log|f^*(\xi)|)\bar{\xi_1}^k dm(\xi) = 0, \quad 0 \leqslant k \leqslant n_1.$$

Parallelly, $f(0,0)=1$ and $\frac{\partial^k f}{\partial w^k}|_{(0,0)}=0$, $1\leqslant k\leqslant n_2$ if and only if

$$\int_{\mathbb{T}^2}(\log|f^*(\xi)|)\bar{\xi_2}^k dm(\xi)=0,\quad 0\leqslant k\leqslant n_2.$$

By Lemma 7.4, Ψ is already an isomorphism from $(H^\infty(\mathbb{D}^2))^{-1}/\mathbb{T}$ to $P(L^\infty_{\mathbb{R}}(\mathbb{T}^2))\cap\mathbb{R}P$, and Ψ is an isomorphism from $\mathcal{G}^{n_1,n_2}$ to $\mathcal{N}^{n_1,n_2}$. □

8 Little Lorentz group

A natural question about Lorentz group $\mathcal{G}(M)$ is whether it contains a nontrivial abelian subgroup, and if it does, whether the subgroup is an invariant for submodules. The following notion seems natural. We recall that R_φ is the restriction of multiplication by φ to M.

Definition 8.1. Let M be a submodule of $H^2(\mathbb{D}^2)$. Then

$$\mathcal{G}_0(M)=\{\varphi\in(H^\infty(\mathbb{D}^2))^{-1}:{R_\varphi}^*C_MR_\varphi=C_M\}$$

is called the little Lorentz group of M.

Clearly, if we identify φ with R_φ then $\mathcal{G}_0(M)$ can be viewed as an abelian subgroup of $\mathcal{G}(M)$.

Proposition 8.2. *$\mathcal{G}_0(M)$ is non-trivial for any M and it is a proper subgroup in $\mathcal{G}(M)$.*

Proof. First, using the fact that $C_M=I-R_zR_z^*-R_wR_w^*+R_zR_wR_z^*R_w^*$ it is not hard to check that $zwM\subset\ker C_M$, or equivalently $\overline{zw}\mathcal{K}$ is orthogonal to M. Fixing an $|\alpha|<1$ and letting $\varphi=1+\alpha zw\in(H^\infty(\mathbb{D}^2))^{-1}$, then we have

$$\begin{aligned}{R_\varphi}^*C_MR_\varphi g&={R_\varphi}^*C_M(g+\alpha zwg)\\&={R_\varphi}^*C_Mg\\&=P_M(1+\overline{\alpha zw})C_Mg\\&=C_Mg,\end{aligned}$$

for any $g\in M$. So $\mathcal{G}_0(M)$ is always non-trivial. The fact that it is proper in $\mathcal{G}(M)$ is not hard to see. But it has interesting details. We consider three cases. If C_M is of rank 1, then M is of the form $\theta H^2(\mathbb{D}^2)$ for some inner function θ, and we shall compute $\mathcal{G}_0(M)$ in the next example, and it will be evident that it is proper. If $1<\mathrm{rank}C_M<\infty$, then $\mathcal{G}(M)$ contains a nontrivial element of order 2 by Section 5 which is of course not in $\mathcal{G}_0(M)$. Hence the latter is proper in the former. In the case $\mathrm{rank}C_M=\infty$, we have no interesting element to display but just resort to the trivial one $I_\mathcal{K}\oplus -I_{\ker C}$. It is clearly in $\mathcal{G}(M)$ and of order 2. □

Now we compute some examples.

