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Entire solution in an ignition nonlocal dispersal equation: Asymmetric kernel

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Abstract This paper mainly focuses on the front-like entire solution of a classical nonlocal dispersal equation with ignition nonlinearity. Especially, the dispersal kernel function J may not be symmetric here. The asymmetry of J has a great influence on the profile of the traveling waves and the sign of the wave speeds, which further makes the properties of the entire solution more diverse. We first investigate the asymptotic behavior of the traveling wave solutions since it plays an essential role in obtaining the front-like entire solution. Due to the impact of $f'(0) = 0$, we can no longer use the common method which mainly depends on Ikehara theorem and bilateral Laplace transform to study the asymptotic rates of the nondecreasing traveling wave and the nonincreasing one tending to 0, respectively, so we adopt another method to investigate them. Afterwards, we establish a new entire solution and obtain its qualitative properties by constructing proper supersolution and subsolution and by classifying the sign and size of the wave speeds.

Keywords entire solution, asymptotic behavior, traveling wave solutions, nonlocal dispersal, asymmetric, ignition

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1 Introduction

In this paper, we are concerned with the following classical nonlocal dispersal equation:

$$u_t(x,t) = (J * u)(x,t) - u(x,t) + f(u(x,t)), \quad (x,t) \in \mathbb{R}^2, \tag{1.1}$$

where $(J * u)(x,t) - u(x,t) = \int_{\mathbb{R}} J(y)[u(x-y,t) - u(x,t)]dy$ is a nonlocal dispersal term, and f is an ignition nonlinearity. They satisfy the following:

(J1) $J \in C(\mathbb{R})$, $J(x) \geqslant 0$, $\int_{\mathbb{R}} J(x)dx = 1$ and $\exists\, \lambda > 0$ such that $\int_{\mathbb{R}} J(x)\mathrm{e}^{\lambda|x|}dx < \infty$.

(FI) $f \in C^2(\mathbb{R})$, $f(0) = f(1) = 0$, $f'(1) < 0$, and there exists $\rho \in (0,1)$ such that $f|_{[0,\rho]} \equiv 0$, $f|_{(\rho,1)} > 0$.

It is well known that the long range dispersal phenomenon is very widespread in the natural life, so in recent years, the nonlocal equations and systems have been widely concerned and studied by more and more researchers. Propagation dynamics is one of the hot research topics; more precisely, it includes the

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traveling wave solution, spreading speed, entire solution and so on. Especially, the entire solution has been widely studied because of its important theoretical and practical significance. From the perspective of dynamical system, the solution of the initial value problem can be seen as a semi-flow or a half-orbit, so we can only determine the state of the solution in $t \in [0, +\infty)$. But an entire solution can be viewed as a full-flow or a full-orbit, so it allows us to study the information of the solution in any space point $x \in \mathbb{R}$ and any time $t \in \mathbb{R}$, which further helps us to understand the transient dynamics and the structure of the global attractors. For more rich entire solutions, one can refer to [5–8, 10, 12, 17, 18] for classical Laplace diffusion equations, and [9, 11, 13, 14, 19, 23] for nonlocal dispersal equations.

Note that the above results for nonlocal equations and systems are all based on the symmetry of the kernel function. However, as we mentioned in the previous work [16, 22], the asymmetric dispersal phenomenon is very common and widespread in reality since there is a formal analogy between asymmetric-nonlocal-dispersal equations and reaction-advection-diffusion equations [4, 16]. For the entire solutions of reaction-advection-diffusion equations, we refer to [5, 8] and the references therein.

In [16, 22], we have investigated the entire solutions of the asymmetric dispersal equation (1.1) with monostable and bistable nonlinearities, respectively. Comparing the symmetric case [9, 14], we found that there exist new types of entire solutions due to the influence of the asymmetry of the kernel function. In addition, catching up phenomenon between traveling wave solutions is very common and qualitative properties of some entire solutions also become very different.

In this paper, we focus our attention on the entire solution of (1.1) with asymmetric kernel function and ignition nonlinearity since there is no result on this issue. We use the front-like entire solution to describe the propagation of flame since different types of entire solutions correspond to different ways of flame propagation. What should be pointed out is that the key step to construct such an entire solution which behaves like interactions of different traveling waves is having a precise information on the asymptotic behaviors of the traveling wave solutions at infinity. Therefore, we take half the length of this paper to investigate them.

A traveling wave solution of (1.1) is a special translation invariant solution of the form $u(x,t) = \phi(\xi)(\xi = x + ct)$ which satisfies

$$\begin{cases} c\phi'(\xi) = J * \phi(\xi) - \phi(\xi) + f(\phi(\xi)), \\ \phi(-\infty) = 0, \quad \phi(+\infty) = 1, \end{cases} \tag{1.2}$$

where $\phi(\pm\infty)$ denotes the limit of $\phi(\xi)$ as $\xi \to \pm\infty$. For asymmetric kernel function J and bistable or ignition nonlinearity, Coville [3] has proved the existence of the traveling wave solution ϕ and the uniqueness of the speed c. The specific results can be stated as follows. About the traveling wave solutions of the asymmetric equation (1.1) with monostable nonlinearity, one can refer to Coville et al. [4], Sun et al. [15] and Yagisita [20].

Theorem 1.1 (See [3])**.** *Assume that J satisfies* (J1) *and let f be of bistable or ignition type. Then there exists a constant $c \in \mathbb{R}$ and a nondecreasing function ϕ such that (ϕ, c) is a solution of* (1.2). *Moreover the speed c is unique.*

In virtue of the asymmetry of the kernel function J, we cannot get a nonincreasing traveling wave solution directly, although we can do this for the symmetric case. However, note that $\hat{u}(x,t) := u(-x,t)$ is a solution of the following nonlocal equation:

$$\hat{u}_t(x,t) = \int_{\mathbb{R}} J(y)\hat{u}(x+y,t)dy - \hat{u}(x,t) + f(\hat{u}(x,t)) \tag{1.3}$$

whenever $u(x,t)$ is a solution of (1.1). Thus the existence of the nondecreasing traveling wave solution of (1.3) from 0 to 1 immediately implies the existence of the nonincreasing traveling wave solution of (1.1) from 1 to 0. Therefore, by using a similar method with that in [3], we obtain a nonincreasing front $\hat{\phi}(\hat{\xi}) := \hat{\phi}(x + \hat{c}t)$ with speed $\hat{c} \in \mathbb{R}$ satisfying

$$\begin{cases} \hat{c}\hat{\phi}' = J * \hat{\phi} - \hat{\phi} + f(\hat{\phi}), \\ \hat{\phi}(-\infty) = 1, \quad \hat{\phi}(+\infty) = 0. \end{cases} \tag{1.4}$$

But here, the two waves $\phi(\xi)$ and $\hat{\phi}(\hat{\xi})$ are likely to be no longer symmetric in shape although they always are symmetric in shape when the kernel function is symmetric.

In order to construct a proper supersolution and further obtain the existence and qualitative properties of a front-like entire solution of (1.1), we must make clear the asymptotic behaviors of the traveling wave solutions $\phi(\xi)$ and $\hat{\phi}(\hat{\xi})$ at infinity. As $\xi \to +\infty$ (or $\hat{\xi} \to -\infty$), since $f'(1) < 0$, we can obtain a precise exponential decay rate of $\phi(\xi)$ (or $\hat{\phi}(\hat{\xi})$) by using similar arguments with those in [22] for the bistable case. However, as $\xi \to -\infty$ (or $\hat{\xi} \to +\infty$), due to the effect of $f'(0) = 0$, we cannot use the same method anymore, so by combining the methods provided in [1,21], we find another way which is mainly based on the constant use of a comparison principle and the construction of appropriate barrier functions. But due to the asymmetry of the kernel function J, the details of the proof are very different. Moreover, in order to ensure J satisfies a proper comparison principle, we give the following assumption. For this comment, one can refer to [2,3] for more details.

