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Anisotropic Hardy-Lorentz spaces and their applications

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Abstract Let $p \in (0, 1]$, $q \in (0, \infty]$ and A be a general expansive matrix on $\mathbb{R}^n$. We introduce the anisotropic Hardy-Lorentz space $H^{p,q}_A(\mathbb{R}^n)$ associated with A via the non-tangential grand maximal function and then establish its various real-variable characterizations in terms of the atomic and the molecular decompositions, the radial and the non-tangential maximal functions, and the finite atomic decompositions. All these characterizations except the ∞-atomic characterization are new even for the classical isotropic Hardy-Lorentz spaces on $\mathbb{R}^n$. As applications, we first prove that $H^{p,q}_A(\mathbb{R}^n)$ is an intermediate space between $H^{p_1,q_1}_A(\mathbb{R}^n)$ and $H^{p_2,q_2}_A(\mathbb{R}^n)$ with $0 < p_1 < p < p_2 < \infty$ and q_1, q, $q_2 \in (0, \infty]$, and also between $H^{p,q_1}_A(\mathbb{R}^n)$ and $H^{p,q_2}_A(\mathbb{R}^n)$ with $p \in (0, \infty)$ and $0 < q_1 < q < q_2 \leqslant \infty$ in the real method of interpolation. We then establish a criterion on the boundedness of sublinear operators from $H^{p,q}_A(\mathbb{R}^n)$ into a quasi-Banach space; moreover, we obtain the boundedness of δ-type Calderón-Zygmund operators from $H^p_A(\mathbb{R}^n)$ to the weak Lebesgue space $L^{p,\infty}(\mathbb{R}^n)$ (or to $H^{p,\infty}_A(\mathbb{R}^n)$) in the critical case, from $H^{p,q}_A(\mathbb{R}^n)$ to $L^{p,q}(\mathbb{R}^n)$ (or to $H^{p,q}_A(\mathbb{R}^n)$) with $\delta \in (0, \frac{\ln \lambda_-}{\ln b}]$, $p \in (\frac{1}{1+\delta}, 1]$ and $q \in (0, \infty]$, as well as the boundedness of some Calderón-Zygmund operators from $H^{p,q}_A(\mathbb{R}^n)$ to $L^{p,\infty}(\mathbb{R}^n)$, where $b := |\det A|$, $\lambda_- := \min\{|\lambda| : \lambda \in \sigma(A)\}$ and $\sigma(A)$ denotes the set of all eigenvalues of A.

Keywords Lorentz space, anisotropic Hardy-Lorentz space, expansive matrix, Calderón reproducing formula, grand maximal function, atom, molecule, Calderón-Zygmund operator

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1 Introduction

The study of Lorentz spaces originated from Lorentz [50, 51] in the early 1950's. As a generalization of $L^p(\mathbb{R}^n)$, Lorentz spaces are known to be the intermediate spaces of Lebesgue spaces in the real interpolation method; see [14, 46, 61]. For a systematic treatment of Lorentz spaces as well as their dual spaces, we refer the reader to Hunt [38], Cwikel [21] and Cwikel and Fefferman [22, 23]; see also [7, 8, 35, 65, 70]. It is well known that, due to their fine structures, Lorentz spaces play an irreplaceable role in the study on various critical or endpoint analysis problems from many different research fields and there exist a lot of literatures on this subject, here we only mention several recent papers from harmonic analysis (see, for example, [56, 59, 67, 73]) and partial differential equations (see, for example, [39, 54, 62]).

On the other hand, the theory of Hardy spaces has been well developed and these spaces play an important role in many branches of analysis and partial differential equations; see, for example, [18, 30,

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32, 36, 49, 57, 58, 68, 69, 71]. It is well known that Hardy spaces are good substitutes of Lebesgue spaces when $p \in (0,1]$, particularly, for the study on the boundedness of maximal functions and singular integral operators. Moreover, for the study on the boundedness of operators, the weak Hardy space $H^{1,\infty}(\mathbb{R}^n)$ is also a good substitute of $L^{1,\infty}(\mathbb{R}^n)$. Recall that the weak Hardy spaces $H^{p,\infty}(\mathbb{R}^n)$ with $p \in (0,1)$ were first introduced by Fefferman et al. [29] in 1974, which naturally appear as the intermediate spaces of Hardy spaces $H^p(\mathbb{R}^n)$ with $p \in (0,1]$ via the real interpolation. Later on, to find out the biggest space from which the Hilbert transform is bounded to the weak Lebesgue space $L^{1,\infty}(\mathbb{R}^n)$, Fefferman and Soria [31] introduced the weak Hardy space $H^{1,\infty}(\mathbb{R}^n)$, in which they also obtained the ∞-atomic decomposition of $H^{1,\infty}(\mathbb{R}^n)$ and the boundedness of some Calderón-Zygmund operators from $H^{1,\infty}(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$. In 1994, Álvarez [3] considered the Calderón-Zygmund theory related to $H^{p,\infty}(\mathbb{R}^n)$ with $p \in (0,1]$, while Liu [47] studied the weak Hardy spaces $H^{p,\infty}(\mathbb{R}^n)$ with $p \in (0,1]$ on homogeneous groups. Nowadays, it is well known that the weak Hardy spaces $H^{p,\infty}(\mathbb{R}^n)$, with $p \in (0,1]$, play a key role when studying the boundedness of operators in the critical case; see, for example, [3, 4, 25–27, 34, 77]. Moreover, it is known that the weak Hardy spaces $H^{p,\infty}(\mathbb{R}^n)$ are special cases of the Hardy-Lorentz spaces $H^{p,q}(\mathbb{R}^n)$ which, when $p = 1$ and $q \in (1,\infty)$, were introduced and investigated by Parilov [60]. In 2007, Abu-Shammala and Torchinsky [1] studied the Hardy-Lorentz spaces $H^{p,q}(\mathbb{R}^n)$ for the full range $p \in (0,1]$ and $q \in (0,\infty]$, and obtained their ∞-atomic characterization, real interpolation properties over parameter q, and the boundedness of singular integrals and some other operators on these spaces. In 2010, Almeida and Caetano [2] studied the generalized Hardy spaces which include the classical Hardy-Lorentz spaces $H^{p,q}(\mathbb{R}^n)$ investigated in [1] as special cases. To be more precise, Almeida and Caetano [2] obtained some maximal characterizations of these generalized Hardy spaces and some real interpolation results with function parameters and, as applications, they studied the behavior of some classical operators in this generalized setting.

As the series of works (see, for example, [1–3, 29, 31, 47, 60]) reveal, the Hardy-Lorentz spaces (as well the weak Hardy spaces) serve as a more subtle research object than the usual Hardy spaces when considering the boundedness of singular integrals, especially, in some endpoint cases, due to the fact that these function spaces own finer structures. However, the real-variable theory of these spaces is still not complete. For example, the r-atomic, with $r \in (1,\infty)$, or the molecular characterizations, the characterizations in terms of the radial or the non-tangential maximal functions, and the finite atomic characterizations of Hardy-Lorentz spaces are still unknown.

On the other hand, from 1970's, there has been an increasing interesting in extending classical function spaces arising in harmonic analysis from Euclidean spaces to anisotropic settings and some other domains; see, for example, [16, 17, 32, 33, 45, 66, 72, 74, 75]. The study of function spaces on $\mathbb{R}^n$ associated with anisotropic dilations was originally started from these celebrated articles [15–17] of Calderón and Torchinsky on anisotropic Hardy spaces. Since then, the theory of anisotropic function spaces was well developed by many authors; see, for example, [32, 69, 74]. In 2003, Bownik [9] introduced and investigated the anisotropic Hardy spaces associated with general expansive dilations, which were extended to the weighted setting in [13]. For further developments of function spaces on the anisotropic Euclidean spaces, we refer the reader to [11–13, 24, 41–43, 76].

To give a complete theory of Hardy-Lorentz spaces and also to establish this theory in a more general anisotropic setting, in this paper, we systematically develop a theory of Hardy-Lorentz spaces associated with anisotropic dilations A. To be precise, in this paper, for all $p \in (0,1]$ and $q \in (0,\infty]$, we introduce the anisotropic Hardy-Lorentz spaces $H^{p,q}_A(\mathbb{R}^n)$ associated with a general expansive matrix A via the non-tangential grand maximal function. Then we characterize $H^{p,q}_A(\mathbb{R}^n)$ in terms of the atomic and the molecular decompositions, the radial and the non-tangential maximal functions, and the finite atomic decompositions. All these results except the ∞-atomic characterization are new even for the classical isotropic Hardy-Lorentz spaces on $\mathbb{R}^n$. As applications, we first prove that the space $H^{p,q}_A(\mathbb{R}^n)$ is an intermediate space between $H^{p_1,q_1}_A(\mathbb{R}^n)$ and $H^{p_2,q_2}_A(\mathbb{R}^n)$ with $0 < p_1 < p < p_2 < \infty$ and $q_1,\, q,\, q_2 \in (0,\infty]$, and also between $H^{p,q_1}_A(\mathbb{R}^n)$ and $H^{p,q_2}_A(\mathbb{R}^n)$ with $p \in (0,\infty)$ and $0 < q_1 < q < q_2 \leqslant \infty$ in the real method of interpolation. We then establish a criterion on the boundedness of sublinear operators from $H^{p,q}_A(\mathbb{R}^n)$ into a quasi-Banach space. Moreover, we obtain the boundedness of δ-type Calderón-Zygmund operators from $H_A^p(\mathbb{R}^n)$ to the weak Lebesgue space $L^{p,\infty}(\mathbb{R}^n)$ (or to $H_A^{p,\infty}(\mathbb{R}^n)$) in the critical case, from $H_A^{p,q}(\mathbb{R}^n)$ to $L^{p,q}(\mathbb{R}^n)$ (or to $H_A^{p,q}(\mathbb{R}^n)$) with $\delta \in (0, \frac{\ln \lambda_-}{\ln b}]$, $p \in (\frac{1}{1+\delta}, 1]$ and $q \in (0, \infty]$, as well as the boundedness of some Calderón-Zygmund operators from $H_A^{p,q}(\mathbb{R}^n)$ to $L^{p,\infty}(\mathbb{R}^n)$.

To be precise, this paper is organized as follows.

In Section 2, we first present some basic notions and notation appearing in this paper, including Lorentz spaces and their properties and also some known facts on expansive matrixes in [9]. Then we introduce the anisotropic Hardy-Lorentz spaces $H_A^{p,q}(\mathbb{R}^n)$, with $p \in (0, 1]$ and $q \in (0, \infty]$, via the non-tangential grand maximal function (see Definition 2.5 below). These anisotropic Hardy-Lorentz spaces include the classical Hardy spaces of Fefferman and Stein [30], the classical Hardy-Lorentz spaces of Abu-Shammala and Torchinsky [1], the anisotropic Hardy spaces of Bownik [9] and the anisotropic weak Hardy spaces of Ding and Lan [24] as special cases. Some basic properties of $H_A^{p,q}(\mathbb{R}^n)$ are also obtained in this section (see Propositions 2.7 and 2.8 below).

Section 3 is devoted to the atomic and the molecular characterizations of $H_A^{p,q}(\mathbb{R}^n)$. These characterizations of $H_A^{p,q}(\mathbb{R}^n)$ are obtained by using the Calderón-Zygmund decomposition associated with non-tangential grand maximal functions on anisotropic $\mathbb{R}^n$ from [9, p. 9, Lemma 2.7], as well as a criterion for affirming some functions being in Lorentz spaces $L^{p,q}(\mathbb{R}^n)$ from [1, Lemma 1.2]. Recall that, for the classical Hardy-Lorentz spaces $H^{p,q}(\mathbb{R}^n)$, only their ∞-atomic characterizations are known (see [1]). Thus, the r-atomic characterizations of $H_A^{p,q}(\mathbb{R}^n)$, with $r \in (1, \infty)$, presented in Theorem 3.6 below are new even for the classical Hardy-Lorentz spaces. We also point out that the molecular characterizations in Theorem 3.9 below are new even when $p = q$ for the anisotropic Hardy spaces $H_A^p(\mathbb{R}^n)$ with $p \in (0, 1]$. Moreover, the approach used to prove the r-atomic characterization in this paper is much more complicated than that used to prove the ∞-atomic characterization in [1]. Indeed, in the proof of the ∞-atomic characterization, an $L^\infty(\mathbb{R}^n)$ estimate of an infinite combination of ∞-atoms can be easily obtained by the size condition and the finite overlapping property of ∞-atoms (see [1, p. 291]), but this approach fails for the corresponding $L^r(\mathbb{R}^n)$ estimate with $r \in (1, \infty)$. To overcome this difficulty, we employ a distributional estimate (see (3.23) below) instead of the $L^r(\mathbb{R}^n)$ estimate in this paper, which relies on some subtle applications of the boundedness of the grand maximal function on $L^r(\mathbb{R}^n)$ and the finite overlapping property of r-atoms.

In Section 4, we characterize $H_A^{p,q}(\mathbb{R}^n)$ by means of the radial and the non-tangential maximal functions (see Theorem 4.9 below). To this end, via the Aoki-Rolewicz theorem (see [6, 63]), we first prove that the $L^{p,q}(\mathbb{R}^n)$ quasi-norm of the tangential maximal function $T_\varphi^{N(K,L)}(f)$ can be controlled by that of the non-tangential maximal function $M_\varphi^{(K,L)}(f)$ for all $f \in \mathcal{S}'(\mathbb{R}^n)$ (see Lemma 4.6 below), where K is the truncation level, L is the decay level and $\mathcal{S}'(\mathbb{R}^n)$ denotes the set of all tempered distributions on $\mathbb{R}^n$. Then we obtain the boundedness of the maximal function $M_{\mathcal{F}}(f)$ on $L^{p,q}(\mathbb{R}^n)$ with $p \in (1, \infty)$ and $q \in (0, \infty]$ (see Lemma 4.7 below), where $M_{\mathcal{F}}(f)$ is defined as in (2.17) below. As a consequence of Lemma 4.7, both the non-tangential grand maximal function and the Hardy-Littlewood maximal function given by (2.20) are also bounded on $L^{p,q}(\mathbb{R}^n)$ (see Remark 4.8 below). We point out that Lemmas 4.6 and 4.7, Remark 4.8 and the monotone property of the non-increasing rearrangement (see [35, Proposition 1.4.5(8)]) play a key role in proving Theorem 4.9.

In Section 5, we obtain the finite atomic decomposition characterizations of $H_A^{p,q}(\mathbb{R}^n)$. In what follows, $C_c^\infty(\mathbb{R}^n)$ denotes the space of all smooth functions with compact supports. For any admissible anisotropic triplet (p, r, s), via proving that $H_A^{p,q}(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)$, with $r \in [1, \infty]$, and $H_A^{p,q}(\mathbb{R}^n) \cap C_c^\infty(\mathbb{R}^n)$ are dense in $H_A^{p,q}(\mathbb{R}^n)$ (see Lemma 5.2 below), we establish the finite atomic decomposition characterizations of $H_A^{p,q}(\mathbb{R}^n)$ (see Theorem 5.7 below). This extends [53, Theorem 3.1 and Remark 3.3] and [37, Theorem 5.6] to the present setting of anisotropic Hardy-Lorentz spaces.

Section 6 is devoted to the interpolation of $H_A^{p,q}(\mathbb{R}^n)$ and the boundedness of Calderón-Zygmund operators. As an application, in Subsection 6.1, we show that $H_A^{p,q}(\mathbb{R}^n)$ is an intermediate space between $H_A^{p_1,q_1}(\mathbb{R}^n)$ and $H_A^{p_2,q_2}(\mathbb{R}^n)$ with $0 < p_1 < p < p_2 < \infty$ and q_1, q, $q_2 \in (0, \infty]$, and also between $H_A^{p,q_1}(\mathbb{R}^n)$ and $H_A^{p,q_2}(\mathbb{R}^n)$ with $p \in (0, \infty)$ and $0 < q_1 < q < q_2 \leqslant \infty$ in the sense of real interpolation (see Theorem 6.1 below), whose isotropic version includes [1, Theorem 2.5] as a special case (see Remark 6.7(ii) below). In Subsection 6.2, by using the atomic characterization of $H_A^p(\mathbb{R}^n)$, we first obtain the boundedness of δ-type Calderón-Zygmund operators from $H_A^p(\mathbb{R}^n)$ to the weak Lebesgue space $L^{p,\infty}(\mathbb{R}^n)$ (or to $H_A^{p,\infty}(\mathbb{R}^n)$) in the critical case (see Theorem 6.8 below). In this case, even for the classical isotropic setting, δ-type Calderón-Zygmund operators are not bounded from $H^p(\mathbb{R}^n)$ to itself. Moreover, for all $p \in (0,1]$ and $q \in (p,\infty]$, employing the atomic characterizations of $H_A^{p,q}(\mathbb{R}^n)$, we also obtain the boundedness of some Calderón-Zygmund operators from $H_A^{p,q}(\mathbb{R}^n)$ to $L^{p,\infty}(\mathbb{R}^n)$ (see Theorem 6.11 below). In addition, as an application of the finite atomic decomposition characterizations for $H_A^{p,q}(\mathbb{R}^n)$ (see Theorem 5.7 below) obtained in Section 5, we establish a criterion on the boundedness of sublinear operators from $H_A^{p,q}(\mathbb{R}^n)$ into a quasi-Banach space (see Theorem 6.13 below), which is of independent interest; by this criterion, we further conclude that, if T is a sublinear operator and maps all (p,r,s)-atoms with $r \in (1,\infty)$ (or all continuous (p,∞,s)-atoms) into uniformly bounded elements of some quasi-Banach space $\mathcal{B}$, then T has a unique bounded sublinear extension from $H_A^{p,q}(\mathbb{R}^n)$ into $\mathcal{B}$ (see Corollary 6.14 below). This extends the corresponding results of Meda et al. [53], Yang and Zhou [79] and Grafakos et al. [37] to the present setting. Finally, via the criterion established in Theorem 6.13, we also obtain the boundedness of δ-type Calderón-Zygmund operators from $H_A^{p,q}(\mathbb{R}^n)$ to $L^{p,q}(\mathbb{R}^n)$ (or to $H_A^{p,q}(\mathbb{R}^n)$) with $\delta \in (0, \frac{\ln \lambda_-}{\ln b}]$, $p \in (\frac{1}{1+\delta}, 1]$ and $q \in (0,\infty]$ (see Theorem 6.16 below).

We point out that we also obtain the Littlewood-Paley characterizations of anisotropic Hardy-Lorentz spaces $H_A^{p,q}(\mathbb{R}^n)$, respectively, in terms of the Lusin-area functions, the Littlewood-Paley g-functions or the g_λ^*-functions; to restrict the length of this article, we present these characterizations in [48]. More applications of these anisotropic Hardy-Lorentz spaces $H_A^{p,q}(\mathbb{R}^n)$ are expectable.

Finally, we make some conventions on notation. Throughout this paper, we always let $\mathbb{N} := \{1,2,\ldots\}$ and $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$. We denote by C a *positive constant* which is independent of the main parameters, but its value may change from line to line. *Constants with subscripts*, such as C_1, are the same in different occurrences. We also use $C_{(\alpha,\beta,\ldots)}$ to denote a positive constant depending on the indicated parameters α, $\beta,\ldots$ For any multi-index $\beta := (\beta_1,\ldots,\beta_n) \in \mathbb{Z}_+^n$, let $|\beta| := \beta_1 + \cdots + \beta_n$ and $\partial^\beta := (\frac{\partial}{\partial x_1})^{\beta_1} \cdots (\frac{\partial}{\partial x_n})^{\beta_n}$. We use $f \lesssim g$ to denote $f \leqslant Cg$ and, if $f \lesssim g \lesssim f$, then we write $f \sim g$. For every index $r \in [1,\infty]$, we use r' to denote its *conjugate index*, i.e., $1/r + 1/r' = 1$. Moreover, for any set $F \subset \mathbb{R}^n$, we denote by χ_F its *characteristic function*, by $F^\complement$ the set $\mathbb{R}^n \setminus F$, and by $\sharp F$ the cardinality of F. The symbol $\lfloor s \rfloor$, for any $s \in \mathbb{R}$, denotes the biggest integer less than or equal to s.

2 Anisotropic Hardy-Lorentz spaces

In this section, we introduce the anisotropic Hardy-Lorentz spaces via grand maximal functions and give out some basic properties of these spaces.

First we recall the definition of Lorentz spaces. Let $p \in (0,\infty)$ and $q \in (0,\infty]$. The *Lorentz space* $L^{p,q}(\mathbb{R}^n)$ is defined to be the space of all measurable functions f with finite $L^{p,q}(\mathbb{R}^n)$ quasi-norm $\|f\|_{L^{p,q}(\mathbb{R}^n)}$ given by

$$\|f\|_{L^{p,q}(\mathbb{R}^n)} := \begin{cases} \left[\dfrac{q}{p}\displaystyle\int_0^\infty \{t^{\frac{1}{p}} f^*(t)\}^q \frac{dt}{t}\right]^{\frac{1}{q}}, & \text{when } q \in (0,\infty), \\ \displaystyle\sup_{t \in (0,\infty)} [t^{\frac{1}{p}} f^*(t)], & \text{when } q = \infty, \end{cases}$$

where f^* denotes the non-increasing rearrangement of f, namely,

$$f^*(t) := \{\alpha \in (0,\infty) : d_f(\alpha) \leqslant t\}, \quad t \in (0,\infty).$$

Here and hereafter, for any $\alpha \in (0,\infty)$, $d_f(\alpha) := |\{x \in \mathbb{R}^n : |f(x)| > \alpha\}|$. It is well known that, if $q \in (0,\infty)$, then

$$\|f\|_{L^{p,q}(\mathbb{R}^n)} \sim \left\{\int_0^\infty \alpha^{q-1}[d_f(\alpha)]^{\frac{q}{p}} d\alpha\right\}^{\frac{1}{q}} \sim \left\{\sum_{k\in\mathbb{Z}} [2^k\{d_f(2^k)\}^{\frac{1}{p}}]^q\right\}^{\frac{1}{q}} \tag{2.1}$$

and

$$\|f\|_{L^{p,\infty}(\mathbb{R}^n)} \sim \sup_{\alpha\in(0,\infty)} \{\alpha[d_f(\alpha)]^{\frac{1}{p}}\} \sim \sup_{k\in\mathbb{Z}} \{2^k[d_f(2^k)]^{\frac{1}{p}}\}, \tag{2.2}$$

where the implicit equivalent positive constants are independent of f; see [35]. By [35, Remark 1.4.7], for all p, $r \in (0,\infty)$, $q \in (0,\infty]$ and all measurable functions g, we have

$$\||g|^r\|_{L^{p,q}(\mathbb{R}^n)} = \|g\|^r_{L^{pr,qr}(\mathbb{R}^n)}. \tag{2.3}$$

Now let us recall the notion of expansive matrixes (see, for example, [9]).

Definition 2.1. An *expansive matrix* (for short, a *dilation*) is an $n \times n$ real matrix A such that all eigenvalues λ of A satisfy $|\lambda| > 1$.

Throughout this paper, we always let A be a fixed dilation and $b := |\det A|$. By [9, p. 6, (2.7)], we know that $b \in (1,\infty)$. Let λ_- and λ_+ be *positive numbers* such that

$$1 < \lambda_- < \min\{|\lambda| : \lambda \in \sigma(A)\} \leqslant \max\{|\lambda| : \lambda \in \sigma(A)\} < \lambda_+,$$

where $\sigma(A)$ denotes the set of all eigenvalues of A. Then there exists a positive constant C, independent of x and j, such that, for all $x \in \mathbb{R}^n$, when $j \in \mathbb{Z}_+$,

$$C^{-1}(\lambda_-)^j|x| \leqslant |A^j x| \leqslant C(\lambda_+)^j|x| \tag{2.4}$$

and, when $j \in \mathbb{Z} \setminus \mathbb{Z}_+$,

$$C^{-1}(\lambda_+)^j|x| \leqslant |A^j x| \leqslant C(\lambda_-)^j|x|. \tag{2.5}$$

In the case when A is diagonalizable over $\mathbb{C}$, we can even take

$$\lambda_- := \min\{|\lambda| : \lambda \in \sigma(A)\} \quad \text{and} \quad \lambda_+ := \max\{|\lambda| : \lambda \in \sigma(A)\}.$$

Otherwise, we need to choose them sufficiently close to these equalities according to what we need in our arguments.

It was proved in [9, p. 5, Lemma 2.2] that, for a given dilation A, there exist an open ellipsoid Δ and $r \in (1,\infty)$ such that $\Delta \subset r\Delta \subset A\Delta$, and one can additionally assume that $|\Delta| = 1$, where $|\Delta|$ denotes the n-dimensional Lebesgue measure of the set Δ. Let $B_k := A^k\Delta$ for all $k \in \mathbb{Z}$. An ellipsoid $x + B_k$ for some $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$ is called a *dilated ball*. Let $\mathfrak{B}$ be the set of all such dilated balls, namely,

$$\mathfrak{B} := \{x + B_k : x \in \mathbb{R}^n, k \in \mathbb{Z}\}. \tag{2.6}$$

Then B_k is open, $B_k \subset rB_k \subset B_{k+1}$ and $|B_k| = b^k$. Throughout this paper, let τ be the *minimal integer* such that $r^\tau \geqslant 2$. Then, for all $k \in \mathbb{Z}$, it holds true that

$$B_k + B_k \subset B_{k+\tau}, \tag{2.7}$$

$$B_k + (B_{k+\tau})^{\complement} \subset (B_k)^{\complement}, \tag{2.8}$$

where $E + F$ denotes the algebraic sum $\{x + y : x \in E, y \in F\}$ of sets E, $F \subset \mathbb{R}^n$.

Define the *step homogeneous quasi-norm* ρ on $\mathbb{R}^n$ associated to A and Δ as

$$\rho(x) := \begin{cases} b^j, & \text{when } x \in B_{j+1}\backslash B_j, \\ 0, & \text{when } x = \vec{0}_n, \text{ here and hereafter, } \vec{0}_n = (0,\ldots,0) \in \mathbb{R}^n. \end{cases} \tag{2.9}$$

Obviously, for all $k \in \mathbb{Z}$, $B_k = \{x \in \mathbb{R}^n : \rho(x) < b^k\}$. By (2.7) and (2.8), we know that, for all x, $y \in \mathbb{R}^n$,

$$\max\{1, \rho(x+y)\} \leqslant b^\tau(\max\{1,\rho(x)\})(\max\{1,\rho(y)\}) \tag{2.10}$$

and, for all $j \in \mathbb{Z}_+$ and $x \in \mathbb{R}^n$, $\max\{1,\rho(A^j x)\} \leqslant b^j \max\{1,\rho(x)\}$; see [9, p. 8]. Moreover, $(\mathbb{R}^n, \rho, dx)$ is a space of homogeneous type in the sense of Coifman and Weiss [19, 20], here and hereafter, dx denotes the n-dimensional Lebesgue measure.

Recall that the homogeneous quasi-norm induced by A was introduced in [9, p. 6, Definition 2.3] as follows.

Definition 2.2. A *homogeneous quasi-norm* associated with a dilation A is a measurable mapping $\rho : \mathbb{R}^n \to [0, \infty]$ satisfying that

(i) $\rho(x) = 0 \Leftrightarrow x = \vec{0}_n$;

(ii) $\rho(Ax) = b\rho(x)$ for all $x \in \mathbb{R}^n$;

(iii) there exists a positive constant $H \in [1, \infty)$ such that, for all $x,\ y \in \mathbb{R}^n$,

$$\rho(x + y) \leqslant H[\rho(x) + \rho(y)].$$

In the standard dyadic case $A := 2\mathrm{I}_{n\times n}$, $\rho(x) := |x|^n$ for all $x \in \mathbb{R}^n$ is an example of the homogeneous quasi-norm associated with A, here and hereafter, $\mathrm{I}_{n\times n}$ denotes the $n\times n$ *unit matrix* and $|\cdot|$ the Euclidean norm in $\mathbb{R}^n$. It was proved in [9, p. 6, Lemma 2.4] that all homogeneous quasi-norms associated with A are equivalent. Therefore, in what follows, we always use the step homogeneous quasi-norm induced by the given dilation A for convenience.

A $C^\infty(\mathbb{R}^n)$ function φ is said to belong to the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ if, for every integer $\ell \in \mathbb{Z}_+$ and multi-index α,

$$\|\varphi\|_{\alpha,\ell} := \sup_{x\in\mathbb{R}^n} [\rho(x)]^\ell |\partial^\alpha \varphi(x)| < \infty.$$

The dual space of $\mathcal{S}(\mathbb{R}^n)$, namely, the space of all tempered distributions on $\mathbb{R}^n$ equipped with the weak-$*$ topology, is denoted by $\mathcal{S}'(\mathbb{R}^n)$. For any $N \in \mathbb{Z}_+$, define $\mathcal{S}_N(\mathbb{R}^n)$ as

$$\mathcal{S}_N(\mathbb{R}^n) := \{\varphi \in \mathcal{S}(\mathbb{R}^n) : \|\varphi\|_{\alpha,\ell} \leqslant 1, |\alpha| \leqslant N, \ell \leqslant N\},$$

equivalently,

$$\varphi \in \mathcal{S}_N(\mathbb{R}^n) \Leftrightarrow \|\varphi\|_{\mathcal{S}_N(\mathbb{R}^n)} := \sup_{|\alpha|\leqslant N} \sup_{x\in\mathbb{R}^n} [|\partial^\alpha \varphi(x)| \max\{1, [\rho(x)]^N\}] \leqslant 1. \tag{2.11}$$

In what follows, for $\varphi \in \mathcal{S}(\mathbb{R}^n), k \in \mathbb{Z}$ and $x \in \mathbb{R}^n$, let $\varphi_k(x) := b^{-k}\varphi(A^{-k}x)$.

Definition 2.3. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}'(\mathbb{R}^n)$. The *non-tangential maximal function* $M_\varphi(f)$ of f with respect to φ is defined as

$$M_\varphi(f)(x) := \sup_{y\in x+B_k, k\in\mathbb{Z}} |f * \varphi_k(y)|, \quad \forall\, x \in \mathbb{R}^n. \tag{2.12}$$

The *radial maximal function* $M^0_\varphi(f)$ of f with respect to φ is defined as

$$M^0_\varphi(f)(x) := \sup_{k\in\mathbb{Z}} |f * \varphi_k(x)|, \quad \forall\, x \in \mathbb{R}^n. \tag{2.13}$$

For $N \in \mathbb{N}$, the *non-tangential grand maximal function* $M_N(f)$ of $f \in \mathcal{S}'(\mathbb{R}^n)$ is defined as

$$M_N(f)(x) := \sup_{\varphi\in\mathcal{S}_N(\mathbb{R}^n)} M_\varphi(f)(x), \quad \forall\, x \in \mathbb{R}^n \tag{2.14}$$

and the *radial grand maximal function* $M^0_N(f)$ of $f \in \mathcal{S}'(\mathbb{R}^n)$ is defined as

$$M^0_N(f)(x) := \sup_{\varphi\in\mathcal{S}_N(\mathbb{R}^n)} M^0_\varphi(f)(x), \quad \forall\, x \in \mathbb{R}^n.$$

The following proposition is just [9, p. 17, Proposition 3.10].

Proposition 2.4. *For every given $N \in \mathbb{N}$, there exists a positive constant $C_{(N)}$, depending only on N, such that, for all $f \in \mathcal{S}'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,*

$$M^0_N(f)(x) \leqslant M_N(f)(x) \leqslant C_{(N)} M^0_N(f)(x).$$

We now introduce the notion of anisotropic Hardy-Lorentz spaces.

Definition 2.5. Suppose $p\in(0,\infty)$, $q\in(0,\infty]$ and

$$N_{(p)}:=\begin{cases}\left\lfloor\left(\frac{1}{p}-1\right)\frac{\ln b}{\ln\lambda_-}\right\rfloor+2, & \text{when } p\in(0,1],\\ 2, & \text{when } p\in(1,\infty).\end{cases}$$

For every $N\in\mathbb{N}\cap[N_{(p)},\infty)$, the *anisotropic Hardy-Lorentz space* $H_A^{p,q}(\mathbb{R}^n)$ is defined by

$$H_A^{p,q}(\mathbb{R}^n):=\{f\in\mathcal{S}'(\mathbb{R}^n):M_N(f)\in L^{p,q}(\mathbb{R}^n)\}$$

and, for any $f\in H_A^{p,q}(\mathbb{R}^n)$, let $\|f\|_{H_A^{p,q}(\mathbb{R}^n)}:=\|M_N(f)\|_{L^{p,q}(\mathbb{R}^n)}$.

Remark 2.6. Even though the quasi-norm of $H_A^{p,q}(\mathbb{R}^n)$ in Definition 2.5 depends on N, it follows from Theorem 3.6 below that the space $H_A^{p,q}(\mathbb{R}^n)$ is independent of the choice of N as long as $N\in\mathbb{N}\cap[N_{(p)},\infty)$.

Obviously, when $p=q$, $H_A^{p,q}(\mathbb{R}^n)$ becomes the anisotropic Hardy space $H_A^p(\mathbb{R}^n)$ introduced by Bownik [9] and, when $q=\infty$, $H_A^{p,q}(\mathbb{R}^n)$ is the anisotropic weak Hardy space $H_A^{p,\infty}(\mathbb{R}^n)$ investigated by Ding and Lan [24].

Now let us give some basic properties of $H_A^{p,q}(\mathbb{R}^n)$.

Proposition 2.7. *Let $p\in(0,\infty)$, $q\in(0,\infty]$ and $N\in\mathbb{N}\cap[N_{(p)},\infty)$. Then $H_A^{p,q}(\mathbb{R}^n)\subset\mathcal{S}'(\mathbb{R}^n)$ and the inclusion is continuous.*

Proof. Let $f\in H_A^{p,q}(\mathbb{R}^n)$. Then, for any $\varphi\in\mathcal{S}(\mathbb{R}^n)$ and $x\in B_0$, we have

$$\beta:=|\langle f,\varphi\rangle|=|f*\widetilde{\varphi}(\vec{0}_n)|\leqslant M_{\widetilde{\varphi}}(f)(x),\tag{2.15}$$

where $\widetilde{\varphi}(\cdot):=\varphi(-\cdot)$ and $M_{\widetilde{\varphi}}$ is as in (2.12) with φ replaced by $\widetilde{\varphi}$. Notice that, for $q\in(0,\infty]$, by the definitions of $M_{\widetilde{\varphi}}$ and M_N,

$$\|M_{\widetilde{\varphi}}(f)\|_{L^{p,q}(\mathbb{R}^n)}\leqslant\|\widetilde{\varphi}\|_{\mathcal{S}_N(\mathbb{R}^n)}\|M_N(f)\|_{L^{p,q}(\mathbb{R}^n)}=\|\widetilde{\varphi}\|_{\mathcal{S}_N(\mathbb{R}^n)}\|f\|_{H_A^{p,q}(\mathbb{R}^n)}.$$

Thus, to show Proposition 2.7, it suffices to prove that $\beta\lesssim\|M_{\widetilde{\varphi}}(f)\|_{L^{p,q}(\mathbb{R}^n)}$.

To this end, by (2.15) and (2.1), for $q\in(0,\infty)$, we have

$$\begin{aligned}\beta&\lesssim\left\{\sum_{k\in\mathbb{Z},\,k<\log_2\beta}2^{kq}\right\}^{\frac{1}{q}}\sim\left\{\sum_{k\in\mathbb{Z},\,k<\log_2\beta}2^{kq}|\{x\in B_0:\beta\chi_{B_0}(x)>2^k\}|^{\frac{q}{p}}\right\}^{\frac{1}{q}}\\&\lesssim\left\{\sum_{k\in\mathbb{Z}}2^{kq}|\{x\in B_0:M_{\widetilde{\varphi}}(f)(x)>2^k\}|^{\frac{q}{p}}\right\}^{\frac{1}{q}}\lesssim\|M_{\widetilde{\varphi}}(f)\|_{L^{p,q}(\mathbb{R}^n)}.\end{aligned}\tag{2.16}$$

Similar to (2.16), we also conclude that

$$\beta\lesssim\|M_{\widetilde{\varphi}}(f)\|_{L^{p,\infty}(\mathbb{R}^n)}.$$

This finishes the proof of Proposition 2.7. □

Proposition 2.8. *For all $p\in(0,\infty)$, $q\in(0,\infty]$ and $N\in\mathbb{N}\cap[N_{(p)},\infty)$, $H_A^{p,q}(\mathbb{R}^n)$ is complete.*

Proof. To prove that $H_A^{p,q}(\mathbb{R}^n)$ is complete, it suffices to show that, for any sequence $\{f_k\}_{k\in\mathbb{N}}\subset H_A^{p,q}(\mathbb{R}^n)$ such that $\|f_k\|_{H_A^{p,q}(\mathbb{R}^n)}\leqslant 2^{-k}$ for $k\in\mathbb{N}$, the series $\{\sum_{k=1}^m f_k\}_{m\in\mathbb{N}}$ converges in $H_A^{p,q}(\mathbb{R}^n)$. Since $\{\sum_{k=1}^m f_k\}_{m\in\mathbb{N}}$ is a Cauchy sequence in $H_A^{p,q}(\mathbb{R}^n)$, from Proposition 2.7, it follows that $\{\sum_{k=1}^m f_k\}_{m\in\mathbb{N}}$ is also a Cauchy sequence in $\mathcal{S}'(\mathbb{R}^n)$ which, together with the completeness of $\mathcal{S}'(\mathbb{R}^n)$, implies that there exists some $f\in\mathcal{S}'(\mathbb{R}^n)$ such that $\sum_{k=1}^m f_k$ converges to f in $\mathcal{S}'(\mathbb{R}^n)$ as $m\to\infty$. Thus, for any $\varphi\in\mathcal{S}(\mathbb{R}^n)$, the series $\sum_{k=1}^m f_k*\varphi(x)$ converges pointwise to $f*\varphi(x)$ for all $x\in\mathbb{R}^n$ as $m\to\infty$. Therefore, for all $x\in\mathbb{R}^n$, we have

$$M_N(f)(x)\leqslant\sum_{k\in\mathbb{N}}M_N(f_k)(x).$$

By this and the Aoki-Rolewicz theorem (see [6, 63]), we know that there exists $\upsilon \in (0,1]$ such that

$$\|M_N(f)\|_{L^{p,q}(\mathbb{R}^n)}^{\upsilon} \leqslant \left\|\sum_{k\in\mathbb{N}} M_N(f_k)\right\|_{L^{p,q}(\mathbb{R}^n)}^{\upsilon} \lesssim \sum_{k\in\mathbb{N}} \|M_N(f_k)\|_{L^{p,q}(\mathbb{R}^n)}^{\upsilon}.$$

From this, it follows that, for all $m \in \mathbb{N}$,

$$\begin{aligned}\left\|f - \sum_{k=1}^{m} f_k\right\|_{H_A^{p,q}(\mathbb{R}^n)} &= \left\|\sum_{k=m+1}^{\infty} f_k\right\|_{H_A^{p,q}(\mathbb{R}^n)} = \left\|M_N\left(\sum_{k=m+1}^{\infty} f_k\right)\right\|_{L^{p,q}(\mathbb{R}^n)} \\ &\lesssim \left[\sum_{k=m+1}^{\infty} \|M_N(f_k)\|_{L^{p,q}(\mathbb{R}^n)}^{\upsilon}\right]^{\frac{1}{\upsilon}} \\ &\lesssim \left(\sum_{k=m+1}^{\infty} 2^{-k\upsilon}\right)^{\frac{1}{\upsilon}} \sim 2^{-m} \to 0 \quad \text{as} \quad m \to \infty.\end{aligned}$$

Thus, $\sum_{k=1}^{m} f_k \to f$ in $H_A^{p,q}(\mathbb{R}^n)$ as $m \to \infty$. This finishes the proof of Proposition 2.8. □

The following Proposition 2.9 is just [9, p. 13, Theorem 3.6].