Example 8.3. Consider $M=\theta H^2(\mathbb{D}^2)$ for some inner function θ. Since

$$K^M(\lambda,\eta;z,w)=\frac{\overline{\theta(\lambda,\eta)}\theta(z,w)}{(1-\overline{\lambda}z)(1-\overline{\eta}w)},$$

$G^M(\lambda,z)=\overline{\theta(\lambda)}\theta(z)$, and hence $C_Mf=\langle f,\theta\rangle\theta$. By Proposition 6.2, $T^*C_MT=C_M$ if and only if $T^*\theta=\eta\theta$, where $|\eta|=1$. In other words, $\mathcal{G}(M)$ is the group of all $T\in B^{-1}(M)$ such that T^* fixes θ up to a unimodular scalar. In particular, it is non-abelian. Let $f=\theta h\in M$ for some $h\in H^2(\mathbb{D}^2)$. Then

$$C_Mf=\langle\theta h,\theta\rangle\theta=h(0,0)\theta.$$

In addition,

$$\begin{aligned}{R_\varphi}^*C_MR_\varphi f&={R_\varphi}^*(\langle\varphi\theta h,\theta\rangle\theta)\\&=\varphi(0,0)h(0,0)P_M(\overline{\varphi}\theta)\\&=|\varphi(0,0)|^2h(0,0)\theta.\end{aligned}$$

Since ${R_\varphi}^* C_M R_\varphi = C_M$, we have $\mathcal{G}_0(M) = \{\varphi \in (H^\infty(\mathbb{D}^2))^{-1} : |\varphi(0,0)| = 1\}$, which is $\mathbb{T} \times \mathcal{G}^{0,0}$.

Now we take another look at the submodule in Example 5.1, i.e.,

$$M = p(z)H^2(\mathbb{D}^2) + q(w)H^2(\mathbb{D}^2),$$

where $q(z)$ and $q(w)$ are nontrival one-variable inner functions. In this case $\mathrm{rank} C_M = 3$ and

$$C_M f = \langle f, p\rangle p + \langle f, q\rangle q - \langle f, pq\rangle pq.$$

To make the computation simpler, we assume that $p(0) = q(0) = 0$, in which case $\{p, q, pq\}$ is an orthonormal basis for $\mathcal{K}$. To proceed, we need to introduce two evaluation operators mapping $H^2(\mathbb{D}^2)$ into $H^2(\mathbb{D})$. Define

$$L(0)h = h(0, w), \quad R(0)h = h(z, w), \quad h \in H^2(\mathbb{D}^2).$$

The evaluation operators were defined and studied in [22], and they played important roles in proving the Hilbert-Schmidtness of C_M (see [19, 20]). For simplicity, we denote $L(0)$ by L and $R(0)$ by R when there is no confusion. Note that R and R_φ are different operators. It is easy to see that $L(\varphi h) = L\varphi Lh$, and $R(\varphi h) = R\varphi Rh$. Also recall that T_f stands for the Toeplitz operator. We state the result as follows.

Theorem 8.4. *Assume M, p and q are as above. Consider $\varphi \in (H^\infty)^{-1}$ with $\varphi(0,0) = 1$. Then $\varphi \in \mathcal{G}_0(M)$ if and only if*

$$T^*_{L\varphi} q = q, \quad T^*_{R\varphi} p = p.$$

Proof. For $f = ph_1 + qh_2 \in M$, where h_1 and h_2 are arbitrary functions in $H^2(\mathbb{D}^2)$, we have

$$\begin{aligned} C_M f &= \langle ph_1 + qh_2, p\rangle p + \langle ph_1 + qh_2, q\rangle q - \langle ph_1 + qh_2, pq\rangle pq \\ &= (h_1(0,0) + q(0)\langle Rh_2, p\rangle)p + (h_2(0,0) + p(0)\langle Lh_1, q\rangle)q \\ &\quad - (\langle Lh_1, q\rangle + \langle Rh_2, p\rangle)pq \\ &= h_1(0,0)p + h_2(0,0)q - (\langle Lh_1, q\rangle + \langle Rh_2, p\rangle)pq. \end{aligned}$$

Then

$$\begin{aligned} R^*_\varphi C_M R_\varphi f &= R^*_\varphi C_M(\varphi f) \\ &= h_1(0,0)R^*_\varphi p + h_2(0,0)R^*_\varphi q - (\langle L(\varphi h_1), q\rangle + \langle R(\varphi h_2), p\rangle)R^*_\varphi pq. \end{aligned} \tag{8.1}$$