(J2) $\exists\, a \leqslant 0 \leqslant b, a \neq b$ such that $J(a) > 0$ and $J(b) > 0$, which is equivalent to $\mathrm{supp}(J) \cap \mathbb{R}^+ \neq \emptyset$ and $\mathrm{supp}(J) \cap \mathbb{R}^- \neq \emptyset$.

From now on, if there is no special note, we always assume the wave speeds $c \neq 0$ and $\hat{c} \neq 0$. In Section 2, we give a special asymmetric kernel function as an example to illustrate that the assumption is reasonable and meaningful (see Example 2.4).

Now we state our main result.

Theorem 1.2. *Assume that J satisfies* (J1)–(J2) *and* (2.3), *f satisfies $\max_{u\in[0,1]} f'(u) < 1$ and* (FI). *Let $\phi(x+ct)$ and $\hat{\phi}(x+\hat{c}t)$ be the monotone solutions of* (1.2) *and* (1.4) *which satisfy* (4.1), *respectively. Then for any constant $\theta \in \mathbb{R}$,* (1.1) *admits an entire solution $u(x,t) : \mathbb{R}^2 \to [0,1]$ satisfying*

$$\lim_{t\to-\infty}\left\{\sup_{x\leqslant -\frac{c+\hat{c}}{2}t}|u(x,t)-\hat{\phi}(x+\hat{c}t-\theta)| + \sup_{x\geqslant -\frac{c+\hat{c}}{2}t}|u(x,t)-\phi(x+ct+\theta)|\right\} = 0. \tag{1.5}$$

In addition, there exist the positive constants C_1 and C_2 such that for any $\eta > 0$,

$$|u(x+\eta,t)-u(x,t)| \leqslant C_1\eta, \quad \left|\frac{\partial u}{\partial t}(x+\eta,t)-\frac{\partial u}{\partial t}(x,t)\right| \leqslant C_2\eta. \tag{1.6}$$

In addition, $u(x,t)$ possesses different properties according to the sign and size of the speeds c and $\hat{c}$ (*see Lemma* 2.5).

(a) *If $c > 0$ and $\hat{c} > 0$, then $u(x,t)$ is increasing with respect to θ and for some constants $a, N, t_0 \in \mathbb{R}$,*

$$\lim_{\theta\to+\infty} u(x,t) = 1 \quad \text{uniformly for} \quad (x,t) \in [-a,+\infty)^2 \cup (-\infty,a]^2, \tag{1.7}$$

$$\lim_{x\to+\infty}\sup_{t\geqslant t_0}|u(x,t)-1| = 0, \qquad \lim_{x\to-\infty}\sup_{t\leqslant t_0}|u(x,t)-1| = 0, \tag{1.8}$$

$$\lim_{t\to+\infty}\sup_{x\in[-N,+\infty)}|u(x,t)-1| = 0. \tag{1.9}$$

Moreover, the traveling wave solutions ϕ and $\hat{\phi}$ propagate in the same direction and from the left of the x-axis as $t \to -\infty$. Since $c > \hat{c}$ (*see Lemma* 2.5), *the two waves get closer and closer as time goes on, and finally ϕ is likely to catch up $\hat{\phi}$.*

(b) *If $c < 0$ and $\hat{c} < 0$, then $u(x,t)$ is increasing with respect to θ and*

$$\lim_{\theta\to+\infty} u(x,t) = 1 \quad \text{uniformly for} \quad (x,t) \in [-a,+\infty)\times(-\infty,a] \cup (-\infty,a]\times[-a,+\infty),$$

$$\lim_{x\to+\infty}\sup_{t\leqslant t_0}|u(x,t)-1| = 0, \qquad \lim_{x\to-\infty}\sup_{t\geqslant t_0}|u(x,t)-1| = 0,$$

$$\lim_{t\to+\infty}\sup_{x\in(-\infty,N]}|u(x,t)-1| = 0.$$

A similar statement to (a) *also holds for this case, i.e., the two waves ϕ and $\hat{\phi}$ propagate in the same direction and from the right of the x-axis as $t \to -\infty$, and they get closer and closer as time goes on, and finally $\hat{\phi}$ is likely to catch up ϕ.*

(c) *If $c > 0 > \hat{c}$, then $\frac{\partial u}{\partial t}(x,t) > 0$ and $u(x,t)$ is increasing with respect to θ and*

$$\lim_{\theta\to+\infty} u(x,t) = 1 \quad \text{uniformly for} \quad (x,t) \in [-a,+\infty)^2 \cup (-\infty,a] \times [-a,+\infty),$$

$$\lim_{x\to\pm\infty} \sup_{t\geqslant t_0} |u(x,t)-1| = 0, \qquad \lim_{t\to+\infty} \sup_{x\in\mathbb{R}} |u(x,t)-1| = 0.$$

Here, the two waves ϕ and $\hat{\phi}$ propagate from the opposite ends of the x-axis as $t \to -\infty$, and are likely to merge with each other eventually.

The remainder of this paper is organized as follows. In Section 2, we give some useful lemmas which mainly focus on the initial value problem of (1.1) and the speeds of traveling wave solutions. Sections 3 and 4 are respectively devoted to proving the exponential behaviors of traveling wave solutions and the existence and qualitative properties of the entire solution of (1.1). In Section 5, we give the discussion.

2 Preliminaries

In this section, we make some preparations for getting our main results later. Since the main theorem is proved by considering the solving sequences of Cauchy problems starting at time $-n$ with suitable initial values, we first consider the following Cauchy problem of (1.1):

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = (J*u)(x,t) - u(x,t) + f(u(x,t)), & x \in \mathbb{R}, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{2.1}$$

Definition 2.1. A function $\bar{u}(x,t)$ is called a supersolution of (1.1) on $(x,t) \in \mathbb{R} \times [\tau,T)$, $\tau < T$, if $\bar{u}(x,t) \in C^{0,1}(\mathbb{R} \times [\tau,T), \mathbb{R})$ and satisfies

$$\frac{\partial}{\partial t}\bar{u}(x,t) \geqslant (J*\bar{u})(x,t) - \bar{u}(x,t) + f(\bar{u}(x,t)), \quad \forall\, (x,t) \in \mathbb{R} \times [\tau,T). \tag{2.2}$$

Furthermore, if for any $\tau < T$, $\bar{u}$ is a supersolution of (1.1) on $(x,t) \in \mathbb{R} \times [\tau,T)$, then $\bar{u}$ is called a supersolution of (1.1) on $(x,t) \in \mathbb{R} \times (-\infty,T)$. Similarly, a subsolution $\underline{u}(x,t)$ can be defined by reversing the inequality (2.2).

