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Small prime solutions to cubic equations

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Abstract Let $a_1,\dots,a_9$ be nonzero integers not of the same sign, and let b be an integer. Suppose that $a_1,\dots,a_9$ are pairwise coprime and $a_1+\cdots+a_9\equiv b \pmod 2$. We apply the p-adic method of Davenport to find an explicit $P=P(a_1,\dots,a_9,n)$ such that the cubic equation $a_1p_1^3+\cdots+a_9p_9^3=b$ is solvable with $p_j\ll P$ for all $1\leqslant j\leqslant 9$. It is proved that one can take $P=\max\{|a_1|,\dots,|a_9|\}^c+|b|^{1/3}$ with $c=2$. This improves upon the earlier result with $c=14$ due to Liu (2013).

Keywords circle method, prime number, cubic equation

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1 Introduction

The works of Vinogradov [17] and Hua [7] established that for any natural number k, there exists $s=s(k)$ such that any sufficiently large integer n satisfying certain congruence conditions can be represented as

$$n=p_1^k+\cdots+p_s^k,$$

where $p_1,\dots,p_s$ are prime numbers. In general, one would ask how to find an explicit $P=P(a_1,\dots,a_s,n)$ such that the equation

$$a_1p_1^k+\cdots+a_sp_s^k=n$$

is solvable in the box $p_j\leqslant P$ $(1\leqslant j\leqslant s)$. Following the pioneer work of Baker [1], Liu and Tsang [12] made substantial progress for the linear case $k=1$ with $s=3$ and the quadratic case $k=2$ with $s=5$ in [13]. Their results were considerably improved by Li [10], Choi and Liu [3,4], Liu and Tsang [11], Choi and Kumchev [2], and Harman and Kumchev [6]. In the cubic case, the classical result of Hua asserts that every sufficiently large odd number can be represented in the form

$$n=p_1^3+\cdots+p_9^3.$$

Leung [9] considered the cubic equation

$$a_1p_1^3+\cdots+a_9p_9^3=n, \tag{1.1}$$

where $p_1, \ldots, p_9$ are prime variables, $a_1, \ldots, a_9$ are nonzero integer coefficients and $n \in \mathbb{Z}$. Throughout, we suppose that

$$(a_i, a_j) = 1 \quad \text{for} \quad 1 \leqslant i < j \leqslant 9, \tag{1.2}$$

and

$$a_1 + \cdots + a_9 \equiv n \pmod 2. \tag{1.3}$$

Leung established that if $a_1, \ldots, a_9$ are not of the same sign, then (1.1) is solvable in primes p_j $(1 \leqslant j \leqslant 9)$ with

$$p_j \ll \max\{|a_1|, \ldots, |a_9|\}^{20+\varepsilon} + |n|^{1/3}, \tag{1.4}$$

where the implied constant depends only on ε. This was refined by Liu [14], who showed that the exponent 20 in (1.4) can be reduced to 14. In this paper, we prove the following theorem.

Theorem 1.1. *Let $a_1, \ldots, a_9$ be nonzero integers, and let $n \in \mathbb{Z}$. Suppose that $a_1, \ldots, a_9$ and n satisfy* (1.2) *and* (1.3). *If $a_1, \ldots, a_9$ are not of the same sign, then there are prime solutions to the equation* (1.1) *with*

$$p_j \ll \max\{|a_1|, \ldots, |a_9|\}^2 + |n|^{1/3} \tag{1.5}$$

for all $1 \leqslant j \leqslant 9$.

The proof can be applied to establish the following parallel result. We omit the details.

Theorem 1.2. *Suppose that $a_1, \ldots, a_9, n$ are positive integers satisfying* (1.2) *and* (1.3). *Then there exists an absolute constant $\mathcal{K} > 0$ such that the equation* (1.1) *is solvable whenever*

$$n \geqslant \mathcal{K} \max\{|a_1|, \ldots, |a_9|\}^7.$$

Our improvement comes from the application of the p-adic method of Davenport [5]. In particular, our treatment of the mean value estimate makes use of the condition that $a_1, \ldots, a_9$ are pairwise coprime.

As usual, we abbreviate $\mathrm{e}^{2\pi \mathrm{i}z}$ to $e(z)$. The letter p, with or without a subscript, always denotes a prime number. We use ε to denote a sufficiently small positive number. Denote by $\phi(n)$ the Euler function.