Proposition 2.9. *For any given $s \in (1,\infty)$, let*

$$\mathcal{F} := \{\varphi \in L^{\infty}(\mathbb{R}^n) : |\varphi(x)| \leqslant [1+\rho(x)]^{-s}, \forall\, x \in \mathbb{R}^n\}.$$

For $p \in [1,\infty]$ and $f \in L^p(\mathbb{R}^n)$, the maximal function associated with $\mathcal{F}$, $M_{\mathcal{F}}(f)$, is defined by setting

$$M_{\mathcal{F}}(f)(x) := \sup_{\varphi\in\mathcal{F}} M_{\varphi}(f)(x), \quad \forall\, x \in \mathbb{R}^n. \tag{2.17}$$

Then there exists a positive constant $C_{(s)}$, depending on s, such that, for all $\lambda \in (0,\infty)$ and $f \in L^1(\mathbb{R}^n)$,

$$|\{x \in \mathbb{R}^n : M_{\mathcal{F}}(f)(x) > \lambda\}| \leqslant C_{(s)} \|f\|_{L^1(\mathbb{R}^n)}/\lambda \tag{2.18}$$

and, for all $p \in (1,\infty]$ and $f \in L^p(\mathbb{R}^n)$,

$$\|M_{\mathcal{F}}(f)\|_{L^p(\mathbb{R}^n)} \leqslant \frac{C_{(s)}}{1-1/p} \|f\|_{L^p(\mathbb{R}^n)}. \tag{2.19}$$

Remark 2.10. Clearly, by Proposition 2.9, we know that the non-tangential grand maximal function $M_N(f)$, defined in (2.14), and the Hardy-Littlewood maximal function $M_{\mathrm{HL}}(f)$, defined by setting, for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$M_{\mathrm{HL}}(f)(x) := \sup_{k\in\mathbb{Z}} \sup_{y\in x+B_k} \frac{1}{|B_k|} \int_{y+B_k} |f(z)|\, dz = \sup_{x\in B\in\mathfrak{B}} \frac{1}{|B|} \int_B |f(z)|\, dz, \tag{2.20}$$

where $\mathfrak{B}$ is as in (2.6), satisfy (2.18) and (2.19).

3 Atomic and molecular characterizations of $H_A^{p,q}(\mathbb{R}^n)$

In this section, we establish the atomic and the molecular characterizations of $H_A^{p,q}(\mathbb{R}^n)$.

3.1 Atomic characterizations of $H_A^{p,q}(\mathbb{R}^n)$

In this subsection, by using the Calderón-Zygmund decomposition associated with the non-tangential grand maximal function on anisotropic $\mathbb{R}^n$ established in [9], we obtain the atomic characterizations of $H_A^{p,q}(\mathbb{R}^n)$.

We begin with the following notion of anisotropic (p,r,s)-atoms from [9, p. 19, Definition 4.1].

Definition 3.1. An anisotropic triplet (p,r,s) is said to be *admissible* if $p\in(0,1]$, $r\in(1,\infty]$ and $s\in\mathbb{N}$ with $s\geqslant\lfloor(1/p-1)\ln b/\ln\lambda_-\rfloor$. For an admissible anisotropic triplet (p,r,s), a measurable function a on $\mathbb{R}^n$ is called an *anisotropic* (p,r,s)*-atom* if

(i) $\operatorname{supp} a\subset B$, where $B\in\mathfrak{B}$ and $\mathfrak{B}$ is as in (2.6);

(ii) $\|a\|_{L^r(\mathbb{R}^n)}\leqslant|B|^{1/r-1/p}$;

(iii) $\int_{\mathbb{R}^n}a(x)x^\alpha\,dx=0$ for any $\alpha\in\mathbb{Z}_+^n$ with $|\alpha|\leqslant s$.

Throughout this paper, an anisotropic (p,r,s)-atom is simply called a (p,r,s)-atom. Now, via (p,r,s)-atoms, we give the definition of the anisotropic atomic Hardy-Lorentz space $H_A^{p,r,s,q}(\mathbb{R}^n)$ as follows.

Definition 3.2. For an anisotropic triplet (p,r,s) as in Definition 3.1, $q\in(0,\infty]$ and a dilation A, the *anisotropic atomic Hardy-Lorentz space* $H_A^{p,r,s,q}(\mathbb{R}^n)$ is defined to be the set of all distributions $f\in\mathcal{S}'(\mathbb{R}^n)$ satisfying that there exist a sequence of (p,r,s)-atoms, $\{a_i^k\}_{i\in\mathbb{N},\,k\in\mathbb{Z}}$, supported on $\{x_i^k+B_i^k\}_{i\in\mathbb{N},\,k\in\mathbb{Z}}\subset\mathfrak{B}$, respectively, and a positive constant $\widetilde{C}$ such that

$$\sum_{i\in\mathbb{N}}\chi_{x_i^k+B_i^k}(x)\leqslant\widetilde{C}$$

for all $x\in\mathbb{R}^n$ and $k\in\mathbb{Z}$, and

$$f=\sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}}\lambda_i^ka_i^k\quad\text{in}\quad\mathcal{S}'(\mathbb{R}^n),$$

where $\lambda_i^k\sim2^k|B_i^k|^{1/p}$ for all $k\in\mathbb{Z}$ and $i\in\mathbb{N}$ with the implicit equivalent positive constants independent of k and i.

Moreover, for all $f\in H_A^{p,r,s,q}(\mathbb{R}^n)$, define

$$\|f\|_{H_A^{p,r,s,q}(\mathbb{R}^n)}:=\inf\left\{\left[\sum_{k\in\mathbb{Z}}\left(\sum_{i\in\mathbb{N}}|\lambda_i^k|^p\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}:f=\sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}}\lambda_i^ka_i^k\right\}$$

with the usual interpretation for $q=\infty$, where the infimum is taken over all decompositions of f as above.

In order to establish the atomic decomposition of $H_A^{p,q}(\mathbb{R}^n)$, we need the following several technical lemmas, which are [9, p. 9, Lemma 2.7; p. 19, Theorem 4.2] and [1, Lemma 1.2], respectively.

Lemma 3.3. *Suppose that $\Omega\subset\mathbb{R}^n$ is open and $|\Omega|<\infty$. For any $d\in\mathbb{Z}_+$, there exist a sequence of points, $\{x_j\}_{j\in\mathbb{N}}\subset\Omega$, and a sequence of integers, $\{\ell_j\}_{j\in\mathbb{N}}\subset\mathbb{Z}$, such that*

(i) $\Omega=\bigcup_{j\in\mathbb{N}}(x_j+B_{\ell_j})$;

(ii) $\{x_j+B_{\ell_j-\tau}\}_{j\in\mathbb{N}}$ *are pairwise disjoint, where τ is as in* (2.7) *and* (2.8);

(iii) *for every $j\in\mathbb{N}$, $(x_j+B_{\ell_j+d})\cap\Omega^\complement=\emptyset$, but $(x_j+B_{\ell_j+d+1})\cap\Omega^\complement\neq\emptyset$;*

(iv) *if $(x_i+B_{\ell_i+d-2\tau})\cap(x_j+B_{\ell_j+d-2\tau})\neq\emptyset$, then $|\ell_i-\ell_j|\leqslant\tau$;*

(v) *for all $i\in\mathbb{N}$, $\sharp\{j\in\mathbb{N}:(x_i+B_{\ell_i+d-2\tau})\cap(x_j+B_{\ell_j+d-2\tau})\neq\emptyset\}\leqslant L$, where L is a positive constant independent of Ω, f and i.*

Lemma 3.4. *Let (p,r,s) be an admissible anisotropic triplet as in Definition* 3.1 *and $N\in\mathbb{N}\cap[N_{(p)},\infty)$. Then there exists a positive constant C, depending only on p and r, such that, for all (p,r,s)-atoms a,*

$$\|M_N(a)\|_{L^p(\mathbb{R}^n)}\leqslant C.$$

Lemma 3.5. *Suppose that $p\in(0,\infty)$, $q\in(0,\infty]$, $\{\mu_k\}_{k\in\mathbb{Z}}$ is a non-negative sequence of complex numbers such that $\{2^k\mu_k\}_{k\in\mathbb{Z}}\in\ell^q$ and φ is a non-negative function having the following property: There exists $\delta\in(0,\min\{1,q/p\})$ such that, for any $k_0\in\mathbb{N}$, $\varphi\leqslant\psi_{k_0}+\eta_{k_0}$, where ψ_{k_0} and η_{k_0} are functions, depending on k_0 and satisfying*

$$2^{k_0p}[d_{\psi_{k_0}}(2^{k_0})]^\delta\leqslant\widetilde{C}\sum_{k=-\infty}^{k_0-1}[2^k(\mu_k)^\delta]^p\quad\textit{and}\quad2^{k_0\delta p}d_{\eta_{k_0}}(2^{k_0})\leqslant\widetilde{C}\sum_{k=k_0}^{\infty}[2^{k\delta}\mu_k]^p$$

for some positive constant $\widetilde{C}$ independent of k_0. Then $\varphi \in L^{p,q}(\mathbb{R}^n)$ and

$$\|\varphi\|_{L^{p,q}(\mathbb{R}^n)} \leqslant C\|\{2^k\mu_k\}_{k\in\mathbb{Z}}\|_{\ell^q},$$

where C is a positive constant independent of φ and $\{\mu_k\}_{k\in\mathbb{Z}}$.

Now, it is a position to state the main result of this subsection.

Theorem 3.6. *Let (p,r,s) be an admissible anisotropic triplet as in Definition 3.1, $q \in (0,\infty]$ and $N \in \mathbb{N} \cap [N_{(p)},\infty)$. Then $H_A^{p,q}(\mathbb{R}^n) = H_A^{p,r,s,q}(\mathbb{R}^n)$ with equivalent quasi-norms.*

Proof. First, we show that

$$H_A^{p,q}(\mathbb{R}^n) \subset H_A^{p,r,s,q}(\mathbb{R}^n). \tag{3.1}$$

Observe that, by Definition 3.1, for any $r \in (1,\infty)$, a (p,∞,s)-atom is also a (p,r,s)-atom and hence $H_A^{p,\infty,s,q}(\mathbb{R}^n) \subset H_A^{p,r,s,q}(\mathbb{R}^n)$. Thus, to prove (3.1), we only need to show that

$$H_A^{p,q}(\mathbb{R}^n) \subset H_A^{p,\infty,s,q}(\mathbb{R}^n). \tag{3.2}$$

Now we prove (3.2) by three steps.

Step 1. To show (3.2), for any $f \in H_A^{p,q}(\mathbb{R}^n)$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n}\varphi(x)\,dx = 1$, and $m \in \mathbb{N}$, let $f^{(m)} := f * \varphi_{-m}$. Then, by [9, p. 15, Lemma 3.8], we have $f^{(m)} \to f$ in $\mathcal{S}'(\mathbb{R}^n)$ as $m \to \infty$. Moreover, by [9, p. 39, Lemma 6.6], we know that, for all $m \in \mathbb{N}$ and $x \in \mathbb{R}^n$,

$$M_{N+2}(f^{(m)})(x) \leqslant C_{(N,\varphi)}M_N(f)(x), \tag{3.3}$$

where $C_{(N,\varphi)}$ is a positive constant depending on N and φ, but independent of f. Therefore, $f^{(m)} \in H_A^{p,q}(\mathbb{R}^n)$ and $\|f^{(m)}\|_{H_A^{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{H_A^{p,q}(\mathbb{R}^n)}$ with the implicit positive constant independent of m and f.

In what follows of this step, we show that, for any $m \in \mathbb{N}$,

$$f^{(m)} = \sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}} h_i^{m,k} \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n), \tag{3.4}$$

where, for all m, $i \in \mathbb{N}$ and $k \in \mathbb{Z}$, $h_i^{m,k}$ is a (p,∞,s)-atom multiplied by a constant depending on k and i but, independent of f and m.

To show (3.4), we borrow some ideas from the proof of [9, p. 38, Theorem 6.4]. For $k \in \mathbb{Z}$ and $N \in \mathbb{N} \cap [N_{(p)},\infty)$, let

$$\Omega_k := \{x \in \mathbb{R}^n : M_N(f)(x) > 2^k\}.$$

Then Ω_k is open. Applying Lemma 3.3 to Ω_k with $d = 6\tau$, we obtain a sequence $\{x_i^k\}_{i\in\mathbb{N}} \subset \Omega_k$ and a sequence of integers, $\{\ell_i^k\}_{i\in\mathbb{N}}$, satisfying, with τ and L same as in Lemma 3.3,

$$\Omega_k = \bigcup_{i\in\mathbb{N}}(x_i^k + B_{\ell_i^k}), \tag{3.5}$$

$$(x_i^k + B_{\ell_i^k-\tau}) \cap (x_j^k + B_{\ell_j^k-\tau}) = \emptyset \quad \text{for all} \quad i,\ j \in \mathbb{N} \quad \text{with} \quad i \neq j, \tag{3.6}$$

$$(x_i^k + B_{\ell_i^k+6\tau}) \cap \Omega_k^{\complement} = \emptyset, \quad (x_i^k + B_{\ell_i^k+6\tau+1}) \cap \Omega_k^{\complement} \neq \emptyset \quad \text{for all} \quad i \in \mathbb{N},$$

$$\text{if} \quad (x_i^k + B_{\ell_i^k+4\tau}) \cap (x_j^k + B_{\ell_j^k+4\tau}) \neq \emptyset, \quad \text{then} \quad |\ell_i^k - \ell_j^k| \leqslant \tau,$$

$$\sharp\{j \in \mathbb{N} : (x_i^k + B_{\ell_i^k+4\tau}) \cap (x_j^k + B_{\ell_j^k+4\tau}) \neq \emptyset\} \leqslant L \quad \text{for all} \quad i \in \mathbb{N}. \tag{3.7}$$

Fix $\theta \in \mathcal{S}(\mathbb{R}^n)$ such that $\operatorname{supp}\theta \subset B_\tau$, $0 \leqslant \theta \leqslant 1$, and $\theta \equiv 1$ on B_0. For each $i \in \mathbb{N}$, $k \in \mathbb{Z}$ and all $x \in \mathbb{R}^n$, define $\theta_i^k(x) := \theta(A^{-\ell_i^k}(x - x_i^k))$ and

$$\zeta_i^k(x) := \frac{\theta_i^k(x)}{\Sigma_{j\in\mathbb{N}}\theta_j^k(x)}.$$

Then $\zeta_i^k \in \mathcal{S}(\mathbb{R}^n)$, $\operatorname{supp}\zeta_i^k \subset x_i^k + B_{\ell_i^k+\tau}$, $0 \leqslant \zeta_i^k \leqslant 1$, $\zeta_i^k \equiv 1$ on $x_i^k + B_{\ell_i^k-\tau}$ by (3.6), and $\sum_{i\in\mathbb{N}} \zeta_i^k = \chi_{\Omega_k}$. Therefore, the family $\{\zeta_i^k\}_{i\in\mathbb{N}}$ forms a smooth partition of unity on Ω_k.

For $\ell \in \mathbb{Z}_+$, let $\mathcal{P}_\ell(\mathbb{R}^n)$ denote the linear space of all polynomials on $\mathbb{R}^n$ with degree not more than ℓ. For each i and $P \in \mathcal{P}_\ell(\mathbb{R}^n)$, define

$$\|P\|_{i,k} := \left[\frac{1}{\int_{\mathbb{R}^n} \zeta_i^k(x)dx} \int_{\mathbb{R}^n} |P(x)|^2 \zeta_i^k(x)dx\right]^{1/2}, \tag{3.8}$$

which induces a finite dimensional Hilbert space $(\mathcal{P}_\ell(\mathbb{R}^n), \|\cdot\|_{i,k})$. For each i, via

$$Q \mapsto \frac{1}{\int_{\mathbb{R}^n} \zeta_i^k(x)dx} \langle f^{(m)}, Q\zeta_i^k\rangle, \quad Q \in \mathcal{P}_\ell(\mathbb{R}^n),$$

the function $f^{(m)}$ induces a linear bounded functional on $\mathcal{P}_\ell(\mathbb{R}^n)$. By the Riesz lemma, there exists a unique polynomial $P_i^{m,k} \in \mathcal{P}_\ell(\mathbb{R}^n)$ such that, for all $Q \in \mathcal{P}_\ell(\mathbb{R}^n)$,

$$\frac{1}{\int_{\mathbb{R}^n} \zeta_i^k(x)dx} \langle f^{(m)}, Q\zeta_i^k\rangle = \frac{1}{\int_{\mathbb{R}^n} \zeta_i^k(x)dx} \langle P_i^{m,k}, Q\zeta_i^k\rangle = \frac{1}{\int_{\mathbb{R}^n} \zeta_i^k(x)dx} \int_{\mathbb{R}^n} P_i^{m,k}(x)Q(x)\zeta_i^k(x)dx.$$

For every $i \in \mathbb{N}$, $k \in \mathbb{Z}$ and $m \in \mathbb{N}$, define a distribution

$$b_i^{m,k} := [f^{(m)} - P_i^{m,k}]\zeta_i^k.$$

From the fact that, for all $k \in \mathbb{Z}$ and $x \in \mathbb{R}^n$, $\sum_{i\in\mathbb{N}} \chi_{x_i^k + B_{\ell_i^k+4\tau}}(x) \leqslant L$ and $\operatorname{supp} b_i^{m,k} \subset x_i^k + B_{\ell_i^k+4\tau}$, it follows that $\{\sum_{i=1}^I b_i^{m,k}\}_{I\in\mathbb{N}}$ converges in $\mathcal{S}'(\mathbb{R}^n)$. Let

$$g^{m,k} := f^{(m)} - \sum_{i\in\mathbb{N}} b_i^{m,k} = f^{(m)} - \sum_{i\in\mathbb{N}} [f^{(m)} - P_i^{m,k}]\zeta_i^k = f^{(m)}\chi_{\Omega_k^\complement} + \sum_{i\in\mathbb{N}} P_i^{m,k}\zeta_i^k.$$

Notice that, for any $k \in \mathbb{Z}$ and $x \in \mathbb{R}^n$, the number i satisfying $\zeta_i^k(x) \neq 0$ is less than L, where L is the same as in (3.7). Therefore, by a proof similar to that of [9, p. 25, Lemma 5.3], we easily conclude that, for all $x \in \mathbb{R}^n$,

$$\left|\sum_{i\in\mathbb{N}} P_i^{m,k}(x)\zeta_i^k(x)\right| \lesssim 2^k,$$

where the implicit positive constant is independent of k. Clearly, by (3.3), for all $x \in \mathbb{R}^n$, we have

$$|f^{(m)}(x)\chi_{\Omega_k^\complement}(x)| \lesssim M_N(f)(x)\chi_{\Omega_k^\complement}(x) \lesssim 2^k,$$

where the implicit positive constants are independent of k. Thus, $\|g^{m,k}\|_{L^\infty(\mathbb{R}^n)} \lesssim 2^k$ and $\|g^{m,k}\|_{L^\infty(\mathbb{R}^n)} \to 0$ as $k \to -\infty$.

Following the proof of [9, p. 31, Lemma 5.7], we conclude that, for any $k \in \mathbb{Z}$ and $p_0 \in (0, p)$ satisfying $\lfloor (1/p_0 - 1)\ln b/\ln \lambda_-\rfloor \leqslant s$,

$$\int_{\mathbb{R}^n} \left[M_N\left(\sum_{i\in\mathbb{N}} b_i^{m,k}\right)(x)\right]^{p_0} dx \lesssim \int_{\widetilde{\Omega}_k} [M_N(f^{(m)})(x)]^{p_0}\, dx, \tag{3.9}$$

where

$$\widetilde{\Omega}_k := \{x \in \mathbb{R}^n : M_N(f^{(m)})(x) > 2^k\}.$$

Since $f^{(m)} \in H_A^{p,q}(\mathbb{R}^n)$, it follows that there exists an integer k_0 such that, for any $k \in [k_0, \infty) \cap \mathbb{Z}$, $|\widetilde{\Omega}_k| < \infty$. Noticing that, for all $\alpha \in (0, \infty)$,

$$|\widetilde{\Omega}_k \cap \{x \in \mathbb{R}^n : M_N(f^{(m)})(x) > \alpha\}| \leqslant \min\{|\widetilde{\Omega}_k|, \alpha^{-p}\|M_N(f^{(m)})\|_{L^{p,\infty}(\mathbb{R}^n)}^p\},$$

we have

$$\int_{\widetilde{\Omega}_k} [M_N(f^{(m)})(x)]^{p_0}\, dx = \int_0^\infty p_0\alpha^{p_0-1}|\{x \in \widetilde{\Omega}_k : M_N(f^{(m)})(x) > \alpha\}|\, d\alpha$$

$$
\begin{aligned}
&\leqslant \int_0^\gamma p_0|\widetilde{\Omega}_k|\alpha^{p_0-1}\,d\alpha+\int_\gamma^\infty p_0\alpha^{p_0-p-1}\|M_N(f^{(m)})\|^p_{L^{p,\infty}(\mathbb{R}^n)}\,d\alpha\\
&=\frac{p}{p-p_0}|\widetilde{\Omega}_k|^{1-p_0/p}\|M_N(f^{(m)})\|^{p_0}_{L^{p,\infty}(\mathbb{R}^n)}\\
&\lesssim|\widetilde{\Omega}_k|^{1-p_0/p}\|M_N(f^{(m)})\|^{p_0}_{L^{p,q}(\mathbb{R}^n)},
\end{aligned}\tag{3.10}
$$

where

$$\gamma:=\|M_N(f^{(m)})\|_{L^{p,\infty}(\mathbb{R}^n)}|\widetilde{\Omega}_k|^{-1/p}.$$

By (3.9) and (3.10), we find that

$$\Bigg\|\sum_{i\in\mathbb{N}}b_i^{m,k}\Bigg\|_{H_A^{p_0}(\mathbb{R}^n)}:=\Bigg\|M_N\Bigg(\sum_{i\in\mathbb{N}}b_i^{m,k}\Bigg)\Bigg\|_{L^{p_0}(\mathbb{R}^n)}\lesssim|\widetilde{\Omega}_k|^{\frac{1}{p_0}-\frac{1}{p}}\|M_N(f^{(m)})\|_{L^{p,q}(\mathbb{R}^n)}\to 0\quad\text{as}\quad k\to\infty,$$

where the implicit positive constant is independent of k and $H_A^{p_0}(\mathbb{R}^n)$ denotes the anisotropic Hardy space introduced by Bownik [9]. From the above estimates, we further deduce that

$$\Bigg\|f^{(m)}-\sum_{k=-N}^{N}(g^{m,k+1}-g^{m,k})\Bigg\|_{H_A^{p_0}(\mathbb{R}^n)+L^\infty(\mathbb{R}^n)}\lesssim\Bigg\|\sum_{i\in\mathbb{N}}b_i^{m,N+1}\Bigg\|_{H_A^{p_0}(\mathbb{R}^n)}+\|g^{m,-N}\|_{L^\infty(\mathbb{R}^n)}\to 0$$

as $N\to\infty$. Here, the implicit positive constant is independent of N and, for any $f\in H_A^{p_0}(\mathbb{R}^n)+L^\infty(\mathbb{R}^n)$, let

$$\|f\|_{H_A^{p_0}(\mathbb{R}^n)+L^\infty(\mathbb{R}^n)}:=\inf\{\|f_1\|_{H_A^{p_0}(\mathbb{R}^n)}+\|f_2\|_{L^\infty(\mathbb{R}^n)}:f=f_1+f_2,\ f_1\in H_A^{p_0}(\mathbb{R}^n),f_2\in L^\infty(\mathbb{R}^n)\},$$

where the infimum is taken over all decompositions of f as above. Therefore,

$$f^{(m)}=\sum_{k=-\infty}^{\infty}(g^{m,k+1}-g^{m,k})\quad\text{in}\quad\mathcal{S}'(\mathbb{R}^n).\tag{3.11}$$

Moreover, for $i\in\mathbb{N}$, $k\in\mathbb{Z}$ and $j\in\mathbb{N}$, define a polynomial $P_{i,j}^{m,k+1}$ as the orthogonal projection of $[f^{(m)}-P_j^{m,k+1}]\zeta_i^k$ on $\mathcal{P}_\ell(\mathbb{R}^n)$ with respect to the norm defined by (3.8), namely, $P_{i,j}^{m,k+1}$ is the unique element of $\mathcal{P}_\ell(\mathbb{R}^n)$ satisfying that, for all $Q\in\mathcal{P}_\ell(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n}[f^{(m)}(x)-P_j^{m,k+1}(x)]\zeta_i^k(x)Q(x)\zeta_j^{k+1}(x)\,dx=\int_{\mathbb{R}^n}P_{i,j}^{m,k+1}(x)Q(x)\zeta_j^{k+1}(x)\,dx.$$

By an argument parallel to the proof of [9, p. 37, Lemma 6.3], we find that

$$\sum_{j\in\mathbb{N}}\sum_{i\in\mathbb{N}}P_{j,i}^{m,k+1}\zeta_i^{k+1}=0.$$

Then, for any $k\in\mathbb{Z}$, by the facts that $\sum_{j\in\mathbb{N}}\zeta_j^k=\chi_{\Omega_k}$,

$$\sum_{i\in\mathbb{N}}b_i^{m,k+1}=f^{(m)}\chi_{\Omega_{k+1}}-\sum_{i\in\mathbb{N}}P_i^{m,k+1}\zeta_i^{k+1},\quad\operatorname{supp}\Bigg(\sum_{i\in\mathbb{N}}P_i^{m,k+1}\zeta_i^{k+1}\Bigg)\subset\Omega_{k+1}$$

and $\Omega_{k+1}\subset\Omega_k$, we have

$$
\begin{aligned}
g^{m,k+1}-g^{m,k}&=\Bigg[f^{(m)}-\sum_{i\in\mathbb{N}}b_i^{m,k+1}\Bigg]-\Bigg[f^{(m)}-\sum_{j\in\mathbb{N}}b_j^{m,k}\Bigg]\\
&=\sum_{j\in\mathbb{N}}b_j^{m,k}-\sum_{j\in\mathbb{N}}\sum_{i\in\mathbb{N}}b_i^{m,k+1}\zeta_j^k+\sum_{j\in\mathbb{N}}\sum_{i\in\mathbb{N}}P_{j,i}^{m,k+1}\zeta_i^{k+1}\\
&=\sum_{i\in\mathbb{N}}\Bigg[b_i^{m,k}-\sum_{j\in\mathbb{N}}(b_j^{m,k+1}\zeta_i^k-P_{i,j}^{m,k+1}\zeta_j^{k+1})\Bigg]=:\sum_{i\in\mathbb{N}}h_i^{m,k},
\end{aligned}\tag{3.12}
$$

where all the series converge in $\mathcal{S}'(\mathbb{R}^n)$. Furthermore, for all $i\in\mathbb{N}$ and $k\in\mathbb{Z}$,

$$h_i^{m,k}=[f^{(m)}-P_i^{m,k}]\zeta_i^k-\sum_{j\in\mathbb{N}}\{[f^{(m)}-P_j^{m,k+1}]\zeta_i^k-P_{i,j}^{m,k+1}\}\zeta_j^{k+1}.$$

By the definitions of $P_i^{m,k}$ and $P_{i,j}^{m,k+1}$, we know that

$$\int_{\mathbb{R}^n}h_i^{m,k}(x)Q(x)\,dx=0\quad\text{for all}\quad Q\in\mathcal{P}_\ell(\mathbb{R}^n).\tag{3.13}$$

In addition, recall that $P_{i,j}^{m,k+1}\neq 0$ implies

$$(x_j^{k+1}+B_{\ell_j^{k+1}+\tau})\cap(x_i^k+B_{\ell_i^k+\tau})\neq\emptyset.$$

Then, by a proof similar to that of [9, p. 35, Lemma 6.1(i)], we find that

$$\operatorname{supp}\zeta_j^{k+1}\subset(x_j^{k+1}+B_{\ell_j^{k+1}+\tau})\subset(x_i^k+B_{\ell_i^k+4\tau}).$$

Therefore,

$$\operatorname{supp}h_i^{m,k}\subset(x_i^k+B_{\ell_i^k+4\tau}).\tag{3.14}$$

Since $\sum_{j\in\mathbb{N}}\zeta_j^{k+1}=\chi_{\Omega_{k+1}}$, it follows that

$$h_i^{m,k}=\zeta_i^kf^{(m)}\chi_{\Omega_{k+1}^\complement}-P_i^{m,k}\zeta_i^k+\zeta_i^k\sum_{j\in\mathbb{N}}P_j^{m,k+1}\zeta_j^{k+1}+\sum_{j\in\mathbb{N}}P_{i,j}^{m,k+1}\zeta_j^{k+1}.\tag{3.15}$$

By a proof similar to that of [9, p. 35, Lemma 6.1(ii); p. 36, Lemma 6.2], we find that, for all $j\in\mathbb{N}$,

$$\sharp\{i\in\mathbb{N}:(x_j^{k+1}+B_{\ell_j^{k+1}+\tau})\cap(x_i^k+B_{\ell_i^k+\tau})\neq\emptyset\}\lesssim 1$$

and, for all $i,\ j,\ m\in\mathbb{N}$ and $k\in\mathbb{Z}$,

$$\sup_{x\in\mathbb{R}^n}|P_{i,j}^{m,k+1}(x)\zeta_j^{k+1}(x)|\lesssim 2^{k+1},$$

which, combined with $\sup_{x\in\mathbb{R}^n}|P_i^{m,k}(x)\zeta_i^k(x)|\lesssim 2^k$, $\|f^{(m)}\chi_{\Omega_{k+1}^\complement}\|_{L^\infty(\mathbb{R}^n)}\lesssim 2^k$ and (3.15), further implies that, for all $i,\,m\in\mathbb{N}$ and $k\in\mathbb{Z}$,

$$\|h_i^{m,k}\|_{L^\infty(\mathbb{R}^n)}\lesssim 2^k.\tag{3.16}$$

By (3.13), (3.14) and (3.16), we know that, for all $k\in\mathbb{Z}$ and $m,\ i\in\mathbb{N}$, $h_i^{m,k}$ is a multiple of a (p,∞,s)-atom, which, together with (3.11) and (3.12), implies that (3.4) holds true.

Step 2. By (3.16) and the Alaoglu theorem (see, for example, [64, Theorem 3.17]), there exists a subsequence $\{m_\iota\}_{\iota=1}^\infty\subset\mathbb{N}$ such that, for every $i\in\mathbb{N}$ and $k\in\mathbb{Z}$, $h_i^{m_\iota,k}\to h_i^k$ weak-$*$ in $L^\infty(\mathbb{R}^n)$ as $\iota\to\infty$. It is easy to see that $\operatorname{supp}h_i^k\subset(x_i^k+B_{\ell_i^k+4\tau})$, $\|h_i^k\|_{L^\infty(\mathbb{R}^n)}\lesssim 2^k$ and $\int_{\mathbb{R}^n}h_i^k(x)Q(x)dx=0$ for all $Q\in\mathcal{P}_\ell(\mathbb{R}^n)$. Thus, h_i^k is a multiple of a (p,∞,s)-atom a_i^k. Let $h_i^k:=\lambda_i^ka_i^k$, where $\lambda_i^k\sim 2^k|B_{\ell_i^k+4\tau}|^{1/p}$. Then, by (3.7), (3.5) and (2.1), for $q\in(0,\infty)$, we have

$$\begin{aligned}\sum_{k=-\infty}^{\infty}\left(\sum_{i\in\mathbb{N}}|\lambda_i^k|^p\right)^{\frac{q}{p}}&\sim\sum_{k=-\infty}^{\infty}\left(\sum_{i\in\mathbb{N}}2^{kp}|B_{\ell_i^k+4\tau}|\right)^{\frac{q}{p}}\\&\sim\sum_{k=-\infty}^{\infty}2^{kq}|\Omega_k|^{\frac{q}{p}}\sim\|M_N(f)\|_{L^{p,q}(\mathbb{R}^n)}^q\sim\|f\|_{H_A^{p,q}(\mathbb{R}^n)}^q.\end{aligned}\tag{3.17}$$

Similar to (3.17), we conclude that $\sup_{k\in\mathbb{Z}}(\sum_{i\in\mathbb{N}}|\lambda_i^k|^p)^{1/p}\lesssim\|f\|_{H_A^{p,\infty}(\mathbb{R}^n)}$.

Step 3. By Step 2, we easily know that, to prove (3.2), it suffices to show that $f = \sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}} h_i^k$ in $\mathcal{S}'(\mathbb{R}^n)$.

To show this, for all $k\in\mathbb{Z}$, let

$$f_k := \sum_{i\in\mathbb{N}} h_i^k.$$

Then $f_k^{(m_\iota)} \to f_k$ in $\mathcal{S}'(\mathbb{R}^n)$ as $\iota\to\infty$, where, for all $m\in\mathbb{N}$ and $k\in\mathbb{Z}$, $f_k^{(m)} := g^{m,k+1} - g^{m,k}$. Indeed, by the finite intersection property of $\{x_i^k + B_{\ell_i^k+4\tau}\}_{i\in\mathbb{N}}$ for each $k\in\mathbb{Z}$ (see (3.7)), and the support conditions of h_i^k and $h_i^{m,k}$, we know that, for any $\phi\in\mathcal{S}(\mathbb{R}^n)$,

$$\langle f_k^{(m_\iota)},\phi\rangle = \left\langle \sum_{i\in\mathbb{N}} h_i^{m_\iota,k},\phi\right\rangle = \sum_{i\in\mathbb{N}}\langle h_i^{m_\iota,k},\phi\rangle \to \sum_{i\in\mathbb{N}}\langle h_i^k,\phi\rangle = \left\langle \sum_{i\in\mathbb{N}} h_i^k,\phi\right\rangle = \langle f_k,\phi\rangle \quad \text{as} \quad \iota\to\infty.$$

We next show that, for all $m\in\mathbb{N}$,

$$\sum_{|k|\geqslant K_1} f_k^{(m)} \to 0 \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n) \quad \text{as} \quad K_1\to\infty. \tag{3.18}$$

To show (3.18), we consider two cases.