Lemma 2.2. *Assume* (J1)–(J2) *and* (FI) *hold. Then*

(i) *For any $u_0(x) \in C(\mathbb{R},[0,1])$, (2.1) admits a unique solution $u(x,t;u_0) \in C^{0,1}(\mathbb{R} \times [0,\infty),[0,1])$.*

(ii) *For any pair of supersolution $\bar{u}(x,t)$ and subsolution $\underline{u}(x,t)$ of (1.1) on $\mathbb{R} \times [0,+\infty)$ with $\underline{u}(x,0) \leqslant \bar{u}(x,0)$ and $0 \leqslant \underline{u}(x,t), \bar{u}(x,t) \leqslant 1$ for $(x,t) \in \mathbb{R} \times [0,+\infty)$, there holds $0 \leqslant \underline{u}(x,t) \leqslant \bar{u}(x,t) \leqslant 1$ for all $(x,t) \in \mathbb{R} \times [0,+\infty)$.*

Lemma 2.3. *Assume* (J1) *and* (FI) *hold. Let $u(x,t)$ be a solution of (2.1) with $u_0(x) \in C(\mathbb{R},[0,1])$. Then there exists a constant $K_0 > 0$, independent of x,t and $u_0(x)$, such that*

$$|u_t(x,t)|,\ |u_{tt}(x,t)| \leqslant K_0 \quad \text{for any} \quad x \in \mathbb{R}, \quad t > 0.$$

In addition, if we further assume $\max_{u\in[0,1]} f'(u) < 1$ and there exists a constant $K_1 > 0$ such that for any $\eta > 0$,

$$\int_{\mathbb{R}} |J(x+\eta) - J(x)| dx \leqslant K_1\eta \tag{2.3}$$

and

$$|u_0(x+\eta) - u_0(x)| \leqslant K_1\eta, \tag{2.4}$$

then for any $x \in \mathbb{R}$, $t > 0$ and $\eta > 0$, one has

$$|u(x+\eta,t) - u(x,t)| \leqslant K_2\eta, \quad \left|\frac{\partial u}{\partial t}(x+\eta,t) - \frac{\partial u}{\partial t}(x,t)\right| \leqslant K_2\eta, \tag{2.5}$$

where K_2 is some positive constant independent of u_0 and η.

We omit the details of the proofs of Lemmas 2.2 and 2.3 because they are standard and common (see [9, Theorem 2.4] and [11, Lemma 3.2]).

Next, we discuss the sign and size of the wave speeds. Due to the asymmetry of the kernel function J, the speed c of the nondecreasing traveling wave solution ϕ may be nonpositive and the speed $\hat{c}$ of the nonincreasing one $\hat{\phi}$ may be nonnegative, which are very different from the case of the symmetric kernel function. Precisely, integrating the first equation of (1.2) from $-\infty$ to $+\infty$, we have

$$\begin{aligned} c &= \int_{\mathbb{R}}\int_{\mathbb{R}} J(y)[\phi(\xi-y)-\phi(\xi)]dyd\xi + \int_{\mathbb{R}} f(\phi(\xi))d\xi \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} J(y)\int_0^1 \phi'(\xi-\theta y)(-y)d\theta dyd\xi + \int_{\mathbb{R}} f(\phi(\xi))d\xi \\ &= -\int_{\mathbb{R}} J(y)ydy + \int_{\mathbb{R}} f(\phi(\xi))d\xi. \end{aligned} \tag{2.6}$$

Similarly,

$$\hat{c} = -\int_{\mathbb{R}} J(y)ydy - \int_{\mathbb{R}} f(\hat{\phi}(\hat{\xi}))d\hat{\xi}. \tag{2.7}$$

If J is symmetric, then $\int_{\mathbb{R}} J(y)ydy = 0$. It is obvious that $c > 0$ and $\hat{c} < 0$ since $0 \leqslant \phi(\cdot), \hat{\phi}(\cdot) \leqslant 1$ and $f(u) \geqslant 0$ for $u \in [0,1]$. However, if J is asymmetric, the sign and size of the integral $\int_{\mathbb{R}} J(y)ydy$ cannot be determined exactly, thus the signs of speeds c and $\hat{c}$ are uncertain. Actually, from (2.6) and (2.7), we know that the signs of c and $\hat{c}$ partially depend on the properties of the kernel function J. More specifically, we give an example of the asymmetric kernel functions to illustrate such dependence or influence.

Example 2.4. The kernel function given below can make $c \neq 0$ and $\hat{c} \neq 0$, i.e.,

$$J(x) = \begin{cases} \dfrac{2}{15}\mathrm{e}^{-(x-2)}, & x \geqslant 2, \\ \dfrac{8}{15}\mathrm{e}^{2(x+1)}, & x \leqslant -1, \\ \dfrac{4}{45}x^2 - \dfrac{2}{9}x + \dfrac{2}{9}, & -1 < x < 2. \end{cases}$$

Through some calculations, we verify that $J(x)$ satisfies (J1), (J2) and $\int_{\mathbb{R}} J(x)xdx = 0$. Then (2.6) and (2.7) show that $c \neq 0$ and $\hat{c} \neq 0$ hold (more precisely, such a kernel function can ensure $c > 0$ and $\hat{c} < 0$). In fact, more generally speaking, in the case of this paper, as long as the kernel function $J(x)$ satisfies (J1), (J2), $\int_{\mathbb{R}} J(y)ydy \neq \int_{\mathbb{R}} f(\phi(\xi))d\xi$ and $\int_{\mathbb{R}} J(y)ydy \neq \int_{\mathbb{R}} f(\hat{\phi}(\hat{\xi}))d\hat{\xi}$, the wave speeds $c \neq 0$ and $\hat{c} \neq 0$ are naturally established. One of the simplest case is that J satisfies $\int_{\mathbb{R}} J(y)ydy = 0$ besides (J1) and (J2).

Now we give a lemma to classify the sign of the wave speeds.

Lemma 2.5. *Assume* (J1) *and* (FI) *hold. As for the signs of speeds c and $\hat{c}$, there are only three possibilities*:

(i) $c > 0$ *and* $\hat{c} > 0$;

(ii) $c > 0$ *and* $\hat{c} < 0$;

(iii) $c < 0$ *and* $\hat{c} < 0$,

which means the case of $c < 0$ and $\hat{c} > 0$ is impossible. Moreover, $c > \hat{c}$.

Proof. If $c < 0$, (2.6) implies $\int_{\mathbb{R}} f(\phi(\xi))d\xi < \int_{\mathbb{R}} J(y)ydy$. Then from (2.7), we get

$$\hat{c} < -\int_{\mathbb{R}} f(\phi(\xi))d\xi - \int_{\mathbb{R}} f(\hat{\phi}(\hat{\xi}))d\hat{\xi}.$$

Combining (2.6) and (2.7) with the properties of $\phi(\cdot)$, $\hat{\phi}(\cdot)$ and $f|_{[\rho,1]} > 0$, we conclude that $\hat{c} < 0$. Similarly, from (2.6) and (2.7), one has $c - \hat{c} = \int_{\mathbb{R}} f(\phi(\xi))d\xi + \int_{\mathbb{R}} f(\hat{\phi}(\hat{\xi}))d\hat{\xi}$ which yields $c > \hat{c}$. □

3 Asymptotic behaviors of traveling wave solutions

In the front part of this subsection, we prove the following asymptotic behaviors of traveling waves.

Theorem 3.1. *Assume* (J1)–(J2) *and* (FI) *hold. Let* $\phi(x+ct)$ *and* $\hat{\phi}(x+\hat{c}t)$ *be the nondecreasing and nonincreasing solutions of* (1.2) *and* (1.4), *respectively. Then the following conclusions hold*:

(i) *As for* $\phi(x+ct)$, *one has the following*:

(a) *There exist two positive constants* A_0 *and* μ_1 *such that* $0<\phi'(\xi)\leqslant A_0\mathrm{e}^{\mu_1\xi}$ *for* $\xi\leqslant 0$.

(b) *There exist positive constants* $A_2\leqslant A_1$ *and* $\mu_2\geqslant\mu_1$ *such that* $A_2\mathrm{e}^{\mu_2\xi}\leqslant\phi(\xi)\leqslant A_1\mathrm{e}^{\mu_1\xi}$ *for* $\xi\leqslant 0$.