Observe that in order that $R^*_\varphi C_M R_\varphi = C_M$, $\mathcal{K}$ should be invariant under R^*_φ. Hence, $R^*_\varphi p$, $R^*_\varphi q$ and $R^*_\varphi(pq)$ are linear combinations of p, q and pq. Since $\{p, q, pq\}$ is an orthonormal basis of $\mathcal{K}$ and $\varphi(0,0) = 1$, one verifies that

$$\begin{aligned} &R^*_\varphi p = p, \\ &R^*_\varphi q = q, \\ &R^*_\varphi pq = \langle q, L\varphi\rangle p + \langle p, R\varphi\rangle q + pq. \end{aligned}$$

Putting these into (8.1), we have

$$\begin{aligned} &R^*_\varphi C_M R_\varphi f \\ &\quad = h_1(0,0)p + h_2(0,0)q - (\langle L(\varphi h_1), q\rangle + \langle R(\varphi h_2), p\rangle)(\langle q, L\varphi\rangle p + \langle p, R\varphi\rangle q + pq) \\ &\quad = [h_1(0,0) - (\langle L(\varphi h_1), q\rangle + \langle R(\varphi h_2), p\rangle)\langle q, L\varphi\rangle]p \\ &\qquad + [h_2(0,0) - (\langle L(\varphi h_1), q\rangle + \langle R(\varphi h_2), p\rangle)\langle p, R\varphi\rangle]q \\ &\qquad - (\langle L(\varphi h_1), q\rangle + \langle R(\varphi h_2), p\rangle)pq. \end{aligned}$$

Comparing the coefficients of pq in $C_M f$ with those in $R^*_\varphi C_M R_\varphi f$, we have

$$\langle L(\varphi h_1), q\rangle + \langle R(\varphi h_2), p\rangle = \langle Lh_1, q\rangle q + \langle Rh_2, p\rangle.$$

Since h_1 and h_2 are arbitrary, we have

$$\langle L(\varphi h_1), q\rangle = \langle Lh_1, q\rangle,$$
$$\langle R(\varphi h_2), p\rangle = \langle Rh_2, p\rangle,$$

or equivalently,

$$L(\varphi - 1)Lh_1 \perp q, \quad R(\varphi - 1)Rh_2 \perp p, \quad \forall\, h_1,\, h_2 \in H^2(\mathbb{D}^2).$$

Since $L(H^2(\mathbb{D}^2)) = H^2(\mathbb{D})$ and $R(H^2(\mathbb{D}^2)) = H^2(\mathbb{D})$ (in different variables), the Toeplitz operators satisfy

$$T^*_{L\varphi-1}q = 0, \quad T^*_{R\varphi-1}p = 0,$$

e.g.,

$$T^*_{L\varphi}q = q, \quad T^*_{R\varphi}p = p. \tag{8.2}$$

For the other direction, if (8.2) holds, then tracing the above arguments upward, we have

$$\langle L(\varphi h_1), q\rangle = \langle Lh_1, q\rangle,$$
$$\langle R(\varphi h_2), p\rangle = \langle Rh_2, p\rangle,$$

for every $h_1,\, h_2 \in H^2(\mathbb{D}^2)$. Hence, the coefficients of pq in $C_M f$ and those in $R^*_\varphi C_M R_\varphi f$ agree. In particular, if we set $h_1 = h_2 = 1$ in the above two equations, then they imply

$$\langle L\varphi,\ q\rangle = \overline{q(0)} = 0$$

and

$$\langle R\varphi,\ p\rangle = \overline{p(0)} = 0.$$

Putting these into $R^*_\varphi C_M R_\varphi f$ we see that the coefficients of p and q agree with those in $C_M f$. Hence $\varphi \in \mathcal{G}_0(M)$. □

It is worth noting that in Theorem 8.4 if we write $\varphi = 1 + \tau$, then $T^*_{L\tau}q = 0$ and $T^*_{R\tau}p = 0$. It is not hard to check that the set of τ satisfying these equations forms an ideal, say J in H^∞. Then by the remarks preceding Theorem 7.1, $(1 + J) \cap (H^\infty)^{-1}$ is indeed a group.