For $p\in(0,1)$, it suffices to show that, for all $m\in\mathbb{N}$,

$$\lim_{K_2\to-\infty}\left\|\sum_{k\leqslant K_2} f_k^{(m)}\right\|_{H_A^1(\mathbb{R}^n)} = 0 \quad \text{and} \quad \lim_{K_3\to\infty}\left\|\sum_{k\geqslant K_3} f_k^{(m)}\right\|_{H_A^{p_0}(\mathbb{R}^n)} = 0, \tag{3.19}$$

where $p_0\in(0,p)$ satisfies that $\lfloor(1/p_0-1)\ln b/\ln\lambda_-\rfloor\leqslant s$. Indeed, by (3.13), (3.14) and (3.16), we find that $(2^k|B_{\ell_i^k+4\tau}|)^{-1}h_i^{m,k}$ is a $(1,\infty,s)$-atom multiplied by a constant. Therefore, by Lemma 3.4, (3.7), (3.5) and (2.2), we have

$$\begin{aligned}\left\|\sum_{k\leqslant K_2} f_k^{(m)}\right\|_{H_A^1(\mathbb{R}^n)} &\leqslant \sum_{k\leqslant K_2}\sum_{i\in\mathbb{N}}\|h_i^{m,k}\|_{H_A^1(\mathbb{R}^n)} \lesssim \sum_{k\leqslant K_2}\sum_{i\in\mathbb{N}} 2^k|B_{\ell_i^k+4\tau}|\\ &\lesssim \sum_{k\leqslant K_2} 2^k|\Omega_k| \lesssim \sum_{k\leqslant K_2} 2^{k(1-p)}\|f\|_{H_A^{p,\infty}(\mathbb{R}^n)}^p \lesssim \sum_{k\leqslant K_2} 2^{k(1-p)}\|f\|_{H_A^{p,q}(\mathbb{R}^n)}^p,\end{aligned}$$

which converges to 0 as $K_2\to-\infty$. Similarly, $(2^k|B_{\ell_i^k+4\tau}|^{1/p_0})^{-1}h_i^{m,k}$ is a (p_0,∞,s)-atom multiplied by a constant. Since $\lfloor(1/p_0-1)\ln b/\ln\lambda_-\rfloor\leqslant s$, by Lemma 3.4, (3.7), (3.5) and (2.2) again, we find that

$$\begin{aligned}\left\|\sum_{k\geqslant K_3} f_k^{(m)}\right\|_{H_A^{p_0}(\mathbb{R}^n)}^{p_0} &\leqslant \sum_{k\geqslant K_3}\sum_{i\in\mathbb{N}}\|h_i^{m,k}\|_{H_A^{p_0}(\mathbb{R}^n)}^{p_0} \lesssim \sum_{k\geqslant K_3}\sum_{i\in\mathbb{N}} 2^{kp_0}|B_{\ell_i^k+4\tau}| \lesssim \sum_{k\geqslant K_3} 2^{kp_0}|\Omega_k|\\ &\lesssim \sum_{k\geqslant K_3} 2^{k(p_0-p)}\|f\|_{H_A^{p,\infty}(\mathbb{R}^n)}^p \lesssim \sum_{k\geqslant K_3} 2^{k(p_0-p)}\|f\|_{H_A^{p,q}(\mathbb{R}^n)}^p,\end{aligned}$$

which converges to 0 as $K_3\to\infty$. These prove (3.19) and hence (3.18).

For $p=1$, we replace $H_A^1(\mathbb{R}^n)$ by $L^2(\mathbb{R}^n)$. Notice that

$$\begin{aligned}\left\|\sum_{k\leqslant K_2} f_k^{(m)}\right\|_{L^2(\mathbb{R}^n)} &\leqslant \sum_{k\leqslant K_2}\left\|\sum_{i\in\mathbb{N}} h_i^{m,k}\right\|_{L^2(\mathbb{R}^n)} \lesssim \sum_{k\leqslant K_2} 2^k\left|\bigcup_{i\in\mathbb{N}}(x_i^k + B_{\ell_i^k+4\tau})\right|^{1/2}\\ &\lesssim \sum_{k\leqslant K_2} 2^k|\Omega_k|^{1/2} \lesssim \sum_{k\leqslant K_2} 2^{k/2}\|f\|_{H_A^{1,\infty}(\mathbb{R}^n)}^{1/2} \lesssim \sum_{k\leqslant K_2} 2^{k/2}\|f\|_{H_A^{1,q}(\mathbb{R}^n)}^{1/2},\end{aligned}$$

which converges to 0 as $K_3\to\infty$. This implies that (3.18) also holds true in this case.

An argument similar to that used in the proof of (3.18) also shows that $\sum_{|k|\geqslant K_1} f_k\to 0$ in $\mathcal{S}'(\mathbb{R}^n)$ as $K_1\to\infty$. From this and (3.18), it follows that, for any $\phi\in\mathcal{S}(\mathbb{R}^n)$ and $\varepsilon\in(0,\infty)$, there exists some $\widetilde{K}_1\in\mathbb{N}$, independent of m_ι, such that

$$\left|\left\langle \sum_{|k|\geqslant\widetilde{K}_1} f_k^{(m_\iota)},\phi\right\rangle\right| < \frac{\varepsilon}{3} \quad \text{and} \quad \left|\left\langle \sum_{|k|\geqslant\widetilde{K}_1} f_k,\phi\right\rangle\right| < \frac{\varepsilon}{3}. \tag{3.20}$$

Fixing this $\widetilde{K}_1$, by the fact that $f_k^{(m_\iota)} \to f_k$ in $\mathcal{S}'(\mathbb{R}^n)$ as $\iota \to \infty$ for all $k \in \mathbb{Z}$, we know that there exists an integer $\widetilde{\iota}$ such that, if $\iota > \widetilde{\iota}$, then, for all integers k with $|k| \leqslant \widetilde{K}_1$, we have

$$|\langle f_k^{(m_\iota)} - f_k, \phi\rangle| < \frac{\varepsilon}{6\widetilde{K}_1 + 3},$$

which, combined with (3.20), implies that, if $\iota > \widetilde{\iota}$,

$$\begin{aligned}\left|\left\langle \sum_{k\in\mathbb{Z}} f_k^{(m_\iota)}, \phi\right\rangle - \left\langle \sum_{k\in\mathbb{Z}} f_k, \phi\right\rangle\right| &\leqslant \left|\left\langle \sum_{|k|\geqslant\widetilde{K}_1} f_k^{(m_\iota)}, \phi\right\rangle\right| + \left|\left\langle \sum_{|k|\geqslant\widetilde{K}_1} f_k, \phi\right\rangle\right| + \left|\sum_{|k|\leqslant\widetilde{K}_1} \langle f_k^{(m_\iota)} - f_k, \phi\rangle\right| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

Thus, $\lim_{\iota\to\infty}\langle\sum_{k\in\mathbb{Z}} f_k^{(m_\iota)}, \phi\rangle = \langle\sum_{k\in\mathbb{Z}} f_k, \phi\rangle$, which, together with the fact that $f^{(m)} \to f$ in $\mathcal{S}'(\mathbb{R}^n)$ as $m \to \infty$, further implies that

$$\langle f, \phi\rangle = \lim_{\iota\to\infty}\langle f^{(m_\iota)}, \phi\rangle = \lim_{\iota\to\infty}\left\langle \sum_{k\in\mathbb{Z}} f_k^{(m_\iota)}, \phi\right\rangle = \left\langle \sum_{k\in\mathbb{Z}} f_k, \phi\right\rangle.$$

This shows

$$f = \sum_{k\in\mathbb{Z}} f_k = \sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}} h_i^k \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n),$$

which completes the proof of (3.2) and hence the proof of the statement that $H_A^{p,q}(\mathbb{R}^n) \subset H_A^{p,r,s,q}(\mathbb{R}^n)$.

We now prove $H_A^{p,r,s,q}(\mathbb{R}^n) \subset H_A^{p,q}(\mathbb{R}^n)$. To this end, for any $f \in H_A^{p,r,s,q}(\mathbb{R}^n)$, by Definition 3.2, we know that there exists a sequence of (p, r, s)-atoms, $\{a_i^k\}_{i\in\mathbb{N},\, k\in\mathbb{Z}}$, supported on $\{x_i^k + B_i^k\}_{i\in\mathbb{N},\, k\in\mathbb{Z}} \subset \mathfrak{B}$, respectively, such that $f = \sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}} \lambda_i^k a_i^k$ in $\mathcal{S}'(\mathbb{R}^n)$, where $\lambda_i^k \sim 2^k|B_i^k|^{1/p}$ for all $k \in \mathbb{Z}$ and $i \in \mathbb{N}$, $\sum_{i\in\mathbb{N}} \chi_{x_i^k + B_i^k}(x) \lesssim 1$ for all $k \in \mathbb{Z}$ and $x \in \mathbb{R}^n$, and

$$\|f\|_{H_A^{p,r,s,q}(\mathbb{R}^n)} \sim \left[\sum_{k\in\mathbb{Z}}\left(\sum_{i\in\mathbb{N}} |\lambda_i^k|^p\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}. \tag{3.21}$$

Clearly, there exists a sequence, $\{\ell_i^k\}_{i\in\mathbb{N},\, k\in\mathbb{Z}}$, of integers such that $x_i^k + B_{\ell_i^k} = x_i^k + B_i^k$ for $i \in \mathbb{N}$ and $k \in \mathbb{Z}$. It suffices to only consider the case when $N = N_{(p)} := \lfloor(\frac{1}{p} - 1)\frac{\ln b}{\ln\lambda_-}\rfloor + 2$. Let $\mu_k := (\sum_{i\in\mathbb{N}} |B_{\ell_i^k}|)^{1/p}$ and

$$\beta := \left(\frac{\ln b}{\ln\lambda_-} + N - 1\right)\frac{\ln\lambda_-}{\ln b} > \frac{1}{p}.$$

Then, for $r \in (1, \infty]$, there exists $\delta \in (1/r, 1)$ such that $\frac{1}{\beta} < \delta p < 1$. Notice that, for any fixed $k_0 \in \mathbb{Z}$ and all $x \in \mathbb{R}^n$,

$$M_N(f)(x) \leqslant M_N\left(\sum_{k=-\infty}^{k_0-1}\sum_{i\in\mathbb{N}} \lambda_i^k a_i^k\right)(x) + \sum_{k=k_0}^{\infty}\sum_{i\in\mathbb{N}} |\lambda_i^k| M_N(a_i^k)(x) =: \psi_{k_0}(x) + \eta_{k_0}(x).$$

To prove $H_A^{p,r,s,q}(\mathbb{R}^n) \subset H_A^{p,q}(\mathbb{R}^n)$, we now consider two cases: $q/p \in [1, \infty]$ and $q/p \in (0, 1)$.

Case 1. $q/p \in [1, \infty]$. In this case, to show the desired conclusion, we claim that

$$2^{k_0 p}[d_{\psi_{k_0}}(2^{k_0})]^{\delta} \lesssim \sum_{k=-\infty}^{k_0-1} [2^k \mu_k^{\delta}]^p \quad \text{and} \quad 2^{k_0\delta p} d_{\eta_{k_0}}(2^{k_0}) \lesssim \sum_{k=k_0}^{\infty} [2^{k\delta}\mu_k]^p. \tag{3.22}$$

Assume that (3.22) holds true for the time being. Notice that $\delta \in (0, q/p)$. Then, by Lemma 3.5, the fact that $|B_{\ell_i^k}| \sim \frac{|\lambda_i^k|^p}{2^{kp}}$ and (3.21), we have

$$\|f\|_{H_A^{p,q}(\mathbb{R}^n)} = \|M_N(f)\|_{L^{p,q}(\mathbb{R}^n)} \lesssim \|\{2^k\mu_k\}_{k\in\mathbb{Z}}\|_{\ell^q} \lesssim \left[\sum_{k\in\mathbb{Z}}\left(\sum_{i\in\mathbb{N}} |\lambda_i^k|^p\right)^{\frac{q}{p}}\right]^{\frac{1}{q}} \sim \|f\|_{H_A^{p,r,s,q}(\mathbb{R}^n)}$$

with the usual interpretation for $q=\infty$, which implies that $\|f\|_{H_A^{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{H_A^{p,r,s,q}(\mathbb{R}^n)}$ and hence $H_A^{p,r,s,q}(\mathbb{R}^n) \subset H_A^{p,q}(\mathbb{R}^n)$.

Now, let us give the proof of the claim (3.22). To this end, we first estimate ψ_{k_0}. Notice that a_i^k is a (p,r,s)-atom, $\operatorname{supp} a_i^k \subset (x_i^k + B_{\ell_i^k})$, $\sum_{i\in\mathbb{N}} \chi_{x_i^k+B_{\ell_i^k}} \lesssim 1$ and $\lambda_i^k \sim 2^k |B_{\ell_i^k}|^{1/p}$. For $r\in(1,\infty)$, by Hölder's inequality, we find that, for $\sigma := 1-\frac{p}{r\delta} > 0$ and all $x\in\mathbb{R}^n$,

$$\begin{aligned}
\psi_{k_0}(x) &\leqslant \sum_{k=-\infty}^{k_0-1} M_N\left(\sum_{i\in\mathbb{N}} \lambda_i^k a_i^k\right)(x)\\
&\leqslant \left(\sum_{k=-\infty}^{k_0-1} 2^{k\sigma r'}\right)^{1/r'} \left\{\sum_{k=-\infty}^{k_0-1} 2^{-k\sigma r}\left[M_N\left(\sum_{i\in\mathbb{N}} \lambda_i^k a_i^k\right)(x)\right]^r\right\}^{1/r}\\
&= \widetilde{C} 2^{k_0\sigma}\left\{\sum_{k=-\infty}^{k_0-1} 2^{-k\sigma r}\left[M_N\left(\sum_{i\in\mathbb{N}} \lambda_i^k a_i^k\right)(x)\right]^r\right\}^{1/r},
\end{aligned}$$

where $\widetilde{C} := (\frac{1}{2^{\sigma r'}-1})^{1/r'}$, which, combined with Proposition 2.9, Remark 2.10 and the finite intersection property of $\{x_i^k + B_{\ell_i^k}\}_{i\in\mathbb{N}}$ for each $k\in\mathbb{Z}$, further implies that

$$\begin{aligned}
2^{k_0 p}[d_{\psi_{k_0}}(2^{k_0})]^\delta &= 2^{k_0 p}|\{x\in\mathbb{R}^n : \psi_{k_0}(x) > 2^{k_0}\}|^\delta\\
&\leqslant 2^{k_0 p}\left|\left\{x\in\mathbb{R}^n : \widetilde{C}^r \sum_{k=-\infty}^{k_0-1} 2^{-k\sigma r}\left[M_N\left(\sum_{i\in\mathbb{N}} \lambda_i^k a_i^k\right)(x)\right]^r > 2^{k_0 r(1-\sigma)}\right\}\right|^\delta\\
&= 2^{k_0 p}\left\{\int_{\{x\in\mathbb{R}^n : \widetilde{C}^r \sum_{k=-\infty}^{k_0-1} 2^{-k\sigma r}[M_N(\sum_{i\in\mathbb{N}} \lambda_i^k a_i^k)(x)]^r > 2^{k_0 r(1-\sigma)}\}} dx\right\}^\delta\\
&\leqslant \widetilde{C}^{r\delta} 2^{k_0 p} 2^{-k_0 r\delta(1-\sigma)}\left\{\int_{\mathbb{R}^n} \sum_{k=-\infty}^{k_0-1} 2^{-k\sigma r}\left[M_N\left(\sum_{i\in\mathbb{N}} \lambda_i^k a_i^k\right)(x)\right]^r dx\right\}^\delta\\
&\lesssim \left[\sum_{k=-\infty}^{k_0-1} 2^{-k\sigma r}\int_{\mathbb{R}^n}\left|\sum_{i\in\mathbb{N}} \lambda_i^k a_i^k(x)\right|^r dx\right]^\delta\\
&\lesssim \left[\sum_{k=-\infty}^{k_0-1} 2^{-k\sigma r}\sum_{i\in\mathbb{N}} |\lambda_i^k|^r \int_{x_i^k+B_{\ell_i^k}} |a_i^k(x)|^r dx\right]^\delta\\
&\lesssim \left[\sum_{k=-\infty}^{k_0-1} 2^{-k\sigma r}\sum_{i\in\mathbb{N}} 2^{kr}|B_{\ell_i^k}|^{\frac{r}{p}}|B_{\ell_i^k}|^{(\frac{1}{r}-\frac{1}{p})r}\right]^\delta\\
&\lesssim \sum_{k=-\infty}^{k_0-1} 2^{kp}\left(\sum_{i\in\mathbb{N}} |B_{\ell_i^k}|\right)^\delta \sim \sum_{k=-\infty}^{k_0-1} [2^k \mu_k^\delta]^p, \qquad (3.23)
\end{aligned}$$

which is the desired estimate of ψ_{k_0} for $r\in(1,\infty)$ in (3.22).

For $r=\infty$, by Proposition 2.9, Remark 2.10 and the finite intersection property of $\{x_i^k + B_{\ell_i^k}\}_{i\in\mathbb{N}}$ for each $k\in\mathbb{Z}$ again, we have

$$\begin{aligned}
2^{k_0 p}[d_{\psi_{k_0}}(2^{k_0})]^\delta &= 2^{k_0 p}|\{x\in\mathbb{R}^n : \psi_{k_0}(x) > 2^{k_0}\}|^\delta\\
&\leqslant 2^{k_0(p-\delta\widetilde{r})}\left\{\sum_{k=-\infty}^{k_0-1}\int_{\mathbb{R}^n}\left[M_N\left(\sum_{i\in\mathbb{N}} \lambda_i^k a_i^k\right)(x)\right]^{\widetilde{r}} dx\right\}^\delta\\
&\lesssim 2^{k_0(p-\delta\widetilde{r})}\left\{\sum_{k=-\infty}^{k_0-1}\sum_{i\in\mathbb{N}}\int_{x_i^k+B_{\ell_i^k}} |\lambda_i^k a_i^k(x)|^{\widetilde{r}} dx\right\}^\delta\\
&\lesssim \sum_{k=-\infty}^{k_0-1} 2^{kp}\left(\sum_{i\in\mathbb{N}} |B_{\ell_i^k}|\right)^\delta \sim \sum_{k=-\infty}^{k_0-1} [2^k \mu_k^\delta]^p, \qquad (3.24)
\end{aligned}$$

where $\widetilde{r} \in (1, \infty)$ such that $\delta\widetilde{r} > p$, which, together with (3.23), implies the desired estimate of ψ_{k_0} in (3.22).

In order to estimate η_{k_0}, it suffices to prove that, for all $i \in \mathbb{N}$ and $k \in \mathbb{Z}$,

$$\int_{\mathbb{R}^n} [M_N(a_i^k)(x)]^{\delta p} dx \lesssim |B_{\ell_i^k}|^{1-\delta} \tag{3.25}$$

with the implicit positive constant independent of i and k. Indeed, by (3.25), we have

$$\begin{aligned}
2^{k_0\delta p} d_{\eta_{k_0}}(2^{k_0}) &= 2^{k_0\delta p} \left| \left\{ x \in \mathbb{R}^n : \left[\sum_{k=k_0}^{\infty} \sum_{i\in\mathbb{N}} |\lambda_i^k| M_N(a_i^k)(x) \right]^{\delta p} > 2^{k_0\delta p} \right\} \right| \\
&\leqslant \int_{\mathbb{R}^n} \left[\sum_{k=k_0}^{\infty} \sum_{i\in\mathbb{N}} |\lambda_i^k| M_N(a_i^k)(x) \right]^{\delta p} dx \leqslant \sum_{k=k_0}^{\infty} \sum_{i\in\mathbb{N}} |\lambda_i^k|^{\delta p} \int_{\mathbb{R}^n} [M_N(a_i^k)(x)]^{\delta p}\, dx \\
&\lesssim \sum_{k=k_0}^{\infty} \sum_{i\in\mathbb{N}} |\lambda_i^k|^{\delta p} |B_{\ell_i^k}|^{1-\delta} \lesssim \sum_{k=k_0}^{\infty} 2^{k\delta p} \sum_{i\in\mathbb{N}} |B_{\ell_i^k}| \sim \sum_{k=k_0}^{\infty} [2^{k\delta} \mu_k]^p,
\end{aligned} \tag{3.26}$$

which is the desired estimate of η_{k_0} in (3.22).

To show (3.25), we write

$$\int_{\mathbb{R}^n} [M_N(a_i^k)(x)]^{\delta p}\, dx = \int_{x_i^k + B_{\ell_i^k+\tau}} [M_N(a_i^k)(x)]^{\delta p}\, dx + \int_{(x_i^k + B_{\ell_i^k+\tau})^\complement} \cdots =: \mathrm{I}_1 + \mathrm{I}_2.$$

For $r \in (1, \infty]$, by Hölder's inequality, Proposition 2.9 and Remark 2.10, we find that

$$\begin{aligned}
\mathrm{I}_1 &= \int_{x_i^k + B_{\ell_i^k+\tau}} [M_N(a_i^k)(x)]^{\delta p}\, dx \\
&\leqslant \left\{ \int_{x_i^k + B_{\ell_i^k+\tau}} [M_N(a_i^k)(x)]^r\, dx \right\}^{\frac{\delta p}{r}} |B_{\ell_i^k+\tau}|^{1-\frac{\delta p}{r}} \lesssim \|a_i^k\|_{L^r(\mathbb{R}^n)}^{\delta p} |B_{\ell_i^k+\tau}|^{1-\frac{\delta p}{r}} \\
&\lesssim |B_{\ell_i^k+\tau}|^{\delta p(\frac{1}{r}-\frac{1}{p})} |B_{\ell_i^k+\tau}|^{1-\frac{\delta p}{r}} \sim |B_{\ell_i^k+\tau}|^{1-\delta} \sim |B_{\ell_i^k}|^{1-\delta},
\end{aligned}$$

where the implicit positive constants are independent of i and k.

To estimate I_2, it suffices to show that, for all $i \in \mathbb{N}$, $k \in \mathbb{Z}$ and $x \in (x_i^k + B_{\ell_i^k+\tau})^\complement$,

$$M_N^0(a_i^k)(x) \lesssim |B_{\ell_i^k}|^{-\frac{1}{p}} \frac{|B_{\ell_i^k}|^\beta}{[\rho(x - x_i^k)]^\beta} \tag{3.27}$$

with the implicit positive constant independent of i and k, where $M_N^0(f)$ denotes the radial grand maximal function of f as in Definition 2.3, $\beta := (\frac{\ln b}{\ln \lambda_-} + N - 1)\frac{\ln \lambda_-}{\ln b}$ and ρ denotes the homogeneous quasi-norm associated with the dilation A in Definition 2.2. Indeed, assuming that (3.27) holds true for the time being, noticing that $\beta\delta p > 1$, then, by Proposition 2.4 and (3.27), we have

$$\begin{aligned}
\mathrm{I}_2 &\lesssim \int_{(x_i^k + B_{\ell_i^k+\tau})^\complement} [M_N^0(a_i^k)(x)]^{\delta p}\, dx \lesssim \int_{\rho(x-x_i^k) \geqslant |B_{\ell_i^k+\tau}|} |B_{\ell_i^k}|^{-\delta} \frac{|B_{\ell_i^k}|^{\beta\delta p}}{[\rho(x - x_i^k)]^{\beta\delta p}}\, dx \\
&\sim \sum_{j=0}^{\infty} \int_{2^j |B_{\ell_i^k+\tau}| \leqslant \rho(x-x_i^k) < 2^{j+1} |B_{\ell_i^k+\tau}|} |B_{\ell_i^k}|^{-\delta} \frac{|B_{\ell_i^k}|^{\beta\delta p}}{[\rho(x - x_i^k)]^{\beta\delta p}}\, dx \\
&\sim \sum_{j=0}^{\infty} \int_{\rho(x-x_i^k) \sim 2^j |B_{\ell_i^k+\tau}|} |B_{\ell_i^k}|^{-\delta} \frac{|B_{\ell_i^k}|^{\beta\delta p}}{(2^j |B_{\ell_i^k+\tau}|)^{\beta\delta p}}\, dx \\
&\sim |B_{\ell_i^k}|^{-\delta} |B_{\ell_i^k+\tau}| \sum_{j=0}^{\infty} 2^j 2^{-j\beta\delta p} \sim |B_{\ell_i^k}|^{1-\delta}
\end{aligned} \tag{3.28}$$

with the implicit positive constants independent of i and k, which completes the proof of (3.25).

Thus, to obtain the desired conclusion of Case 1, we still need to prove (3.27). To this end, take $x \in (x_i^k + B_{\ell_i^k+\tau})^{\complement}$, $\varphi \in \mathcal{S}_N(\mathbb{R}^n)$ and $t \in \mathbb{Z}$. Suppose that P is a polynomial with degree not more than s, which will be determined later. Then, for all $i \in \mathbb{N}$ and $k \in \mathbb{Z}$, by Hölder's inequality, we have

$$\begin{aligned}
|(a_i^k * \varphi_t)(x)| &= b^{-t}\left|\int_{\mathbb{R}^n} a_i^k(y)\varphi(A^{-t}(x-y))\,dy\right| \\
&= b^{-t}\left|\int_{x_i^k+B_{\ell_i^k}} a_i^k(y)[\varphi(A^{-t}(x-y)) - P(A^{-t}(x-y))]\,dy\right| \\
&\leqslant b^{-t}\|a_i^k\|_{L^r(\mathbb{R}^n)}\left[\int_{x_i^k+B_{\ell_i^k}} |\varphi(A^{-t}(x-y)) - P(A^{-t}(x-y))|^{r'}\,dy\right]^{1/r'} \\
&\leqslant b^{-t}|B_{\ell_i^k}|^{1/r-1/p} b^{t/r'}\left\{\int_{A^{-t}(x-x_i^k)+B_{\ell_i^k-t}} |\varphi(y)-P(y)|^{r'}\,dy\right\}^{1/r'} \\
&\leqslant b^{-t}|B_{\ell_i^k}|^{1/r-1/p} b^{t/r'} b^{-t/r'}|B_{\ell_i^k}|^{1/r'} \sup_{y\in A^{-t}(x-x_i^k)+B_{\ell_i^k-t}} |\varphi(y)-P(y)| \\
&= |B_{\ell_i^k}|^{-1/p} b^{\ell_i^k-t} \sup_{y\in A^{-t}(x-x_i^k)+B_{\ell_i^k-t}} |\varphi(y)-P(y)|.
\end{aligned}\tag{3.29}$$

Suppose that $x \in [x_i^k + (B_{\ell_i^k+\tau+m+1}\setminus B_{\ell_i^k+\tau+m})]$ for some integer $m \in \mathbb{Z}_+$. Then, by (2.8), we obtain

$$\begin{aligned}
A^{-t}(x-x_i^k) + B_{\ell_i^k-t} &\subset A^{-t}(B_{\ell_i^k+\tau+m+1}\setminus B_{\ell_i^k+\tau+m}) + B_{\ell_i^k-t} \\
&= A^{\ell_i^k-t}([B_{\tau+m+1}\setminus B_{\tau+m}] + B_0) \subset A^{\ell_i^k-t}(B_m)^{\complement}.
\end{aligned}\tag{3.30}$$

If $\ell_i^k \geqslant t$, we choose $P \equiv 0$. Then, by (3.30), we know that

$$\sup_{y\in A^{-t}(x-x_i^k)+B_{\ell_i^k-t}} |\varphi(y)| \leqslant \sup_{y\in A^{-t}(x-x_i^k)+B_{\ell_i^k-t}} \min\{1,\rho(y)^{-N}\} \leqslant b^{-N(\ell_i^k-t+m)}. \tag{3.31}$$

If $\ell_i^k < t$, then we let P be the Taylor expansion of φ at the point $A^{-t}(x-x_i^k)$ of order s. By the Taylor remainder theorem, (2.5) and (3.30), we have

$$\begin{aligned}
\sup_{y\in A^{-t}(x-x_i^k)+B_{\ell_i^k-t}} |\varphi(y)-P(y)| &\lesssim \sup_{z\in B_{\ell_i^k-t}} \sup_{|\alpha|=s+1} |\partial^\alpha\varphi(A^{-t}(x-x_i^k)+z)||z|^{s+1} \\
&\lesssim \lambda_-^{(s+1)(\ell_i^k-t)} \sup_{y\in A^{-t}(x-x_i^k)+B_{\ell_i^k-t}} \min\{1,\rho(y)^{-N}\} \\
&\lesssim \lambda_-^{(s+1)(\ell_i^k-t)} \min\{1, b^{-N(\ell_i^k-t+m)}\}.
\end{aligned}\tag{3.32}$$

Combining (3.29), (3.31) and (3.32), for all $x \in [x_i^k + (B_{\ell_i^k+\tau+m+1}\setminus B_{\ell_i^k+\tau+m})]$ with $m \in \mathbb{Z}_+$, we further conclude that

$$\begin{aligned}
M_N^0(a_i^k)(x) &= \sup_{\varphi\in\mathcal{S}_N(\mathbb{R}^n)} \sup_{t\in\mathbb{Z}} |(a_i^k * \varphi_t)(x)| \\
&\lesssim |B_{\ell_i^k}|^{-1/p} \max\bigg\{\sup_{t\in\mathbb{Z},\, t\leqslant\ell_i^k} b^{\ell_i^k-t} b^{-N(\ell_i^k-t+m)}, \\
&\qquad \sup_{t\in\mathbb{Z},\, t>\ell_i^k} b^{\ell_i^k-t}\lambda_-^{(s+1)(\ell_i^k-t)} \min\{1, b^{-N(\ell_i^k-t+m)}\}\bigg\}.
\end{aligned}$$

Notice that the supremum over $t \leqslant \ell_i^k$ has the largest value when $t = \ell_i^k$. Without loss of generality, we may assume that $s := \lfloor(\frac{1}{p}-1)\frac{\ln b}{\ln\lambda_-}\rfloor$. Since $N = s+2$ implies $b\lambda_-^{s+1} \leqslant b^N$ and the above supremum over $t > \ell_i^k$ is attained when $\ell_i^k - t + m = 0$, it follows that

$$M_N^0(a_i^k)(x) \lesssim |B_{\ell_i^k}|^{-1/p} \max\{b^{-Nm}, (b\lambda_-^{s+1})^{-m}\} \lesssim |B_{\ell_i^k}|^{-1/p}(b\lambda_-^{s+1})^{-m}$$

$$
\begin{aligned}
&\sim |B_{\ell_i^k}|^{-1/p} b^{-m} b^{-(s+1)m\frac{\ln\lambda_-}{\ln b}} \lesssim |B_{\ell_i^k}|^{-1/p} b^{\ell_i^k[(s+1)\frac{\ln\lambda_-}{\ln b}+1]} b^{-(\ell_i^k+\tau+m)[(s+1)\frac{\ln\lambda_-}{\ln b}+1]}\\
&\lesssim |B_{\ell_i^k}|^{-1/p} |B_{\ell_i^k}|^{(s+1)\frac{\ln\lambda_-}{\ln b}+1} [\rho(x-x_i^k)]^{-[(s+1)\frac{\ln\lambda_-}{\ln b}+1]}\\
&\sim |B_{\ell_i^k}|^{-1/p} \frac{|B_{\ell_i^k}|^{\beta}}{[\rho(x-x_i^k)]^{\beta}} \qquad (3.33)
\end{aligned}
$$

with the implicit positive constants independent of i and k, which is (3.27). This finishes the proof of Case 1.

Case 2. $q/p \in (0,1)$. In this case, when $r\in(1,\infty)$, similar to (3.23), we have

$$
2^{k_0 p}[d_{\psi_{k_0}}(2^{k_0})]^{\delta} \lesssim \left[\sum_{k=-\infty}^{k_0-1} 2^{-k\sigma r} \sum_{i\in\mathbb{N}} 2^{kr} |B_{\ell_i^k}|^{\frac{r}{p}} |B_{\ell_i^k}|^{(\frac{1}{r}-\frac{1}{p})r}\right]^{\delta} \sim \left(\sum_{k=-\infty}^{k_0-1} 2^{\frac{kp}{\delta}} \mu_k^p\right)^{\delta}. \qquad (3.34)
$$

By a similar calculation to (3.24), we easily know that (3.34) also holds true for $r=\infty$. This further implies that

$$
\begin{aligned}
\sum_{k_0\in\mathbb{Z}} 2^{k_0 q} |\{x\in\mathbb{R}^n : \psi_{k_0}(x) > 2^{k_0}\}|^{\frac{q}{p}} &\lesssim \sum_{k_0\in\mathbb{Z}} 2^{k_0(q-\frac{q}{\delta})} \sum_{k=-\infty}^{k_0-1} 2^{\frac{kq}{\delta}} \mu_k^q\\
&\sim \sum_{k\in\mathbb{Z}} \sum_{k_0=k+1}^{\infty} 2^{k_0(q-\frac{q}{\delta})} 2^{\frac{kq}{\delta}} \mu_k^q \lesssim \sum_{k\in\mathbb{Z}} 2^{kq} \mu_k^q. \qquad (3.35)
\end{aligned}
$$

On the other hand, similar to (3.26), we obtain

$$
2^{k_0\delta p} d_{\eta_{k_0}}(2^{k_0}) \lesssim \sum_{k=k_0}^{\infty} [2^{k\delta}\mu_k]^p,
$$

which, together with $q<p$, implies that

$$
2^{k_0\delta p} |\{x\in\mathbb{R}^n : \eta_{k_0}(x) > 2^{k_0}\}| \lesssim \sum_{k=k_0}^{\infty} 2^{-k\widetilde{\delta}p} [2^{k(1-\widetilde{\delta})}\mu_k]^p \lesssim 2^{-k_0\widetilde{\delta}p} \left\{\sum_{k=k_0}^{\infty} [2^{k(1-\widetilde{\delta})}\mu_k]^q\right\}^{\frac{p}{q}},
$$

where $\widetilde{\delta} := \frac{1-\delta}{2}$. Therefore, we have

$$
\begin{aligned}
&\sum_{k_0\in\mathbb{Z}} 2^{k_0 q} |\{x\in\mathbb{R}^n : \eta_{k_0}(x) > 2^{k_0}\}|^{\frac{q}{p}}\\
&\lesssim \sum_{k_0\in\mathbb{Z}} 2^{k_0\widetilde{\delta}q} \sum_{k=k_0}^{\infty} [2^{k(1-\widetilde{\delta})}\mu_k]^q \sim \sum_{k\in\mathbb{Z}} [2^{k(1-\widetilde{\delta})}\mu_k]^q \sum_{k_0=-\infty}^{k} 2^{k_0\widetilde{\delta}q} \lesssim \sum_{k\in\mathbb{Z}} 2^{kq}\mu_k^q. \qquad (3.36)
\end{aligned}
$$

Notice that $\mu_k := (\sum_{i\in\mathbb{N}} |B_{\ell_i^k}|)^{1/p}$ and $\lambda_i^k \sim 2^k |B_{\ell_i^k}|^{1/p}$. Combining (2.1), (3.35), (3.36) and (3.21), we further conclude that

$$
\begin{aligned}
\|M_N(f)\|_{L^{p,q}(\mathbb{R}^n)}^q &\sim \sum_{k_0\in\mathbb{Z}} 2^{k_0 q} |\{x\in\mathbb{R}^n : M_N(f)(x) > 2^{k_0}\}|^{\frac{q}{p}}\\
&\lesssim \sum_{k_0\in\mathbb{Z}} 2^{k_0 q} |\{x\in\mathbb{R}^n : \psi_{k_0}(x) > 2^{k_0}\}|^{\frac{q}{p}} + \sum_{k_0\in\mathbb{Z}} 2^{k_0 q} |\{x\in\mathbb{R}^n : \eta_{k_0}(x) > 2^{k_0}\}|^{\frac{q}{p}}\\
&\lesssim \sum_{k\in\mathbb{Z}} 2^{kq}\mu_k^q \sim \sum_{k\in\mathbb{Z}} \left[\sum_{i\in\mathbb{N}} |\lambda_i^k|^p\right]^{\frac{q}{p}} \sim \|f\|_{H_A^{p,r,s,q}(\mathbb{R}^n)}^q,
\end{aligned}
$$

which implies that $\|f\|_{H_A^{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{H_A^{p,r,s,q}(\mathbb{R}^n)}$ and $H_A^{p,r,s,q}(\mathbb{R}^n) \subset H_A^{p,q}(\mathbb{R}^n)$. This finishes the proof of Case 2 and hence Theorem 3.6. ∎

3.2 Molecular characterizations of $H_A^{p,q}(\mathbb{R}^n)$

In this subsection, we establish the molecular characterizations of $H_A^{p,q}(\mathbb{R}^n)$. We begin with the following notion of anisotropic (p,r,s,ε)-molecules associated with dilated balls.

Definition 3.7. An anisotropic quadruple (p,r,s,ε) is said to be *admissible* if $p\in(0,1]$, $r\in(1,\infty]$, $s\in\mathbb{N}$ with $s\geqslant\lfloor(1/p-1)\ln b/\ln\lambda_-\rfloor$ and $\varepsilon\in(0,\infty)$. For an admissible anisotropic quadruple (p,r,s,ε), a measurable function m is called an *anisotropic* (p,r,s,ε)*-molecule* associated with a dilated ball $B\in\mathfrak{B}$ if

(i) for each $j\in\mathbb{Z}_+$, $\|m\|_{L^r(U_j(B))}\leqslant b^{-j\varepsilon}|B|^{1/r-1/p}$, where $U_0(B):=B$ and, for any $j\in\mathbb{N}$,

$$U_j(B):=(A^jB)\setminus(A^{j-1}B);$$

(ii) $\int_{\mathbb{R}^n}m(x)x^\alpha\,dx=0$ for any $\alpha\in\mathbb{Z}_+^n$ with $|\alpha|\leqslant s$.

Throughout this paper, an *anisotropic* (p,r,s,ε)*-molecule* is called a (p,r,s,ε)*-molecule* for simplicity. Via (p,r,s,ε)-molecules, we introduce the following anisotropic molecular Hardy-Lorentz space $H_A^{p,r,s,\varepsilon,q}(\mathbb{R}^n)$.

Definition 3.8. For an admissible anisotropic quadruple (p,r,s,ε), $q\in(0,\infty]$ and a dilation A, the *anisotropic molecular Hardy-Lorentz space* $H_A^{p,r,s,\varepsilon,q}(\mathbb{R}^n)$ is defined to be the set of all distributions $f\in\mathcal{S}'(\mathbb{R}^n)$ satisfying that there exist a sequence of (p,r,s,ε)-molecules, $\{m_i^k\}_{i\in\mathbb{N},\,k\in\mathbb{Z}}$, associated with dilated balls $\{x_i^k+B_i^k\}_{i\in\mathbb{N},\,k\in\mathbb{Z}}\subset\mathfrak{B}$, respectively, and a positive constant $\widetilde{C}$ such that $\sum_{i\in\mathbb{N}}\chi_{x_i^k+B_i^k}(x)\leqslant\widetilde{C}$ for all $k\in\mathbb{Z}$ and $x\in\mathbb{R}^n$, and $f=\sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}}\lambda_i^km_i^k$ in $\mathcal{S}'(\mathbb{R}^n)$, where $\lambda_i^k\sim2^k|B_i^k|^{1/p}$ for all $k\in\mathbb{Z}$ and $i\in\mathbb{N}$.