(c) *There exist two constants* $A_3>0$ *and* $\mu_{21}<0$ *such that*

$$\lim_{\xi\to+\infty}(1-\phi(\xi))\mathrm{e}^{-\mu_{21}\xi}=A_3,\quad \lim_{\xi\to+\infty}\phi'(\xi)\mathrm{e}^{-\mu_{21}\xi}=-A_3\mu_{21}.$$

(ii) *As for* $\hat{\phi}(x+\hat{c}t)$, *one has the following*:

(a) *There exist two positive constants* $\hat{A}_0$ *and* $\hat{\mu}_1$ *such that* $|\hat{\phi}'(\hat{\xi})|\leqslant\hat{A}_0\mathrm{e}^{-\hat{\mu}_1\hat{\xi}}$ *for* $\hat{\xi}\geqslant 0$.

(b) *There exist positive constants* $\hat{A}_2\leqslant\hat{A}_1$ *and* $\hat{\mu}_2\geqslant\hat{\mu}_1$ *such that* $\hat{A}_2\mathrm{e}^{-\hat{\mu}_2\hat{\xi}}\leqslant\hat{\phi}(\hat{\xi})\leqslant\hat{A}_1\mathrm{e}^{-\hat{\mu}_1\hat{\xi}}$ *for* $\hat{\xi}\geqslant 0$.

(c) *There exist positive constants* $\hat{A}_3$ *and* $\hat{\mu}_{22}$ *such that*

$$\lim_{\hat{\xi}\to-\infty}(1-\hat{\phi}(\hat{\xi}))\mathrm{e}^{-\hat{\mu}_{22}\hat{\xi}}=\hat{A}_3,\quad \lim_{\hat{\xi}\to-\infty}\hat{\phi}'(\hat{\xi})\mathrm{e}^{-\hat{\mu}_{22}\hat{\xi}}=-\hat{A}_3\hat{\mu}_{22}.$$

To obtain the precise exponential asymptotic behaviors of $\phi(\xi)$ as $\xi\to+\infty$ and $\hat{\phi}(\hat{\xi})$ as $\hat{\xi}\to-\infty$ (described as (c) of (i) and (ii) in Theorem 3.1), we use a similar argument with [22, Lemma 3.4] since $f'(1)<0$. But as $\xi\to-\infty$ for $\phi(\xi)$ and $\hat{\xi}\to+\infty$ for $\hat{\phi}(\hat{\xi})$, we can no longer use the same method due to the fact $f'(0)=0$. Here, we adopt a proper comparison principle which can be proved by similar arguments to [2, Theorem 3.1] or [1, Theorem 1.5.1] and construct appropriate barrier functions to prove Theorem 3.1. The comparison principle is stated as follows.

Theorem 3.2. *Assume J satisfies* (J1)–(J2), $a(x)\in C(\mathbb{R},[0,+\infty))$ *and* $b(x)\in L^\infty(\mathbb{R})\cap C(\mathbb{R})$. *Let u and v be two smooth functions* ($C^1(\mathbb{R})$) *and* ω *be a connected subset of* $\mathbb{R}$. *Assume that u and v satisfy the following conditions*:

$$\begin{aligned}
&Lu=J*u-u-a(x)u-b(x)u'\leqslant 0\quad \textit{in}\quad \omega\subset\mathbb{R},\\
&Lv=J*v-v-a(x)v-b(x)v'\geqslant 0\quad \textit{in}\quad \omega\subset\mathbb{R},\\
&u\geqslant_{\not\equiv} v\quad \textit{in}\quad \mathbb{R}-\omega,\\
&\textit{if }\ \omega \textit{ is an unbounded domain, also assume that } \lim_{\pm\infty}(u-v)\geqslant 0.
\end{aligned}$$

Then $u\geqslant v$ *in* $\mathbb{R}$.

Now define four complex functions $F_i(\mu)$ and $\hat{F}_i(\mu)$ for $i=1,2$ by

$$\begin{aligned}
F_1(\mu)&=\int_{\mathbb{R}}J(-y)\mathrm{e}^{\mu y}dy-1-c\mu+f'(0)=\int_{\mathbb{R}}J(-y)\mathrm{e}^{\mu y}dy-1-c\mu,\\
F_2(\mu)&=\int_{\mathbb{R}}J(-y)\mathrm{e}^{\mu y}dy-1-c\mu+f'(1),\\
\hat{F}_1(\hat{\mu})&=\int_{\mathbb{R}}J(y)\mathrm{e}^{\hat{\mu}y}dy-1+\hat{c}\hat{\mu},\\
\hat{F}_2(\hat{\mu})&=\int_{\mathbb{R}}J(y)\mathrm{e}^{\hat{\mu}y}dy-1+\hat{c}\hat{\mu}+f'(1).
\end{aligned}$$

Then the following lemma holds.

Lemma 3.3. *Assume J satisfies* (J1) *and* (J2). *Then the following conclusions hold*:

(i) *The equation* $F_1(\mu)=0$ (*or* $\hat{F}_1(\hat{\mu})=0$) *has a positive real root* μ_1 (*or* $\hat{\mu}_1$) *such that*

$$F_1(\mu)\ (or\ \hat{F}_1(\hat{\mu}))\begin{cases}<0 & for \quad \mu\in(0,\mu_1)\ (or\ \hat{\mu}\in(0,\hat{\mu}_1)),\\ >0 & for \quad \mu>\mu_1\ (or\ \hat{\mu}>\hat{\mu}_1).\end{cases}$$

(ii) *The equation* $F_2(\mu)=0$ (*or* $\hat{F}_2(\hat{\mu})=0$) *has two real roots* $\mu_{21}<0$ (*or* $\hat{\mu}_{21}<0$) *and* $\mu_{22}>0$ (*or* $\hat{\mu}_{22}>0$) *such that*

$$F_2(\mu)\ (or\ \hat{F}_2(\hat{\mu}))\begin{cases}>0 & for \quad \mu<\mu_{21}\ (or\ \hat{\mu}<\hat{\mu}_{21}),\\ <0 & for \quad \mu\in(\mu_{21},\mu_{22})\ (or\ \hat{\mu}\in(\hat{\mu}_{21},\hat{\mu}_{22})),\\ >0 & for \quad \mu>\mu_{22}\ (or\ \hat{\mu}>\hat{\mu}_{22}).\end{cases} \tag{3.1}$$

Proof. (i) Direct computations show that $F_1(0)=0$ and

$$F_1'(\mu)=\int_{\mathbb{R}}J(-y)y\mathrm{e}^{\mu y}dy-c,\quad F_1''(\mu)=\int_{\mathbb{R}}J(-y)y^2\mathrm{e}^{\mu y}dy\geqslant 0.$$

From (2.6), we have

$$F_1'(0)=\int_{\mathbb{R}}J(-y)ydy-\int_{\mathbb{R}}J(-y)ydy-\int_{\mathbb{R}}f(\phi(\xi))d\xi<0.$$

Moreover, from (J2) we know that $J(y)\not\equiv 0$ in $\mathbb{R}^-$, then

$$F_1(\mu)=\int_{-\infty}^{0}J(-y)\mathrm{e}^{\mu y}dy+\int_{0}^{+\infty}J(-y)\mathrm{e}^{\mu y}dy-1-c\mu\to+\infty \quad \text{as} \quad \mu\to+\infty.$$

Therefore, there exists a positive constant μ_1 such that $F_1(\mu_1)=0$. Other conclusions can be proved similarly and we omit the details. □

Proof of Theorem 3.1. We only prove (i) since (ii) can be proved by similar arguments.