2 Preliminaries

For $X \geqslant 1$, we define

$$g_X(\alpha) = \sum_{1 \leqslant x \leqslant X} e(x^3 \alpha).$$

By Hua's lemma, one has

$$\int_0^1 |g_X(\alpha)|^6 d\alpha \ll X^{7/2+\varepsilon}. \tag{2.1}$$

Let

$$\mathcal{M}(X) = \bigcup_{q \leqslant X} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} \left[\frac{a}{q} - \frac{1}{qX^2}, \frac{a}{q} + \frac{1}{qX^2} \right].$$

In view of the proof of [16, Theorem 4.4], one has

$$\int_{\mathcal{M}(X)} |g_X(\alpha)|^6 d\alpha \ll X^3. \tag{2.2}$$

We define

$$T(P, X) = \sum_{q \leqslant P} \sum_{1 \leqslant a \leqslant Aq} \int_{|\beta| \leqslant \frac{1}{qAHP}} \frac{1}{\sqrt{q(1 + AHP^2|\beta|)}} \left| g_X\left(\frac{a}{Aq} + \beta \right) \right|^2 d\beta. \tag{2.3}$$

Lemma 2.1. *Let $T(P,X)$ be defined as above. Then one has*

$$T(P,X) \ll X^{2+\varepsilon}H^{-1}P^{-3/2} + XH^{-1}P^{-1/2}.$$

Proof. The argument is routine. We include the details for completeness. Note that

$$\sum_{1\leqslant a\leqslant Aq} \left| g_X\left(\frac{a}{Aq}+\beta\right)\right|^2 \leqslant A \sum_{1\leqslant x\neq y\leqslant X} (q, x^3-y^3) + XAq.$$

Therefore,

$$\begin{aligned} T(P,X) &\leqslant \sum_{q\leqslant P}\left(A\sum_{1\leqslant x\neq y\leqslant X}\frac{(q,x^3-y^3)}{\sqrt{q}} + XA\sqrt{q}\right)\int_{|\beta|\leqslant\frac{1}{qAHP}}\frac{1}{\sqrt{(1+AHP^2|\beta|)}}d\beta \\ &\ll \sum_{q\leqslant P}\left(\sum_{1\leqslant x\neq y\leqslant X}\frac{(q,x^3-y^3)}{\sqrt{q}} + X\sqrt{q}\right)(HP^2)^{-1}\sqrt{P/q} \\ &\ll X^{2+\varepsilon}H^{-1}P^{-3/2} + XH^{-1}P^{-1/2}. \end{aligned}$$

This completes the proof. □

We define

$$F(\alpha) = \sum_{1\leqslant h\leqslant H}\sum_{1\leqslant x\leqslant P} e((3hx^2+3h^2Bx+h^3B^2)\alpha),$$

where $H \geqslant 1$ and B is a natural number satisfying $1 \leqslant BH \leqslant P$.

Lemma 2.2. *Suppose that $|\alpha - a/q| \leqslant (qHP)^{-1}$ for some $a\in\mathbb{Z}$ and $q\in\mathbb{N}$ with $1\leqslant q\leqslant HP$. Then one has*

$$F(\alpha) \ll HP^{1+\varepsilon}\left(\frac{1}{q(1+HP^2|\alpha-a/q|)} + \frac{1}{P} + \frac{q(1+HP^2|\alpha-a/q|)}{HP^2}\right)^{1/2}.$$

Proof. In view of the proof of the lemma in [15], one has

$$F(\alpha) \ll HP^{1+\varepsilon}\left(\frac{1}{q} + \frac{1}{P} + \frac{q}{HP^2}\right)^{1/2}. \tag{2.4}$$

It follows from (2.4) by the standard argument in Waring's problem (see [16, Exercise 2 of Chapter 2]) that

$$F(\alpha) \ll HP^{1+\varepsilon}\left(\frac{1}{q(1+HP^2|\alpha-a/q|)} + \frac{1}{P} + \frac{q(1+HP^2|\alpha-a/q|)}{HP^2}\right)^{1/2}.$$

We complete the proof. □

For any natural number A, we define

$$\mathcal{R}(P;A) = \bigcup_{q\leqslant P}\bigcup_{\substack{1\leqslant a\leqslant Aq\\(a,q)=1}}\left[\frac{a}{q}-\frac{1}{qHP}, \frac{a}{q}+\frac{1}{qHP}\right].$$

In light of Lemma 2.2, we have

$$F(\alpha) \ll \frac{HP^{1+\varepsilon}}{\sqrt{q(1+HP^2|\alpha-a/q|)}} \quad \text{for} \quad \alpha\in\mathcal{R}(P;A). \tag{2.5}$$

We define

$$\mathcal{R}_A(P) = \{\alpha\in[(AHP^2)^{-1}, 1+(AHP^2)^{-1}] : A\alpha\in\mathcal{R}(P;A)\}. \tag{2.6}$$

Now we conclude the following.