Moreover, define

$$\|f\|_{H_A^{p,r,s,\varepsilon,q}(\mathbb{R}^n)}:=\inf\left\{\left[\sum_{k\in\mathbb{Z}}\left(\sum_{i\in\mathbb{N}}|\lambda_i^k|^p\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}:f=\sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}}\lambda_i^km_i^k\right\}$$

with the usual modification made when $q=\infty$, where the infimum is taken over all decompositions of f as above.

Now we state the main theorem of this subsection as follows.

Theorem 3.9. *Let (p,r,s,ε) be an admissible anisotropic quadruple defined as in Definition* 3.7 *with $\varepsilon\in(\max\{1,(s+1)\log_b(\lambda_+)\},\infty)$, $q\in(0,\infty]$ and $N\in\mathbb{N}\cap[N_{(p)},\infty)$. Then $H_A^{p,q}(\mathbb{R}^n)=H_A^{p,r,s,\varepsilon,q}(\mathbb{R}^n)$ with equivalent quasi-norms.*

Proof. By the definitions of (p,∞,s)-atoms and (p,r,s,ε)-molecules, we know that each (p,∞,s)-atom is also a (p,r,s,ε)-molecule, which implies that

$$H_A^{p,\infty,s,q}(\mathbb{R}^n)\subset H_A^{p,r,s,\varepsilon,q}(\mathbb{R}^n).$$

This, combined with Theorem 3.6, further implies that, to prove Theorem 3.9, it suffices to show $H_A^{p,r,s,\varepsilon,q}(\mathbb{R}^n)\subset H_A^{p,q}(\mathbb{R}^n)$.

To prove this, for any $f\in H_A^{p,r,s,\varepsilon,q}(\mathbb{R}^n)$, by Definition 3.8, we know that there exists a sequence of (p,r,s,ε)-molecules, $\{m_i^k\}_{i\in\mathbb{N},\,k\in\mathbb{Z}}$, associated with $\{x_i^k+B_i^k\}_{i\in\mathbb{N},\,k\in\mathbb{Z}}\subset\mathfrak{B}$, respectively, such that $f=\sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}}\lambda_i^km_i^k$ in $\mathcal{S}'(\mathbb{R}^n)$, $\lambda_i^k\sim2^k|B_i^k|^{1/p}$ for all $k\in\mathbb{Z}$ and $i\in\mathbb{N}$, $\sum_{i\in\mathbb{N}}\chi_{x_i^k+B_i^k}(x)\lesssim1$ for all $k\in\mathbb{Z}$ and $x\in\mathbb{R}^n$, and

$$\|f\|_{H_A^{p,r,s,\varepsilon,q}(\mathbb{R}^n)}\sim\left[\sum_{k\in\mathbb{Z}}\left(\sum_{i\in\mathbb{N}}|\lambda_i^k|^p\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}. \tag{3.37}$$

Take a sequence $\{\ell_i^k\}_{i\in\mathbb{N},\,k\in\mathbb{Z}}$ of integers satisfying that, for all $i\in\mathbb{N}$ and $k\in\mathbb{Z}$, $x_i^k+B_{\ell_i^k}:=x_i^k+B_i^k$. It suffices to only consider the case when $N=N_{(p)}:=\lfloor(\frac{1}{p}-1)\frac{\ln b}{\ln\lambda_-}\rfloor+2$. Let

$$\beta:=\left(\frac{\ln b}{\ln\lambda_-}+N-1\right)\frac{\ln\lambda_-}{\ln b}>\frac{1}{p}$$

and $\mu_k := (\sum_{i\in\mathbb{N}} |B_{\ell_i^k}|)^{1/p}$ for all $k \in \mathbb{Z}$. Then, for $r \in (1,\infty]$, there exist $\widetilde{r} \in (1,r)$ and $\delta \in (0,1)$ such that $\frac{1}{\widetilde{r}} < \delta < 1$ and $\frac{1}{\beta} < \delta p < 1$. Notice that, for any fixed $k_0 \in \mathbb{Z}$ and all $x \in \mathbb{R}^n$,

$$M_N(f)(x) \leqslant M_N\left(\sum_{k=-\infty}^{k_0-1}\sum_{i\in\mathbb{N}} \lambda_i^k m_i^k\right)(x) + \sum_{k=k_0}^{\infty}\sum_{i\in\mathbb{N}} |\lambda_i^k| M_N(m_i^k)(x) =: \widetilde{\psi}_{k_0}(x) + \widetilde{\eta}_{k_0}(x). \tag{3.38}$$

To finish the proof that $H_A^{p,r,s,\varepsilon,q}(\mathbb{R}^n) \subset H_A^{p,q}(\mathbb{R}^n)$, it suffices to show

$$\|M_N(f)\|_{L^{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{H_A^{p,r,s,\varepsilon,q}(\mathbb{R}^n)}.$$

To this end, we now consider two cases: $q/p \in [1,\infty]$ and $q/p \in (0,1)$.

Case 1. $q/p \in [1,\infty]$. In this case, if we can prove that

$$2^{k_0 p}[d_{\widetilde{\psi}_{k_0}}(2^{k_0})]^{\delta} \lesssim \sum_{k=-\infty}^{k_0-1} [2^k \mu_k^{\delta}]^p \quad \text{and} \quad 2^{k_0\delta p} d_{\widetilde{\eta}_{k_0}}(2^{k_0}) \lesssim \sum_{k=k_0}^{\infty} [2^{k\delta}\mu_k]^p, \tag{3.39}$$

then, noticing that $\delta \in (0, q/p)$, by Lemma 3.5, $|B_{\ell_i^k}| \sim \frac{|\lambda_i^k|^p}{2^{kp}}$ and (3.37), we have

$$\|f\|_{H_A^{p,q}(\mathbb{R}^n)} = \|M_N(f)\|_{L^{p,q}(\mathbb{R}^n)} \lesssim \|\{2^k\mu_k\}_{k\in\mathbb{Z}}\|_{\ell^q} \lesssim \left[\sum_{k\in\mathbb{Z}}\left(\sum_{i\in\mathbb{N}} |\lambda_i^k|^p\right)^{\frac{q}{p}}\right]^{\frac{1}{q}} \sim \|f\|_{H_A^{p,r,s,\varepsilon,q}(\mathbb{R}^n)},$$

which is the desired conclusion.

Now, we show (3.39). Notice that, for all $k \in \mathbb{Z}$ and $i \in \mathbb{N}$, m_i^k is a (p,r,s,ε)-molecule associated with $x_i^k + B_{\ell_i^k}$, $\sum_{i\in\mathbb{N}} \chi_{x_i^k + B_{\ell_i^k}} \lesssim 1$, $\lambda_i^k \sim 2^k |B_{\ell_i^k}|^{1/p}$ and $\widetilde{r} \in (1,r)$. By Hölder's inequality, we find that, for $\sigma := 1 - \frac{p}{\widetilde{r}\delta} > 0$ and all $x \in \mathbb{R}^n$,

$$\begin{aligned}
\widetilde{\psi}_{k_0}(x) &\leqslant \sum_{k=-\infty}^{k_0-1} M_N\left(\sum_{i\in\mathbb{N}} \lambda_i^k m_i^k\right)(x) \\
&\leqslant \left(\sum_{k=-\infty}^{k_0-1} 2^{k\sigma\widetilde{r}'}\right)^{1/\widetilde{r}'} \left\{\sum_{k=-\infty}^{k_0-1} 2^{-k\sigma\widetilde{r}}\left[M_N\left(\sum_{i\in\mathbb{N}} \lambda_i^k m_i^k\right)(x)\right]^{\widetilde{r}}\right\}^{1/\widetilde{r}} \\
&\sim 2^{k_0\sigma}\left\{\sum_{k=-\infty}^{k_0-1} 2^{-k\sigma\widetilde{r}}\left[M_N\left(\sum_{i\in\mathbb{N}} \lambda_i^k m_i^k\right)(x)\right]^{\widetilde{r}}\right\}^{1/\widetilde{r}}.
\end{aligned}$$

This further implies that

$$\begin{aligned}
2^{k_0 p}[d_{\widetilde{\psi}_{k_0}}(2^{k_0})]^{\delta} &\lesssim 2^{k_0 p} 2^{-k_0\widetilde{r}\delta(1-\sigma)}\left\{\int_{\mathbb{R}^n}\sum_{k=-\infty}^{k_0-1} 2^{-k\sigma\widetilde{r}}\left[M_N\left(\sum_{i\in\mathbb{N}} \lambda_i^k m_i^k\right)(x)\right]^{\widetilde{r}} dx\right\}^{\delta} \\
&\lesssim \left[\sum_{k=-\infty}^{k_0-1} 2^{-k\sigma\widetilde{r}}\int_{\mathbb{R}^n}\left|\sum_{i\in\mathbb{N}} \lambda_i^k m_i^k(x)\right|^{\widetilde{r}} dx\right]^{\delta} =: \left(\sum_{k=-\infty}^{k_0-1} 2^{-k\sigma\widetilde{r}} F_k\right)^{\delta}.
\end{aligned} \tag{3.40}$$

Moreover, by Hölder's inequality, we know that, for all $k \in \mathbb{Z} \cap (-\infty, k_0 - 1]$,

$$\begin{aligned}
(F_k)^{1/\widetilde{r}} &\sim \sup_{\|g\|_{L^{\widetilde{r}'}(\mathbb{R}^n)}=1}\left|\int_{\mathbb{R}^n}\left[\sum_{i\in\mathbb{N}} \lambda_i^k m_i^k(x)\right] g(x)\,dx\right| \\
&\lesssim \sup_{\|g\|_{L^{\widetilde{r}'}(\mathbb{R}^n)}=1}\left\{\sum_{i\in\mathbb{N}} |\lambda_i^k| \sum_{j\in\mathbb{Z}_+}\int_{U_j(x_i^k + B_{\ell_i^k})} |m_i^k(x)||g(x)|\,dx\right\} \\
&\lesssim \sup_{\|g\|_{L^{\widetilde{r}'}(\mathbb{R}^n)}=1}\left\{\sum_{i\in\mathbb{N}} |\lambda_i^k| \sum_{j\in\mathbb{Z}_+}\left[\int_{U_j(x_i^k + B_{\ell_i^k})} |m_i^k(x)|^r\,dx\right]^{1/r}\left[\int_{U_j(x_i^k + B_{\ell_i^k})} |g(x)|^{r'}\,dx\right]^{1/r'}\right\}
\end{aligned}$$

$$\sim \sup_{\|g\|_{L^{\widetilde{r}'}(\mathbb{R}^n)}=1} \left\{ \sum_{i\in\mathbb{N}} |\lambda_i^k| \sum_{j\in\mathbb{Z}_+} \|m_i^k\|_{L^r(U_j(x_i^k+B_{\ell_i^k}))} F_{i,j}^k \right\}, \tag{3.41}$$

where, for all $k \in \mathbb{Z}\cap(-\infty, k_0-1]$, $i\in\mathbb{N}$ and $j\in\mathbb{Z}_+$,

$$F_{i,j}^k := \left[\int_{U_j(x_i^k+B_{\ell_i^k})} |g(x)|^{r'}\, dx \right]^{1/r'}, \tag{3.42}$$

$U_0(x_i^k+B_{\ell_i^k}) := x_i^k+B_{\ell_i^k}$ and, for any $j\in\mathbb{N}$,

$$U_j(x_i^k+B_{\ell_i^k}) := [A^j(x_i^k+B_{\ell_i^k})] \setminus [A^{j-1}(x_i^k+B_{\ell_i^k})].$$

By this, (3.42) and (2.20), we find that, for all $k \in \mathbb{Z}\cap(-\infty, k_0-1]$, $i\in\mathbb{N}$ and $j\in\mathbb{Z}_+$,

$$\begin{aligned} F_{i,j}^k &\leqslant |A^jB_{\ell_i^k}|^{1/r'} \left[\frac{1}{|A^jB_{\ell_i^k}|} \int_{A^j(x_i^k+B_{\ell_i^k})} |g(x)|^{r'}\, dx \right]^{1/r'} \\ &\lesssim |A^jB_{\ell_i^k}|^{1/r'} \inf_{x\in x_i^k+B_{\ell_i^k}} \{M_{\rm HL}(|g|^{r'})(x)\}^{1/r'} \\ &\lesssim |A^jB_{\ell_i^k}|^{1/r'} \left\{ \frac{1}{|B_{\ell_i^k}|} \int_{x_i^k+B_{\ell_i^k}} [M_{\rm HL}(|g|^{r'})(x)]^{\widetilde{r}'/r'}\, dx \right\}^{1/\widetilde{r}'}, \end{aligned} \tag{3.43}$$

where $M_{\rm HL}(f)$ denotes the Hardy-Littlewood maximal function as in (2.20). Notice that $M_{\rm HL}$ is bounded on $L^r(\mathbb{R}^n)$ for all $r\in(1,\infty)$ (see Lemma 2.9 and Remark 2.10), $\{x_i^k+B_{\ell_i^k}\}_{i\in\mathbb{N}}$ is finitely overlapped, $|\lambda_i^k| \sim 2^k|B_{\ell_i^k}|^{1/p}$,

$$\|m_i^k\|_{L^r(U_j(x_i^k+B_{\ell_i^k}))} \leqslant b^{-j\varepsilon}|B_{\ell_i^k}|^{1/r-1/p},$$

$\varepsilon > 1 > 1/r'$ and $r > \widetilde{r}$. Then, by (3.41), (3.43) and Hölder's inequality, for all $k \in \mathbb{Z}\cap(-\infty, k_0-1]$, we have

$$\begin{aligned} F_k &\lesssim \sup_{\|g\|_{L^{\widetilde{r}'}(\mathbb{R}^n)}=1} \left\{ \sum_{i\in\mathbb{N}} 2^k|B_{\ell_i^k}|^{1/p} \sum_{j\in\mathbb{Z}_+} b^{-j\varepsilon}|B_{\ell_i^k}|^{1/r-1/p} b^{j/r'} |B_{\ell_i^k}|^{1/r'} \right. \\ &\quad \left. \times \left[\frac{1}{|B_{\ell_i^k}|} \int_{x_i^k+B_{\ell_i^k}} \{M_{\rm HL}(|g|^{r'})(x)\}^{\widetilde{r}'/r'}\, dx \right]^{1/\widetilde{r}'} \right\}^{\widetilde{r}} \\ &\lesssim \sup_{\|g\|_{L^{\widetilde{r}'}(\mathbb{R}^n)}=1} \left\{ \left[\sum_{i\in\mathbb{N}} 2^{k\widetilde{r}}|B_{\ell_i^k}| \right]^{1/\widetilde{r}} \left[\sum_{i\in\mathbb{N}} \int_{x_i^k+B_{\ell_i^k}} \{M_{\rm HL}(|g|^{r'})(x)\}^{\widetilde{r}'/r'}\, dx \right]^{1/\widetilde{r}'} \right\}^{\widetilde{r}} \\ &\lesssim \left[\sum_{i\in\mathbb{N}} 2^{k\widetilde{r}}|B_{\ell_i^k}| \right] \sup_{\|g\|_{L^{\widetilde{r}'}(\mathbb{R}^n)}=1} \left[\int_{\mathbb{R}^n} \{M_{\rm HL}(|g|^{r'})(x)\}^{\widetilde{r}'/r'}\, dx \right]^{\widetilde{r}/\widetilde{r}'} \\ &\lesssim \left[\sum_{i\in\mathbb{N}} 2^{k\widetilde{r}}|B_{\ell_i^k}| \right] \sup_{\|g\|_{L^{\widetilde{r}'}(\mathbb{R}^n)}=1} \left[\int_{\mathbb{R}^n} |g(x)|^{\widetilde{r}'}\, dx \right]^{\widetilde{r}/\widetilde{r}'} \lesssim \sum_{i\in\mathbb{N}} 2^{k\widetilde{r}}|B_{\ell_i^k}|. \end{aligned}$$

By this and (3.40), we know that

$$\begin{aligned} 2^{k_0p}[d_{\widetilde{\psi}_{k_0}}(2^{k_0})]^\delta &\lesssim \left[\sum_{k=-\infty}^{k_0-1} 2^{-k\sigma\widetilde{r}} \sum_{i\in\mathbb{N}} 2^{k\widetilde{r}}|B_{\ell_i^k}| \right]^\delta \\ &\lesssim \sum_{k=-\infty}^{k_0-1} 2^{k\widetilde{r}\delta(1-\sigma)} \left[\sum_{i\in\mathbb{N}} |B_{\ell_i^k}| \right]^\delta \sim \sum_{k=-\infty}^{k_0-1} [2^k\mu_k^\delta]^p, \end{aligned} \tag{3.44}$$

which is the first desired estimate in (3.39).

Now we establish the desired estimate of $\widetilde{\eta}_{k_0}$ in (3.39). It suffices to show that, for all $i \in \mathbb{N}$ and $k \in \mathbb{Z}$,

$$\int_{\mathbb{R}^n} [M_N(m_i^k)(x)]^{\delta p} dx \lesssim |B_{\ell_i^k}|^{1-\delta}, \tag{3.45}$$

where the implicit positive constant is independent of i and k. Indeed, as in (3.26), by (3.45), we find that

$$\begin{aligned} 2^{k_0 \delta p} d_{\widetilde{\eta}_{k_0}}(2^{k_0}) &\leqslant \sum_{k=k_0}^{\infty} \sum_{i \in \mathbb{N}} |\lambda_i^k|^{\delta p} \int_{\mathbb{R}^n} [M_N(m_i^k)(x)]^{\delta p}\, dx \\ &\lesssim \sum_{k=k_0}^{\infty} \sum_{i \in \mathbb{N}} |\lambda_i^k|^{\delta p} |B_{\ell_i^k}|^{1-\delta} \lesssim \sum_{k=k_0}^{\infty} 2^{k\delta p} \sum_{i \in \mathbb{N}} |B_{\ell_i^k}| \sim \sum_{k=k_0}^{\infty} [2^{k\delta} \mu_k]^p, \end{aligned} \tag{3.46}$$

which is the desired conclusion.

In order to show (3.45), we write

$$\int_{\mathbb{R}^n} [M_N(m_i^k)(x)]^{\delta p}\, dx = \int_{x_i^k + B_{\ell_i^k+\tau}} [M_N(m_i^k)(x)]^{\delta p}\, dx + \int_{(x_i^k + B_{\ell_i^k+\tau})^{\complement}} \cdots =: \widetilde{\mathrm{I}}_1 + \widetilde{\mathrm{I}}_2.$$

For $r \in (1, \infty]$, by Hölder's inequality, Proposition 2.9 and Remark 2.10, we have

$$\begin{aligned} \widetilde{\mathrm{I}}_1 &= \int_{x_i^k + B_{\ell_i^k+\tau}} [M_N(m_i^k)(x)]^{\delta p}\, dx \leqslant \left\{ \int_{x_i^k + B_{\ell_i^k+\tau}} [M_N(m_i^k)(x)]^{r}\, dx \right\}^{\frac{p\delta}{r}} |B_{\ell_i^k+\tau}|^{1-\frac{p\delta}{r}} \\ &\lesssim \sum_{j \in \mathbb{Z}_+} \|m_i^k\|_{L^r(U_j(x_i^k + B_{\ell_i^k}))}^{p\delta} |B_{\ell_i^k+\tau}|^{1-\frac{p\delta}{r}} \\ &\lesssim \sum_{j \in \mathbb{Z}_+} b^{-j\varepsilon p\delta} |B_{\ell_i^k}|^{(\frac{1}{r}-\frac{1}{p})p\delta} |B_{\ell_i^k+\tau}|^{1-\frac{p\delta}{r}} \sim |B_{\ell_i^k}|^{1-\delta} \end{aligned} \tag{3.47}$$

with the implicit positive constants independent of i and k. To estimate $\widetilde{\mathrm{I}}_2$, we only need to prove that, for all $i \in \mathbb{N}$, $k \in \mathbb{Z}$ and $x \in (x_i^k + B_{\ell_i^k+\tau})^{\complement}$,

$$M_N^0(m_i^k)(x) \lesssim |B_{\ell_i^k}|^{-\frac{1}{p}} \frac{|B_{\ell_i^k}|^{\beta}}{[\rho(x - x_i^k)]^{\beta}} \tag{3.48}$$

with the implicit positive constant independent of i and k, where $M_N^0(f)$ denotes the radial grand maximal function of f as in Definition 2.3, ρ denotes the homogeneous quasi-norm associated with dilation A and $\beta := (\frac{\ln b}{\ln \lambda_-} + N - 1)\frac{\ln \lambda_-}{\ln b}$. Indeed, noticing that $\beta\delta p > 1$, as in (3.28), by Proposition 2.4 and (3.48), we have

$$\begin{aligned} \widetilde{\mathrm{I}}_2 &= \int_{(x_i^k + B_{\ell_i^k+\tau})^{\complement}} [M_N(m_i^k)(x)]^{\delta p}\, dx \\ &\lesssim \int_{\rho(x - x_i^k) \geqslant |B_{\ell_i^k+\tau}|} |B_{\ell_i^k}|^{-\delta} \frac{|B_{\ell_i^k}|^{\beta\delta p}}{[\rho(x - x_i^k)]^{\beta\delta p}}\, dx \sim |B_{\ell_i^k}|^{1-\delta} \end{aligned}$$

with the implicit positive constants independent of i and k, which, combined with (3.47), completes the proof of (3.45).

Thus, to obtain the desired conclusion of Case 1, we only need to prove (3.48). To this end, for any $i \in \mathbb{N}$ and $k \in \mathbb{Z}$, take $x \in (x_i^k + B_{\ell_i^k+\tau})^{\complement}$, $\varphi \in \mathcal{S}_N(\mathbb{R}^n)$ and $t \in \mathbb{Z}$. Suppose that P is a polynomial of degree no more than s which will be determined later. Then, for all $i \in \mathbb{N}$ and $k \in \mathbb{Z}$, by Hölder's inequality, we find that

$$|(m_i^k * \varphi_t)(x)| = b^{-t} \left| \int_{\mathbb{R}^n} m_i^k(y) \varphi(A^{-t}(x - y))\, dy \right|$$

$$
\begin{aligned}
&\leqslant b^{-t}\sum_{j\in\mathbb{Z}_+}\left|\int_{U_j(x_i^k+B_{\ell_i^k})} m_i^k(y)[\varphi(A^{-t}(x-y))-P(A^{-t}(x-y))]\,dy\right| \\
&\leqslant b^{-t}\sum_{j\in\mathbb{Z}_+}\|m_i^k\|_{L^r(U_j(x_i^k+B_{\ell_i^k}))}\left[\int_{U_j(x_i^k+B_{\ell_i^k})}|\varphi(A^{-t}(x-y))-P(A^{-t}(x-y))|^{r'}\,dy\right]^{1/r'} \\
&\leqslant b^{-t}\sum_{j\in\mathbb{Z}_+} b^{-j\varepsilon}|B_{\ell_i^k}|^{1/r-1/p}b^{t/r'}\left\{\int_{A^{-t+j}(x-x_i^k)+A^jB_{\ell_i^k-t}}|\varphi(y)-P(y)|^{r'}\,dy\right\}^{1/r'} \\
&\leqslant b^{-t}|B_{\ell_i^k}|^{1/r-1/p}b^{t/r'}b^{-t/r'}|B_{\ell_i^k}|^{1/r'} \\
&\quad\times\left\{\sum_{j\in\mathbb{Z}_+} b^{-j\varepsilon}b^{j/r'}\sup_{y\in A^{-t+j}(x-x_i^k)+A^jB_{\ell_i^k-t}}|\varphi(y)-P(y)|\right\} \\
&=|B_{\ell_i^k}|^{-1/p}b^{\ell_i^k-t}\sum_{j\in\mathbb{Z}_+} b^{-j\varepsilon}b^{j/r'}\sup_{y\in A^{-t+j}(x-x_i^k)+A^jB_{\ell_i^k-t}}|\varphi(y)-P(y)|. \qquad (3.49)
\end{aligned}
$$

Suppose that $x\in[x_i^k+(B_{\ell_i^k+\tau+m+1}\setminus B_{\ell_i^k+\tau+m})]$ for some $m\in\mathbb{Z}_+$. Then, by (2.8), we obtain

$$
\begin{aligned}
A^{-t+j}(x-x_i^k)+A^jB_{\ell_i^k-t}&\subset A^{-t+j}(B_{\ell_i^k+\tau+m+1}\setminus B_{\ell_i^k+\tau+m})+A^jB_{\ell_i^k-t} \\
&=A^{\ell_i^k-t+j}([B_{\tau+m+1}\setminus B_{\tau+m}]+B_0)\subset A^{\ell_i^k-t+j}(B_m)^{\complement}. \qquad (3.50)
\end{aligned}
$$

If $\ell_i^k\geqslant t$, we choose $P\equiv 0$. Then

$$
\sup_{y\in A^{-t+j}(x-x_i^k)+A^jB_{\ell_i^k-t}}|\varphi(y)|\leqslant\sup_{y\in A^{\ell_i^k-t+j}(B_m)^{\complement}}\min\{1,\rho(y)^{-N}\}\leqslant b^{-N(\ell_i^k-t+j+m)}. \qquad (3.51)
$$

If $\ell_i^k<t$, then we let P be the Taylor expansion of φ at the point $A^{-t+j}(x-x_i^k)$ of order s. By the Taylor remainder theorem, (2.4), (2.5) and (3.50), we have

$$
\begin{aligned}
\sup_{y\in A^{-t+j}(x-x_i^k)+A^jB_{\ell_i^k-t}}|\varphi(y)-P(y)|&\lesssim\sup_{z\in A^jB_{\ell_i^k-t}}\sup_{|\alpha|=s+1}|\partial^\alpha\varphi(A^{-t+j}(x-x_i^k)+z)||z|^{s+1} \\
&\lesssim b^{j(s+1)\log_b(\lambda_+)}\lambda_-^{(s+1)(\ell_i^k-t)}\sup_{y\in A^{-t+j}(x-x_i^k)+A^jB_{\ell_i^k-t}}\min\{1,\rho(y)^{-N}\} \\
&\lesssim b^{j(s+1)\log_b(\lambda_+)}\lambda_-^{(s+1)(\ell_i^k-t)}\sup_{y\in A^{\ell_i^k-t+j}(B_m)^{\complement}}\min\{1,\rho(y)^{-N}\} \\
&\lesssim b^{j(s+1)\log_b(\lambda_+)}\lambda_-^{(s+1)(\ell_i^k-t)}\min\{1,b^{-N(\ell_i^k-t+j+m)}\}. \qquad (3.52)
\end{aligned}
$$

Take $s:=\lfloor(\frac{1}{p}-1)\frac{\ln b}{\ln\lambda_-}\rfloor$. Since $N=s+2$, it follows that $b\lambda_-^{s+1}\leqslant b^N$. By this, (3.49), (3.51), (3.52) and $\varepsilon>(s+1)\log_b(\lambda_+)$, for all $x\in[x_i^k+(B_{\ell_i^k+\tau+m+1}\setminus B_{\ell_i^k+\tau+m})]$ with $m\in\mathbb{Z}_+$, we find that

$$
\begin{aligned}
[M_N^0(m_i^k)(x)]^p&=\sup_{\varphi\in\mathcal{S}_N(\mathbb{R}^n)}\sup_{t\in\mathbb{Z}}|(m_i^k*\varphi_t)(x)|^p \\
&\lesssim|B_{\ell_i^k}|^{-1}\sum_{j\in\mathbb{Z}_+}b^{-jp(\varepsilon-1/r')}\max\bigg\{\sup_{t\in\mathbb{Z},\,t\leqslant\ell_i^k}b^{p(\ell_i^k-t)}b^{-Np(\ell_i^k-t+j+m)}, \\
&\qquad\sup_{t\in\mathbb{Z},\,t>\ell_i^k}b^{p(\ell_i^k-t)}b^{jp(s+1)\log_b(\lambda_+)}\lambda_-^{p(s+1)(\ell_i^k-t)}\min\{1,b^{-Np(\ell_i^k-t+j+m)}\}\bigg\} \\
&\lesssim|B_{\ell_i^k}|^{-1}\sum_{j\in\mathbb{Z}_+}b^{-jp[\varepsilon-(s+1)\log_b(\lambda_+)+1-1/r']}\max\{b^{-Npm},(b\lambda_-^{s+1})^{-pm}\} \\
&\lesssim|B_{\ell_i^k}|^{-1}(b\lambda_-^{s+1})^{-pm}.
\end{aligned}
$$

Form this, as in (3.33), we easily deduce that (3.48) holds true for $q/p\in[1,\infty]$. This finishes the proof of Case 1.

Case 2. $q/p \in (0,1)$. In this case, let $\widetilde{\psi}_{k_0}$ and $\widetilde{\eta}_{k_0}$ be as in (3.38). Similar to (3.44), we have

$$2^{k_0 p}[d_{\widetilde{\psi}_{k_0}}(2^{k_0})]^{\delta} \lesssim \left[\sum_{k=-\infty}^{k_0-1} 2^{-k\sigma\widetilde{r}} \sum_{i\in\mathbb{N}} 2^{k\widetilde{r}}|B_{\ell_i^k}|\right]^{\delta} \sim \left(\sum_{k=-\infty}^{k_0-1} 2^{\frac{kp}{\delta}} \mu_k^p\right)^{\delta},$$

which further implies that

$$\sum_{k_0\in\mathbb{Z}} 2^{k_0 q}|\{x\in\mathbb{R}^n : \widetilde{\psi}_{k_0}(x) > 2^{k_0}\}|^{\frac{q}{p}} \lesssim \sum_{k_0\in\mathbb{Z}} 2^{k_0(q-\frac{q}{\delta})} \sum_{k=-\infty}^{k_0-1} 2^{\frac{kq}{\delta}} \mu_k^q$$
$$\sim \sum_{k\in\mathbb{Z}} \sum_{k_0=k+1}^{\infty} 2^{k_0(q-\frac{q}{\delta})} 2^{\frac{kq}{\delta}} \mu_k^q \lesssim \sum_{k\in\mathbb{Z}} 2^{kq}\mu_k^q. \tag{3.53}$$

On the other hand, similar to (3.46), we find that

$$2^{k_0\delta p} d_{\widetilde{\eta}_{k_0}}(2^{k_0}) \lesssim \sum_{k=k_0}^{\infty} [2^{k\delta}\mu_k]^p,$$

which implies that

$$2^{k_0\delta p}|\{x\in\mathbb{R}^n : \widetilde{\eta}_{k_0}(x) > 2^{k_0}\}| \lesssim \sum_{k=k_0}^{\infty} 2^{-k\widetilde{\delta}p}[2^{k(1-\widetilde{\delta})}\mu_k]^p \lesssim 2^{-k_0\widetilde{\delta}p}\left\{\sum_{k=k_0}^{\infty} [2^{k(1-\widetilde{\delta})}\mu_k]^q\right\}^{\frac{p}{q}},$$

where $\widetilde{\delta} := \frac{1-\delta}{2}$. Therefore, we have

$$\sum_{k_0\in\mathbb{Z}} 2^{k_0 q}|\{x\in\mathbb{R}^n : \widetilde{\eta}_{k_0}(x) > 2^{k_0}\}|^{\frac{q}{p}} \lesssim \sum_{k_0\in\mathbb{Z}} 2^{k_0\widetilde{\delta}q} \sum_{k=k_0}^{\infty} [2^{k(1-\widetilde{\delta})}\mu_k]^q$$
$$\sim \sum_{k\in\mathbb{Z}} [2^{k(1-\widetilde{\delta})}\mu_k]^q \sum_{k_0=-\infty}^{k} 2^{k_0\widetilde{\delta}q} \lesssim \sum_{k\in\mathbb{Z}} 2^{kq}\mu_k^q. \tag{3.54}$$

Notice that $\mu_k := (\sum_{i\in\mathbb{N}} |B_{\ell_i^k}|)^{1/p}$ and $\lambda_i^k \sim 2^k|B_{\ell_i^k}|^{1/p}$. Combining (2.1), (3.53), (3.54) and (3.37), we further conclude that

$$\begin{aligned}\|M_N(f)\|_{L^{p,q}(\mathbb{R}^n)}^q &\sim \sum_{k_0\in\mathbb{Z}} 2^{k_0 q}|\{x\in\mathbb{R}^n : M_N(f)(x) > 2^{k_0}\}|^{\frac{q}{p}}\\ &\lesssim \sum_{k_0\in\mathbb{Z}} 2^{k_0 q}|\{x\in\mathbb{R}^n : \widetilde{\psi}_{k_0}(x) > 2^{k_0}\}|^{\frac{q}{p}} + \sum_{k_0\in\mathbb{Z}} 2^{k_0 q}|\{x\in\mathbb{R}^n : \widetilde{\eta}_{k_0}(x) > 2^{k_0}\}|^{\frac{q}{p}}\\ &\lesssim \sum_{k\in\mathbb{Z}} 2^{kq}\mu_k^q \sim \sum_{k\in\mathbb{Z}}\left[\sum_{i\in\mathbb{N}} |\lambda_i^k|^p\right]^{\frac{q}{p}} \sim \|f\|_{H_A^{p,r,s,\varepsilon,q}(\mathbb{R}^n)}^q,\end{aligned}$$

which implies that

$$\|f\|_{H_A^{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{H_A^{p,r,s,\varepsilon,q}(\mathbb{R}^n)}.$$

This finishes the proof of Case 2 and hence Theorem 3.9. □

4 Maximal function characterizations of $H_A^{p,q}(\mathbb{R}^n)$

In this section, we characterize $H_A^{p,q}(\mathbb{R}^n)$ in terms of the radial maximal function M_φ^0 (see (2.13)) and the non-tangential maximal function M_φ (see (2.12)). We begin with the following Definitions 4.1 and 4.2 from [9].

Definition 4.1. For any function $F:\mathbb{R}^n\times\mathbb{Z}\to[0,\infty)$, $K\in\mathbb{Z}\cup\{\infty\}$ and $\ell\in\mathbb{Z}$, the maximal function of F with *aperture* ℓ is defined by

$$F_\ell^{*K}(x):=\sup_{k\in\mathbb{Z},\,k\leqslant K}\sup_{y\in x+B_{k+\ell}}F(y,k),\quad\forall\,x\in\mathbb{R}^n.$$

Definition 4.2. Let $K\in\mathbb{Z}$ and $L\in[0,\infty)$. For $\varphi\in\mathcal{S}$, the *radial maximal function* $M_\varphi^{0(K,L)}(f)$, the *non-tangential maximal function* $M_\varphi^{(K,L)}(f)$ and the *tangential maximal function* $T_\varphi^{N(K,L)}(f)$ of $f\in\mathcal{S}'(\mathbb{R}^n)$ are, respectively, defined by setting, for all $x\in\mathbb{R}^n$,

$$M_\varphi^{0(K,L)}(f)(x):=\sup_{k\in\mathbb{Z},\,k\leqslant K}|(f*\varphi_k)(x)|[\max\{1,\rho(A^{-K}x)\}]^{-L}(1+b^{-k-K})^{-L},$$
$$M_\varphi^{(K,L)}(f)(x):=\sup_{k\in\mathbb{Z},\,k\leqslant K}\sup_{y\in x+B_k}|(f*\varphi_k)(y)|[\max\{1,\rho(A^{-K}y)\}]^{-L}(1+b^{-k-K})^{-L}$$

and

$$T_\varphi^{N(K,L)}(f)(x):=\sup_{k\in\mathbb{Z},k\leqslant K}\sup_{y\in\mathbb{R}^n}\frac{|(f*\varphi_k)(y)|}{[\max\{1,\rho(A^{-k}(x-y))\}]^N}\frac{(1+b^{-k-K})^{-L}}{[\max\{1,\rho(A^{-K}y)\}]^L}.$$

Furthermore, the *radial grand maximal function* $M_N^{0(K,L)}(f)$ and the *non-tangential grand maximal function* $M_N^{(K,L)}(f)$ of $f\in\mathcal{S}'(\mathbb{R}^n)$ are, respectively, defined by setting, for all $x\in\mathbb{R}^n$,

$$M_N^{0(K,L)}(f)(x):=\sup_{\varphi\in\mathcal{S}_N(\mathbb{R}^n)}M_\varphi^{0(K,L)}(f)(x)$$

and

$$M_N^{(K,L)}(f)(x):=\sup_{\varphi\in\mathcal{S}_N(\mathbb{R}^n)}M_\varphi^{(K,L)}(f)(x).$$

Lemma 4.3 through Lemma 4.5 are just [9, p. 42, Lemma 7.2; p. 45, Lemma 7.5; p. 46, Lemma 7.6], respectively.