(a) The conclusion $\phi'(\xi)>0$ can be obtained directly from the monotonicity of ϕ which was proved by Coville [2]. Let $\psi(\xi)=A_0\mathrm{e}^{\mu_1\xi}$ and L be the following integro-differential operator:

$$Lu=J*u-u-cu'. \tag{3.2}$$

A quick computation shows that ψ satisfies

$$\begin{aligned}L\psi(\xi)&=A_0\int_{\mathbb{R}}J(y)\mathrm{e}^{\mu_1(\xi-y)}dy-\psi(\xi)-c\mu_1\psi(\xi)\\&=\psi(\xi)\bigg(\int_{\mathbb{R}}J(-y)\mathrm{e}^{\mu_1 y}dy-1-c\mu_1\bigg)\\&=0.\end{aligned}$$

Recalling from [3, Lemma 2.1], we know that $\phi'(\xi)\to 0$ as $\xi\to\pm\infty$. Then by taking $A_0=\|\phi'\|_\infty$, we achieve $\psi(\xi)>\phi'(\xi)$ on $\mathbb{R}^+$. According to the translation invariance, without loss of generality, we can assume $\phi(0)=\rho$. Then by the monotonicity of $\phi(\xi)$ and the first equation of (1.2), we have $L\phi=0$ on $\mathbb{R}^-$. Now by using Theorem 3.2 with ϕ' and ψ on $\mathbb{R}^-$, we obtain the desired inequality

$$\phi'(\xi)\leqslant\|\phi'\|_\infty\mathrm{e}^{\mu_1\xi} \quad \text{for} \quad \xi\leqslant 0, \tag{3.3}$$

which ends the proof of (a).

(b) Due to $\phi(-\infty)=0$ and $\phi(+\infty)=1$, one has the following asymptotic behavior of ϕ by integrating (3.3) from $-\infty$ to ξ with $\xi\leqslant 0$,

$$\phi(\xi)\leqslant A_1\mathrm{e}^{\mu_1\xi} \quad \text{for} \quad \xi\leqslant 0,$$

where $A_1 = \|\phi'\|_\infty/\mu_1$.

To complete the proof, we just need to establish the following inequality:

$$A_2 e^{\mu_2 \xi} \leqslant \phi(\xi) \tag{3.4}$$

for $\xi \leqslant 0$ and some positive constants A_2 and μ_2.

Define a function $\varphi_1(\xi, \mu)$ for $\mu > 0$ by

$$\varphi_1(\xi,\mu) := \begin{cases} \dfrac{\rho}{2} e^{\mu\xi} & \text{for} \quad \xi \leqslant 0, \\ \dfrac{\rho}{2} & \text{for} \quad \xi > 0. \end{cases} \tag{3.5}$$

Redefine L as the following operator:

$$Lu = J * u - u - cu' - ku.$$

Choose $k > 0$ such that $-k < \frac{f(u)}{u} < k$ for any $u \in [0,1]$. Such a k exists because f is Lipschitz continuous on $[0,1]$. Now we compute $L\varphi_1$ for $\xi \leqslant 0$,

$$\begin{aligned} L\varphi_1(\xi,\mu) &= \varphi_1(\xi,\mu)\left[e^{-\mu\xi}\int_{-\infty}^{\xi} J(y)dy + \int_{\xi}^{+\infty} J(y)e^{-\mu y}dy - 1 - c\mu - k\right] \\ &\geqslant \varphi_1(\xi,\mu)\left[e^{-\mu\xi}\int_{-\infty}^{\xi} J(y)dy + \int_{\xi}^{0} J(y)e^{-\mu y}dy - 1 - c\mu - k\right]. \end{aligned} \tag{3.6}$$

Since $\operatorname{supp}(J) \not\subset \mathbb{R}^+$, we can choose r and μ_2 large enough such that

$$\int_{-r}^{0} J(y)e^{-\mu_2 y}dy - 1 - c\mu_2 - k \geqslant 0. \tag{3.7}$$

Then from (3.6) and (3.7), for $\xi \leqslant -r$ we get

$$L\varphi_1(\xi,\mu_2) \geqslant \varphi_1(\xi,\mu_2)\left[e^{-\mu_2\xi}\int_{-\infty}^{\xi} J(y)dy + \int_{-r}^{0} J(y)e^{-\mu_2 y}dy - 1 - c\mu_2 - k\right] \geqslant 0. \tag{3.8}$$

Note that ϕ and any translation of ϕ satisfy the following relationship on $\mathbb{R}$:

$$L\phi = J * \phi - \phi - c\phi' - k\phi = -\left(\frac{f(\phi)}{\phi} + k\right)\phi \leqslant 0.$$

Assume that $\phi(-r) = \frac{\rho}{2}$. Then $\varphi_1(\xi,\mu_2) \leqslant \frac{\rho}{2} \leqslant \phi(\xi)$ for $\xi > -r$ since ϕ is nondecreasing. Now by using Theorem 3.2 with ϕ and φ_1 on $(-\infty, -r)$, one has $\varphi_1(\xi,\mu_2) \leqslant \phi$ on $\mathbb{R}$, and then (b) is proved by taking $A_2 = \min\{\frac{\rho}{2}, A_1\}$.

The main idea of the proof of (c) and [4, Section 5] is almost the same except for some detailed estimates, so we will not repeat it here. The proof is complete. □

Theorem 3.1 has given some exponential estimates of $\phi(\xi)$ as $\xi \to -\infty$ and $\hat{\phi}(\xi)$ as $\xi \to +\infty$, but in order to obtain the entire solution that we desired, we need to further research the asymptotic behaviors of $\phi'(\xi)/\phi(\xi)$ as $\xi \to -\infty$ and $\hat{\phi}'(\hat{\xi})/\hat{\phi}(\hat{\xi})$ as $\hat{\xi} \to +\infty$. The main idea comes from Zhang et al. [21], which is used efficiently to investigate the detailed asymptotic behavior of the traveling wave solution for the nonlocal dispersal equations with degenerate monostable nonlinearity. However in this paper, due to the asymmetry of the kernel function, we cannot determine the signs of c and $\hat{c}$ exactly, which further makes the details more difficult and complex. Firstly, we prove the following essential lemma.

Lemma 3.4. *Assume J satisfies* (J1)–(J2). *Let $c \neq 0$ and $B(\xi)$ be a continuous function having finite limits at infinity, $B(\pm\infty) := \lim_{x\to\pm\infty} B(x)$, and $z(\cdot)$ be a measurable function satisfying*

$$cz(x) = \int_{\mathbb{R}} J(y)e^{\int_x^{x-y} z(s)ds}dy + B(x), \quad x \in \mathbb{R}. \tag{3.9}$$

Then z is uniformly continuous and bounded. Moreover, $\mu^\pm = \lim_{x\to\pm\infty} z(x)$ exist and are real roots of the characteristic equation $c\mu = \int_{\mathbb{R}} J(y)e^{-\mu y}dy + B(\pm\infty)$.

Proof. We discuss it by dividing into two cases: (i) $c>0$; (ii) $c<0$.