Lemma 2.3. *Let $\mathcal{R}_A(P)$ be defined as above. Then one has*

$$\int_{\mathcal{R}_A(P)} F(A\alpha)|g_X(\alpha)|^2 d\alpha \ll X^{2+\varepsilon}P^{-1/2+\varepsilon} + XP^{1/2+\varepsilon}. \tag{2.7}$$

Proof. Since $A\alpha \in \mathcal{R}(P;A)$, there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that $1 \leqslant q \leqslant P$, $1 \leqslant a \leqslant Aq$, $(a,q)=1$ and

$$\left|\alpha - \frac{a}{Aq}\right| \leqslant \frac{1}{qHPA}.$$

By (2.5), one has

$$F(A\alpha) \ll \frac{HP^{1+\varepsilon}}{\sqrt{q(1+HP^2|A\alpha - a/q|)}} = \frac{HP^{1+\varepsilon}}{\sqrt{q(1+AHP^2|\alpha - \frac{a}{Aq}|)}}.$$

We deduce that

$$\int_{\mathcal{R}_A(P)} F(A\alpha)|g_X(\alpha)|^2 d\alpha \ll HP^{1+\varepsilon}T(P,X),$$

where $T(P,X)$ is given by (2.3). We establish (2.7) by applying Lemma 2.1. □

Lemma 2.4. *Let $\mathcal{R}_A(P)$ be defined in* (2.6). *Then one has*

$$\int_{\mathcal{R}_A(P)} F(A\alpha)|g_X(\alpha)|^6 d\alpha \ll HPX^3 + X^{5+\varepsilon}P^{-1/2+\varepsilon} + X^{4+\varepsilon}P^{1/2+\varepsilon}.$$

Proof. Let

$$\mathcal{M} = \bigcup_{q\leqslant X} \bigcup_{\substack{-q\leqslant a\leqslant 2q \\ (a,q)=1}} \left[\frac{a}{q} - \frac{1}{qX^2}, \frac{a}{q} + \frac{1}{qX^2}\right].$$

For $\alpha \in \mathcal{R}_A(P)$, by Dirichlet's approximation theorem, there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $-q \leqslant a \leqslant 2q$, $1 \leqslant q \leqslant X^2$, $(a,q)=1$ and $|\alpha - a/q| \leqslant (qX^2)^{-1}$. For $\alpha \notin \mathcal{M}$, one has $q > X$ and thus by [16, Lemma 2.4], $g_X(\alpha) \ll X^{3/4+\varepsilon}$. We conclude that

$$g_X(\alpha) \ll X^{3/4+\varepsilon} \quad \text{for} \quad \alpha \in \mathcal{R}_A(P) \setminus \mathcal{M}.$$

Then we deduce

$$\int_{\mathcal{R}_A(P)\setminus\mathcal{M}} F(A\alpha)|g_X(\alpha)|^6 d\alpha \ll X^{3+\varepsilon} \int_{\mathcal{R}_A(P)} F(A\alpha)|g_X(\alpha)|^2 d\alpha,$$

and by Lemma 2.3,

$$\int_{\mathcal{R}_A(P)\setminus\mathcal{M}} F(A\alpha)|g_X(\alpha)|^6 d\alpha \ll X^{5+\varepsilon}P^{-1/2+\varepsilon} + X^{4+\varepsilon}P^{1/2+\varepsilon}. \tag{2.8}$$

On applying (2.2), we obtain

$$\int_{\mathcal{R}_A(P)\cap\mathcal{M}} F(A\alpha)|g_X(\alpha)|^6 d\alpha \ll HP \int_{\mathcal{M}(X)} |g_X(\alpha)|^6 d\alpha \ll HPX^3. \tag{2.9}$$