Lemma 4.3. *There exists a positive constant C such that, for all functions $F:\mathbb{R}^n\times\mathbb{Z}\to[0,\infty)$, $\ell\in[\ell',\infty)\cap\mathbb{Z}$, $K\in\mathbb{Z}\cup\{\infty\}$ and $\lambda\in(0,\infty)$,*

$$|\{x\in\mathbb{R}^n:F_\ell^{*K}(x)>\lambda\}|\leqslant Cb^{\ell-\ell'}|\{x\in\mathbb{R}^n:F_{\ell'}^{*K}(x)>\lambda\}|.$$

Lemma 4.4. *Suppose that $\varphi\in\mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n}\varphi(x)\,dx\neq0$. For any given $N\in\mathbb{N}$ and $L\in[0,\infty)$, there exist an $I\in\mathbb{N}$ and a positive constant $C_{(L)}$, depending on L, such that, for all $K\in\mathbb{Z}_+$ and $f\in\mathcal{S}'(\mathbb{R}^n)$,*

$$M_I^{0(K,L)}(f)(x)\leqslant C_{(L)}T_\varphi^{N(K,L)}(f)(x),\quad\forall\,x\in\mathbb{R}^n.$$

Lemma 4.5. *Suppose that $p\in(0,\infty)$, $\varphi\in\mathcal{S}(\mathbb{R}^n)$ and $K\in\mathbb{Z}_+$. Then, for any given $M\in(0,\infty)$, there exist $L\in(0,\infty)$ and a positive constant $C_{(K,\,M)}$ such that, for all $f\in\mathcal{S}'(\mathbb{R}^n)$ and $x\in\mathbb{R}^n$,*

$$M_\varphi^{(K,L)}(f)(x)\leqslant C_{(K,\,M)}[\max\{1,\rho(x)\}]^{-M}.\tag{4.1}$$

Lemma 4.6. *Let $p\in(0,\infty)$, $N\in(1/p,\infty)\cap\mathbb{N}$, $q\in(0,\infty]$ and $\varphi\in\mathcal{S}(\mathbb{R}^n)$. Then there exists a positive constant C such that, for all $K\in\mathbb{Z}$, $L\in[0,\infty)$ and $f\in\mathcal{S}'(\mathbb{R}^n)$,*

$$\|T_\varphi^{N(K,L)}(f)\|_{L^{p,q}(\mathbb{R}^n)}\leqslant C\|M_\varphi^{(K,L)}(f)\|_{L^{p,q}(\mathbb{R}^n)}.\tag{4.2}$$

Proof. We first prove that, for all $p\in(0,\infty)$, $q\in(0,\infty]$, $K\in\mathbb{Z}$ and $\ell\in[\ell',\infty)\cap\mathbb{Z}$,

$$\|F_\ell^{*K}\|_{L^{p,q}(\mathbb{R}^n)}\lesssim b^{(\ell-\ell')/p}\|F_{\ell'}^{*K}\|_{L^{p,q}(\mathbb{R}^n)},\tag{4.3}$$

where F_ℓ^{*K} is as in Definition 4.1 and, for all $\varphi\in\mathcal{S}(\mathbb{R}^n)$, $f\in\mathcal{S}'(\mathbb{R}^n)$, $k\in\mathbb{Z}$ and $y\in\mathbb{R}^n$,

$$F(y,k):=|(f*\varphi_k)(y)|\max[\{1,\rho(A^{-K}y)\}]^{-L}(1+b^{-k-K})^{-L}.\tag{4.4}$$

To this end, fix $x \in \mathbb{R}^n$. For $k \in (-\infty, K] \cap \mathbb{Z}$, if $x - y \in B_{k+1}$, then

$$F(y,k)[\max\{1, \rho(A^{-k}(x-y))\}]^{-N} \leqslant F_1^{*K}(x); \tag{4.5}$$

if $x - y \in B_{k+j+1} \setminus B_{k+j}$ for some $j \in \mathbb{N}$, then

$$F(y,k)[\max\{1, \rho(A^{-k}(x-y))\}]^{-N} \leqslant F_{j+1}^{*K}(x) b^{-jN}. \tag{4.6}$$

By taking supremum over all $k \in (-\infty, K] \cap \mathbb{Z}$ and $y \in \mathbb{R}^n$ on the both sides of (4.5) and (4.6), together with (4.4) and the definition of $T_\varphi^{N(K,L)}$, we further find that

$$T_\varphi^{N(K,L)}(f)(x) \leqslant \sum_{j=0}^{\infty} F_{j+1}^{*K}(x) b^{-jN}. \tag{4.7}$$

Notice that $F_{\ell'}^{*K}$ is a non-negative function for any $\ell' \in \mathbb{Z}$. By (2.1) and Lemma 4.3, we conclude that, for any $\ell \in [\ell', \infty) \cap \mathbb{Z}$, $K \in \mathbb{Z}$ and p, $q \in (0, \infty)$,

$$\begin{aligned} \|F_\ell^{*K}\|_{L^{p,q}(\mathbb{R}^n)} &\sim \left[\int_0^\infty \lambda^{q-1} |\{x \in \mathbb{R}^n : F_\ell^{*K}(x) > \lambda\}|^{q/p} \, d\lambda\right]^{1/q} \\ &\lesssim b^{(\ell-\ell')/p} \left[\int_0^\infty \lambda^{q-1} |\{x \in \mathbb{R}^n : F_{\ell'}^{*K}(x) > \lambda\}|^{q/p} \, d\lambda\right]^{1/q} \\ &\sim b^{(\ell-\ell')/p} \|F_{\ell'}^{*K}\|_{L^{p,q}(\mathbb{R}^n)}, \end{aligned} \tag{4.8}$$

where the implicit positive constants are independent of ℓ and K. It is easy to see that (4.8) is also true for $q = \infty$ by the definition of the $L^{p,\infty}(\mathbb{R}^n)$ norm. This proves (4.3).

Now we show (4.2). By (4.7), the Aoki-Rolewicz theorem (see [6,63]), (4.3) and $N \in (1/p, \infty) \cap \mathbb{N}$, we know that there exists $\upsilon \in (0, 1]$ such that

$$\begin{aligned} \|T_\varphi^{N(K,L)}(f)\|_{L^{p,q}(\mathbb{R}^n)}^\upsilon &\leqslant \sum_{j=0}^\infty b^{-jN\upsilon} \|F_{j+1}^{*K}\|_{L^{p,q}(\mathbb{R}^n)}^\upsilon \\ &\lesssim \sum_{j=0}^\infty b^{-jN\upsilon} b^{(j+1)\upsilon/p} \|F_0^{*K}\|_{L^{p,q}(\mathbb{R}^n)}^\upsilon \lesssim \|M_\varphi^{(K,L)}(f)\|_{L^{p,q}(\mathbb{R}^n)}^\upsilon \end{aligned}$$

with the implicit positive constants independent of K, L and f, which implies (4.2) and hence completes the proof of Lemma 4.6. □

Lemma 4.7. *Suppose that* $p \in (1, \infty)$ *and* $q \in (0, \infty]$. *Then there exists a positive constant* C *such that, for all* $f \in L^{p,q}(\mathbb{R}^n)$,

$$\|M_{\mathcal{F}}(f)\|_{L^{p,q}(\mathbb{R}^n)} \leqslant C \|f\|_{L^{p,q}(\mathbb{R}^n)}, \tag{4.9}$$

where $M_{\mathcal{F}}(f)$ *is defined as in* (2.17).

Proof. Let $E \subset \mathbb{R}^n$ be an arbitrary measurable set and $|E| < \infty$. By (2.18) and (2.19), we have

$$\|M_{\mathcal{F}}(\chi_E)\|_{L^{1,\infty}(\mathbb{R}^n)} \lesssim \|\chi_E\|_{L^1(\mathbb{R}^n)} \sim |E|$$

and

$$\|M_{\mathcal{F}}(\chi_E)\|_{L^\infty(\mathbb{R}^n)} \lesssim \|\chi_E\|_{L^\infty(\mathbb{R}^n)} \lesssim 1.$$

Thus, applying [44, Theorem 1.1 and Remark 1.4] to $M_{\mathcal{F}}$ and $f \in L^{p,q}(\mathbb{R}^n)$ with $p_0 = q_0 = 1$ and $p_1 = q_1 = \infty$, we obtain (4.9). This finishes the proof of Lemma 4.7. □

Remark 4.8. As a corollary of Lemma 4.7, the operators $M_N(f)$ in (2.14) and $M_{\mathrm{HL}}(f)$ in (2.20) also satisfy (4.9).

Now we state the main result of this section.

Theorem 4.9. *Suppose that* $p \in (0,\infty)$, $q \in (0,\infty]$ *and* $\varphi \in \mathcal{S}(\mathbb{R}^n)$ *with* $\int_{\mathbb{R}^n} \varphi(x)\,dx \neq 0$. *Then, for all* $f \in \mathcal{S}'(\mathbb{R}^n)$, *the following statements are mutually equivalent*:

$$f \in H_A^{p,q}(\mathbb{R}^n), \tag{4.10}$$

$$M_\varphi(f) \in L^{p,q}(\mathbb{R}^n), \tag{4.11}$$

$$M_\varphi^0(f) \in L^{p,q}(\mathbb{R}^n). \tag{4.12}$$

In this case, it holds true that

$$\|f\|_{H_A^{p,q}(\mathbb{R}^n)} \leqslant C_1 \|M_\varphi^0(f)\|_{L^{p,q}(\mathbb{R}^n)} \leqslant C_1 \|M_\varphi(f)\|_{L^{p,q}(\mathbb{R}^n)} \leqslant C_2 \|f\|_{H_A^{p,q}(\mathbb{R}^n)},$$

where C_1 *and* C_2 *are positive constants independent of* f.

Proof. Clearly, (4.10) implies (4.11) and (4.11) implies (4.12). Thus, to prove Theorem 4.9, it suffices to show that (4.11) implies (4.10) and that (4.12) implies (4.11).

We first prove that (4.11) implies (4.10). To this end, notice that, by Lemma 4.4 with $N \in (1/p,\infty)\cap\mathbb{N}$ and $L = 0$, we find that there exists an $I \in \mathbb{N}$ such that $M_I^{0(K,0)}(f)(x) \lesssim T_\varphi^{N(K,0)}(f)(x)$ for all $K \in \mathbb{Z}_+$, $f \in \mathcal{S}'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. From this and Lemma 4.6, we further deduce that, for all $K \in \mathbb{Z}_+$ and $f \in \mathcal{S}'(\mathbb{R}^n)$,

$$\|M_I^{0(K,0)}(f)\|_{L^{p,q}(\mathbb{R}^n)} \lesssim \|M_\varphi^{(K,0)}(f)\|_{L^{p,q}(\mathbb{R}^n)}. \tag{4.13}$$

Letting $K \to \infty$ in (4.13), by [35, Proposition 1.4.5(8)] and the Fatou lemma, we know that

$$\|M_I^0(f)\|_{L^{p,q}(\mathbb{R}^n)} \lesssim \|M_\varphi(f)\|_{L^{p,q}(\mathbb{R}^n)},$$

which, together with Proposition 2.4, shows that (4.11) implies (4.10).

Now we show that (4.12) implies (4.11). Suppose now that $M_\varphi^0(f) \in L^{p,q}(\mathbb{R}^n)$. By Lemma 4.5, we find that there exists $L \in (0,\infty)$ such that (4.1) holds true, which further implies that $M_\varphi^{(K,L)}(f) \in L^{p,q}(\mathbb{R}^n)$ for all $K \in \mathbb{Z}_+$. Indeed, for $q/p \in (0,1]$, by (2.1) and (4.1), we have

$$\begin{aligned}
\|M_\varphi^{(K,L)}(f)\|_{L^{p,q}(\mathbb{R}^n)}^q &\sim \int_0^\infty \lambda^{q-1} |\{x \in \mathbb{R}^n : M_\varphi^{(K,L)}(f)(x) > \lambda\}|^{q/p}\,d\lambda \\
&\lesssim \int_0^\infty \lambda^{q-1} |\{x \in B_1 : M_\varphi^{(K,L)}(f)(x) > \lambda\}|^{q/p}\,d\lambda \\
&\quad + \sum_{j=1}^\infty \int_0^\infty \lambda^{q-1} |\{x \in B_{j+1} \setminus B_j : M_\varphi^{(K,L)}(f)(x) > \lambda\}|^{q/p}\,d\lambda \\
&\lesssim \int_0^1 \lambda^{q-1} |B_1|^{q/p}\,d\lambda + \sum_{j=1}^\infty \int_0^{b^{-jM}} \lambda^{q-1} |B_{j+1}|^{q/p}\,d\lambda \\
&\sim \sum_{j=0}^\infty b^{-jMq} b^{(j+1)q/p} \sim 1 \quad \text{as} \quad M > 1/p.
\end{aligned}$$

For another case when $q/p \in (1,\infty)$, by (2.1), the Minkowski integral inequality and (4.1), we find that

$$\begin{aligned}
\|M_\varphi^{(K,L)}(f)\|_{L^{p,q}(\mathbb{R}^n)}^q &\lesssim \|M_\varphi^{(K,L)}(f)\|_{L^{p,q}(B_1)}^q + \|M_\varphi^{(K,L)}(f)\|_{L^{p,q}(\mathbb{R}^n \setminus B_1)}^q \\
&\sim \int_0^\infty \lambda^{q-1} |\{x \in B_1 : M_\varphi^{(K,L)}(f)(x) > \lambda\}|^{q/p}\,d\lambda \\
&\quad + \left[\sum_{k\in\mathbb{Z}} 2^{kq} \left(\sum_{j=1}^\infty |\{x \in B_{j+1} \setminus B_j : M_\varphi^{(K,L)} f(x) > 2^k\}|\right)^{\frac{q}{p}}\right]^{\frac{p}{q}\cdot\frac{q}{p}} \\
&\lesssim \int_0^\infty \lambda^{q-1} |\{x \in B_1 : M_\varphi^{(K,L)}(f)(x) > \lambda\}|^{q/p}\,d\lambda
\end{aligned}$$

$$
\begin{aligned}
&\quad + \left[\sum_{j=1}^{\infty}\left(\sum_{k\in\mathbb{Z}} 2^{kq}|\{x\in B_{j+1}\setminus B_j : M_{\varphi}^{(K,L)}(f)(x) > 2^k\}|^{\frac{q}{p}}\right)^{\frac{p}{q}}\right]^{\frac{q}{p}}\\
&\sim \int_0^{\infty} \lambda^{q-1}|\{x\in B_1 : M_{\varphi}^{(K,L)}(f)(x) > \lambda\}|^{q/p}\,d\lambda\\
&\quad + \left[\sum_{j=1}^{\infty}\left(\int_0^{\infty} \lambda^{q-1}|\{x\in B_{j+1}\setminus B_j : M_{\varphi}^{(K,L)}(f)(x) > \lambda\}|^{\frac{q}{p}}\,d\lambda\right)^{\frac{p}{q}}\right]^{\frac{q}{p}}\\
&\lesssim \int_0^{1} \lambda^{q-1}|B_1|^{q/p}\,d\lambda + \left[\sum_{j=1}^{\infty}\left(\int_0^{b^{-jM}} \lambda^{q-1}|B_{j+1}|^{q/p}\,d\lambda\right)^{\frac{p}{q}}\right]^{\frac{q}{p}}\\
&\lesssim \left[\sum_{j=0}^{\infty} b^{-jMp} b^{(j+1)}\right]^{\frac{q}{p}} \sim 1 \quad \text{as} \quad M > 1/p. \qquad (4.14)
\end{aligned}
$$

Clearly, (4.14) also holds true for $q=\infty$, since

$$
\|f\|_{L^{p,\infty}(\mathbb{R}^n)} = \sup_{\lambda\in(0,\infty)}\{\lambda d_f^{1/p}(\lambda)\} \sim \sup_{k\in\mathbb{Z}} 2^k[d_f(2^k)]^{\frac{1}{p}}.
$$

On the other hand, by Lemmas 4.4 and 4.6, we know that, for any $L\in(0,\infty)$, there exists some $I\in\mathbb{N}$ such that, for all $K\in\mathbb{Z}_+$ and $f\in\mathcal{S}'(\mathbb{R}^n)$,

$$
\|M_I^{0(K,L)}(f)\|_{L^{p,q}(\mathbb{R}^n)} \leqslant C_3\|M_{\varphi}^{(K,L)}(f)\|_{L^{p,q}(\mathbb{R}^n)},
$$

where C_3 is a positive constant independent of K. For any given $K\in\mathbb{Z}_+$, let

$$
\Omega_K := \{x\in\mathbb{R}^n : M_I^{0(K,L)}(f)(x) \leqslant C_4 M_{\varphi}^{(K,L)}(f)(x)\} \qquad (4.15)
$$

with $C_4 := 2C_3$. Then

$$
\|M_{\varphi}^{(K,L)}(f)\|_{L^{p,q}(\mathbb{R}^n)} \lesssim \|M_{\varphi}^{(K,L)}(f)\|_{L^{p,q}(\Omega_K)}, \qquad (4.16)
$$

because

$$
\|M_{\varphi}^{(K,L)}(f)\|_{L^{p,q}(\Omega_K^{\complement})} \leqslant C_4^{-1}\|M_I^{0(K,L)}(f)\|_{L^{p,q}(\Omega_K^{\complement})} \leqslant C_3/C_4\|M_{\varphi}^{(K,L)}(f)\|_{L^{p,q}(\mathbb{R}^n)}.
$$

To finish the proof that (4.12) implies (4.11), for any given $L\in[0,\infty)$, it suffices to show that, for all $t\in(0,p)$, $K\in\mathbb{Z}_+$ and $f\in\mathcal{S}'(\mathbb{R}^n)$,

$$
M_{\varphi}^{(K,L)}(f)(x) \lesssim \{M_{\mathrm{HL}}([M_{\varphi}^{0(K,L)}(f)]^t)(x)\}^{1/t}, \quad \forall\, x\in\Omega_K. \qquad (4.17)
$$

Indeed, if (4.17) holds true for the time being, then, by (4.16), (2.3), (4.17) and Remark 4.8, for all $K\in\mathbb{Z}_+$ and $f\in\mathcal{S}'(\mathbb{R}^n)$, we have

$$
\begin{aligned}
\|M_{\varphi}^{(K,L)}(f)\|_{L^{p,q}(\mathbb{R}^n)}^t &\lesssim \|M_{\varphi}^{(K,L)}(f)\|_{L^{p,q}(\Omega_K)}^t \sim \|[M_{\varphi}^{(K,L)}(f)]^t\|_{L^{\frac{p}{t},\frac{q}{t}}(\Omega_K)}\\
&\lesssim \|M_{\mathrm{HL}}([M_{\varphi}^{0(K,L)}(f)]^t)\|_{L^{\frac{p}{t},\frac{q}{t}}(\Omega_K)}\\
&\lesssim \|[M_{\varphi}^{0(K,L)}(f)]^t\|_{L^{\frac{p}{t},\frac{q}{t}}(\mathbb{R}^n)} \sim \|M_{\varphi}^{0(K,L)}(f)\|_{L^{p,q}(\mathbb{R}^n)}^t. \qquad (4.18)
\end{aligned}
$$

Noticing that $M_{\varphi}^{(K,L)}(f)(x)$ and $M_{\varphi}^{0(K,L)}(f)(x)$ converge pointwise and monotonically to $M_{\varphi}(f)(x)$ and $M_{\varphi}^0(f)(x)$ for all $f\in\mathcal{S}'(\mathbb{R}^n)$ and $x\in\mathbb{R}^n$, respectively, as $K\to\infty$, by [35, Proposition 1.4.5(8)], the monotone convergence theorem and (4.18), we have

$$
\|M_{\varphi}(f)\|_{L^{p,q}(\mathbb{R}^n)} \lesssim \|M_{\varphi}^0(f)\|_{L^{p,q}(\mathbb{R}^n)},
$$

which shows that (4.12) implies (4.11).

Thus, to complete the proof of Theorem 4.9, we only need to prove (4.17). For this purpose, for all $f \in \mathcal{S}'(\mathbb{R}^n)$, $k \in \mathbb{Z}$ and $y \in \mathbb{R}^n$, let

$$F(y,k) := |(f * \varphi_k)(y)|[\max\{1, \rho(A^{-K}y)\}]^{-L}[1 + b^{-k-K}]^{-L}. \tag{4.19}$$

Let $x \in \Omega_K$. By the definitions of F_0^{*K} and $M_\varphi^{(K,L)}$ and (4.19), it is easy to see that there exist $k \in (-\infty, K] \cap \mathbb{N}$ and $y \in x + B_k$ such that

$$F(y,k) \geqslant F_0^{*K}(x)/2 = M_\varphi^{(K,L)}(f)(x)/2. \tag{4.20}$$

For this y, consider $\widetilde{x} \in (y + B_{k-\ell})$ for some integer $\ell \in \mathbb{Z}_+$ to be specified later. We write

$$f * \varphi_k(\widetilde{x}) - f * \varphi_k(z) = f * \Psi_k(z), \quad \forall\, z \in \mathbb{R}^n, \tag{4.21}$$

where $\Psi(z) := \varphi(z + A^{-k}(\widetilde{x} - y)) - \varphi(z)$ for all $z \in \mathbb{R}^n$. By (2.11), (2.10), the mean value theorem and (2.5), we conclude that

$$\begin{aligned}
\|\Psi\|_{\mathcal{S}_I(\mathbb{R}^n)} &\leqslant \sup_{h \in B_{-\ell}} \|\varphi(\cdot + h) - \varphi(\cdot)\|_{\mathcal{S}_I(\mathbb{R}^n)} \\
&= \sup_{h \in B_{-\ell}} \sup_{z \in \mathbb{R}^n} \sup_{|\alpha| \leqslant I} \max\{1, [\rho(z)]^I\}|\partial^\alpha \varphi(z + h) - \partial^\alpha \varphi(z)| \\
&\lesssim \Big[\sup_{h \in B_{-\ell}} \sup_{z \in \mathbb{R}^n} \sup_{|\alpha| \leqslant I+1} \max\{1, [\rho(z+h)]^I\}|\partial^\alpha \varphi(z+h)|\Big]\Big[\max_{h \in B_{-\ell}} |h|\Big] \\
&\lesssim C_5 \lambda_-^{-\ell},
\end{aligned} \tag{4.22}$$

where the positive constant C_5 is independent of L. Notice that, by a proof similar to that of [9, p. 17, Proposition 3.10], we find that, for all $x \in \mathbb{R}^n$,

$$M_I^{(K,L)}(f)(x) \leqslant b^{\tau I} M_I^{0(K,L)}(f)(x). \tag{4.23}$$

Moreover, by (2.10), we know that, for all $\widetilde{x} \in y + B_{k-\ell}$,

$$\max\{1, \rho(A^{-K}\widetilde{x})\} \leqslant b^\tau \max\{1, \rho(A^{-K}y)\},$$

which, combined with (4.20)–(4.23) and (4.15), implies that

$$\begin{aligned}
b^{\tau L} F(\widetilde{x}, k) &\geqslant [|f * \varphi_k(y)| - |f * \Psi_k(y)|][\max\{1, \rho(A^{-K}y)\}]^{-L}(1 + b^{-k-K})^{-L} \\
&\geqslant F(y,k) - M_I^{(K,L)}(f)(x)\|\Psi\|_{\mathcal{S}_I(\mathbb{R}^n)} \gtrsim M_\varphi^{(K,L)}(f)(x)/2 - C_5 \lambda_-^{-\ell} b^{\tau I} M_I^{0(K,L)}(f)(x) \\
&\gtrsim M_\varphi^{(K,L)}(f)(x)/2 - C_4 C_5 \lambda_-^{-\ell} b^{\tau I} M_\varphi^{(K,L)}(f)(x) \gtrsim M_\varphi^{(K,L)}(f)(x)/4,
\end{aligned} \tag{4.24}$$

where ℓ is chosen to be the smallest integer such that $C_4 C_5 \lambda_-^{-\ell} b^{\tau I} \leqslant 1/4$. Therefore, by (4.24) and (2.7), we conclude that, for all $t \in (0, p)$ and $x \in \Omega_K$,

$$\begin{aligned}
[M_\varphi^{(K,L)}(f)(x)]^t &\lesssim \frac{4^t b^{\tau L t}}{|B_{k-\ell}|} \int_{y + B_{k-\ell}} [F(z,k)]^t \, dz \lesssim 4^t b^{\tau L t} \frac{b^{\tau+\ell}}{|B_{k+\tau}|} \int_{x + B_{k+\tau}} [M_\varphi^{0(K,L)}(f)]^t(z) \, dz \\
&\lesssim 4^t b^{\tau L t} M_{\rm HL}([M_\varphi^{0(K,L)}(f)]^t)(x)
\end{aligned}$$

with the implicit positive constants independent of t, K and f, which implies (4.17). This finishes the proof of Theorem 4.9. □

5 Finite atomic decomposition characterizations of $H_A^{p,q}(\mathbb{R}^n)$

In this section, we obtain the finite atomic decomposition characterizations of $H_A^{p,q}(\mathbb{R}^n)$. To be precise, we prove that, for any given finite linear combination of (p, r, s)-atoms with $r \in (1, \infty)$ (or continuous (p, ∞, s)-atoms), its quasi-norm in $H_A^{p,q}(\mathbb{R}^n)$ can be achieved via all its finite atomic decompositions.

Definition 5.1. For an admissible anisotropic triplet (p, r, s), $q \in (0, \infty]$ and a dilation A, denote by $H^{p,r,s,q}_{A,\mathrm{fin}}(\mathbb{R}^n)$ the set of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfying that there exist $K, I \in \mathbb{N}$, a finite sequence of (p, r, s)-atoms, $\{a^k_i\}_{i\in[1,I]\cap\mathbb{N},\,k\in[1,K]\cap\mathbb{Z}}$, supported on $\{x^k_i + B^k_i\}_{i\in[1,I]\cap\mathbb{N},\,k\in[1,K]\cap\mathbb{Z}} \subset \mathfrak{B}$, respectively, and a positive constant $\widetilde{C}$, independent of I and K, such that $\sum_{i=1}^{I} \chi_{x^k_i+B^k_i}(x) \leqslant \widetilde{C}$ for all $x \in \mathbb{R}^n$ and $k \in [1, K] \cap \mathbb{Z}$, and

$$f = \sum_{k=1}^{K} \sum_{i=1}^{I} \lambda^k_i a^k_i \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n),$$

where $\lambda^k_i \sim 2^k |B^k_i|^{1/p}$ for all $k \in [1, K]\cap\mathbb{Z}$ and $i \in [1, I]\cap\mathbb{N}$ with the implicit equivalent positive constants independent of k, K and i, I. Moreover, the quasi-norm of f in $H^{p,r,s,q}_{A,\mathrm{fin}}(\mathbb{R}^n)$ is defined by

$$\|f\|_{H^{p,r,s,q}_{A,\mathrm{fin}}(\mathbb{R}^n)} := \inf\left\{ \left[\sum_{k=1}^{K} \left(\sum_{i=1}^{I} |\lambda^k_i|^p \right)^{\frac{q}{p}} \right]^{\frac{1}{q}} : f = \sum_{k=1}^{K} \sum_{i=1}^{I} \lambda^k_i a^k_i, K, I \in \mathbb{N} \right\}$$

with the usual modification made when $q = \infty$, where the infimum is taken over all decompositions of f as above.

Obviously, by Theorem 3.6, we know that, for any admissible anisotropic triplet (p, r, s) and $q \in (0, \infty)$, the set $H^{p,r,s,q}_{A,\mathrm{fin}}(\mathbb{R}^n)$ is dense in $H^{p,q}_A(\mathbb{R}^n)$ with respect to the quasi-norm $\|\cdot\|_{H^{p,q}_A(\mathbb{R}^n)}$. From this, we deduce the following density of $H^{p,q}_A(\mathbb{R}^n)$.

Lemma 5.2. *If $p, q \in (0, \infty)$, then*

(i) *for any $r \in [1, \infty]$, $H^{p,q}_A(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)$ is dense in $H^{p,q}_A(\mathbb{R}^n)$;*

(ii) *$H^{p,q}_A(\mathbb{R}^n) \cap C^\infty_c(\mathbb{R}^n)$ is dense in $H^{p,q}_A(\mathbb{R}^n)$.*

Proof. We first prove (i). If $p \in (1, \infty)$ and $q \in (0, \infty)$, then $H^{p,q}_A(\mathbb{R}^n) = L^{p,q}(\mathbb{R}^n)$ (see Remark 6.6(ii) below). By [35, Theorem 1.4.13], we know that the set of simple functions is dense in $L^{p,q}(\mathbb{R}^n)$. Thus, $L^{p,q}(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)$ is also dense in $L^{p,q}(\mathbb{R}^n)$ for all $r \in [1, \infty]$. If $p \in (0, 1]$ and $q \in (0, \infty)$, by the density of the set $H^{p,\infty,s,q}_{A,\mathrm{fin}}(\mathbb{R}^n)$ in $H^{p,q}_A(\mathbb{R}^n)$ and $H^{p,\infty,s,q}_{A,\mathrm{fin}}(\mathbb{R}^n) \subset L^r(\mathbb{R}^n)$ for all $r \in [1, \infty]$, we easily find that $H^{p,q}_A(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)$ is dense in $H^{p,q}_A(\mathbb{R}^n)$. This finishes the proof of (i).

Now we prove (ii). To this end, we claim that, for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \varphi(x)\,dx \neq 0$ and $f \in H^{p,q}_A(\mathbb{R}^n)$,

$$f * \varphi_k \to f \quad \text{in} \quad H^{p,q}_A(\mathbb{R}^n) \quad \text{as} \quad k \to -\infty. \tag{5.1}$$

To show this, we first assume that $f \in H^{p,q}_A(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. In this case, to prove (5.1), it suffices to show that

$$M_N(f * \varphi_k - f)(x) \to 0 \quad \text{for almost every} \quad x \in \mathbb{R}^n \quad \text{as} \quad k \to -\infty, \tag{5.2}$$

where $N := N_{(p)} + 2$. Indeed, it is easy to see that $f * \varphi_k - f \in L^2(\mathbb{R}^n)$ for all $k \in \mathbb{Z}$, which, together with Proposition 2.9 and Remark 2.10, implies that $M_N(f * \varphi_k - f) \in L^2(\mathbb{R}^n)$ for all $k \in \mathbb{Z}$. By this, [9, p. 39, Lemma 6.6], (5.2), (2.1) and the Lebesgue dominated convergence theorem, we know that (5.1) holds true for all $f \in H^{p,q}_A(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.

Now, we show (5.2). Notice that, if g is continuous and has compact support, then g is uniformly continuous on $\mathbb{R}^n$. Thus, for any $\delta \in (0, \infty)$, there exists $\eta \in (0, \infty)$ such that, for all $y \in \mathbb{R}^n$ satisfying $\rho(y) < \eta$ and $x \in \mathbb{R}^n$,

$$|g(x-y) - g(x)| < \frac{\delta}{2\|\varphi\|_{L^1(\mathbb{R}^n)}}.$$

Without loss of generality, we may assume that $\int_{\mathbb{R}^n} \varphi(x)\,dx = 1$. Then $\int_{\mathbb{R}^n} \varphi_k(x)\,dx = 1$ for all $k \in \mathbb{Z}$. From this, we deduce that, for all $k \in \mathbb{Z}$ and $x \in \mathbb{R}^n$,

$$|g * \varphi_k(x) - g(x)| \leqslant \int_{\rho(y)<\eta} |g(x-y) - g(x)||\varphi_k(y)|\,dy + \int_{\rho(y)\geqslant\eta} |g(x-y) - g(x)||\varphi_k(y)|\,dy$$

$$< \frac{\delta}{2} + 2\|g\|_{L^\infty(\mathbb{R}^n)} \int_{\rho(y)\geqslant b^{-k}\eta} |\varphi(y)|\,dy. \tag{5.3}$$

On the other hand, by the integrability of φ, we know that there exists $\widetilde{k} \in \mathbb{Z}$ such that, for all $k \in [\widetilde{k}, \infty) \cap \mathbb{Z}$,

$$2\|g\|_{L^\infty(\mathbb{R}^n)} \int_{\rho(y)\geqslant b^{-k}\eta} |\varphi(y)|\,dy < \frac{\delta}{2},$$

which, combined with (5.3), implies that $\lim_{k\to-\infty} |g * \varphi_k(x) - g(x)| = 0$ holds true uniformly for all $x \in \mathbb{R}^n$. Therefore, $\|g * \varphi_k - g\|_{L^\infty(\mathbb{R}^n)} \to 0$ as $k \to -\infty$, which, together with Proposition 2.9 and Remark 2.10, further implies that

$$\|M_N(g * \varphi_k - g)\|_{L^\infty(\mathbb{R}^n)} \lesssim \|g * \varphi_k - g\|_{L^\infty(\mathbb{R}^n)} \to 0 \quad \text{as} \quad k \to -\infty. \tag{5.4}$$

For any given $\epsilon \in (0, \infty)$, there exists a continuous function g with compact support such that

$$\|f - g\|^2_{L^2(\mathbb{R}^n)} < \epsilon.$$

By (5.4) and [9, p. 39, Lemma 6.6], there exists a positive constant C_6 such that, for all $x \in \mathbb{R}^n$,

$$\begin{aligned}
&\limsup_{k\to-\infty} M_N(f * \varphi_k - f)(x) \\
&\quad\leqslant \sup_{k\in\mathbb{Z}} M_N((f - g) * \varphi_k)(x) + \limsup_{k\to-\infty} M_N(g * \varphi_k - g)(x) + M_N(g - f)(x) \\
&\quad\leqslant C_6 M_{N_{(p)}}(g - f)(x).
\end{aligned}$$

Therefore, by Proposition 2.9 and Remark 2.10 again, we find that there exists a positive constant C_7 such that, for any $\lambda \in (0, \infty)$,

$$\begin{aligned}
&\left|\left\{x \in \mathbb{R}^n : \limsup_{k\to-\infty} M_N(f * \varphi_k - f)(x) > \lambda\right\}\right| \\
&\quad\leqslant \left|\left\{x \in \mathbb{R}^n : M_{N_{(p)}}(g - f)(x) > \frac{\lambda}{C_6}\right\}\right| \leqslant C_7 \frac{\|f - g\|^2_{L^2(\mathbb{R}^n)}}{\lambda^2} \leqslant C_7 \frac{\epsilon}{\lambda^2},
\end{aligned}$$

which implies that (5.2) holds true for all $f \in H^{p,q}_A(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.

Assume now $f \in H^{p,q}_A(\mathbb{R}^n)$. By (i), we know that $H^{p,q}_A(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is dense in $H^{p,q}_A(\mathbb{R}^n)$. Thus, for any given $\epsilon \in (0, \infty)$, there exists a function $g \in H^{p,q}_A(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ such that

$$\|f - g\|^q_{H^{p,q}_A(\mathbb{R}^n)} < \epsilon.$$

Moreover, by [9, p. 39, Lemma 6.6] again and $f \in H^{p,q}_A(\mathbb{R}^n)$, we find that $\{f * \varphi_k\}_{k\in\mathbb{Z}}$ are uniformly bounded in $H^{p,q}_A(\mathbb{R}^n)$ and

$$\sup_{k\in\mathbb{Z}} \|(f - g) * \varphi_k\|_{H^{p,q}_A(\mathbb{R}^n)} \lesssim \|f - g\|_{H^{p,q}_A(\mathbb{R}^n)}.$$

Therefore, by (5.1) being true for all $f \in H^{p,q}_A(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, we further conclude that

$$\begin{aligned}
&\limsup_{k\to-\infty} \|f * \varphi_k - f\|^q_{H^{p,q}_A(\mathbb{R}^n)} \\
&\quad\leqslant \sup_{k\in\mathbb{Z}} \|(f - g) * \varphi_k\|^q_{H^{p,q}_A(\mathbb{R}^n)} + \limsup_{k\to-\infty} \|g * \varphi_k - g\|^q_{H^{p,q}_A(\mathbb{R}^n)} + \|g - f\|^q_{H^{p,q}_A(\mathbb{R}^n)} \\
&\quad\lesssim \|g - f\|^q_{H^{p,q}_A(\mathbb{R}^n)} \lesssim \epsilon.
\end{aligned}$$

This implies that the claim (5.1) holds true.

Notice that, if $f \in H^{p,r,s,q}_{A,\mathrm{fin}}(\mathbb{R}^n)$ and $\varphi \in C^\infty_c(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \varphi(x)\,dx \neq 0$, then, for all $k \in \mathbb{Z}$,

$$f * \varphi_k \in C^\infty_c(\mathbb{R}^n) \cap H^{p,q}_A(\mathbb{R}^n)$$

and, by (5.1),

$$f * \varphi_k \to f \quad \text{in} \quad H^{p,q}_A(\mathbb{R}^n) \quad \text{as} \quad k \to -\infty.$$

From this and the density of the set $H^{p,r,s,q}_{A,\mathrm{fin}}(\mathbb{R}^n)$ in $H^{p,q}_A(\mathbb{R}^n)$, we further deduce that $C^\infty_c(\mathbb{R}^n) \cap H^{p,q}_A(\mathbb{R}^n)$ is dense in $H^{p,q}_A(\mathbb{R}^n)$. This finishes the proof of (ii) and hence Lemma 5.2. □

The following conclusion is from Theorem 3.6 and its proof. We state it here for the later application.