(i) $c>0$. Let $w(x)=\mathrm{e}^{mx+\int_0^x z(s)ds}$ for some constant m. Then

$$cw'(x)=(cm+B(x))w(x)+\int_{\mathbb{R}}J(y)\mathrm{e}^{my}w(x-y)dy. \tag{3.10}$$

Choosing $m=\|B(x)\|_{L^\infty(\mathbb{R})}/c$, we get $w'(x)\geqslant 0$. Integrating (3.10) over $[x-\delta,x]$, one has

$$\begin{aligned}cw(x)&\geqslant c\omega(x)-cw(x-\delta)\\&\geqslant\int_{\mathbb{R}}J(y)\mathrm{e}^{my}\int_{x-\delta}^{x}w(s-y)dsdy\\&\geqslant\delta\int_{\mathbb{R}}J(y)\mathrm{e}^{my}w(x-\delta-y)dy\\&\geqslant\delta\int_{-\infty}^{-2\delta}J(y)\mathrm{e}^{my}w(x-\delta-y)dy\\&\geqslant\delta w(x+\delta)\int_{-\infty}^{-2\delta}J(y)\mathrm{e}^{my}dy.\end{aligned}$$

Since $\mathrm{supp}(J)\not\subset\mathbb{R}^+$ from (J2), we can choose $\delta>0$ such that $\int_{-\infty}^{-2\delta}J(y)\mathrm{e}^{my}dy>0$. Let $M_1:=\int_{-\infty}^{-2\delta}J(y)\mathrm{e}^{my}dy$ and (J1) ensures $M_1<+\infty$. Then we have

$$\frac{w(x+\delta)}{w(x)}\leqslant\frac{c}{\delta M_1}. \tag{3.11}$$

Similarly, integrating (3.10) over $[x,x+\delta]$, we obtain

$$cw(x+\delta)\geqslant c\omega(x+\delta)-cw(x)\geqslant\delta\int_{\mathbb{R}}J(y)\mathrm{e}^{my}w(x-y)dy,$$

which implies

$$c\frac{w(x+\delta)}{w(x)}\geqslant\delta\int_{\mathbb{R}}J(y)\mathrm{e}^{my}\frac{w(x-y)}{w(x)}dy. \tag{3.12}$$

Note that

$$cz(x)=\int_{\mathbb{R}}J(y)\mathrm{e}^{my}\frac{w(x-y)}{w(x)}dy+B(x).$$

By combining (3.11) with (3.12), we obtain $z(\cdot)\in L^\infty(\mathbb{R})$.

(ii) $c<0$. From (3.10) we know that $w'(x)\leqslant 0$. Integrating (3.10) on $[x,x+\delta]$, one has

$$\begin{aligned}-cw(x)&\geqslant c\omega(x+\delta)-cw(x)\\&\geqslant\delta\int_{\mathbb{R}}J(y)\mathrm{e}^{my}w(x+\delta-y)dy\\&\geqslant\delta\int_{2\delta}^{+\infty}J(y)\mathrm{e}^{my}w(x+\delta-y)dy\\&\geqslant\delta w(x-\delta)\int_{2\delta}^{+\infty}J(y)\mathrm{e}^{my}dy.\end{aligned}$$

Letting $M_2:=\int_{2\delta}^{+\infty}J(y)\mathrm{e}^{my}dy$, from (J1) and (J2), we get $0<M_2<+\infty$. Then

$$\frac{w(x-\delta)}{w(x)}\leqslant\frac{-c}{\delta M_2}.$$

Similarly, we have

$$-c\frac{w(x-\delta)}{w(x)}\geqslant\delta\int_{\mathbb{R}}J(y)\mathrm{e}^{my}\frac{w(x-y)}{w(x)}dy,$$

which yields

$$\int_{\mathbb{R}} J(y)\mathrm{e}^{my}\frac{w(x-y)}{w(x)}dy \leqslant \frac{c^2}{\delta^2 M_2}.$$

Therefore $z(x)$ is bounded. (3.9) implies that z is uniformly continuous. The rest conclusion can be proved by using a completely similar argument with that of [21, Proposition 3.7], because the rest proof does not depend on the sign of c and the symmetry of the kernel function J, so we omit the details. The proof is complete. □

Based on the above discussions, we obtain the following lemma.

Lemma 3.5. *Let $\phi(\xi)$ and $\hat{\phi}(\hat{\xi})$ be the solutions of* (1.2) *and* (1.4), *respectively. Then they satisfy*

$$\lim_{\xi\to-\infty}\frac{\phi'(\xi)}{\phi(\xi)} = \{0,\mu_1\}, \quad \lim_{\hat{\xi}\to+\infty}\frac{\hat{\phi}'(\hat{\xi})}{\hat{\phi}(\hat{\xi})} = \{0,-\hat{\mu}_1\}, \tag{3.13}$$

where μ_1 and $\hat{\mu}_1$ are defined in Lemma 3.3.

Proof. From Theorem 3.1, one knows that $\phi(\xi) > 0$ for any $\xi \in \mathbb{R}$, so we can define

$$z(\xi) := \frac{\phi'(\xi)}{\phi(\xi)}.$$

Dividing the first equation of (1.2) by $\phi(\xi)$, one has

$$cz(\xi) = \int_{\mathbb{R}} J(y)\mathrm{e}^{\int_{\xi}^{\xi-y} z(s)ds}dy + B(\xi),$$

where $B(\xi) = -1 + \frac{f(\phi(\xi))}{\phi(\xi)}$. Then the front part of (3.13) follows from $B(-\infty) = -1$ and Lemmas 3.3 and 3.4. The conclusion for $\hat{\phi}(\hat{\xi})$ can be discussed similarly. □

4 Entire solutions

In this section, we focus our attention on the new entire solution of (1.1) except traveling wave solutions by combining the nondecreasing traveling wave $\phi(x+ct)$ with the nonincreasing one $\hat{\phi}(x+\hat{c}t)$. In order to construct a proper supersolution, we require ϕ and $\hat{\phi}$ to satisfy the following condition:

$$k\phi(\xi) \leqslant \phi'(\xi) \quad \text{for} \quad \xi \leqslant 0 \quad \text{and} \quad \hat{\phi}'(\hat{\xi}) \leqslant -\hat{k}\hat{\phi}(\hat{\xi}) \quad \text{for} \quad \hat{\xi} \geqslant 0 \tag{4.1}$$

with some positive constants k and $\hat{k}$. Actually, (4.1) is easy to be satisfied, e.g., from Lemma 3.5, if ϕ and $\hat{\phi}$ satisfy

$$\lim_{\xi\to-\infty}\frac{\phi'(\xi)}{\phi(\xi)} = \mu_1, \quad \lim_{\hat{\xi}\to+\infty}\frac{\hat{\phi}'(\hat{\xi})}{\hat{\phi}(\hat{\xi})} = -\hat{\mu}_1,$$

then (4.1) can be ensured by combining Theorem 3.1. The simplest case is that the kernel function J has a compact support and the radius of the compact support is very small, such as $\mathrm{supp}(J) \subset [-a,b]$ with $0 < a, b \ll 1$. Without loss of generality, assume $a < b$. Note that $\phi(\xi)$ is smooth. Then the Taylor's formula yields that

$$\begin{aligned} J*\phi(\xi) - \phi(\xi) &= \int_{\mathbb{R}} J(y)[\phi(\xi-y) - \phi(\xi)]dy \\ &= \frac{1}{2}\int_{\mathbb{R}} J(y)y^2dy\phi''(\xi) - \int_{\mathbb{R}} J(y)ydy\phi'(\xi) + o(b^2) \quad \text{as} \quad b\to 0. \end{aligned}$$

Then for $\xi \ll -1$ and $b \ll 1$ small enough, from (1.2) and (FI), one has

$$c\phi'(\xi) = \frac{1}{2}\int_{\mathbb{R}} J(y)y^2dy\phi''(\xi) - \int_{\mathbb{R}} J(y)ydy\phi'(\xi),$$

which implies

$$\lim_{\xi\to-\infty}\frac{\phi'(\xi)}{\phi(\xi)}=\lim_{\xi\to-\infty}\frac{\phi''(\xi)}{\phi'(\xi)}=\frac{2[c+\int_{\mathbb{R}}J(y)ydy]}{\int_{\mathbb{R}}J(y)y^2dy}\neq 0.$$

Then combining Lemma 3.5, we get $\lim_{\xi\to-\infty}\frac{\phi'(\xi)}{\phi(\xi)}=\mu_1$. Similarly, $\lim_{\hat{\xi}\to+\infty}\frac{\hat{\phi}'(\hat{\xi})}{\hat{\phi}(\hat{\xi})}=-\hat{\mu}_1$.