Example 8.5. Let us consider a concrete situation where $p = z^m$ and $q = w^n$, where m and n are positive integers. By Theorem 8.4, $\varphi = 1 + \tau$ is in $\mathcal{G}_0$ if and only if $T^*_{L\tau}w^n = 0$ and $T^*_{R\tau}z^m = 0$. We write $\tau(z, w) = R\tau(z) + w\eta$ for some $\eta \in H^\infty$. Then $T^*_{R\tau}z^m = 0$ if and only if $\langle z^m, (R\tau)h\rangle = 0$ for all $h \in H^2(\mathbb{D})$, and this happens if and only if $R\tau$ has a factor z^k for some $k > m$. Since $\tau(z, w) = R\tau(z) + w\eta$, this is the case if and only if $\frac{\partial^i\tau}{\partial z^i}\,|_{(0,0)} = 0$, for all $0 \leqslant i \leqslant m$. Likewise, $T^*_{L\tau}w^n = 0$ if and only if $\frac{\partial^i\tau}{\partial w^i}\,|_{(0,0)} = 0$, for all $0 \leqslant i \leqslant n$. In conclusion, $\mathcal{G}_0(M) = \mathbb{T} \times \mathcal{G}^{m,n}$ for the submodule $M = z^m H^2(\mathbb{D}^2) + w^n H^2(\mathbb{D}^2)$.

9 Little Lorentz group and unitary equivalence

In this section, we study the relationship between little Lorentz group and unitary equivalence. First, two submodules M_1 and M_2 are said to be unitarily equivalent if there is a unitary module map between them. Unitary equivalence is well studied (see [1, 5, 6, 9]). We will prove that little Lorentz group is an invariant under unitary equivalence of submodules. However, it is not a complete invariant, as we shall see through a somewhat complicated example.

Proposition 9.1. *If two submodules M_1 and M_2 are unitarily equivalent, then $\mathcal{G}_0(M_1) = \mathcal{G}_0(M_2)$.*

Proof. Suppose $U : M_1 \to M_2$ is a unitary module map. For $\varphi \in H^\infty$, we let R^i_φ denote the restrictions of T_φ to submodules M_i, where $i = 1, 2$. Then $UR^1_\varphi = R^2_\varphi U$, and therefore

$$\begin{aligned} C_{M_1} &= I - R^1_z(R^1_z)^* - R^1_w(R^1_w)^* + R^1_z R^1_w (R^1_z)^*(R^1_w)^* \\ &= I - U^*R^2_z(R^2_z)^*U - U^*R^2_w(R^2_w)^*U + U^*R^2_z R^2_w(R^2_z)^*(R^2_w)^*U \\ &= U^*(I - R^2_z R^2_z - R^2_w(R^2_w)^* + R^2_z R^2_w (R^2_z)^*(R^2_w)^*)U \\ &= U^* C_{M_2} U. \end{aligned}$$

For every $\varphi \in \mathcal{G}_0(M_2)$, we have ${R_\varphi^2}^* C_{M_2} R_\varphi^2 = C_{M_2}$. Therefore,

$$\begin{aligned}(R_\varphi^1)^* C_{M_1} R_\varphi^1 &= (R_\varphi^1)^* U^* C_{M_2} U R_\varphi^1 \\ &= U^*(R_\varphi^2)^* C_{M_2} R_\varphi^2 U \\ &= U^* C_{M_2} U = C_{M_1}.\end{aligned}$$

Hence, $\varphi \in \mathcal{G}_0(M_1)$. Similarly, for any $\varphi \in \mathcal{G}_0(M_1)$, φ is also in $\mathcal{G}_0(M_2)$. □

However, the converse of Proposition 9.1 is not true. Let us take another look at Example 5.2. Let

$$M = \theta\bigg(H^2(\mathbb{D}^2) \oplus \bigoplus_{j=0}^{\infty} \mathbb{C} z^j \overline{w} K_{rw}(z)\bigg),$$

where

$$K_{rw}(z) = \frac{\sqrt{1-r^2}}{1 - r\overline{w}z}, \quad 0 \leqslant |r| < 1$$

and θ is an inner function satisfying $\theta/(w - rz) \in H^2(\mathbb{D}^2)$.