Lemma 5.3. *If $p \in (0,1]$, $q \in (0,\infty]$, $r \in (1,\infty]$ and $s \in \mathbb{N}$ with $s \geqslant \lfloor (1/p-1)\ln b/\ln \lambda_-\rfloor$, then, for any $f \in H^{p,q}_A(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)$, there exist $\{\lambda^k_i\}_{k\in\mathbb{Z}, i\in\mathbb{N}} \subset \mathbb{C}$, $\{x^k_i\}_{k\in\mathbb{Z}, i\in\mathbb{N}} \subset \mathbb{R}^n$, balls $\{B_{\ell^k_i}\}_{k\in\mathbb{Z}, i\in\mathbb{N}}$ and (p,∞,s)-atoms $\{a^k_i\}_{k\in\mathbb{Z}, i\in\mathbb{N}}$ such that*

$$f = \sum_{k\in\mathbb{Z}} \sum_{i\in\mathbb{N}} \lambda^k_i a^k_i,$$

where the series converges almost everywhere and also converges in $\mathcal{S}'(\mathbb{R}^n)$,

$$\operatorname{supp} a^k_i \subset B_{\ell^k_i+4\tau}, \quad \Omega_k = \bigcup_{i\in\mathbb{N}} (x^k_i + B_{\ell^k_i+4\tau}) \quad \textit{for all } k \in \mathbb{Z} \textit{ and } i \in \mathbb{N}, \tag{5.5}$$

where

$$\Omega_k := \{x \in \mathbb{R}^n : M_N(f)(x) > 2^k\},$$

$$(x^k_i + B_{\ell^k_i-\tau}) \cap (x^k_j + B_{\ell^k_j-\tau}) = \emptyset \quad \textit{for all } k \in \mathbb{Z} \textit{ and } i, j \in \mathbb{N} \textit{ with } i \neq j, \tag{5.6}$$

$$\sharp\{j \in \mathbb{N} : (x^k_i + B_{\ell^k_i+4\tau}) \cap (x^k_j + B_{\ell^k_j+4\tau}) \neq \emptyset\} \leqslant L \quad \textit{for all} \quad i \in \mathbb{N}, \tag{5.7}$$

where L is a positive constant independent of Ω_k and f. Moreover, there exists a positive constant C, independent of f, such that, for all $k \in \mathbb{Z}$ and $i \in \mathbb{N}$,

$$|\lambda^k_i a^k_i| \leqslant C 2^k \quad \textit{almost everywhere} \tag{5.8}$$

and

$$\sum_{k\in\mathbb{Z}} \left(\sum_{i\in\mathbb{N}} |\lambda^k_i|^p \right)^{\frac{q}{p}} \leqslant C \|f\|^q_{H^{p,q}_A(\mathbb{R}^n)}. \tag{5.9}$$

Remark 5.4. For all $i \in \mathbb{N}$, $k \in \mathbb{Z}$ and $\ell \in [0,\infty)$, let ζ^k_i and $\mathcal{P}_\ell(\mathbb{R}^n)$ be the same as in the proof of Theorem 3.6. For any $f \in H^{p,q}_A(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)$, by an argument similar to that used in the proof of Theorem 3.6, we also find that there exists a unique polynomial $P^k_i \in \mathcal{P}_\ell(\mathbb{R}^n)$ such that, for all $Q \in \mathcal{P}_\ell(\mathbb{R}^n)$,

$$\langle f, Q\zeta^k_i \rangle = \langle P^k_i, Q\zeta^k_i \rangle = \int_{\mathbb{R}^n} P^k_i(x) Q(x) \zeta^k_i(x)\, dx. \tag{5.10}$$

Moreover, for any $i, j \in \mathbb{N}$ and $k \in \mathbb{Z}$, we let the polynomial $P^{k+1}_{i,j}$ be the orthogonal projection of $(f - P^{k+1}_j)\zeta^k_i$ on $\mathcal{P}_\ell(\mathbb{R}^n)$ with respect to the norm defined by (3.8), namely, $P^{k+1}_{i,j}$ is the unique element of $\mathcal{P}_\ell(\mathbb{R}^n)$ such that, for all $Q \in \mathcal{P}_\ell(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} [f(x) - P^{k+1}_j(x)] \zeta^k_i(x) Q(x) \zeta^{k+1}_j(x)\, dx = \int_{\mathbb{R}^n} P^{k+1}_{i,j}(x) Q(x) \zeta^{k+1}_j(x)\, dx \tag{5.11}$$

and, for all $i \in \mathbb{N}$ and $k \in \mathbb{Z}$,

$$\lambda^k_i a^k_i = (f - P^k_i)\zeta^k_i - \sum_{j\in\mathbb{N}} [(f - P^{k+1}_j)\zeta^k_i - P^{k+1}_{i,j}] \zeta^{k+1}_j. \tag{5.12}$$

Lemmas 5.5 and 5.6 are just [13, Lemmas 4.4 and 5.2], respectively; see also [9, p. 25, Lemma 5.3 and p. 36, Lemma 6.2].

Lemma 5.5. *There exists a positive constant C, independent of f, such that, for all $i\in\mathbb{N}$ and $k\in\mathbb{Z}$,*

$$\sup_{y\in\mathbb{R}^n}|P_i^k(y)\zeta_i^k(y)|\leqslant C\sup_{y\in U_i^k}M_N(f)(y)\leqslant C2^k,$$

where $U_i^k:=(x_i^k+B_{\ell_i^k+4\tau+1})\cap(\Omega_k)^{\complement}$.

Lemma 5.6. *There exists a positive constant C, independent of f, such that, for all i, $j\in\mathbb{N}$ and $k\in\mathbb{Z}$,*

$$\sup_{y\in\mathbb{R}^n}|P_{i,j}^{k+1}(y)\zeta_j^{k+1}(y)|\leqslant C\sup_{y\in\widetilde{U}_i^k}M_N(f)(y)\leqslant C2^{k+1},$$

where $\widetilde{U}_i^k:=(x_j^{k+1}+B_{\ell_j^{k+1}+4\tau+1})\cap(\Omega_{k+1})^{\complement}$.

The following Theorem 5.7 extends [53, Theorem 3.1 and Remark 3.3] to the setting of anisotropic Hardy-Lorentz spaces.

Theorem 5.7. *Let $q\in(0,\infty]$ and (p,r,s) be an admissible anisotropic triplet.*

(i) *If $r\in(1,\infty)$, then $\|\cdot\|_{H_{A,\mathrm{fin}}^{p,r,s,q}(\mathbb{R}^n)}$ and $\|\cdot\|_{H_A^{p,q}(\mathbb{R}^n)}$ are equivalent quasi-norms on $H_{A,\mathrm{fin}}^{p,r,s,q}(\mathbb{R}^n)$;*

(ii) *$\|\cdot\|_{H_{A,\mathrm{fin}}^{p,\infty,s,q}(\mathbb{R}^n)}$ and $\|\cdot\|_{H_A^{p,q}(\mathbb{R}^n)}$ are equivalent quasi-norms on $H_{A,\mathrm{fin}}^{p,\infty,s,q}(\mathbb{R}^n)\cap C(\mathbb{R}^n)$.*

Proof. Obviously, by Theorem 3.6, $H_{A,\mathrm{fin}}^{p,r,s,q}(\mathbb{R}^n)\subset H_A^{p,q}(\mathbb{R}^n)$ and, for all $f\in H_{A,\mathrm{fin}}^{p,r,s,q}(\mathbb{R}^n)$,

$$\|f\|_{H_A^{p,q}(\mathbb{R}^n)}\lesssim\|f\|_{H_{A,\mathrm{fin}}^{p,r,s,q}(\mathbb{R}^n)}.$$

Thus, we only need to prove that, for all $f\in H_{A,\mathrm{fin}}^{p,r,s,q}(\mathbb{R}^n)$ when $r\in(1,\infty)$ and for all $f\in[H_{A,\mathrm{fin}}^{p,r,s,q}(\mathbb{R}^n)\cap C(\mathbb{R}^n)]$ when $r=\infty$, $\|f\|_{H_{A,\mathrm{fin}}^{p,r,s,q}(\mathbb{R}^n)}\lesssim\|f\|_{H_A^{p,q}(\mathbb{R}^n)}$. We prove this by five steps.

Step 1. Let $r\in(1,\infty]$. In this case, without loss of generality, we may assume that $f\in H_{A,\mathrm{fin}}^{p,r,s,q}(\mathbb{R}^n)$ and $\|f\|_{H_A^{p,q}(\mathbb{R}^n)}=1$. Notice that f has compact support. Then there exists some $k_0\in\mathbb{Z}$ such that $\operatorname{supp}f\subset B_{k_0}$, where B_{k_0} is as in Section 2. For any $k\in\mathbb{Z}$, let

$$\Omega_k:=\{x\in\mathbb{R}^n:M_N(f)(x)>2^k\},$$

here and hereafter in this section, we let $N\equiv N_{(p)}$. Since $f\in H_A^{p,q}(\mathbb{R}^n)\cap L^{\widetilde{r}}(\mathbb{R}^n)$, where $\widetilde{r}:=r$ if $r\in(1,\infty)$ and $\widetilde{r}:=2$ if $r=\infty$, by Lemma 5.3, we know that there exist $\{\lambda_i^k\}_{k\in\mathbb{Z},\,i\in\mathbb{N}}\subset\mathbb{C}$ and a sequence of (p,∞,s)-atoms, $\{a_i^k\}_{k\in\mathbb{Z},\,i\in\mathbb{N}}$, such that $f=\sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}}\lambda_i^ka_i^k$ holds true almost everywhere and also in $\mathcal{S}'(\mathbb{R}^n)$ and, moreover, (5.5) through (5.9) of Lemma 5.3 also hold true.

Step 2. In this step, we prove that there exists a positive constant $\widetilde{C}$ such that, for all $x\in(B_{k_0+4\tau})^{\complement}$,

$$M_N(f)(x)\leqslant\widetilde{C}|B_{k_0}|^{-1/p}.\tag{5.13}$$

To this end, for any fixed $x\in(B_{k_0+4\tau})^{\complement}$, by Proposition 2.4, we have

$$M_N(f)(x)\lesssim M_N^0(f)(x)\lesssim\sup_{\phi\in\mathcal{S}_N(\mathbb{R}^n)}\sup_{k\in[k_0,\infty)\cap\mathbb{Z}}|f*\phi_k(x)|+\sup_{\phi\in\mathcal{S}_N(\mathbb{R}^n)}\sup_{k\in(-\infty,k_0)\cap\mathbb{Z}}\cdots=:\mathrm{I}_1+\mathrm{I}_2.$$

For I_1, assume that $\theta\in\mathcal{S}(\mathbb{R}^n)$ satisfyies that $\operatorname{supp}\theta\subset B_\tau$, $0\leqslant\theta\leqslant1$ and $\theta\equiv1$ on B_0. For $k\in[k_0,\infty)\cap\mathbb{Z}$, from $\operatorname{supp}f\subset B_{k_0}$, we deduce that

$$f*\phi_k(x)=\int_{\mathbb{R}^n}\phi_k(x-y)\theta(A^{-k_0}y)f(y)\,dy=:f*\varphi_{k_0}(\vec{0}_n),\tag{5.14}$$

where $\varphi(y):=b^{k_0-k}\phi(A^{-k}x+A^{k_0-k}y)\theta(-y)$ for all $y\in\mathbb{R}^n$. Noticing that, for any $\alpha\in\mathbb{Z}_+^n$ with $|\alpha|\leqslant N$, by (2.5), $\lambda_-\in(1,\infty)$, $k\in[k_0,\infty)\cap\mathbb{Z}$ and $\|\phi\|_{\mathcal{S}_N(\mathbb{R}^n)}\leqslant1$, we find that, for all $y\in\mathbb{R}^n$,

$$|\partial^\alpha[\phi(A^{k_0-k}\cdot)](y)|\lesssim(\lambda_-)^{(k_0-k)|\alpha|}\|\phi\|_{\mathcal{S}_N(\mathbb{R}^n)}\lesssim1,$$

which, combined with the product rule and $\operatorname{supp}\theta\subset B_\tau$, further implies that

$$\|\varphi\|_{\mathcal{S}_N(\mathbb{R}^n)}=\sup_{|\alpha|\leqslant N}\sup_{y\in B_\tau}|\partial_y^\alpha[\phi(A^{-k}x+A^{k_0-k}y)\theta(-y)]|[1+\rho(y)]^N\lesssim1.\tag{5.15}$$

Thus, noticing that $[\|\varphi\|_{\mathcal{S}_N(\mathbb{R}^n)}]^{-1}\varphi \in \mathcal{S}_N(\mathbb{R}^n)$ and, for all $z \in B_{k_0}$, $\vec{0}_n \in z+B_{k_0}$, by the definition of M_N, we know that, for all $z \in B_{k_0}$,

$$M_N(f)(z) \geqslant \sup_{u\in z+B_{k_0}} \left|\left(\frac{\varphi}{\|\varphi\|_{\mathcal{S}_N(\mathbb{R}^n)}}\right)_{k_0} * f(u)\right| \geqslant \frac{1}{\|\varphi\|_{\mathcal{S}_N(\mathbb{R}^n)}}|\varphi_{k_0} * f(\vec{0}_n)|. \tag{5.16}$$

Combining (5.14)–(5.16), we further conclude that, for all $x \in (B_{k_0+4\tau})^\complement$,

$$|f * \phi_k(x)| \leqslant \|\varphi\|_{\mathcal{S}_N(\mathbb{R}^n)} \inf_{z\in B_{k_0}} M_N(f)(z) \lesssim \inf_{z\in B_{k_0}} M_N(f)(z)$$

and hence $\mathrm{I}_1 \lesssim \inf_{z\in B_{k_0}} M_N(f)(z) =: C_8 \inf_{z\in B_{k_0}} M_N(f)(z)$. Let

$$\widetilde{\mathrm{I}} := \frac{\mathrm{I}_1 + C_8 \inf_{z\in B_{k_0}} M_N(f)(z)}{2}.$$

Then, it is easy to see that $\mathrm{I}_1 < \widetilde{\mathrm{I}} < C_8 \inf_{z\in B_{k_0}} M_N(f)(z)$. Therefore, we have

$$\begin{aligned}\|f\|_{H^{p,q}_A(\mathbb{R}^n)} \geqslant \|f\|_{H^{p,\infty}_A(\mathbb{R}^n)} &\gtrsim \sup_{\lambda\in(0,\infty)} \lambda|\{z \in B_{k_0} : C_8 M_N(f)(z) > \lambda\}|^{\frac{1}{p}}\\ &\gtrsim \mathrm{I}_1|\{z \in B_{k_0} : \widetilde{\mathrm{I}} > \mathrm{I}_1\}|^{\frac{1}{p}} \sim \mathrm{I}_1|B_{k_0}|^{\frac{1}{p}},\end{aligned}$$

which, together with $\|f\|_{H^{p,q}_A(\mathbb{R}^n)} = 1$, further implies that $\mathrm{I}_1 \lesssim |B_{k_0}|^{-\frac{1}{p}}$.

For I_2, by $\operatorname{supp} f \subset B_{k_0}$ and $\theta \equiv 1$ on B_0, we find that, for all $k \in (-\infty, k_0) \cap \mathbb{Z}$, $x \in (B_{k_0+4\tau})^\complement$ and $z \in B_{k_0}$,

$$f * \phi_k(x) = \int_{\mathbb{R}^n} \phi_k(x-y)\theta(A^{-k_0}y)f(y)\,dy =: f * \psi_k(z),$$

where $\psi(u) := \phi(A^{-k}(x-z)+u)\theta(A^{-k_0}z - A^{k-k_0}u)$ for all $u \in \mathbb{R}^n$. Notice that, if $u \in \operatorname{supp}\psi$, then $A^{-k_0}z - A^{k-k_0}u \in B_\tau$ and hence $u \in B_{k_0-k+2\tau}$. Therefore, by (2.7) and (2.8), we have

$$\begin{aligned}A^{-k}(x-z)+u &\in (B_{k_0-k+4\tau})^\complement + B_{k_0-k} + B_{k_0-k+2\tau}\\ &\subset (B_{k_0-k+4\tau})^\complement + B_{k_0-k+3\tau} \subset (B_{k_0-k+3\tau})^\complement,\end{aligned}$$

which implies that $\rho(A^{-k}(x-z)+u) \geqslant b^{k_0-k+3\tau}$. From this, (2.5), $\lambda_- \in (1,\infty)$, $k \in (-\infty, k_0) \cap \mathbb{Z}$ and $\phi \in \mathcal{S}_N(\mathbb{R}^n)$, we further deduce that

$$\|\psi\|_{\mathcal{S}_N(\mathbb{R}^n)} \lesssim \sup_{|\alpha|\leqslant N} \sup_{u\in\operatorname{supp}\psi} (\lambda_-)^{(k-k_0)|\alpha|}\left[\frac{1+\rho(u)}{1+\rho(A^{-k}(x-z)+u)}\right]^N \lesssim 1.$$

Thus, by an argument similar to that used for I_1, we have

$$\mathrm{I}_2 \lesssim \inf_{z\in B_{k_0}} M_N(f)(z) \lesssim |B_{k_0}|^{-\frac{1}{p}}.$$

Combining the above estimates of I_1 and I_2, we show that (5.13) holds true.

Step 3. We now denote by $\widetilde{k}$ the largest integer k such that $2^k < \widetilde{C}|B_{k_0}|^{-\frac{1}{p}}$, where $\widetilde{C}$ is the same as in (5.13). Then, by (5.13), we have

$$\Omega_k \subset B_{k_0+4\tau} \quad \text{for all} \quad k \in (\widetilde{k}, \infty] \cap \mathbb{Z}. \tag{5.17}$$

Let

$$h := \sum_{k=-\infty}^{\widetilde{k}} \sum_{i\in\mathbb{N}} \lambda_i^k a_i^k \quad \text{and} \quad \ell := \sum_{k=\widetilde{k}+1}^{\infty} \sum_{i\in\mathbb{N}} \lambda_i^k a_i^k,$$

where the series converge almost everywhere and also in $\mathcal{S}'(\mathbb{R}^n)$. Clearly, $f = h + \ell$. In what follows of this step, we show that h is a constant multiple of a (p,∞,s)-atom with the constant independent of f. To this end, observe that $\operatorname{supp}\ell \subset \bigcup_{k=\widetilde{k}+1}^{\infty}\Omega_k \subset B_{k_0+4\tau}$, which, combined with $\operatorname{supp} f \subset B_{k_0+4\tau}$, further implies that $\operatorname{supp} h \subset B_{k_0+4\tau}$.

Notice that, for any $r \in (1,\infty]$ and $r_1 \in (1,r)$, by Hölder's inequality, we have

$$\int_{\mathbb{R}^n} |f(x)|^{r_1}\,dx \leqslant |B_{k_0}|^{1-\frac{r_1}{r}}\|f\|_{L^r(\mathbb{R}^n)}^{r_1} < \infty.$$

Observing that $\operatorname{supp} f \subset B_{k_0}$ and f has vanishing moments up to order s, we know that f is a constant multiple of a $(1,r_1,0)$-atom and therefore, by Lemma 3.4, $M_N(f) \in L^1(\mathbb{R}^n)$. Then, by (5.7), (5.5), (5.17) and (5.8), we have

$$\int_{\mathbb{R}^n} \sum_{k=\widetilde{k}+1}^{\infty}\sum_{i\in\mathbb{N}} |\lambda_i^k a_i^k(x)x^\alpha|\,dx \lesssim \sum_{k\in\mathbb{Z}} 2^k|\Omega_k| \lesssim \|M_N(f)\|_{L^1(\mathbb{R}^n)} < \infty.$$

This, together with the vanishing moments of a_i^k, implies that ℓ has vanishing moments up to s and hence so does h by $h = f - \ell$. Moreover, by (5.7), (5.8) and the fact that $2^{\widetilde{k}} < \widetilde{C}|B_{k_0}|^{-\frac{1}{p}}$, we find that, for all $x \in \mathbb{R}^n$,

$$|h(x)| \lesssim \sum_{k=-\infty}^{\widetilde{k}} 2^k \lesssim |B_{k_0}|^{-\frac{1}{p}}.$$

Thus, there exists a positive constant C_9, independent of f, such that h/C_9 is a (p,∞,s)-atom and, by Definition 3.1, it is also a (p,r,s)-atom for any admissible anisotropic triplet (p,r,s).

Step 4. In this step, we show (i). To this end, assume that $r \in (1,\infty)$. We first show that

$$\sum_{k=\widetilde{k}+1}^{\infty}\sum_{i\in\mathbb{N}} \lambda_i^k a_i^k \in L^r(\mathbb{R}^n).$$

For all $x \in \mathbb{R}^n$, since $\mathbb{R}^n = \bigcup_{j\in\mathbb{Z}}(\Omega_j\backslash\Omega_{j+1})$, it follows that there exists a $j_0 \in \mathbb{Z}$ such that $x \in (\Omega_{j_0}\backslash\Omega_{j_0+1})$. Notice that $\operatorname{supp} a_i^k \subset B_{\ell_i^k+\tau} \subset \Omega_k \subset \Omega_{j_0+1}$ for all $k \in (j_0,\infty)\cap\mathbb{Z}$, using (5.7) and (5.8), we conclude that, for all $x \in (\Omega_{j_0}\setminus\Omega_{j_0+1})$,

$$\sum_{k=\widetilde{k}+1}^{\infty}\sum_{i\in\mathbb{N}} |\lambda_i^k a_i^k(x)| \lesssim \sum_{k\leqslant j_0} 2^k \lesssim 2^{j_0} \lesssim M_N(f)(x).$$

Since $f \in L^r(\mathbb{R}^n)$, from Proposition 2.9 and Remark 2.10, it follows that $M_N(f) \in L^r(\mathbb{R}^n)$. Thus, by the Lebesgue dominated convergence theorem, we further have $\sum_{k=\widetilde{k}+1}^{K}\sum_{i\in\mathbb{N}}\lambda_i^k a_i^k$ converges to ℓ in $L^r(\mathbb{R}^n)$ as $K \geqslant \widetilde{k}+1$ and $K \to \infty$.

Now, for any positive integer $K > \widetilde{k}$ and $k \in [\widetilde{k}+1,K]\cap\mathbb{Z}$, let

$$I_{(K,k)} := \{i \in \mathbb{N} : |i| + |k| \leqslant K\} \quad \text{and} \quad \ell_{(K)} := \sum_{k=\widetilde{k}+1}^{K}\sum_{i\in I_{(K,k)}} \lambda_i^k a_i^k.$$

Since $\ell \in L^r(\mathbb{R}^n)$, it follows that, for any given $\epsilon \in (0,1)$, there exists $K \in [\widetilde{k}+1,\infty)\cap\mathbb{Z}$ large enough, depending on ϵ, such that $(\ell-\ell_{(K)})/\epsilon$ is a (p,r,s)-atom. Therefore, $f = h + \ell_{(K)} + (\ell - \ell_{(K)})$ is a finite linear combination of (p,r,s)-atoms. By Step 3 and (5.9), we further conclude that

$$\|f\|^q_{H^{p,r,s,q}_{A,\mathrm{fin}}(\mathbb{R}^n)} \lesssim (C_0)^q + \sum_{k=\widetilde{k}+1}^{K}\left(\sum_{i\in I_{(K,k)}} |\lambda_i^k|^p\right)^{\frac{q}{p}} + \epsilon^q \lesssim 1,$$

which completes the proof of (i).

Step 5. In this step, we show (ii). To this end, assume that f is a continuous function in $H^{p,\infty,s,q}_{A,\mathrm{fin}}(\mathbb{R}^n)$. Then a_i^k is also continuous due to its construction (see also (3.15)). Since

$$M_N(f)(x) \leqslant C_{(n,N)}\|f\|_{L^\infty(\mathbb{R}^n)} \quad \text{for all} \quad x \in \mathbb{R}^n,$$

where $C_{(n,N)}$ is a positive constant only depending on n and N, it follows that the level set Ω_k is empty for all k satisfying that

$$2^k \geqslant C_{(n,N)}\|f\|_{L^\infty(\mathbb{R}^n)}. \tag{5.18}$$

Let $\widehat{k}$ be the largest integer for which (5.18) does not hold true. Then the index k in the sum defining ℓ runs only over $k \in \{\widetilde{k}+1,\dots,\widehat{k}\}$.

Let $\epsilon \in (0,\infty)$. Since f is uniformly continuous, it follows that there exists a $\delta \in (0,\infty)$ such that $|f(x)-f(y)|<\epsilon$ whenever $\rho(x-y)<\delta$. Write $\ell = \ell_1^\epsilon + \ell_2^\epsilon$ with

$$\ell_1^\epsilon := \sum_{k=\widetilde{k}+1}^{\widehat{k}} \sum_{i\in F_1^{(k,\delta)}} \lambda_i^k a_i^k \quad \text{and} \quad \ell_2^\epsilon := \sum_{k=\widetilde{k}+1}^{\widehat{k}} \sum_{i\in F_2^{(k,\delta)}} \lambda_i^k a_i^k,$$

where, for $k \in \{\widetilde{k}+1,\dots,\widehat{k}\}$,

$$F_1^{(k,\delta)} := \{i\in\mathbb{N} : b^{\ell_i^k+\tau} \geqslant \delta\} \quad \text{and} \quad F_2^{(k,\delta)} := \{i\in\mathbb{N} : b^{\ell_i^k+\tau} < \delta\}.$$

Notice that, for any fixed $k \in \{\widetilde{k}+1,\dots,\widehat{k}\}$, by (5.6) and (5.17), we know that $F_1^{(k,\delta)}$ is a finite set and hence ℓ_1^ϵ is continuous.

On the other hand, for any $k \in \{\widetilde{k}+1,\dots,\widehat{k}\}$, $i\in\mathbb{N}$ such that $b^{\ell_i^k+\tau}<\delta$ and $x \in x_i^k + B_{\ell_i^k+\tau}$, $|f(x)-f(x_i^k)|<\epsilon$. By (5.10) and $\operatorname{supp}\zeta_i^k \subset x_i^k + B_{\ell_i^k+\tau}$, we find that, for all $Q\in\mathcal{P}_\ell(\mathbb{R}^n)$,

$$\frac{1}{\int_{\mathbb{R}^n}\zeta_i^k(x)\,dx}\int_{\mathbb{R}^n}[\widetilde{f}(x)-\widetilde{P}_i^k(x)]Q(x)\zeta_i^k(x)\,dx = 0,$$

where, for all $x\in\mathbb{R}^n$,

$$\widetilde{f}(x) := [f(x)-f(x_i^k)]\chi_{\{x_i^k+B_{\ell_i^k+\tau}\}}(x) \quad \text{and} \quad \widetilde{P}_i^k(x) := P_i^k(x) - f(x_i^k).$$

Since $|\widetilde{f}(x)|<\epsilon$ for all $x\in\mathbb{R}^n$ implies $M_N(\widetilde{f})(x)\lesssim\epsilon$ for all $x\in\mathbb{R}^n$, from Lemma 5.5, it follows that

$$\sup_{y\in\mathbb{R}^n}|\widetilde{P}_i^k(y)\zeta_i^k(y)| \lesssim \sup_{y\in\mathbb{R}^n} M_N(\widetilde{f})(y) \lesssim \epsilon. \tag{5.19}$$

Similar to Remark 5.4, for all $k \in \{\widetilde{k}+1,\dots,\widehat{k}\}$, $i\in F_2^{(k,\delta)}$ and $j\in\mathbb{N}$, let $\widetilde{P}_{i,j}^{k+1}$ be the orthogonal projection of $(\widetilde{f}-\widetilde{P}_j^{k+1})\zeta_i^k$ on $\mathcal{P}_\ell(\mathbb{R}^n)$ with respect to the norm defined by (3.8), then, for all $Q\in\mathcal{P}_\ell(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n}[\widetilde{f}(x)-\widetilde{P}_j^{k+1}(x)]\zeta_i^k(x)Q(x)\zeta_j^{k+1}(x)\,dx = \int_{\mathbb{R}^n}\widetilde{P}_{i,j}^{k+1}(x)Q(x)\zeta_j^{k+1}(x)\,dx. \tag{5.20}$$

By $\operatorname{supp}\zeta_i^k \subset x_i^k + B_{\ell_i^k+\tau}$, we have $[\widetilde{f}-\widetilde{P}_j^{k+1}]\zeta_i^k = [f-P_j^{k+1}]\zeta_i^k$. From this, (5.11) and (5.20), we further deduce that $\widetilde{P}_{i,j}^{k+1} = P_{i,j}^{k+1}$. Then, by Lemma 5.6, we find that

$$\sup_{y\in\mathbb{R}^n}|\widetilde{P}_{i,j}^{k+1}(y)\zeta_j^{k+1}(y)| \lesssim \sup_{y\in\mathbb{R}^n} M_N(\widetilde{f})(y) \lesssim \epsilon. \tag{5.21}$$

Furthermore, by (5.12) and $\sum_{j\in\mathbb{N}}\zeta_j^{k+1} = \chi_{\Omega_{k+1}}$, we have

$$\begin{aligned}\lambda_i^k a_i^k &= (f-P_i^k)\zeta_i^k - \sum_{j\in\mathbb{N}}[(f-P_j^{k+1})\zeta_i^k - P_{i,j}^{k+1}]\zeta_j^{k+1}\\ &= \zeta_i^k\widetilde{f}\chi_{\Omega_{k+1}^\complement} - \widetilde{P}_i^k\zeta_i^k + \zeta_i^k\sum_{j\in\mathbb{N}}\widetilde{P}_j^{k+1}\zeta_j^{k+1} + \sum_{j\in\mathbb{N}}\widetilde{P}_{i,j}^{k+1}\zeta_j^{k+1},\end{aligned}$$

which, combined with (5.19), (5.21) and [9, p. 35, Lemma 6.1(ii)], further implies that $|\lambda_i^k a_i^k(x)|\lesssim\epsilon$ for all $k \in \{\widetilde{k}+1,\dots,\widehat{k}\}$, $i\in F_2^{(k,\delta)}$ and $x\in x_i^k+B_{\ell_i^k+\tau}$.

Moreover, using (5.5) and (5.7), we conclude that there exists a positive constant C_{10}, independent of f, such that

$$|\ell_2^\epsilon| \leqslant C_{10} \sum_{k=\widetilde{k}+1}^{\widehat{k}} \epsilon = C_{10}(\widehat{k}-\widetilde{k})\epsilon. \tag{5.22}$$

Since ϵ is arbitrary, we hence split ℓ into a continuous part and a part which is uniformly arbitrarily small. This fact implies that ℓ is continuous. Therefore, $h = f - \ell$ is a C_9 multiple of a continuous (p,∞,s)-atom by Step 3.

Now we give a finite decomposition of f. To this end, we use again the splitting $\ell := \ell_1^\epsilon + \ell_2^\epsilon$. Obviously, for any $\epsilon \in (0,\infty)$, ℓ_1^ϵ is a finite linear combination of continuous (p,∞,s)-atoms and, by (5.9), we have

$$\sum_{k=\widetilde{k}+1}^{\widehat{k}} \left(\sum_{i \in F_1^{(k,\delta)}} |\lambda_i^k|^p \right)^{\frac{q}{p}} \lesssim \|f\|^q_{H_A^{p,q}(\mathbb{R}^n)}. \tag{5.23}$$

Observe that ℓ and ℓ_1^ϵ are continuous and have vanishing moments up to s and hence so does ℓ_2^ϵ. Moreover, $\operatorname{supp} \ell_2^\epsilon \subset B_{k_0+4\tau}$ and $\|\ell_2^\epsilon\|_{L^\infty(\mathbb{R}^n)} \leqslant C_{10}(\widehat{k}-\widetilde{k})\epsilon$ by (5.22). Therefore, we choose ϵ small enough such that ℓ_2^ϵ becomes an arbitrarily small multiple of a continuous (p,∞,s)-atom. Indeed, $\ell_2^\epsilon = \lambda^\epsilon a^\epsilon$, where

$$\lambda^\epsilon := C_{10}(\widehat{k}-\widetilde{k})\epsilon |B_{k_0+4\tau}|^{1/p}$$

and a^ϵ is a continuous (p,∞,s)-atom. Thus, $f = h + \ell_1^\epsilon + \ell_2^\epsilon$ gives the desired finite atomic decomposition of f. Then, by (5.23) and the fact that h/C_9 is a (p,∞,s)-atom, we have

$$\|f\|_{H^{p,\infty,s,q}_{A,\mathrm{fin}}(\mathbb{R}^n)} \leqslant \|h\|_{H^{p,\infty,s,q}_{A,\mathrm{fin}}(\mathbb{R}^n)} + \|\ell_1^\epsilon\|_{H^{p,\infty,s,q}_{A,\mathrm{fin}}(\mathbb{R}^n)} + \|\ell_2^\epsilon\|_{H^{p,\infty,s,q}_{A,\mathrm{fin}}(\mathbb{R}^n)} \lesssim 1.$$

This finishes the proof of (ii) and hence Theorem 5.7. □

6 Some applications

In this section, we give some applications. In Subsection 6.1, we consider the interpolation properties of the anisotropic Hardy-Lorentz space $H_A^{p,q}(\mathbb{R}^n)$ via the real method. In Subsection 6.2, we first obtain the boundedness of the δ-type Calderón-Zygmund operators from $H_A^p(\mathbb{R}^n)$ to $L^{p,\infty}(\mathbb{R}^n)$ (or to $H_A^{p,\infty}(\mathbb{R}^n)$) in the critical case. Then we prove that some Calderón-Zygmund operators are bounded from $H_A^{p,q}(\mathbb{R}^n)$ to $L^{p,\infty}(\mathbb{R}^n)$. In addition, as an application of the finite atomic decomposition characterizations of $H_A^{p,q}(\mathbb{R}^n)$ in Theorem 5.7, we establish a criterion for the boundedness of sublinear operators from $H_A^{p,q}(\mathbb{R}^n)$ into a quasi-Banach space, which is of independent interest. Moreover, using this criterion, we further obtain the boundedness of the δ-type Calderón-Zygmund operators from $H_A^{p,q}(\mathbb{R}^n)$ to $L^{p,q}(\mathbb{R}^n)$ (or to $H_A^{p,q}(\mathbb{R}^n)$).

6.1 Interpolation of $H_A^{p,q}(\mathbb{R}^n)$

In this subsection, as an application of the atomic decomposition for the anisotropic Hardy-Lorentz space $H_A^{p,q}(\mathbb{R}^n)$, we prove the real interpolation properties on $H_A^{p,q}(\mathbb{R}^n)$ (see Theorem 6.1 below), whose isotropic version includes [1, Theorem 2.5] as a special case (see Remark 6.7(ii) below).

We first recall some basic notions about the theory of real interpolation. Assume that (X_1, X_2) is a compatible couple of quasi-normed spaces, namely, X_1 and X_2 are two quasi-normed linear spaces which are continuously embedded in some larger topological vector space. Let

$$X_1 + X_2 := \{f_1 + f_2 : f_1 \in X_1,\ f_2 \in X_2\}.$$

For $t \in (0,\infty]$, the Peetre K-functional on $X_1 + X_2$ is defined as

$$K(t,f;X_1,X_2) := \inf\{\|f_1\|_{X_1} + t\|f_2\|_{X_2} : f = f_1 + f_2,\ f_1 \in X_1 \text{ and } f_2 \in X_2\}.$$

Moreover, for $\theta \in (0,1)$ and $q \in (0,\infty]$, the interpolation space $(X_1,\, X_2)_{\theta,q}$ is defined as

$$(X_1,\, X_2)_{\theta,q} := \left\{ f \in X_1 + X_2 : \|f\|_{\theta,q} := \left(\int_0^\infty [t^{-\theta} K(t,f;X_1,X_2)]^q \,\frac{dt}{t} \right)^{1/q} < \infty \right\}.$$

It is well known that

$$(L^{q_1}(\mathbb{R}^n), L^{q_2}(\mathbb{R}^n))_{\theta,q} = L^q(\mathbb{R}^n) \quad \text{and} \quad (L^{p,q_1}(\mathbb{R}^n), L^{p,q_2}(\mathbb{R}^n))_{\theta,q} = L^{p,q}(\mathbb{R}^n),$$

where $p \in (0,\infty)$, $0 < q_1 \leqslant q_2 \leqslant \infty$, $q \in [q_1, q_2]$ and $\theta \in (0,1)$ satisfy that $1/q = (1-\theta)/q_1 + \theta/q_2$ (see [8]).

The main result of this subsection is the following real interpolation properties of $H_A^{p,q}(\mathbb{R}^n)$.

Theorem 6.1. *Let $p \in (0,\infty)$ and $q_1,\, q,\, q_2 \in (0,\infty]$.*

(i) *If $p_1,\, p_2 \in (0,\infty)$, $p_1 \neq p_2$, $\theta \in (0,1)$ and $1/p = (1-\theta)/p_1 + \theta/p_2$, then*

$$(H_A^{p_1,q_1}(\mathbb{R}^n), H_A^{p_2,q_2}(\mathbb{R}^n))_{\theta,q} = H_A^{p,q}(\mathbb{R}^n). \tag{6.1}$$

(ii) *If $\theta \in (0,1)$ and $1/q = (1-\theta)/q_1 + \theta/q_2$, then*

$$(H_A^{p,q_1}(\mathbb{R}^n), H_A^{p,q_2}(\mathbb{R}^n))_{\theta,q} = H_A^{p,q}(\mathbb{R}^n). \tag{6.2}$$

In order to prove Theorem 6.1, we need the following technical lemma on the decomposition of a function into its "good" and "bad" parts, whose proof is similar to the proof of [9, Lemmas 5.7 and 5.10(ii)], the details being omitted.

Lemma 6.2. *Let $p \in (0,1]$, $N \in [N_{(p)},\infty) \cap \mathbb{Z}$, $f \in C_c^\infty(\mathbb{R}^n)$, $\lambda \in (0,\infty)$ and*

$$\Omega_\lambda := \{x \in \mathbb{R}^n : M_N(f)(x) > \lambda\}.$$

Then there exist two functions g_λ and b_λ such that $f = g_\lambda + b_\lambda$ and

$$\|g_\lambda\|_{L^\infty(\mathbb{R}^n)} \leqslant C_{11}\lambda, \quad \|b_\lambda\|^p_{H_A^p(\mathbb{R}^n)} \leqslant C_{12} \int_{\Omega_\lambda} [M_N(f)(x)]^p \,dx,$$

where C_{11} and C_{12} are positive constants independent of f and λ, and $H_A^p(\mathbb{R}^n)$ is the anisotropic Hardy space introduced in [9].

By Lemma 6.2 and an argument parallel to the proof of [29, Theorem 1], we obtain the following real interpolation properties, the details being omitting.