From (FI), we can modify $f(u)$ on $u\in(1,+\infty)$ such that $f'(u)<f'(0)=0$ for any $u\in(1,2)$. Then by the continuity of $f'(u)$ and $f'(1)<0$, there exists $m_0\in(0,1)$ so that

$$f'(u)<f'(0)\quad\text{for any}\quad u\in(1-m_0,2).\tag{4.2}$$

In the remainder of this paper, we always assume $\phi(\xi)$ and $\hat{\phi}(\hat{\xi})$ satisfy

$$\phi(\xi)\geqslant 1-m_0\quad\text{for any}\quad \xi\geqslant 0,\tag{4.3}$$

$$\hat{\phi}(\hat{\xi})\geqslant 1-m_0\quad\text{for any}\quad \hat{\xi}\leqslant 0.\tag{4.4}$$

Indeed, (4.3) and (4.4) can be ensured by translating $\phi(\xi)$ and $\hat{\phi}(\hat{\xi})$ along the x-axis appropriately.

We start with the following ordinary differential problem which plays an important role in the construction of the supersolution:

$$\begin{cases}p'(t)=c_0+N\mathrm{e}^{\sigma p(t)}, & t<0,\\ p(0)<0,\end{cases}\tag{4.5}$$

where c_0,N and σ are positive constants and will be chosen later. (4.5) can be solved explicitly as

$$p(t)-c_0t-\omega=-\frac{1}{\sigma}\ln\left\{1-\frac{r}{1+r}\mathrm{e}^{c_0\sigma t}\right\},\quad r=\frac{N}{c_0}\mathrm{e}^{\sigma p(0)},$$

with

$$\omega:=p(0)-\frac{1}{\sigma}\ln\left\{1+\frac{N}{c_0}\mathrm{e}^{\sigma p(0)}\right\}.\tag{4.6}$$

Then $p(t)\leqslant 0$ for $t\leqslant 0$ and there exists some positive constant K such that

$$0<p(t)-c_0t-\omega\leqslant K\mathrm{e}^{c_0\sigma t}\quad\text{for}\quad t\leqslant 0.\tag{4.7}$$

Now we are ready to construct a supersolution of (1.1). It follows from Lemma 2.5 that $c>\hat{c}$.

Lemma 4.1. *Assume that* (J1)–(J2) *and* (FI) *hold. Let $\phi(x+ct)$ and $\hat{\phi}(x+\hat{c}t)$ be the traveling wave solutions of* (1.1) *satisfying* (1.2), (1.4) *and* (4.1), *respectively. Furthermore, set $\bar{c}=\frac{c+\hat{c}}{2}$ and $c_0=\frac{c-\hat{c}}{2}$. Then for the solution $p(t)$ of* (4.5) *with $N>N^*$* (*it will be given later*) *and $\sigma:=\min\{\mu_1,\hat{\mu}_1\}$, the function*

$$\bar{u}(x,t)=\phi(x+\bar{c}t+p(t))+\hat{\phi}(x+\bar{c}t-p(t))$$

is a supersolution of (1.1) *on $t\in(-\infty,0]$.*

Proof. Define

$$\mathcal{L}(u)(x,t):=u_t(x,t)-(J*u-u)(x,t)-f(u(x,t)).$$

The remaining work is to verify $\mathcal{L}(\bar{u})(x,t)\geqslant 0$ for $(x,t)\in\mathbb{R}\times(-\infty,0]$. For simplification, we write $\phi(x+\bar{c}t+p(t))$ and $\hat{\phi}(x+\bar{c}t-p(t))$ as ϕ and $\hat{\phi}$, respectively. Direct calculations show that

$$\begin{aligned}\mathcal{L}(\bar{u})&=(\bar{c}+p')\phi'+(\bar{c}-p')\hat{\phi}'-(J*\phi-\phi)-(J*\hat{\phi}-\hat{\phi})-f(\phi+\hat{\phi})\\&=(\bar{c}+p'-c)\phi'+(\bar{c}-p'-\hat{c})\hat{\phi}'-[f(\phi+\hat{\phi})-f(\phi)-f(\hat{\phi})]\\&=(\phi'-\hat{\phi}')[N\mathrm{e}^{\sigma p}-F(x,t)],\end{aligned}\tag{4.8}$$

where

$$F(x,t)=\frac{f(\phi+\hat{\phi})-f(\phi)-f(\hat{\phi})}{\phi'-\hat{\phi}'}.$$

Next, we study it by dividing $\mathbb{R}$ into three regions.

(i) $p(t) \leqslant x + \bar{c}t \leqslant -p(t)$. Then $x + \bar{c}t + p(t) \leqslant 0$ and $x + \bar{c}t - p(t) \geqslant 0$ for $t \leqslant 0$. Recalling that $\phi, \hat{\phi} \in [0,1]$, $f(0) = 0$ and $f \in C^2(\mathbb{R})$, one has

$$f(\phi + \hat{\phi}) - f(\phi) - f(\hat{\phi}) = \int_0^1 f'(\phi + s\hat{\phi})\hat{\phi}ds - \int_0^1 f'(s\hat{\phi})\hat{\phi}ds \leqslant L\phi\hat{\phi},$$

with $L := \max_{s\in[0,2]} f''(s)$. From Theorem 3.1 and (4.1), for $0 \leqslant x + \bar{c}t \leqslant -p(t)$, we obtain

$$F(x,t) \leqslant \frac{L\phi\hat{\phi}}{\phi' - \hat{\phi}'} \leqslant \frac{L\hat{\phi}}{\phi'/\phi} \leqslant \frac{L\hat{A}_1 \mathrm{e}^{-\hat{\mu}_1(x+\bar{c}t-p(t))}}{k} \leqslant \frac{L\hat{A}_1}{k}\mathrm{e}^{\hat{\mu}_1 p(t)}. \tag{4.9}$$

Similarly, for $p(t) \leqslant x + \bar{c}t \leqslant 0$, we get

$$F(x,t) \leqslant \frac{L\phi}{-\hat{\phi}'/\hat{\phi}} \leqslant \frac{LA_1 \mathrm{e}^{\mu_1(x+\bar{c}t+p(t))}}{\hat{k}} \leqslant \frac{LA_1}{\hat{k}}\mathrm{e}^{\mu_1 p(t)}. \tag{4.10}$$

(ii) $-\infty < x + \bar{c}t \leqslant p(t)$. Combining (4.2) and (4.4) with $x + \bar{c}t - p(t) \leqslant 0$ for $t \leqslant 0$, one has

$$\begin{aligned} f(\phi + \hat{\phi}) - f(\phi) - f(\hat{\phi}) &= \int_0^1 f'(\hat{\phi} + s\phi)\phi ds - \int_0^1 f'(s\phi)\phi ds \\ &\leqslant \phi \int_0^1 |f'(0) - f'(s\phi)| ds \leqslant L\phi^2. \end{aligned}$$

Then

$$F(x,t) \leqslant \frac{L\phi^2}{\phi' - \hat{\phi}'} \leqslant \frac{L\phi}{\phi'/\phi} \leqslant \frac{LA_1 \mathrm{e}^{\mu_1(x+\bar{c}t+p(t))}}{k} \leqslant \frac{LA_1}{k}\mathrm{e}^{\mu_1 p(t)}. \tag{4.11}$$