Lemma 9.2. *M is not unitarily equivalent to $M^{1,1} := zH^2(\mathbb{D}^2) + wH^2(\mathbb{D}^2)$.*

Proof. Denote $h = \theta/(w - rz)$. Agrawal et al. [1] proved that two submodules M_1 and M_2 satisfying $M_2 \subseteq M_1$ are unitarily equivalent if and only if $M_2 = \eta M_1$, for some inner function θ. Without loss of generality, let $h(0,0) \neq 0$. In [21], it is shown that if M is a submodule that contains two nontrivial one variable functions, then a submodule N is unitarily equivalent to M if and only if $N = \eta M$ for some inner function η. Hence, if $M^{1,1}$ is unitarily equivalent to M, we have $M = \eta M^{1,1}$. So for any $f \in H^2(\mathbb{D}^2)$, there exist $f_1 \in H^2(\mathbb{D}^2)$ and $f_2 \in H^2(\mathbb{D}^2) \ominus zH^2(\mathbb{D}^2)$, such that

$$\bigg(h(w - rz)f + \sum_{j=0}^{\infty} a_j z^j h\bigg) = \eta(zf_1 + wf_2).$$

However, this is not true when $a_0 \neq 0$, contradiction. □

The next fact is a bit unexpected. It indicates that this submodule has the same little Lorent group as that of $M^{1,1}$ (see Example 8.5).

Proposition 9.3. *For the submodule M in Example 5.2, $\mathcal{G}_0(M) = \mathbb{T} \times \mathcal{G}^{1,1}$.*

Proof. In Example 5.2, we have that

$$C_M = \theta \otimes \theta + \theta\overline{w}K_{rw}(z) \otimes \theta\overline{w}K_{rw}(z) - \theta K_{rw}(z) \otimes \theta K_{rw}(z).$$

Consider $f = \theta h_1 + \sum_{j=0}^{\infty} \theta a_j z^j \overline{w} K_{rw}(z) \in M$, where $h_1 \in H^2(\mathbb{D}^2)$.

We have that

$$\begin{aligned} C_M f &= \bigg\langle \theta h_1 + \sum_{j=0}^{\infty} \theta a_j z^j \overline{w} K_{rw}(z), \theta \bigg\rangle \theta \\ &\quad + \bigg\langle \theta h_1 + \sum_{j=0}^{\infty} \theta a_j z^j \overline{w} K_{rw}(z), \theta\overline{w}K_{rw}(z) \bigg\rangle \theta\overline{w}K_{rw}(z) \\ &\quad - \bigg\langle \theta h_1 + \sum_{j=0}^{\infty} \theta a_j z^j \overline{w} K_{rw}(z), \theta K_{rw}(z) \bigg\rangle \theta K_{rw}(z) \\ &= h_1(0,0)\theta + \bigg\langle \sum_{j=0}^{\infty} a_j z^j K_{rw}(z), K_{rw}(z) \bigg\rangle \theta\overline{w}K_{rw}(z) \\ &\quad - \bigg(h_1(0,0)\sqrt{1-r^2} + \bigg\langle \theta h_1 + \sum_{j=0}^{\infty} a_j z^j \overline{w} K_{rw}(z), K_{rw}(z) \bigg\rangle \theta K_{rw}(z) \bigg) \\ &= h_1(0,0)\theta + a_0 \theta\overline{w}K_{rw}(z) - (h_1(0,0)\sqrt{1-r^2} + a_1 r)\theta K_{rw}(z). \end{aligned}$$