Lemma 6.3. *Assume that $q \in (0,\infty]$, $p_0 \in (0,1]$, $\theta \in (0,1)$ and $1/p = (1-\theta)/p_0$. Then*

$$(H_A^{p_0}(\mathbb{R}^n), L^\infty(\mathbb{R}^n))_{\theta,q} = H_A^{p,q}(\mathbb{R}^n). \tag{6.3}$$

Remark 6.4. If $A := d\,\mathrm{I}_{n\times n}$ for some $d \in \mathbb{R}$ with $|d| \in (1,\infty)$, then $H_A^{p_0}(\mathbb{R}^n)$ and $H_A^{p,q}(\mathbb{R}^n)$ in Lemma 6.3 become the classical isotropic Hardy and Hardy-Lorentz spaces, respectively. In this case, if $q \in (0,\infty]$, $p_0 \in (0,1]$, $\theta \in (0,1)$, and $1/p = (1-\theta)/p_0$, then

$$(H^{p_0}(\mathbb{R}^n), L^\infty(\mathbb{R}^n))_{\theta,q} = H^{p,q}(\mathbb{R}^n),$$

which is just [29, Theorem 1].

Now we employ Lemma 6.3 to prove Theorem 6.1(i).

Proof of Theorem 6.1(i). Indeed, if $p_1,\, p_2 \in (0,\infty)$ and $p_1 \neq p_2$, then there exist $r \in (0, \min\{p_1,p_2,1\})$ and $\eta_1,\, \eta_2 \in (0,1)$ such that $1/p_i = (1-\eta_i)/r$, $i \in \{1,2\}$. Let $\eta := (1-\theta)\eta_1 + \theta\eta_2$. Noticing that

$$1/p = (1-\theta)/p_1 + \theta/p_2 = (1-\eta)/r,$$

by Lemma 6.3 and the reiteration theorem (see, for example, [55, Theorem 2]), we know that

$$\begin{aligned}(H_A^{p_1,q_1}(\mathbb{R}^n), H_A^{p_2,q_2}(\mathbb{R}^n))_{\theta,q} &= ((H_A^r(\mathbb{R}^n), L^\infty(\mathbb{R}^n))_{\eta_1,q_1}, (H_A^r(\mathbb{R}^n), L^\infty(\mathbb{R}^n))_{\eta_2,q_2})_{\theta,q}\\ &= (H_A^r(\mathbb{R}^n), L^\infty(\mathbb{R}^n))_{\eta,q} = H_A^{p,q}(\mathbb{R}^n),\end{aligned}$$

which is the desired conclusion (6.1). This finishes the proof of Theorem 6.1(i). □

As an immediate consequence of Theorem 6.1(i), we easily know that the anisotropic Hardy-Lorentz space $H_A^{p,q}(\mathbb{R}^n)$ serves as a median space between two anisotropic Hardy spaces via the real method, which is the following Corollary 6.5.

Corollary 6.5. *Assume that* $q \in (0,\infty]$, p, p_1, $p_2 \in (0,\infty)$, $p_1 \neq p_2$, $\theta \in (0,1)$ *and* $1/p = (1-\theta)/p_1 + \theta/p_2$. *Then*

$$(H_A^{p_1}(\mathbb{R}^n), H_A^{p_2}(\mathbb{R}^n))_{\theta,q} = H_A^{p,q}(\mathbb{R}^n).$$

Remark 6.6. (i) If $A := d\,\mathrm{I}_{n\times n}$ for some $d \in \mathbb{R}$ with $|d| \in (1,\infty)$, then $H_A^{p_1}(\mathbb{R}^n)$, $H_A^{p_2}(\mathbb{R}^n)$ and $H_A^{p,q}(\mathbb{R}^n)$ in Corollary 6.5 become the classical isotropic Hardy and Hardy-Lorentz spaces, respectively. In this case, if $q \in (0,\infty]$, p, p_1, $p_2 \in (0,\infty)$, $p_1 \neq p_2$, $\theta \in (0,1)$ and $1/p = (1-\theta)/p_1 + \theta/p_2$, then, by Corollary 6.5, we have

$$(H^{p_1}(\mathbb{R}^n), H^{p_2}(\mathbb{R}^n))_{\theta,q} = H^{p,q}(\mathbb{R}^n).$$

In particular,

$$(H^{p_1}(\mathbb{R}^n), H^{p_2}(\mathbb{R}^n))_{\theta,p} = H^{p}(\mathbb{R}^n),$$

provided that $\theta \in (0,1)$ and $1/p = (1-\theta)/p_1 + \theta/p_2$.

(ii) If $p \in (1,\infty)$ and $q \in (0,\infty]$, then $H_A^{p,q}(\mathbb{R}^n) = L^{p,q}(\mathbb{R}^n)$. Indeed, for any $p \in (1,\infty)$, there exist p_1, $p_2 \in (1,\infty)$, $p_1 \neq p_2$ and $\theta \in (0,1)$ such that $1/p = (1-\theta)/p_1 + \theta/p_2$. From this, Corollary 6.5, $H_A^r(\mathbb{R}^n) = L^r(\mathbb{R}^n)$ for all $r \in (1,\infty)$ (see [9, p. 16, Remark]) and the corresponding interpolation result of Lorentz spaces (see, for example, [55, Theorem 3]), we deduce that

$$H_A^{p,q}(\mathbb{R}^n) = (H_A^{p_1}(\mathbb{R}^n), H_A^{p_2}(\mathbb{R}^n))_{\theta,q} = (L^{p_1}(\mathbb{R}^n), L^{p_2}(\mathbb{R}^n))_{\theta,q} = L^{p,q}(\mathbb{R}^n).$$

Now we turn to prove Theorem 6.1(ii) via Remark 6.6(ii).

Proof of Theorem 6.1(ii). To show Theorem 6.1(ii), we consider two cases. If $p \in (0,1]$, by a proof similar to that of [1, Theorem 2.5], we easily obtain the desired conclusion (6.2). If $p \in (1,\infty)$, by Remark 6.6(ii) and the interpolation properties of Lorentz spaces (see, for example, [8, Theorem 5.3.1]), we find that (6.2) holds true. This finishes the proof of Theorem 6.1(ii) and hence Theorem 6.1. □

Remark 6.7. (i) If $A := d\,\mathrm{I}_{n\times n}$ for some $d \in \mathbb{R}$ with $|d| \in (1,\infty)$, then $H_A^{p_i,q_i}(\mathbb{R}^n)$, $H_A^{p,q_i}(\mathbb{R}^n)$, $i \in \{1,2\}$, and $H_A^{p,q}(\mathbb{R}^n)$ in Theorem 6.1 become the classical isotropic Hardy-Lorentz spaces. In this case, by Theorem 6.1(i), we know that

$$(H^{p_1,q_1}(\mathbb{R}^n), H^{p_2,q_2}(\mathbb{R}^n))_{\theta,q} = H^{p,q}(\mathbb{R}^n),$$

provided that q_1, q, $q_2 \in (0,\infty]$, p_1, p, $p_2 \in (0,\infty)$, $p_1 \neq p_2$, $\theta \in (0,1)$ and $1/p = (1-\theta)/p_1 + \theta/p_2$, which is a well-known interpolation result for classical isotropic Hardy-Lorentz spaces (see [29, p. 75, (2)]). In addition, by Theorem 6.1(ii), we have

$$(H^{p,q_1}(\mathbb{R}^n), H^{p,q_2}(\mathbb{R}^n))_{\theta,q} = H^{p,q}(\mathbb{R}^n),$$

provided that $p \in (0,\infty)$, $\theta \in (0,1)$ and q_1, q, $q_2 \in (0,\infty]$ satisfy that $1/q = (1-\theta)/q_1 + \theta/q_2$, which generalizes [1, Theorem 2.5].

(ii) Lemma 6.3 also holds true for all $p_0 \in (1,\infty)$ and $q \in (0,\infty]$. Indeed, notice that, if $p_0 \in (1,\infty)$, then $p \in (1,\infty)$. Thus, by Remark 6.6(ii), we have

$$H_A^{p_0}(\mathbb{R}^n) = L^{p_0}(\mathbb{R}^n) \quad \text{and} \quad H_A^{p,q}(\mathbb{R}^n) = L^{p,q}(\mathbb{R}^n).$$

From this and the fact that, for all $q \in (0,\infty]$,

$$(L^{p_0}(\mathbb{R}^n), L^{\infty}(\mathbb{R}^n))_{\theta,q} = L^{p,q}(\mathbb{R}^n) \quad \text{with} \quad \frac{1}{p} = \frac{1-\theta}{p_0} \quad \text{and} \quad \theta \in (0,1)$$

(see [8, Theorem 5.3.1]), we further deduce that (6.3) holds true for all $p_0 \in (1,\infty)$ and $q \in (0,\infty]$.

6.2 Boundedness of Calderón-Zygmund operators

As another application of the atomic decomposition for $H^{p,q}_A(\mathbb{R}^n)$, in this subsection, we first obtain the boundedness of the δ-type Calderón-Zygmund operators from $H^p_A(\mathbb{R}^n)$ to $L^{p,\infty}(\mathbb{R}^n)$ (or to $H^{p,\infty}_A(\mathbb{R}^n)$) in the critical case (see Theorem 6.8 and Remark 6.10 below). As the third application of Theorem 3.6, we also prove that some Calderón-Zygmund operators are bounded from $H^{p,q}_A(\mathbb{R}^n)$ to $L^{p,\infty}(\mathbb{R}^n)$ (see Theorem 6.11 below). This application is a generalization of [1, Theorem 2.2] in the present setting. In addition, as an application of the finite atomic decomposition characterizations for $H^{p,q}_A(\mathbb{R}^n)$ in Theorem 5.7, we establish a criterion for the boundedness of sublinear operators from $H^{p,q}_A(\mathbb{R}^n)$ into a quasi-Banach space (see Theorem 6.13 below), which is of independent interest. Moreover, using this criterion, we further obtain the boundedness of the δ-type Calderón-Zygmund operators from $H^{p,q}_A(\mathbb{R}^n)$ to $L^{p,q}(\mathbb{R}^n)$ (or to $H^{p,q}_A(\mathbb{R}^n)$) with $\delta\in(0,\frac{\ln\lambda_-}{\ln b}]$, $p\in(\frac{1}{1+\delta},1]$ and $q\in(0,\infty]$ (see Theorem 6.16 below).

As the first main result of this subsection, the following Theorem 6.8 is the boundedness of the δ-type Calderón-Zygmund operators from $H^p_A(\mathbb{R}^n)$ to $L^{p,\infty}(\mathbb{R}^n)$ (or to $H^{p,\infty}_A(\mathbb{R}^n)$) in the critical case.

Theorem 6.8. *Let $\delta\in(0,\frac{\ln\lambda_-}{\ln b}]$ and $p=\frac{1}{1+\delta}$. If $k\in\mathcal{S}'(\mathbb{R}^n)$ coincides with a locally integrable function on $\mathbb{R}^n\setminus\{\vec{0}_n\}$ and there exist two positive constants C_{13} and C_{14}, independent of f, x and y, such that*

$$\|k*f\|_{L^2(\mathbb{R}^n)}\leqslant C_{13}\|f\|_{L^2(\mathbb{R}^n)}$$

and, when $\rho(x)\geqslant b^{2\tau}\rho(y)$,

$$|k(x-y)-k(x)|\leqslant C_{14}\frac{[\rho(y)]^{\delta}}{[\rho(x)]^{1+\delta}}, \tag{6.4}$$

then $T(f):=k*f$ for all $f\in L^2(\mathbb{R}^n)\cap H^p_A(\mathbb{R}^n)$ has a unique extension on $H^p_A(\mathbb{R}^n)$ and, moreover, there exist two positive constants C_{15} and C_{16} such that, for all $f\in H^p_A(\mathbb{R}^n)$,

$$\|T(f)\|_{L^{p,\infty}(\mathbb{R}^n)}\leqslant C_{15}\|f\|_{H^p_A(\mathbb{R}^n)} \tag{6.5}$$

and

$$\|T(f)\|_{H^{p,\infty}_A(\mathbb{R}^n)}\leqslant C_{16}\|f\|_{H^p_A(\mathbb{R}^n)}. \tag{6.6}$$

To show Theorem 6.8, we need the following weak-type summable principle, which is from [31, p. 9] (see also [47, p. 114]).

Lemma 6.9. *Let $p\in(0,1)$, (X,μ) be any metric space and $\{f_j\}_{j\in\mathbb{N}}$ be a sequence of measurable functions such that, for all $j\in\mathbb{N}$ and $\lambda\in(0,\infty)$,*

$$\mu(\{x\in X:|f_j(x)|>\lambda\})\leqslant C\lambda^{-p},$$

where C is a positive constant independent of λ and j. If $\{c_j\}_{j\in\mathbb{N}}\subset\mathbb{C}$ satisfies that $\sum_{j\in\mathbb{N}}|c_j|^p<\infty$, then $\sum_{j\in\mathbb{N}}c_jf_j(x)$ is absolutely convergent almost everywhere and there exists a positive constant $\widetilde{C}$ such that, for all $\lambda\in(0,\infty)$,

$$\mu\left(\left\{x\in X:\left|\sum_{j\in\mathbb{N}}c_jf_j(x)\right|>\lambda\right\}\right)\leqslant\widetilde{C}\frac{2-p}{1-p}\left[\sum_{j\in\mathbb{N}}|c_j|^p\right]\lambda^{-p}.$$

Now we show Theorem 6.8.

Proof of Theorem 6.8. We first prove (6.5). By Theorem 3.6, to show (6.5), it suffices to prove that, for h being a constant multiple of a (p,∞,s)-atom associated with ball $B:=x_0+B_\ell$ for some $x_0\in\mathbb{R}^n$ and $\ell\in\mathbb{Z}$,

$$\sup_{k\in\mathbb{Z}}2^{kp}|\{x\in\mathbb{R}^n:|T(h)(x)|>2^k\}|\lesssim\|h\|^p_{L^\infty(\mathbb{R}^n)}|B|. \tag{6.7}$$

Indeed, by Theorem 3.6 with $p = q$, we find that, for any $f \in H^p_A(\mathbb{R}^n)$, there exists a sequence of constant multiples of (p, ∞, s)-atoms, $\{h_j\}_{j\in\mathbb{N}}$, associated with balls $\{B_j\}_{j\in\mathbb{N}}$, such that $f = \sum_{j\in\mathbb{N}} h_j$ in $\mathcal{S}'(\mathbb{R}^n)$ and

$$\|f\|_{H^p_A(\mathbb{R}^n)} \sim \left[\sum_{j\in\mathbb{N}} \|h_j\|^p_{L^\infty(\mathbb{R}^n)}|B_j|\right]^{\frac{1}{p}}.$$

From this, (6.7) and Lemma 6.9, we deduce that

$$\begin{aligned}&\sup_{k\in\mathbb{Z}} 2^{kp}|\{x\in\mathbb{R}^n : |T(f)(x)| > 2^k\}|\\ &\quad\leqslant \sup_{k\in\mathbb{Z}} 2^{kp}\left|\left\{x\in\mathbb{R}^n : \sum_{j\in\mathbb{N}}|T(h_j)(x)| > 2^k\right\}\right| \lesssim \sum_{j\in\mathbb{N}}\|h_j\|^p_{L^\infty(\mathbb{R}^n)}|B_j| \lesssim \|f\|^p_{H^p_A(\mathbb{R}^n)},\end{aligned} \tag{6.8}$$

which implies that $\|T(f)\|_{L^{p,\infty}_A(\mathbb{R}^n)} \lesssim \|f\|_{H^p_A(\mathbb{R}^n)}$. This is as desired.

It remains to prove (6.7). First, by the boundedness of T and Hölder's inequality, we know that

$$\begin{aligned}&\sup_{k\in\mathbb{Z}} 2^{kp}|\{x\in A^{4\tau}B : |T(h)(x)| > 2^k\}|\\ &\quad\leqslant \int_{A^{4\tau}B}|T(h)(x)|^p\,dx \lesssim |B|^{(\frac{p}{2})'}\left\{\int_{A^{4\tau}B}|T(h)(x)|^2\,dx\right\}^{\frac{p}{2}}\\ &\quad\lesssim |B|^{(\frac{p}{2})'}\left\{\int_{\mathbb{R}^n}|h(x)|^2\,dx\right\}^{\frac{p}{2}} \lesssim \|h\|^p_{L^\infty(\mathbb{R}^n)}|B|.\end{aligned} \tag{6.9}$$

On the other hand, by $\int_{\mathbb{R}^n} h(x)\,dx = 0$ and (6.4), we find that, for all $x \in (A^{4\tau}B)^\complement$,

$$\begin{aligned}|T(h)(x)| &\leqslant \int_B |k(x-y) - k(x-x_0)||h(y)|\,dy\\ &\lesssim \|h\|_{L^\infty(\mathbb{R}^n)}\int_B \frac{[\rho(y-x_0)]^\delta}{[\rho(x-x_0)]^{1+\delta}} \lesssim \frac{|B|^{1+\delta}}{[\rho(x-x_0)]^{1+\delta}}\|h\|_{L^\infty(\mathbb{R}^n)},\end{aligned}$$

which further implies that

$$\begin{aligned}&\sup_{k\in\mathbb{Z}} 2^{kp}|\{x\in (A^{4\tau}B)^\complement : |T(h)(x)| > 2^k\}|\\ &\quad\lesssim \sup_{k\in\mathbb{Z}} 2^{kp}\left|\left\{x\in (A^{4\tau}B)^\complement : \frac{|B|^{1+\delta}}{[\rho(x-x_0)]^{1+\delta}}\|h\|_{L^\infty(\mathbb{R}^n)} > 2^k\right\}\right|\\ &\quad\lesssim \sup_{k\in\mathbb{Z}\cap(-\infty,\|h\|_{L^\infty(\mathbb{R}^n)})} 2^{kp}\left[\frac{\|h\|_{L^\infty(\mathbb{R}^n)}}{2^k}\right]^{\frac{1}{1+\delta}}|B| \sim \|h\|^p_{L^\infty(\mathbb{R}^n)}|B|.\end{aligned} \tag{6.10}$$

Then (6.7) follows from (6.9) and (6.10), which completes the proof of (6.5).

Next we prove (6.6). To this end, similar to (6.8), it suffices to prove that

$$\sup_{k\in\mathbb{Z}} 2^{kp}|\{x\in\mathbb{R}^n : M_N(T(h))(x) > 2^k\}| \lesssim \|h\|^p_{L^\infty(\mathbb{R}^n)}|B|. \tag{6.11}$$

First, similar to (6.9), by the boundedness of T and M_N on $L^2(\mathbb{R}^n)$ (see Remark 2.10), we easily conclude that

$$\sup_{k\in\mathbb{Z}} 2^{kp}|\{x\in A^{4\tau}B : M_N(T(h))(x) > 2^k\}| \lesssim \|h\|^p_{L^\infty(\mathbb{R}^n)}|B|. \tag{6.12}$$

By $\int_{\mathbb{R}^n} h(x)\,dx = 0$, we know that $\widehat{T(h)}(\vec{0}_n) = \widehat{k}(\vec{0}_n)\widehat{h}(\vec{0}_n) = 0$ and hence $\int_{\mathbb{R}^n} T(h)(x)\,dx = 0$. By this, we find that, for all $\phi \in \mathcal{S}_N(\mathbb{R}^n)$, $k\in\mathbb{Z}$ and $x \in (A^{4\tau}B)^\complement$,

$$|T(h)*\phi_k(x)| = b^{-k}\left|\int_{\mathbb{R}^n} T(h)(y)[\phi(A^{-k}(x-y)) - \phi(A^{-k}(x-x_0))]\,dy\right|$$

$$
\begin{aligned}
&\leqslant b^{-k}\int_{\mathbb{R}^n}|T(h)(y)||\phi(A^{-k}(x-y))-\phi(A^{-k}(x-x_0))|\,dy\\
&\leqslant b^{-k}\bigg\{\int_{\rho(y-x_0)<b^{2\tau}|B|}+\int_{b^{2\tau}|B|\leqslant\rho(y-x_0)<b^{-2\tau}\rho(x-x_0)}+\int_{\rho(y-x_0)\geqslant b^{-2\tau}\rho(x-x_0)}\bigg\}\\
&\quad\times|T(h)(y)||\phi(A^{-k}(x-y))-\phi(A^{-k}(x-x_0))|\,dy\\
&=:\mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3. \qquad (6.13)
\end{aligned}
$$

For I_1, by the mean value theorem, [9, p. 11, Lemma 3.2], Hölder's inequality and the boundedness of T on $L^2(\mathbb{R}^n)$, we conclude that there exists $\xi(y)\in A^{2\tau}B$ such that, for all $\phi\in\mathcal{S}_N(\mathbb{R}^n)$, $k\in\mathbb{Z}$ and $x\in(A^{4\tau}B)^{\complement}$,

$$
\begin{aligned}
\mathrm{I}_1&=b^{-k}\int_{\rho(y-x_0)<b^{2\tau}|B|}|T(h)(y)||\phi(A^{-k}(x-y))-\phi(A^{-k}(x-x_0))|\,dy\\
&\leqslant b^{-k}\int_{\rho(y-x_0)<b^{2\tau}|B|}|T(h)(y)|\bigg|\sum_{|\beta|=1}\partial^{\beta}\phi(A^{-k}(x-\xi(y)))\bigg||A^{-k}(y-x_0)|\,dy\\
&\lesssim b^{-k}\int_{\rho(y-x_0)<b^{2\tau}|B|}|T(h)(y)|\frac{b^{k(1+\widetilde{\delta})}}{[\rho(x-x_0)]^{1+\widetilde{\delta}}}b^{-k\widetilde{\delta}}[\rho(y-x_0)]^{\widetilde{\delta}}\,dy\\
&\lesssim\frac{|B|^{\widetilde{\delta}}}{[\rho(x-x_0)]^{1+\widetilde{\delta}}}\bigg\{\int_{\mathbb{R}^n}[T(h)]^2(y)\bigg\}^{\frac12}|B|^{\frac12}\lesssim\frac{|B|^{1+\widetilde{\delta}}}{[\rho(x-x_0)]^{1+\widetilde{\delta}}}\|h\|_{L^\infty(\mathbb{R}^n)}, \qquad (6.14)
\end{aligned}
$$

where

$$
\widetilde{\delta}:=\begin{cases}(\ln\lambda_+)/(\ln b), & \text{when } \rho(y-x_0)\geqslant 1,\\ (\ln\lambda_-)/(\ln b), & \text{when } \rho(y-x_0)<1.\end{cases}
$$

For I_2, by $\int_{\mathbb{R}^n}h(x)\,dx=0$, (6.4) and the mean value theorem, we know that, for all $\phi\in\mathcal{S}_N(\mathbb{R}^n)$, $k\in\mathbb{Z}$ and $x\in(A^{4\tau}B)^{\complement}$,

$$
\begin{aligned}
\mathrm{I}_2&=b^{-k}\int_{b^{2\tau}|B|\leqslant\rho(y-x_0)<b^{-2\tau}\rho(x-x_0)}\bigg|\int_B h(z)k(y-z)\,dz\bigg||\phi(A^{-k}(x-y))-\phi(A^{-k}(x-x_0))|\,dy\\
&\lesssim\int_{b^{2\tau}|B|\leqslant\rho(y-x_0)<b^{-2\tau}\rho(x-x_0)}\bigg\{\int_B|h(z)||k(y-z)-k(y-x_0)|\,dz\bigg\}\frac{[\rho(y-x_0)]^{\widetilde{\delta}}}{[\rho(x-x_0)]^{1+\widetilde{\delta}}}\,dy\\
&\lesssim\frac{\|h\|_{L^\infty(\mathbb{R}^n)}}{[\rho(x-x_0)]^{1+\widetilde{\delta}}}\int_{b^{2\tau}|B|\leqslant\rho(y-x_0)<b^{-2\tau}\rho(x-x_0)}\frac{|B|^{1+\delta}}{[\rho(y-x_0)]^{1+\delta-\widetilde{\delta}}}\,dy\\
&\lesssim\frac{|B|^{1+\widetilde{\delta}}}{[\rho(x-x_0)]^{1+\widetilde{\delta}}}\|h\|_{L^\infty(\mathbb{R}^n)}, \qquad (6.15)
\end{aligned}
$$

where $\widetilde{\delta}$ is as in (6.14).

For I_3, by the fact that $\int_{\mathbb{R}^n}b(x)\,dx=0$, (6.4) and $\phi\in\mathcal{S}_N(\mathbb{R}^n)$, we find that, for all $k\in\mathbb{Z}$ and $x\in(A^{4\tau}B)^{\complement}$,

$$
\begin{aligned}
\mathrm{I}_3&=\int_{\rho(y-x_0)\geqslant b^{-2\tau}\rho(x-x_0)}\bigg|\int_B h(z)k(y-z)\,dz\bigg||\phi_k(x-y)|\,dy\\
&\leqslant\int_{\rho(y-x_0)\geqslant b^{-2\tau}\rho(x-x_0)}\bigg[\int_B|h(z)||k(y-z)-k(y-x_0)|\,dz\bigg]|\phi_k(x-y)|\,dy\\
&\lesssim\|h\|_{L^\infty(\mathbb{R}^n)}\int_{\rho(y-x_0)\geqslant b^{-2\tau}\rho(x-x_0)}\bigg\{\int_B\frac{[\rho(z-x_0)]^{\delta}}{[\rho(y-x_0)]^{1+\delta}}\,dz\bigg\}|\phi_k(x-y)|\,dy\\
&\lesssim\frac{|B|^{1+\delta}}{[\rho(x-x_0)]^{1+\delta}}\|h\|_{L^\infty(\mathbb{R}^n)}\int_{\mathbb{R}^n}|\phi_k(x-y)|\,dy\lesssim\frac{|B|^{1+\delta}}{[\rho(x-x_0)]^{1+\delta}}\|h\|_{L^\infty(\mathbb{R}^n)}. \qquad (6.16)
\end{aligned}
$$

Combining (6.13)–(6.16) and Proposition 2.4, we know that, for all $x \in (A^{4\tau}B)^{\complement}$,

$$M_N(T(h))(x) \lesssim \sup_{\phi\in\mathcal{S}_N(\mathbb{R}^n)} \sup_{k\in\mathbb{Z}} |T(h) * \phi_k(x)| \lesssim \frac{|B|^{1+\delta}}{[\rho(x-x_0)]^{1+\delta}} \|h\|_{L^\infty(\mathbb{R}^n)}.$$

From this, together with an argument parallel to (6.10), we further deduce that

$$\sup_{k\in\mathbb{Z}} 2^{kp} |\{x \in (A^{4\tau}B)^{\complement} : M_N(T(h))(x) > 2^k\}| \lesssim \|h\|^p_{L^\infty(\mathbb{R}^n)} |B|,$$

which, combined with (6.12), implies (6.11). This finishes the proof of (6.6) and hence Theorem 6.8. □

Remark 6.10. (i) If $A := d\,\mathrm{I}_{n\times n}$ for some $d \in \mathbb{R}$ with $|d| \in (1,\infty)$, then $\frac{\ln\lambda_-}{\ln b} = \frac{1}{n}$ and $H^p_A(\mathbb{R}^n)$ and $H^{p,\infty}_A(\mathbb{R}^n)$ become the classical isotropic Hardy and weak Hardy spaces, respectively. In this case, we know, by Theorem 6.8, that, if $\delta \in (0,1]$, $p = \frac{n}{n+\delta}$ and T is the Calderón-Zygmund operator satisfying all conditions of Theorem 6.8 with (6.4) replaced by

$$|k(x-y) - k(x)| \lesssim \frac{|y|^\delta}{|x|^{n+\delta}}, \quad |x| \geqslant 2|y|,$$

where the implicit positive constant is independent of x and y, then T is bounded from $H^{\frac{n}{n+\delta}}(\mathbb{R}^n)$ to $H^{\frac{n}{n+\delta},\infty}(\mathbb{R}^n)$, which is just [47, Theorem 1]. Here $\frac{n}{n+\delta}$ is called the *critical index*. In this sense, Theorem 6.8 also establishes the boundedness of Calderón-Zygmund operators from $H^p_A(\mathbb{R}^n)$ to $L^{p,\infty}(\mathbb{R}^n)$ in the critical case under the anisotropic setting.

(ii) Let $\delta \in (0,1]$. A *non-convolutional δ-type Calderón-Zygmund operator* T is a linear operator which is bounded on $L^2(\mathbb{R}^n)$ and satisfies that, for any $f \in L^2(\mathbb{R}^n)$ with compact support and $x \notin \mathrm{supp}(f)$,

$$T(f)(x) = \int_{\mathrm{supp}(f)} \mathcal{K}(x,y) f(y)\,dy,$$

where $\mathcal{K}$ denotes a standard kernel on $(\mathbb{R}^n \times \mathbb{R}^n) \setminus \{(x,x) : x \in \mathbb{R}^n\}$ in the following sense: There exists a positive constant C such that, for all x, y, $z \in \mathbb{R}^n$,

$$|\mathcal{K}(x,y)| \leqslant \frac{C}{\rho(x-y)} \quad \text{if} \quad x \neq y$$

and

$$|\mathcal{K}(x,y) - \mathcal{K}(x,z)| \leqslant C \frac{[\rho(y-z)]^\delta}{[\rho(x-y)]^{1+\delta}} \quad \text{if} \quad \rho(x-y) > b^{2\tau}\rho(y-z). \tag{6.17}$$

By an argument similar to that used in the proof of (6.6) in Theorem 6.8, we find that (6.6) also holds true for non-convolutional δ-type Calderón-Zygmund operators T with the additional assumption that $T^*1 = 0$ (namely, for any $a \in L^1(\mathbb{R}^n)$ with compact support, if $\int_{\mathbb{R}^n} a(x)\,dx = 0$, then $\int_{\mathbb{R}^n} T(a)(x)\,dx = 0$), the details being omitted.

(iii) Following the proof of (6.5) in Theorem 6.8, we know that (6.5) also holds true when T is a non-convolutional δ-type Calderón-Zygmund operator.

(iv) Let $\delta \in (0, \frac{\ln\lambda_-}{\ln b}]$ and $p \in (\frac{1}{1+\delta}, 1]$. If T is either a convolutional δ-type Calderón-Zygmund operator as in Theorem 6.8 or a non-convolutional δ-type Calderón-Zygmund operator with the additional assumption that $T^*1 = 0$, as in (ii) of this remark, then, by a similar proof to that of [9, p. 68, Theorem 9.8], we conclude that T is bounded from $H^p_A(\mathbb{R}^n)$ to $H^p_A(\mathbb{R}^n)$. Moreover, by an argument parallel to the proof of [9, p. 69, Theorem 9.9], we know that, if T is either a convolutional δ-type Calderón-Zygmund operator as in Theorem 6.8 or a non-convolutional δ-type Calderón-Zygmund operator T, then T is bounded from $H^p_A(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. Comparing these with Theorem 6.8 and (ii) and (iii) of this remark, we know that the latter further completes the boundedness of these operators in the critical case by establishing the bounedness from $H^p_A(\mathbb{R}^n)$ to $L^{p,\infty}(\mathbb{R}^n)$ (or to $H^{p,\infty}_A(\mathbb{R}^n)$).

Another interesting application of the atomic decomposition for $H_A^{p,q}(\mathbb{R}^n)$ is to obtain the following boundedness of Calderón-Zygmund operators from $H_A^{p,q}(\mathbb{R}^n)$ to $L^{p,\infty}(\mathbb{R}^n)$ with $p \in (0,1]$ and $q \in (p,\infty]$.

Theorem 6.11. *Suppose that $p \in (0,1]$, $q \in (p,\infty]$, $r \in (1,\infty)$ and T is a Calderón-Zygmund operator associated with the kernel k. Moreover, assume that T is bounded from $L^r(\mathbb{R}^n)$ to $L^{r,\infty}(\mathbb{R}^n)$ and ω_p satisfies a Dini-type condition of order $q/(q-p)$, namely,*

$$A_{(p,q)} := \left\{ \int_0^1 [\omega_p(\delta)]^{q/(q-p)} \frac{d\delta}{\delta} \right\}^{(q-p)/q} < \infty, \tag{6.18}$$

where, for $\delta \in (0,1]$,

$$\omega_p(\delta) := \sup_B \frac{1}{|B|} \int_{\rho(x-y_B) > \frac{b^{2\tau}}{\delta}|B|} \left[\int_B \left| k(x,y) - \sum_{|\beta| \leqslant N} (y-y_B)^\beta k_\beta(x,y_B) \right| dy \right]^p dx,$$

$N := \lfloor (1/p-1) \frac{\ln b}{\ln \lambda_-} \rfloor$, $\beta := (\beta_1, \ldots, \beta_n) \in \mathbb{Z}_+^n$,

$$k_\beta(x,y_B) := \frac{1}{\beta!} D^\beta k(x,y)|_{y=y_B}$$

and the supremum is taken over all dilated balls $B \in \mathfrak{B}$ centered at y_B. Then T is bounded from $H_A^{p,q}(\mathbb{R}^n)$ to $L^{p,\infty}(\mathbb{R}^n)$ and, moreover, there exists a positive constant C such that, for all $f \in H_A^{p,q}(\mathbb{R}^n)$,

$$\|T(f)\|_{L^{p,\infty}(\mathbb{R}^n)} \leqslant C[A_{(p,q)}]^{\frac{1}{p}} \|f\|_{H_A^{p,q}(\mathbb{R}^n)}.$$

Proof. Let $p \in (0,1]$, $q \in (p,\infty]$ and $r \in (1,\infty)$. For all $f \in H_A^{p,q}(\mathbb{R}^n)$, by Theorem 3.6 and Definition 3.2, we know that there exists a sequence of (p,∞,s)-atoms, $\{a_i^k\}_{i\in\mathbb{N},\,k\in\mathbb{Z}}$, respectively supported on $\{x_i^k + B_i^k\}_{i\in\mathbb{N},\,k\in\mathbb{Z}} \subset \mathfrak{B}$ such that $f = \sum_{k\in\mathbb{Z}} \sum_{i\in\mathbb{N}} \lambda_i^k a_i^k$ in $\mathcal{S}'(\mathbb{R}^n)$, $\lambda_i^k \sim 2^k |B_i^k|^{1/p}$ for all $k \in \mathbb{Z}$ and $i \in \mathbb{N}$, $\sum_{i\in\mathbb{N}} \chi_{x_i^k + B_i^k}(x) \lesssim 1$ for all $k \in \mathbb{Z}$ and $x \in \mathbb{R}^n$, and

$$\|f\|_{H_A^{p,q}(\mathbb{R}^n)} \sim \|\{\mu_k\}_{k\in\mathbb{Z}}\|_{\ell^q},$$

where $\mu_k := (\sum_{i\in\mathbb{N}} |\lambda_i^k|^p)^{1/p}$. For all $k_0 \in \mathbb{Z}$, let $f_1 := \sum_{k=-\infty}^{k_0} \sum_{i\in\mathbb{N}} \lambda_i^k a_i^k$ and $f_2 := f - f_1$. Since $\frac{rq}{p} \in (1,\infty]$, from Hölder's inequality, it follows that

$$\begin{aligned}
\|f_1\|_{L^r(\mathbb{R}^n)} &\leqslant \sum_{k=-\infty}^{k_0} \left\| \sum_{i\in\mathbb{N}} \lambda_i^k a_i^k \right\|_{L^r(\mathbb{R}^n)} \sim \sum_{k=-\infty}^{k_0} \left\{ \int_{\bigcup_{i\in\mathbb{N}}(x_i^k + B_i^k)} \left| \sum_{i\in\mathbb{N}} \lambda_i^k a_i^k(x) \right|^r dx \right\}^{1/r} \\
&\lesssim \sum_{k=-\infty}^{k_0} 2^k \left(\sum_{i\in\mathbb{N}} |B_i^k| \right)^{1/r} \lesssim \sum_{k=-\infty}^{k_0} 2^{k(1-\frac{p}{r})} \left(\sum_{i\in\mathbb{N}} |\lambda_i^k|^p \right)^{1/r} \\
&\lesssim 2^{k_0(1-\frac{p}{r})} \|\{\mu_k\}_{k\in\mathbb{Z}}\|_{\ell^q}^{\frac{p}{r}} \sim 2^{k_0(1-\frac{p}{r})} \|f\|_{H_A^{p,q}(\mathbb{R}^n)}^{\frac{p}{r}},
\end{aligned}$$

which, together with the boundedness from $L^r(\mathbb{R}^n)$ to $L^{r,\infty}(\mathbb{R}^n)$ of T, implies that

$$2^{pk_0} |\{x \in \mathbb{R}^n : |T(f_1)(x)| > 2^{k_0}\}| \lesssim \|f\|_{H_A^{p,q}(\mathbb{R}^n)}^p. \tag{6.19}$$

To complete the proof of Theorem 6.11, it suffices to prove that, for all $k_0 \in \mathbb{Z}$,

$$2^{pk_0} |\{x \in \mathbb{R}^n : |T(f_2)(x)| > 2^{k_0}\}| \lesssim A_{(p,q)} \|f\|_{H_A^{p,q}(\mathbb{R}^n)}^p. \tag{6.20}$$

Indeed, if (6.20) is true, then, by (6.19), we further conclude that

$$\begin{aligned}
&2^{pk_0} |\{x \in \mathbb{R}^n : |T(f)(x)| > 2^{k_0}\}| \\
&\quad \leqslant 2^{pk_0} |\{x \in \mathbb{R}^n : |T(f_1)(x)| > 2^{k_0-1}\}| + 2^{pk_0} |\{x \in \mathbb{R}^n : |T(f_2)(x)| > 2^{k_0-1}\}| \\
&\quad \lesssim \|f\|_{H_A^{p,q}(\mathbb{R}^n)}^p + A_{(p,q)} \|f\|_{H_A^{p,q}(\mathbb{R}^n)}^p \lesssim A_{(p,q)} \|f\|_{H_A^{p,q}(\mathbb{R}^n)}^p.
\end{aligned} \tag{6.21}$$

Taking the supremum over all $k_0 \in \mathbb{Z}$ at the left-hand side of (6.21), we find that

$$\|T(f)\|_{L^{p,\infty}(\mathbb{R}^n)} \lesssim [A_{(p,q)}]^{\frac{1}{p}} \|f\|_{H_A^{p,q}(\mathbb{R}^n)},$$

which is the desired conclusion of Theorem 6.11.