(iii) $-p(t) \leqslant x + \bar{c}t \leqslant +\infty$. Similarly, from (4.2), (4.3) and $x + \bar{c}t + p(t) \geqslant 0$, we have $f(\phi + \hat{\phi}) - f(\phi) - f(\hat{\phi}) \leqslant L\hat{\phi}^2$ and

$$F(x,t) \leqslant \frac{L\hat{\phi}^2}{\phi' - \hat{\phi}'} \leqslant \frac{L\hat{\phi}}{-\hat{\phi}'/\hat{\phi}} \leqslant \frac{L\hat{A}_1 \mathrm{e}^{-\hat{\mu}_1(x+\bar{c}t-p(t))}}{\hat{k}} \leqslant \frac{L\hat{A}_1}{\hat{k}}\mathrm{e}^{\hat{\mu}_1 p(t)}. \tag{4.12}$$

Now by taking

$$N \geqslant N^* := \max\left\{\frac{L\hat{A}_1}{k}, \frac{LA_1}{\hat{k}}, \frac{LA_1}{k}, \frac{L\hat{A}_1}{\hat{k}}\right\}, \quad \sigma := \min\{\mu_1, \hat{\mu}_1\}$$

and combining (4.9)–(4.12) with (4.8), we conclude that

$$\mathcal{L}(\bar{u}) = (\phi' - \hat{\phi}')(N\mathrm{e}^{\sigma p} - F(x,t)) \geqslant 0.$$

The proof is complete. □

Proof of Theorem 1.2. For any $n \in \mathbb{N}$, consider the following Cauchy problem:

$$\begin{cases} \dfrac{\partial u_n}{\partial t}(x,t) = (J * u_n)(x,t) - u_n(x,t) + f(u_n(x,t)), & (x,t) \in \mathbb{R} \times (-n, +\infty), \\ u_n(x,-n) := \underline{u}(x,-n) = \max\{\phi(x - cn + \omega), \hat{\phi}(x - \hat{c}n - \omega)\}, & x \in \mathbb{R}. \end{cases} \tag{4.13}$$

Lemma 2.2 shows that (4.13) has a unique classical solution $u_n(x,t)$ satisfying

$$\underline{u}(x,t) = \max\{\phi(x + ct + \omega), \hat{\phi}(x + \hat{c}t - \omega)\} \leqslant u_n(x,t) \leqslant 1 \tag{4.14}$$

for any $(x,t) \in \mathbb{R} \times [-n, +\infty)$ and $n \in \mathbb{N}$. Furthermore,

$$u_n(x,-n) = \max\{\phi(x - cn + \omega), \hat{\phi}(x - \hat{c}n - \omega)\} \leqslant \phi(x - \bar{c}n + p(-n)) + \hat{\phi}(x - \bar{c}n - p(-n))$$

for any $x \in \mathbb{R}$. Then it follows from Lemmas 2.2 and 4.1 that $u_n(x,t) \leqslant \bar{u}(x,t)$ for any $n \in \mathbb{N}$ and $(x,t) \in \mathbb{R} \times [-n,0]$. Moreover, $|\phi'| \leqslant \frac{2+M}{|c|}$ and $|\hat{\phi}'| \leqslant \frac{2+M}{|\hat{c}|}$ with $M := \max_{u\in[0,1]} f(u)$. Then according to Lemma 2.3 and Arzela-Ascoli theorem, there exists a function $u(x,t)$ and a subsequence $\{u_{n_i}(x,t)\}$ of $\{u_n(x,t)\}$ such that $u_{n_i}(x,t)$ and $\frac{\partial}{\partial t}u_{n_i}(x,t)$ converge uniformly in any compact set $S \subset \mathbb{R}^2$ to $u(x,t)$ and $\frac{\partial}{\partial t}u(x,t)$ as $i \to +\infty$, respectively. From the equation satisfied by $u_{n_i}(x,t)$, it is clear that $u(x,t)$ is an entire solution of (1.1) and satisfies

$$\begin{aligned} &\underline{u}(x,t) \leqslant u(x,t) \leqslant \bar{u}(x,t) \quad \text{for} \quad (x,t) \in \mathbb{R} \times (-\infty, 0], \\ &\underline{u}(x,t) \leqslant u(x,t) \leqslant 1 \quad \text{for} \quad (x,t) \in \mathbb{R}^2. \end{aligned}$$

Moreover, by using Theorem 3.1 and (4.7), for $x \geqslant -\bar{c}t$ and $t \leqslant 0$, one has

$$\begin{aligned} u(x,t) - \phi(x+ct+\omega) &\leqslant \phi(x+\bar{c}t+p(t)) + \hat{\phi}(x+\bar{c}t-p(t)) - \phi(x+ct+\omega) \\ &\leqslant \sup_{\xi\in\mathbb{R}} |\phi'(\xi)|(p(t)-c_0t-\omega) + \hat{A}_1 \mathrm{e}^{-\hat{\mu}_1(x+\bar{c}t-p(t))} \\ &\leqslant K \sup_{\xi\in\mathbb{R}} |\phi'(\xi)| \mathrm{e}^{c_0\sigma t} + \hat{A}_1 \mathrm{e}^{\hat{\mu}_1 p(t)}. \end{aligned}$$

Similarly, for $x \leqslant -\bar{c}t$ and $t \leqslant 0$, one has

$$u(x,t) - \hat{\phi}(x+\hat{c}t-\omega) \leqslant K \sup_{\hat{\xi}\in\mathbb{R}} |\hat{\phi}'(\hat{\xi})| \mathrm{e}^{c_0\sigma t} + A_1 \mathrm{e}^{\mu_1 p(t)}.$$

Then (4.7) implies (1.5) of Theorem 1.2. (1.6) can be proved directly by Lemma 2.3.

Next, we prove the entire solution $u(x,t)$ is increasing with respect to ω. Note that the traveling wave ϕ is nondecreasing and $\hat{\phi}$ is nonincreasing. It follows that the functions $u_n(x,-n)$ are nondecreasing in ω for any $n \in \mathbb{N}$, when the other parameters are fixed. Then $u_n(x,t)$, even $u(x,t)$ is nondecreasing in ω. Furthermore, they are increasing in ω from the strong maximum principle established in [2].

The above entire solution established is only for the case $\theta = \omega$ with ω defined by (4.6). For more general $\theta \in \mathbb{R}$, we define $\tilde{u}(x,t) = u(x+x_0, t+t_0)$ with

$$x_0 = \frac{(c+\hat{c})(\omega-\theta)}{c-\hat{c}}, \quad t_0 = \frac{2(\theta-\omega)}{c-\hat{c}}.$$

Denote $\tilde{u}(x,t)$ by $u(x,t)$. Then $u(x,t)$ is the entire solution we desired. The rest of the proof is straightforward and mainly depends on the properties of the subsolution $\underline{u}(x,t)$ defined in (4.14), so we omit the details. The proof is complete. □

5 Discussion

In this paper, we have obtained a new entire solution of the nonlocal dispersal equation (1.1) with asymmetric kernel function and ignition nonlinearity, but we only consider the interactions of the traveling wave solutions with non-zero speeds. However, from the discussion of Section 2, we know that there are special dispersal kernel functions and nonlinearities, which can make $c = 0$ or $\hat{c} = 0$. Therefore, it is natural to ask if there exist some new entire solutions when at least one of c and $\hat{c}$ is equal to zero. We guess that there might exist some entire solutions that come from the interactions of two steady state waves or one traveling wave with non-zero speed and one steady state wave, while it seems very different and difficult to mathematically prove this, and we leave it as a further investigation.

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