It follows that

$$\begin{aligned}C_M R_\varphi f &= \Big\langle \varphi\theta h_1 + \sum_{j=0}^{\infty} \varphi\theta a_j z^j \overline{w} K_{rw}(z), \theta \Big\rangle \theta \\ &\quad + \Big\langle \varphi\theta h_1 + \sum_{j=0}^{\infty} \varphi\theta a_j z^j \overline{w} K_{rw}(z), \theta\overline{w}K_{rw} \Big\rangle \theta\overline{w}K_{rw}(z) \\ &\quad - \Big\langle \varphi\theta h_1 + \sum_{j=0}^{\infty} \varphi\theta a_j z^j \overline{w} K_{rw}(z), \theta K_{rw} \Big\rangle \theta K_{rw}(z) \\ &= \varphi(0,0)h_1(0,0)\theta + \Big\langle \sum_{j=0}^{\infty} \varphi a_j z^j K_{rw}(z), K_{rw} \Big\rangle \theta\overline{w}K_{rw}(z) \\ &\quad - \varphi(0,0)h_1(0,0)\sqrt{1-r^2}\theta K_{rw}(z) - \Big\langle \sum_{j=0}^{\infty} \varphi a_j z^j \overline{w} K_{rw}(z), K_{rw}(z) \Big\rangle \theta K_{rw}(z) \\ &\quad + \frac{\partial\varphi}{\partial w}(0,0) a_0 \sqrt{1-r^2}\theta.\end{aligned}$$

Using the fact that K is the reproducing kernel, we have

$$\begin{aligned}C_M R_\varphi f &= \varphi(0,0)h_1(0,0)\theta + \varphi(0,0)a_0\theta\overline{w}K_{rw}(z) \\ &\quad - \Big(\varphi(0,0)a_1 r + \frac{\partial\varphi}{\partial w}(0,0)a_0 + \varphi(0,0)h_1(0,0)\sqrt{1-r^2} + a_0\frac{\partial\varphi}{\partial z}(0,0)r\Big)\theta K_{rw}(z) \\ &\quad + \frac{\partial\varphi}{\partial w}(0,0)a_0\sqrt{1-r^2}\theta.\end{aligned}$$

In order to compute $R_\varphi^* C_M R_\varphi f$, we verify that

$$\begin{aligned}R_\varphi^*\theta &= \sum_{i,j}\langle\overline{\varphi}\theta, \theta z^i w^j\rangle \theta z^i w^j + \sum_j \langle \overline{\varphi}\theta, \theta z^j \overline{w} K_{rw}(z)\rangle \theta z^j \overline{w} K_{rw}(z) \\ &= \overline{\varphi(0,0)}\theta + \overline{\frac{\partial\varphi}{\partial w}}(0,0)\sqrt{1-r^2}\theta\overline{w}K_{rw}(z), \\ R_\varphi^*\theta\overline{w}K_{rw} &= \sum_{i,j}\langle\overline{\varphi}\theta\overline{w}K_{rw}, \theta z^i w^j\rangle \theta z^i w^j + \sum_j \langle \overline{\varphi}\theta\overline{w}K_{rw}, \theta z^j \overline{w} K_{rw}(z)\rangle \theta z^j \overline{w} K_{rw}(z) \\ &= \overline{\varphi(0,0)}\theta\overline{w}K_{rw}\end{aligned}$$

and

$$\begin{aligned}R_\varphi^*\theta K_{rw} &= \sum_{i,j}\langle\overline{\varphi}\theta K_{rw}, \theta z^i w^j\rangle \theta z^i w^j + \sum_j \langle \overline{\varphi}\theta K_{rw}, \theta z^j \overline{w} K_{rw}(z)\rangle \theta z^j \overline{w} K_{rw}(z) \\ &= \overline{\varphi(0,0)}\sqrt{1-r^2}\theta + \overline{\varphi(0,0)}r\theta z\overline{w}K_{rw} + \overline{\frac{\partial\varphi}{\partial z}}(0,0)r\theta\overline{w}K_{rw} + \overline{\frac{\partial\varphi}{\partial w}}(0,0)\theta\overline{w}K_{rw}.\end{aligned}$$