Finally, we give the proof of (6.20). To this end, for all $i \in \mathbb{N}$ and $k \in \mathbb{Z}$, let $B_{\ell_i^k} := x_i^k + B_i^k$, where $\ell_i^k \in \mathbb{Z}$. For every $k \in (k_0, \infty] \cap \mathbb{Z}$, there exists an $m_k \in \mathbb{N}$ such that

$$b^{m_k-2\tau-1} \leqslant \left(\frac{3}{2}\right)^{p(k-k_0)} < b^{m_k-2\tau}.$$

Let

$$B_{k_0} := \bigcup_{k=k_0+1}^{\infty} \bigcup_{i\in\mathbb{N}} B_{\ell_i^k+m_k+\tau}.$$

Notice that $\lambda_i^k \sim 2^k|B_i^k|^{1/p} \sim 2^k|B_{\ell_i^k}|^{1/p}$. Since $q/p \in (1,\infty]$, from Hölder's inequality, we deduce that, for all $k \in \mathbb{Z}$ and $i \in \mathbb{N}$,

$$\begin{aligned}|B_{k_0}| &\leqslant \sum_{k=k_0+1}^{\infty} \sum_{i\in\mathbb{N}} |B_{\ell_i^k+m_k+\tau}| \lesssim 2^{-pk_0} \sum_{k=k_0+1}^{\infty} \left(\frac{3}{4}\right)^{p(k-k_0)} \sum_{i\in\mathbb{N}} |\lambda_i^k|^p \\ &\lesssim 2^{-pk_0} \|\{\mu_k\}_{k\in\mathbb{Z}}\|_{\ell^q}^p \sim 2^{-pk_0} \|f\|_{H_A^{p,q}(\mathbb{R}^n)}^p. \end{aligned} \tag{6.22}$$

Moreover, by the cancelation condition of a_i^k, we have

$$\begin{aligned}&\int_{\mathbb{R}^n\setminus B_{k_0}} |T(f_2)(x)|^p\,dx \\ &\leqslant \sum_{k=k_0+1}^{\infty} \sum_{i\in\mathbb{N}} |\lambda_i^k|^p \int_{\mathbb{R}^n\setminus B_{\ell_i^k+m_k+\tau}} |T(a_i^k)(x)|^p\,dx \\ &= \sum_{k=k_0+1}^{\infty} \sum_{i\in\mathbb{N}} |\lambda_i^k|^p \left\{ \int_{\mathbb{R}^n\setminus B_{\ell_i^k+m_k+\tau}} \left| \int_{B_{\ell_i^k}} \left[k(x,y) - \sum_{|\beta|\leqslant N} (y-y_{B_{\ell_i^k}})^\beta k_\beta(x, y_{B_{\ell_i^k}}) \right] a_i^k(y)\,dy \right|^p dx \right\}.\end{aligned}$$

Observe that, from $x \in (B_{\ell_i^k+m_k+\tau})^\complement$ and (2.9), we deduce that

$$\rho(x-y_{B_{\ell_i^k}}) > b^{m_k}|B_{\ell_i^k}| > b^{2\tau}\left(\frac{3}{2}\right)^{p(k-k_0)}|B_{\ell_i^k}|.$$

Hence, by Hölder's inequality, we find that

$$\begin{aligned}\int_{\mathbb{R}^n\setminus B_{k_0}} |T(f_2)(x)|^p\,dx &\leqslant \sum_{k=k_0+1}^{\infty} \omega_p\left(\left(\frac{2}{3}\right)^{p(k-k_0)}\right)\mu_k^p \\ &\leqslant \left\{ \sum_{k=k_0+1}^{\infty} \left[\omega_p\left(\left(\frac{2}{3}\right)^{p(k-k_0)}\right)\right]^{\frac{q}{q-p}} \right\}^{\frac{q-p}{q}} \|\{\mu_k\}_{k\in\mathbb{Z}}\|_{\ell^q}^p \\ &\lesssim \left\{ \int_0^1 [\omega_p(\delta)]^{\frac{q}{q-p}} \frac{d\delta}{\delta} \right\}^{\frac{q-p}{q}} \|f\|_{H_A^{p,q}(\mathbb{R}^n)}^p \sim A_{(p,q)}\|f\|_{H_A^{p,q}(\mathbb{R}^n)}^p,\end{aligned}$$

which further implies that

$$2^{pk_0}|\{x \in (B_{k_0})^\complement : |T(f_2)(x)| > 2^{k_0}\}| \lesssim A_{(p,q)}\|f\|_{H_A^{p,q}(\mathbb{R}^n)}^p.$$

By this and (6.22), we conclude that

$$\begin{aligned}2^{pk_0}|\{x \in \mathbb{R}^n : |T(f_2)(x)| > 2^{k_0}\}| &\lesssim 2^{pk_0}[|B_{k_0}| + |\{x \in (B_{k_0})^\complement : |T(f_2)(x)| > 2^{k_0}\}|] \\ &\lesssim (1+A_{(p,q)})\|f\|_{H_A^{p,q}(\mathbb{R}^n)}^p \lesssim A_{(p,q)}\|f\|_{H_A^{p,q}(\mathbb{R}^n)}^p,\end{aligned}$$

which proves (6.20). This finishes the proof of Theorem 6.11. □

Remark 6.12. (i) If A is the same as in Remark 6.10(i), then $N=\lfloor n(1/p-1)\rfloor$, $\frac{\ln b}{\ln\lambda_-}=n$ and $H_A^{p,q}(\mathbb{R}^n)$ and $L^{p,\infty}(\mathbb{R}^n)$ become the classical isotropic Hardy-Lorentz and weak Lebesgue spaces, respectively. In this case, Theorem 6.11 is just [1, Theorem 2.2].

(ii) It is well known that the Hörmander condition implies the boundedness of the Calderón-Zygmund operator T from $H_A^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. Observe that $H_A^1(\mathbb{R}^n)\subsetneqq H_A^{1,q}(\mathbb{R}^n)$ with $q\in(1,\infty]$. Thus, to define T on $H_A^{1,q}(\mathbb{R}^n)$ with $q\in(1,\infty]$, it is natural to require T to satisfy some conditions stronger than the usual Hörmander condition. This is accomplished by the variable dilations (the Dini-type condition (6.18)) of Fefferman and Soria [31] (see also [1]). Moreover, recall that we consider $p=\frac{1}{1+\delta}$ or $p\in(\frac{1}{1+\delta},1]$ with $\delta\in(0,\frac{\ln\lambda_-}{\ln b}]$ in Theorem 6.8 and Remark 6.10, which implies $N=\lfloor\frac{\ln b}{\ln\lambda_-}(1/p-1)\rfloor\leqslant 1$. But, in Theorem 6.11, we consider $p\in(0,1]$. If p becomes smaller, then N becomes larger. Thus, more regularity of the kernel of T is needed. This justifies the definition of $\omega_p(\delta)$ in Theorem 6.11.

(iii) If $\delta\in(0,\frac{\ln\lambda_-}{\ln b}]$, $p\in(\frac{1}{1+\delta},1]$ and T is a non-convolutional δ-type Calderón-Zygmund operator which satisfies all the conditions in Remark 6.10(ii) with (6.17) replaced by

$$|\mathcal{K}(x,y)-\mathcal{K}(x,z)|\leqslant C\frac{[\rho(y-z)]^{\delta}}{[\rho(x-y)]^{1+\delta}}\quad\text{if}\quad \rho(x-y)>b^{\tau}\rho(y-z),$$

where C is a positive constant independent of x, y and z, then $N=\lfloor\frac{\ln b}{\ln\lambda_-}(1/p-1)\rfloor=0$ and $p(1+\delta)>1$. Thus, for $p=q$, we have

$$\begin{aligned}
A_{(p,p)}&=\sup_{\delta\in(0,1]}\{\omega_p(\delta)\}=\sup_B\frac{1}{|B|}\int_{\rho(x-y_B)>b^{2\tau}|B|}\left[\int_B|\mathcal{K}(x,y)-\mathcal{K}(x,y_B)|\,dy\right]^p dx\\
&\lesssim\sup_B\frac{1}{|B|}\int_{\rho(x-y_B)>b^{2\tau}|B|}\left[\int_B\frac{[\rho(y-y_B)]^{\delta}}{[\rho(x-y)]^{1+\delta}}\,dy\right]^p dx\\
&\lesssim\sup_B\frac{1}{|B|}\sum_{k=0}^{\infty}\int_{b^kb^{2\tau}|B|<\rho(x-y_B)\leqslant b^{k+1}b^{2\tau}|B|}\left[\int_B\frac{[\rho(y-y_B)]^{\delta}}{[\rho(x-y_B)]^{1+\delta}}\,dy\right]^p dx\\
&\lesssim\sup_B\frac{1}{|B|}\sum_{k=0}^{\infty}\left[\frac{|B|^{1+\delta}}{(b^k|B|)^{1+\delta}}\right]^p b^k|B|\sim 1,
\end{aligned}$$

where the supremum is taken over all dilated balls $B\in\mathfrak{B}$ centered at y_B and $\mathfrak{B}$ is as in (2.6). This shows that Remark 6.10(iv) is the endpoint (critical) case of Theorem 6.11 in the sense of $p=q$.

Recall that a *quasi-Banach space* $\mathcal{B}$ is a vector space endowed with a quasi-norm $\|\cdot\|_{\mathcal{B}}$, which is non-negative and non-degenerate (namely, $\|f\|_{\mathcal{B}}=\theta$ if and only if $f=\theta$) and satisfyies the quasi-triangle inequality; namely, there exists a positive constant $K\in[1,\infty)$ such that, for all f, $g\in\mathcal{B}$,

$$\|f+g\|_{\mathcal{B}}\leqslant K(\|f\|_{\mathcal{B}}+\|g\|_{\mathcal{B}}).$$

Clearly, a quasi-Banach space $\mathcal{B}$ is called a Banach space if $K=1$.

Let $\mathcal{B}$ be a quasi-Banach space and $\mathcal{Y}$ a linear space. An operator T from $\mathcal{Y}$ to $\mathcal{B}$ is said to be $\mathcal{B}$-*sublinear* if there exists a positive constant C such that, for any λ, $\mu\in\mathbb{C}$ and f, $g\in\mathcal{Y}$,

$$\|T(f+\mu g)\|_{\mathcal{B}}\leqslant C[|\lambda|\|T(f)\|_{\mathcal{B}}+|\mu|\|T(g)\|_{\mathcal{B}}]$$

and $\|T(f)-T(g)\|_{\mathcal{B}}\leqslant C\|T(f-g)\|_{\mathcal{B}}$. Obviously, if T is linear, then T is $\mathcal{B}$-sublinear.

As an application of the finite atomic decomposition characterizations obtained in Section 5 (see Theorem 5.7), as well as Theorem 6.13, we establish the following criterion for the boundedness of sublinear operators from $H_A^{p,q}(\mathbb{R}^n)$ into a quasi-Banach space $\mathcal{B}$, which is a variant of [37, Theorem 5.9]; see also [40, Theorem 3.5] and [78, Theorem 1.1].

Theorem 6.13. *Let (p,r,s) be an admissible anisotropic triplet, $q\in(0,\infty)$ and $\mathcal{B}$ be a quasi-Banach space. If one of the following statements holds true*:

(i) $r \in (1,\infty)$ *and* $T : H^{p,r,s,q}_{A,\mathrm{fin}}(\mathbb{R}^n) \to \mathcal{B}$ *is a* $\mathcal{B}$*-sublinear operator satisfying that there exists a positive constant* C_{17} *such that, for all* $f \in H^{p,r,s,q}_{A,\mathrm{fin}}(\mathbb{R}^n)$,

$$\|T(f)\|_{\mathcal{B}} \leqslant C_{17}\|f\|_{H^{p,r,s,q}_{A,\mathrm{fin}}(\mathbb{R}^n)}; \tag{6.23}$$

(ii) $T : H^{p,\infty,s,q}_{A,\mathrm{fin}}(\mathbb{R}^n) \cap C(\mathbb{R}^n) \to \mathcal{B}$ *is a* $\mathcal{B}$*-sublinear operator satisfying that there exists a positive constant* C_{18} *such that, for all* $f \in H^{p,\infty,s,q}_{A,\mathrm{fin}}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$,

$$\|T(f)\|_{\mathcal{B}} \leqslant C_{18}\|f\|_{H^{p,\infty,s,q}_{A,\mathrm{fin}}(\mathbb{R}^n)},$$

then T *uniquely extends to a bounded sublinear operator from* $H^{p,q}_A(\mathbb{R}^n)$ *into* $\mathcal{B}$*. Moreover, there exists a positive constant* C_{19} *such that, for all* $f \in H^{p,q}_A(\mathbb{R}^n)$,

$$\|T(f)\|_{\mathcal{B}} \leqslant C_{19}\|f\|_{H^{p,q}_A(\mathbb{R}^n)}.$$

Proof. We first prove (i). For any given $r \in (1,\infty)$, assume that (6.23) holds true. Let $f \in H^{p,q}_A(\mathbb{R}^n)$. By Theorem 3.6 and the density of $H^{p,r,s,q}_{A,\mathrm{fin}}(\mathbb{R}^n)$ in $H^{p,r,s,q}_A(\mathbb{R}^n)$, we know that $H^{p,r,s,q}_{A,\mathrm{fin}}(\mathbb{R}^n)$ is dense in $H^{p,q}_A(\mathbb{R}^n)$. Thus, there exists a Cauchy sequence $\{f_j\}_{j\in\mathbb{N}} \subset H^{p,r,s,q}_{A,\mathrm{fin}}(\mathbb{R}^n)$ such that

$$\lim_{j\to\infty}\|f_j - f\|_{H^{p,q}_A(\mathbb{R}^n)} = 0,$$

which, together with (6.23) and Theorem 5.7(i), further implies that

$$\|T(f_i) - T(f_j)\|_{\mathcal{B}} \lesssim \|T(f_i - f_j)\|_{\mathcal{B}} \lesssim \|f_i - f_j\|_{H^{p,r,s,q}_{A,\mathrm{fin}}(\mathbb{R}^n)} \sim \|f_i - f_j\|_{H^{p,q}_A(\mathbb{R}^n)} \to 0$$

as $i,\, j \to \infty$, where the implicit positive constants are independent of i and j. Therefore, $\{T(f_j)\}_{j\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{B}$, which, combined with the completeness of $\mathcal{B}$, implies that there exists $F \in \mathcal{B}$ such that $F = \lim_{j\to\infty} T(f_j)$ in $\mathcal{B}$. Let $T(f) := F$. From (6.23) and Theorem 5.7(i) again, we further deduce that $T(f)$ is well defined and

$$\begin{aligned}\|T(f)\|_{\mathcal{B}} &\lesssim \limsup_{j\to\infty}[\|T(f) - T(f_j)\|_{\mathcal{B}} + \|T(f_j)\|_{\mathcal{B}}] \lesssim \limsup_{j\to\infty}\|T(f_j)\|_{\mathcal{B}}\\ &\lesssim \limsup_{j\to\infty}\|f_j\|_{H^{p,r,s,q}_{A,\mathrm{fin}}(\mathbb{R}^n)} \sim \lim_{j\to\infty}\|f_j\|_{H^{p,q}_A(\mathbb{R}^n)} \sim \|f\|_{H^{p,q}_A(\mathbb{R}^n)},\end{aligned}$$

where the implicit positive constants are independent of f. This finishes the proof of (i).

Now we prove (ii). For $q \in (0,\infty)$, we first claim that $H^{p,\infty,s,q}_{A,\mathrm{fin}}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ is dense in $H^{p,q}_A(\mathbb{R}^n)$. Indeed, by Lemma 5.2(ii), we know that $H^{p,q}_A(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ is dense in $H^{p,q}_A(\mathbb{R}^n)$. Thus, we only need to show that $H^{p,\infty,s,q}_{A,\mathrm{fin}}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ is dense in $H^{p,q}_A(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ with respect to the quasi-norm $\|\cdot\|_{H^{p,q}_A(\mathbb{R}^n)}$. For any $f \in H^{p,q}_A(\mathbb{R}^n) \cap C(\mathbb{R}^n)$, by an argument similar to that used in the proof of Theorem 3.6 (or Lemma 5.3), we find that there exist a sequence of (p,∞,s)-atoms, $\{a^k_i\}_{i\in\mathbb{N},\,k\in\mathbb{Z}}$, and $\{\lambda^k_i\}_{i\in\mathbb{N},\,k\in\mathbb{Z}} \subset \mathbb{C}$ such that $f = \sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}}\lambda^k_i a^k_i$ in $\mathcal{S}'(\mathbb{R}^n)$. Moreover, from definitions of these (p,∞,s)-atoms (see (5.12)) and the continuity of f, we further deduce that all these (p,∞,s)-atoms are continuous. Thus, for any $K \in \mathbb{N}$, if we let $f_K := \sum_{|k|=0}^{K}\sum_{i=1}^{K}\lambda^k_i a^k_i$, then it is easy to see that

$$\{f_K\}_{K\in\mathbb{N}} \subset H^{p,\infty,s,q}_{A,\mathrm{fin}}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$$

and

$$\lim_{K\to\infty}\|f - f_K\|_{H^{p,q}_A(\mathbb{R}^n)} = 0,$$

which implies that $H^{p,\infty,s,q}_{A,\mathrm{fin}}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ is dense in $H^{p,q}_A(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ with respect to the quasi-norm $\|\cdot\|_{H^{p,q}_A(\mathbb{R}^n)}$.

By the density of $H^{p,\infty,s,q}_{A,\mathrm{fin}}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ in $H^{p,q}_A(\mathbb{R}^n)$ and a proof similar to (i), we conclude that (ii) holds true. This finishes the proof of Theorem 6.13. □

By Theorem 6.13, we easily obtain the following conclusion, the details being omitted.

Corollary 6.14. *Let (p, r, s) be an admissible anisotropic triplet, $q \in (0, \infty)$ and $\mathcal{B}$ be a quasi-Banach space. If one of the following statements holds true*:

(i) *$r \in (1, \infty)$ and T is a $\mathcal{B}$-sublinear operator from $H^{p,r,s,q}_{A,\mathrm{fin}}(\mathbb{R}^n)$ to $\mathcal{B}$ satisfying that*

$$\mathfrak{S} := \sup\{\|T(a)\|_{\mathcal{B}} : a \text{ is any } (p, r, s)\text{-atom}\} < \infty;$$

(ii) *T is a $\mathcal{B}$-sublinear operator defined on continuous (p, ∞, s)-atoms satisfying that*

$$\mathfrak{S} := \sup\{\|T(a)\|_{\mathcal{B}} : a \text{ is any continuous } (p, \infty, s)\text{-atom}\} < \infty,$$

then T has a unique bounded $\mathcal{B}$-sublinear extension $\widetilde{T}$ from $H^{p,q}_A(\mathbb{R}^n)$ to $\mathcal{B}$.

Remark 6.15. (i) Obviously, if T is a bounded $\mathcal{B}$-sublinear operator from $H^{p,q}_A(\mathbb{R}^n)$ to $\mathcal{B}$, then, for any admissible anisotropic triplet (p, r, s), T is uniformly bounded on all (p, r, s)-atoms. Corollary 6.14(i) shows that the converse holds true for $r \in (1, \infty)$. However, such converse conclusion is not true in general for $r = \infty$ due to the example in [10, Theorem 2]. Namely, there exists an operator $\mathcal{T}$ which is uniformly bounded on all $(1, \infty, 0)$-atoms, but does not have a bounded extension on $H^1(\mathbb{R}^n)$.

(ii) Corollary 6.14(ii) shows that the uniform boundedness of T on a smaller class of continuous (p, ∞, s)-atoms implies the existence of a bounded extension on the whole space $H^{p,q}_A(\mathbb{R}^n)$. In particular, if we restrict the operator $\mathcal{T}$, in (i) of this remark, to the subspace $H^{1,\infty,0,1}_{A,\mathrm{fin}}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$, then such restriction has a bounded extension, denoted by $\widetilde{\mathcal{T}}$, to the whole space $H^1_A(\mathbb{R}^n)$. However, $\mathcal{T}$ itself does not have such property. Precisely, $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ coincide on all continuous $(1, \infty, 0)$ atoms, while not on all $(1, \infty, 0)$ atoms; see also [53]. This shows that it is necessary to restrict the operator T only on continuous atoms for $r = \infty$ in Corollary 6.14(ii).

Now we use Corollary 6.14 and Theorem 6.1 to show the boundedness of the δ-type Calderón-Zygmund operators from $H^{p,q}_A(\mathbb{R}^n)$ to $L^{p,q}(\mathbb{R}^n)$ (or to $H^{p,q}_A(\mathbb{R}^n)$).

Theorem 6.16. *Let $\delta \in (0, \frac{\ln \lambda_-}{\ln b}]$, $p \in (\frac{1}{1+\delta}, 1]$ and $q \in (0, \infty]$.*

(i) *If T is either a convolutional δ-type Calderón-Zygmund operator as in Theorem* 6.8 *or a non-convolutional δ-type Calderón-Zygmund operator as in Remark* 6.10(ii)*, then there exists a positive constant C_{20} such that, for all $f \in H^{p,q}_A(\mathbb{R}^n)$,*

$$\|T(f)\|_{L^{p,q}(\mathbb{R}^n)} \leqslant C_{20}\|f\|_{H^{p,q}_A(\mathbb{R}^n)}.$$

(ii) *If T is either a convolutional δ-type Calderón-Zygmund operator as in Theorem* 6.8 *or a non-convolutional δ-type Calderón-Zygmund operator satisfying the additional assumption that $T^*1 = 0$ as in Remark* 6.10(ii)*, then there exists a positive constant C_{21} such that, for all $f \in H^{p,q}_A(\mathbb{R}^n)$,*

$$\|T(f)\|_{H^{p,q}_A(\mathbb{R}^n)} \leqslant C_{21}\|f\|_{H^{p,q}_A(\mathbb{R}^n)}.$$

Proof. We first prove (i). When $\delta \in (0, \frac{\ln \lambda_-}{\ln b}]$, $p \in (\frac{1}{1+\delta}, 1)$ and $q \in (0, \infty]$, by the proof of [9, p. 69, Theorem 9.9], we have $\|T(a)\|_{L^p(\mathbb{R}^n)} \lesssim 1$ for any $(p, 2, 0)$-atom a. From this and Corollary 6.14(i), we further deduce that, for all $f \in H^{p,q}_A(\mathbb{R}^n)$,

$$\|T(f)\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{H^{p,q}_A(\mathbb{R}^n)}. \tag{6.24}$$

Notice that T is a linear operator. By (6.24), the corresponding interpolation result of Lebesgue spaces (see, for example, [55, Theorem 3]) and Theorem 6.1(i), we easily conclude that (i) holds true when $p \in (\frac{1}{1+\delta}, 1)$ and $q \in (0, \infty]$.

When $p = 1$ and $q \in (0, \infty]$, combining the linearity of T, (6.24) with $p = q$, the boundedness of T on $L^r(\mathbb{R}^n)$ with $r \in (1, \infty)$ (see, for example, [28, Theorems 5.1 and 5.10]) and Corollary 6.5, we find that (i) also holds true in this case.

Now we turn to show (ii). Notice that, by [9, p. 64, Lemma 9.5] and Theorem 3.6, we know that $\|T(a)\|_{H^p_A(\mathbb{R}^n)} \lesssim 1$ for any $(p, 2, 0)$-atom a. From this, Corollary 6.5 and an argument similar to the proof of (i), we deduce that (ii) holds true. This finishes the proof of Theorem 6.16. □

Remark 6.17. (i) Notice that the δ-type Calderón-Zygmund operators are linear operators. By Remark 6.10(iv), [28, Theorems 5.1 and 5.10], Corollary 6.5 and the corresponding interpolation result of Lorentz spaces (see, for example, [55, Theorem 3]), we also conclude that, as in Theorem 6.16, the boundedness of the δ-type Calderón-Zygmund operators from $H_A^{p,q}(\mathbb{R}^n)$ to $L^{p,q}(\mathbb{R}^n)$ (or to $H_A^{p,q}(\mathbb{R}^n)$) with $\delta \in (0, \frac{\ln \lambda_-}{\ln b}]$, $p \in (\frac{1}{1+\delta}, 1]$ and $q \in (0, \infty]$, the details being omitted.

(ii) We should point out that, in (i) of this remark, the boundedness of the δ-type Calderón-Zygmund operators from $H_A^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ (or to $H_A^p(\mathbb{R}^n)$) is a key tool, namely, [9, p. 68, Theorem 9.8; p. 69, Theorem 9.9] (see Remark 6.10(iv)). Notice that the proofs of [9, p. 68, Theorem 9.8; p. 69, Theorem 9.9] also need to prove that

$$\|T(a)\|_{H_A^p(\mathbb{R}^n)} \lesssim 1 \quad \text{and} \quad \|T(a)\|_{L^p(\mathbb{R}^n)} \lesssim 1$$

for any $(p, 2, 0)$-atom a, respectively, and are more complicated than the proof of Theorem 6.16. Thus, in this sense, the criterion established in Theorem 6.13 is a useful tool.

(iii) If A is the same as in Remark 6.10(i), then $\frac{\ln \lambda_-}{\ln b} = \frac{1}{n}$, $H_A^{p,q}(\mathbb{R}^n)$ and $L^{p,q}(\mathbb{R}^n)$ become the classical isotropic Hardy-Lorentz, respectively, Lorentz spaces and T becomes the classical δ-type Calderón-Zygmund operator correspondingly. In this case, we know that, if $\delta \in (0, 1]$, $p \in (\frac{n}{n+\delta}, 1]$ and $q \in (0, \infty]$, then Theorem 6.16(i) implies that T is bounded from $H^{p,q}(\mathbb{R}^n)$ to $H^{p,q}(\mathbb{R}^n)$ and Theorem 6.16(ii) implies that T is bounded from $H^{p,q}(\mathbb{R}^n)$ to $L^{p,q}(\mathbb{R}^n)$. Moreover, when $p = q$, (i) and (ii) of Theorem 6.16 imply the boundedness of the classical δ-type Calderón-Zygmund operator from $H^p(\mathbb{R}^n)$ to $H^p(\mathbb{R}^n)$, respectively, from $H^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for $\delta \in (0, 1]$ and $p \in (\frac{n}{n+\delta}, 1]$, which is a well-known result (see, for example, [5, 52, 69]).

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References

1 Abu-Shammala W, Torchinsky A. The Hardy-Lorentz spaces $H^{p,q}(\mathbb{R}^n)$. Studia Math, 2007, 182: 283–294

2 Almeida A, Caetano A M. Generalized Hardy spaces. Acta Math Sin Engl Ser, 2010, 26: 1673–1692

3 Álvarez J. H^p and weak H^p continuity of Calderón-Zygmund type operators. In: Fourier Analysis. Lecture Notes in Pure and Appl Math, vol. 157. New York: Dekker, 1994, 17–34

4 Álvarez J. Continuity properties for linear commutators of Calderón-Zygmund operators. Collect Math, 1998, 49: 17–31

5 Álvarez J, Milman M. H^p continuity properties of Calderón-Zygmund-type operators. J Math Anal Appl, 1986, 118: 63–79

6 Aoki T. Locally bounded linear topological spaces. Proc Imp Acad Tokyo, 1942, 18: 588–594

7 Bennett C, Sharpley R. Interpolation of Operators. Orlando: Academic Press, 1988

8 Bergh J, Löfström J. Interpolation Spaces. An Introduction. Berlin-New York: Springer-Verlag, 1976

9 Bownik M. Anisotropic Hardy Spaces and Wavelets. Memoirs of the American Mathematical Society, vol. 164, No. 781. Providence: Amer Math Soc, 2003

10 Bownik M. Boundedness of operators on Hardy spaces via atomic decompositions. Proc Amer Math Soc, 2005, 133: 3535–3542

11 Bownik M. Anisotropic Triebel-Lizorkin spaces with doubling measures. J Geom Anal, 2007, 17: 387–424

12 Bownik M, Ho K-P. Atomic and molecular decompositions of anisotropic Triebel-Lizorkin spaces. Trans Amer Math Soc, 2006, 358: 1469–1510

13 Bownik M, Li B, Yang D, et al. Weighted anisotropic Hardy spaces and their applications in boundedness of sublinear operators. Indiana Univ Math J, 2008, 57: 3065–3100

14 Calderón A-P. Intermediate spaces and interpolation: The complex method. Studia Math, 1964, 24: 113–190

15 Calderón A-P. An atomic decomposition of distributions in parabolic H^p spaces. Adv Math, 1977, 25: 216–225

16 Calderón A-P, Torchinsky A. Parabolic maximal functions associated with a distribution. Adv Math, 1975, 16: 1–64

17 Calderón A-P, Torchinsky A. Parabolic maximal functions associated with a distribution. II. Adv Math, 1977, 24: 101–171

18 Coifman R R, Lions P-L, Meyer Y, et al. Compensated compactness and Hardy spaces. J Math Pures Appl (9), 1993, 72: 247–286

19 Coifman R R, Weiss G. Analyse Harmonique Noncommutative sur Certains Espaces Homogènes. In: Lecture Notes in Mathematics, vol. 242. Berlin: Springer, 1971

20 Coifman R R, Weiss G. Extensions of Hardy spaces and their use in analysis. Bull Amer Math Soc, 1977, 83: 569–645

21 Cwikel M. The dual of weak L^p. Ann Inst Fourier (Grenoble), 1975, 25: 81–126

22 Cwikel M, Fefferman C. Maximal seminorms on weak L^1. Studia Math, 1980/1981, 69: 149–154

23 Cwikel M, Fefferman C. The canonical seminorm on weak L^1. Studia Math, 1984, 78: 275–278

24 Ding Y, Lan S. Anisotropic weak Hardy spaces and interpolation theorems. Sci China Ser A, 2008, 51: 1690–1704

25 Ding Y, Lu S. Hardy spaces estimates for multilinear operators with homogeneous kernels. Nagoya Math J, 2003, 170: 117–133

26 Ding Y, Lu S, Shao S. Integral operators with variable kernels on weak Hardy spaces. J Math Anal Appl, 2006, 317: 127–135

27 Ding Y, Lu S, Xue Q. Parametrized Littlewood-Paley operators on Hardy and weak Hardy spaces. Math Nachr, 2007, 280: 351–363

28 Duoandikoetxea J. Fourier Analysis. In: Graduate Studies in Mathematics, vol. 29. Providence: Amer Math Soc, 2001, xviii+222pp

29 Fefferman C, Rivière N M, Sagher Y. Interpolation between H^p spaces: The real method. Trans Amer Math Soc, 1974, 191: 75–81

30 Fefferman C, Stein E M. H^p spaces of several variables. Acta Math, 1972, 129: 137–193

31 Fefferman R, Soria F. The spaces weak H^1. Studia Math, 1987, 85: 1–16

32 Folland G B, Stein E M. Hardy Spaces on Homogeneous Groups. Princeton: Princeton University Press; Tokyo: University of Tokyo Press, 1982

33 Fu X, Lin H B, Yang D C, et al. Hardy spaces H^p over non-homogeneous metric measure spaces and their applications. Sci China Math, 2015, 58: 309–388

34 Grafakos L. Hardy space estimates for multilinear operators, II. Rev Mat Iberoam, 1992, 8: 69–92

35 Grafakos L. Classical Fourier Analysis, 2nd ed. New York: Springer, 2008

36 Grafakos L. Modern Fourier Analysis, 2nd ed. New York: Springer, 2009

37 Grafakos L, Liu L, Yang D. Maximal function characterizations of Hardy spaces on RD-spaces and their applications. Sci China Ser A, 2008, 51: 2253–2284

38 Hunt R. On $L(p, q)$ spaces. Enseign Math, 1966, 12: 249–276

39 Ioku N, Ishige K, Yanagida E. Sharp decay estimates in Lorentz spaces for nonnegative Schrödinger heat semigroups. J Math Pures Appl (9), 2015, 103: 900–923

40 Ky L D. New Hardy spaces of Musielak-Orlicz type and boundedness of sublinear operators. Integral Equations Operator Theory, 2014, 78: 115–150

41 Li B, Bownik M, Yang D. Littlewood-Paley characterization and duality of weighted anisotropic product Hardy spaces. J Funct Anal, 2014, 266: 2611–2661

42 Li B, Bownik M, Yang D, et al. Duality of weighted anisotropic Besov and Triebel-Lizorkin spaces. Positivity, 2012, 16: 213–244

43 Li B, Bownik M, Yang D, et al. A mean characterization of weighted anisotropic Besov and Triebel-Lizorkin spaces. Z Anal Anwend, 2014, 33: 125–147

44 Liang Y, Liu L, Yang D. An off-diagonal Marcinkiewicz interpolation theorem on Lorentz spaces. Acta Math Sin Engl Ser, 2011, 27: 1477–1488

45 Lin H, Yang D. Equivalent boundedness of Marcinkiewicz integrals on non-homogeneous metric measure spaces. Sci China Math, 2014, 57: 123–144

46 Lions J-L, Peetre J. Sur une classe d'espaces d'interpolation (in French). Inst Hautes études Sci Publ Math, 1964, 19: 5–68

47 Liu H. The weak H^p spaces on homogeneous groups. In: Harmonic Analysis. Lecture Notes in Math, vol. 1494. Berlin: Springer, 1991, 113–118

48 Liu J, Yang D, Yuan W. Littlewood-Paley characterizations of anisotropic Hardy-Lorentz spaces. ArXiv:1601.05242, 2016

49 Liu S, Zhao K. Various characterizations of product Hardy spaces associated to Schrödinger operators. Sci China Math, 2015, 58: 2549–2564

50 Lorentz G G. Some new functional spaces. Ann of Math (2), 1950, 51: 37–55

51 Lorentz G G. On the theory of spaces Λ. Pacific J Math, 1951, 1: 411–429

52 Lu S. Four Lectures on Real H^p Spaces. River Edge: World Scientific, 1995, viii+217pp

53 Meda S, Sjögren P, Vallarino M. On the H^1-L^1 boundedness of operators. Proc Amer Math Soc, 2008, 136: 2921–2931

54 Merker J, Rakotoson J-M. Very weak solutions of Poisson's equation with singular data under Neumann boundary conditions. Calc Var Partial Differential Equations, 2015, 52: 705–726

55 Merucci C. Applications of interpolation with a function parameter to Lorentz, Sobolev and Besov spaces. In: Inter-

polation Spaces and Allied Topics in Analysis. Lecture Notes in Math, vol. 1070. Berlin: Springer, 1984, 183–201
56 Muscalu C, Tao T, Thiele C. A counterexample to a multilinear endpoint question of Christ and Kiselev. Math Res Lett, 2003, 10: 237–246
57 Müller S. Hardy space methods for nonlinear partial differential equations: Equadiff 8. Tatra Mt Math Publ, 1994, 4: 159–168
58 Nakai E, Sawano Y. Orlicz-Hardy spaces and their duals. Sci China Math, 2014, 57: 903–962
59 Oberlin R, Seeger A, Tao T, et al. A variation norm Carleson theorem. J Eur Math Soc, 2012, 14: 421–464
60 Parilov D. Two theorems on the Hardy-Lorentz classes $H^{1,q}$ (in Russian). J Math Sci, 2006, 139: 6447–6456
61 Peetre J. Nouvelles propriétés d'espaces d'interpolation. C R Acad Sci Paris, 1963, 256: 1424–1426
62 Phuc N C. The Navier-Stokes equations in nonendpoint borderline Lorentz spaces. J Math Fluid Mech, 2015, 17: 741–760
63 Rolewicz S. On a certain class of linear metric spaces. Bull Acad Polon Sci Cl Trois, 1957, 5: 471–473
64 Rudin W. Functional Analysis, 2nd ed. New York: McGraw-Hill Inc, 1991
65 Sadosky C. Interpolation of Operators and Singular Integrals. New York: Marcel Dekker, 1976
66 Schmeisser H-J, Triebel H. Topics in Fourier Analysis and Function Spaces. Chichester: John Wiley and Sons Ltd, 1987
67 Seeger A, Tao T. Sharp Lorentz space estimates for rough operators. Math Ann, 2001, 320: 381–415
68 Semmes S. A primer on Hardy spaces, and some remarks on a theorem of Evans and Müller. Comm Partial Differential Equations, 1994, 19: 277–319
69 Stein E M. Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals. Princeton: Princeton University Press, 1993
70 Stein E M, Weiss G. Introduction to Fourier Analysis on Euclidean Spaces. Princeton: Princeton University Press, 1971
71 Strömberg J-O, Torchinsky A. Weighted Hardy Spaces. Berlin: Springer-Verlag, 1989
72 Tan C, Li J. Littlewood-Paley theory on metric spaces with non doubling measures and its applications. Sci China Math, 2015, 58: 983–1004
73 Tao T, Wright J. Endpoint multiplier theorems of Marcinkiewicz type. Rev Mat Iberoam, 2001, 17: 521–558
74 Triebel H. Theory of Function Spaces. Basel: Birkhäuser Verlag, 1983
75 Triebel H. Theory of Function Spaces. II. Basel: Birkhäuser Verlag, 1992
76 Triebel H. Theory of Function Spaces. III. Basel: Birkhäuser Verlag, 2006
77 Wang H. Boundedness of several integral operators with bounded variable kernels on Hardy and weak Hardy spaces. Internat J Math, 2013, 24: 5–6
78 Yang D, Zhou Y. Boundedness of sublinear operators in Hardy spaces on RD-spaces via atoms. J Math Anal Appl, 2008, 339: 622–635
79 Yang D, Zhou Y. A boundedness criterion via atoms for linear operators in Hardy spaces. Constr Approx, 2009, 29: 207–218