Since $R_\varphi^* C_M R_\varphi f = C_M f$, and θ, $\theta\overline{w}K_{rw}$ and $\theta\sum_{n=1}^{\infty}\sqrt{1-r^2}(r\overline{w}z)^n$ are orthogonal, by comparing the corresponding coefficients, we have that for the coefficients of θ,

$$\begin{aligned}&h_1(0,0)r^2 - a_1 r\sqrt{1-r^2} \\ &\quad = |\varphi(0,0)|^2 h_1(0,0)r^2 - |\varphi(0,0)|^2 a_1 r\sqrt{1-r^2} - \overline{\varphi(0,0)}\frac{\partial\varphi}{\partial z}(0,0)a_0 r\sqrt{1-r^2};\end{aligned}$$

for the coefficients of $\theta\overline{w}K_{rw}$,

$$\begin{aligned}a_0 &= |\varphi(0,0)|^2 a_0 - \left|\frac{\partial\varphi}{\partial w}(0,0)\right|^2 a_0 r^2 - \left|\frac{\partial\varphi}{\partial z}(0,0)\right|^2 a_0 r^2 - \overline{\frac{\partial\varphi}{\partial w}(0,0)}\frac{\partial\varphi}{\partial z}(0,0)a_0 r \\ &\quad - \overline{\frac{\partial\varphi}{\partial z}(0,0)}\frac{\partial\varphi}{\partial w}(0,0)a_0 r - \varphi(0,0)\overline{\frac{\partial\varphi}{\partial z}(0,0)}a_1 r^2 - \varphi(0,0)\overline{\frac{\partial\varphi}{\partial w}(0,0)}a_1 r\end{aligned}$$

$$- \varphi(0,0)\overline{\frac{\partial\varphi}{\partial z}(0,0)}h_1(0,0)r\sqrt{1-r^2};$$

and for the coefficients of $\theta\sum_{n=1}^{\infty}\sqrt{1-r^2}(r\overline{w}z)^n$,

$$\begin{aligned} &h_1(0,0)\sqrt{1-r^2}+a_1r \\ &\quad = |\varphi(0,0)|^2h_1(0,0)\sqrt{1-r^2}+|\varphi(0,0)|^2a_1r+\overline{\varphi(0,0)}\frac{\partial\varphi}{\partial z}(0,0)a_0r+\overline{\varphi(0,0)}\frac{\partial\varphi}{\partial w}(0,0)a_0. \end{aligned}$$

By choosing $a_0=0$ and $a_1=0$, we get $|\varphi(0,0)|=1$ from the last equality. Plugging it back into the last equation, and using the fact that a_0 and a_1 are arbitrary, we have $\frac{\partial\varphi}{\partial z}(0,0)=0$ and $\frac{\partial\varphi}{\partial w}(0,0)=0$. Conversely, if $|\varphi(0,0)|=1$, $\frac{\partial\varphi}{\partial z}(0,0)=0$ and $\frac{\partial\varphi}{\partial w}(0,0)=0$, then one checks that all equations above hold. In conclusion, $\mathcal{G}_0(M)=\mathbb{T}\times\mathcal{G}^{1,1}$. □

10 Concluding remarks

The goal of this paper is to introduce Kreĭn space, Lorentz group and little Lorentz group into the study of submodules in $H^2(\mathbb{D}^2)$. Clearly, the defect operator C_M is a pivot in this attempt. C_M can be defined for many other reproducing kernel Hilbert spaces, so at least some parallel work can be done in more general settings. How will the Lorentz group and the little Lorents group change with respect to the change of settings is an appealing question, and some work is on the horizon. However, we conclude this paper by posting some more immediate problems.

(1) Since the defect operator may be non-compact (albeit hard to find), will $\mathcal{G}(M)$ and $\mathcal{G}_0(M)$ be able to detect the non-compactness of C_M?

(2) For an ideal $J\subset H^\infty$, is the rank of J an invariant for the group $\mathcal{G}(J)$?

(3) Is $\mathcal{G}_0(M)$ maximal abelian in $\mathcal{G}(M)$?

(4) Is the converse of Proposition 5.3 true?

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References

1 Agrawal O P, Clark D N, Douglas R G. Invariant subspaces in the polydisk. Pacific J Math, 1986, 121: 1–11
2 Ahmed Z, Jain S R. Pseudounitary symmetry and the Gaussian pesudounitary ensemble of random matrices. Phys Rev E (3), 2003, 67: 045106
3 Beurling A. On two problems concerning linear transformations in Hilbert space. Acta Math, 1949, 81: 239–255
4 Douglas R G, Paulsen V I. Hilbert modules over function algebras. In: Advances in Invariant Subspaces and Other Results of Operator Theory. Operator Theory: Advances and Applications, vol. 17. Basel: Birkhäuser, 1986, 125–139
5 Douglas R G, Paulsen V I, Sah C, et al. Algebraic reduction and rigidity for Hilbert modules. Amer J Math, 1995, 117: 75–92
6 Douglas R G, Sarkar J. On unitarily equivalent submodules. Indiana Univ Math J, 2008, 57: 2729–2743
7 Dritschel M A, Rovnyak J. Operators on indefinite inner product spaces. In: Lectures on Operator Theory and Its Applications. Fields Institute Monographs, vol. 3. Providence: Amer Math Soc, 1996, 141–232
8 Garnett J. Bounded Analytic Functions, 1st ed. New York: Springer-Verlag, 2007
9 Guo K Y. Characteristic spaces and rigidity for analytic Hilbert modules. J Funct Anal, 1999, 163: 133–151
10 Guo K Y, Yang R W. The core function of submodules over the bidisk. Indiana Univ Math J, 2004, 53: 205–222
11 Izuchi K J, Izuchi K H. Rank-one commutators on invariant subspaces of the Hardy space on the bidisk. J Math Anal Appl, 2006, 316: 1–8
12 Izuchi K J, Nakazi T, Seto M. Backward shift invariant subspaces in the bidisc II. J Operator Theory, 2004, 51: 361–376
13 Izuchi K J, Nakazi T, Seto M. Backward shift invariant subspaces in the bidisc III. Acta Sci Math (Szeged), 2004, 70: 727–749
14 Izuchi K J, Ohno S. Selfadjoint commutators and invariant subspaces on the torus. J Operator Theory, 1994, 31: 189–204

15 Rudin W. Function Theory in Polydiscs. New York-Amsterdam: Benjamin, 1969
16 Seto M. Infinite sequences of inner functions and submodules in $H^2(\mathbb{D}^2)$. J Operator Theory, 2009, 61: 75–86
17 Seto M, Yang R W. Inner sequence based invariant subspaces in $H^2(\mathbb{D}^2)$. Proc Amer Math Soc, 2007, 135: 2519–2526
18 Wu Y. Lorentz group of submodules in $H^2(\mathbb{D}^2)$. Dissertation. Albany: State University of New York at Albany, 2015
19 Yang R W. Operator theory in the Hardy space over the bidisk (III). J Funct Anal, 2001, 186: 521–545
20 Yang R W. The core operator and congruent submodules. J Funct Anal, 2005, 228: 469–489
21 Yang R W. Hilbert-Schmidt submodules and issues of unitary equivalence. J Operator Theory, 2005, 53: 169–184
22 Yang R W. Operator theory in the Hardy space over the bidisk (II). Integral Equations Operator Theory, 2006, 56: 431–449
23 Yang R W. A note on classification of submodules in $H^2(\mathbb{D}^2)$. Proc Amer Math Soc, 2009, 137: 2655–2659