We complete the proof by combining (2.8) and (2.9). □

Lemma 2.5. *We have*

$$\int_0^1 F(A\alpha)|g_X(\alpha)|^6 d\alpha \ll HPX^3 + X^{5+\varepsilon}P^{-1/2+\varepsilon} + X^{4+\varepsilon}P^{1/2+\varepsilon} + HP^{1/2}X^{7/2+\varepsilon}.$$

Proof. Let $\mathfrak{r} = [(AHP^2)^{-1}, 1 + (AHP^2)^{-1}] \setminus \mathcal{R}_A(P)$. For any $\alpha \in \mathfrak{r}$, by Dirichlet's approximation theorem, there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a,q)=1$, $1 \leqslant a \leqslant Aq$, $q \leqslant HP$ and $|A\alpha - \frac{a}{q}| \leqslant (qHP)^{-1}$. Since $\alpha \in \mathfrak{r}$, one has $q > P$. By Lemma 2.2, $F(A\alpha) \ll HP^{1/2+\varepsilon}$. Thus by (2.1), we obtain

$$\int_{\mathfrak{r}} F(A\alpha)|g_X(\alpha)|^6 d\alpha \ll HP^{1/2+\varepsilon} \int_0^1 |g_X(\alpha)|^6 d\alpha \ll HP^{1/2+\varepsilon}X^{7/2+\varepsilon}. \tag{2.10}$$

We complete the proof by applying (2.10) and Lemma 2.4. □

3 Mean value estimate via Davenport's method

Let $S(A,B)$ denote the number of solutions to

$$A(x_1^3 - x_2^3) = B(y_1^3 + y_2^3 + y_3^3 - y_4^3 - y_5^3 - y_6^3), \tag{3.1}$$

where $1 \leqslant x_1, x_2 \leqslant P$, $(x_1x_2, B) = 1$, $B \mid x_1^3 - x_2^3$ and $1 \leqslant y_1, y_2, y_3, y_4, y_5, y_6 \leqslant Q$. Similarly, we define $S^-(A,B)$ to be the number of solutions to (3.1) with $1 \leqslant x_1, x_2 \leqslant P$, $(x_1x_2, B) = 1$, $B \mid x_1 - x_2$ and $1 \leqslant y_1, y_2, y_3, y_4, y_5, y_6 \leqslant Q$.

Davenport [5] observed that by choosing $p \equiv 2 \pmod 3$, the equation

$$x_1^3 - x_2^3 = p^3(y_1^3 + y_2^3 - y_3^3 - y_4^3) \tag{3.2}$$

forces $p^3 \mid x_1 - x_2$. In our application of Davenport's method, the parameters B in (3.1) and p^3 in (3.2) play the same role.

Since $B \mid x_1 - x_2$ implies $B \mid x_1^3 - x_2^3$, we have $S^-(A,B) \leqslant S(A,B)$. The first result in this section is as follows.

Lemma 3.1. *For any $\varepsilon > 0$, we have*

$$S(A,B) \leqslant B^{\varepsilon} S^-(A,B).$$

Proof. We introduce

$$g(\alpha; b) = \sum_{\substack{1\leqslant x\leqslant P \\ x\equiv b \pmod B}} e(Ax^3\alpha).$$

On writing $h(\alpha) = g_Q(B\alpha)$, we have

$$S(A,B) = \sum_{\substack{1\leqslant b_1,b_2\leqslant B \\ (b_1b_2,B)=1 \\ b_1^3-b_2^3\equiv 0 \pmod B}} \int_0^1 g(\alpha; b_1)g(-\alpha; b_2)|h(\alpha)^6| d\alpha.$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned}
S(A,B) &\leqslant \sum_{\substack{1\leqslant b_1,b_2\leqslant B \\ (b_1b_2,B)=1 \\ b_1^3-b_2^3\equiv 0 \pmod B}} \left(\int_0^1 |g(\alpha; b_1)^2 h(\alpha)^6| d\alpha\right)^{1/2} \left(\int_0^1 |g(\alpha; b_2)^2 h(\alpha)^6| d\alpha\right)^{1/2} \\
&\leqslant \left(\sum_{\substack{1\leqslant b_1,b_2\leqslant B \\ (b_1b_2,B)=1 \\ b_1^3-b_2^3\equiv 0 \pmod B}} \int_0^1 |g(\alpha; b_1)^2 h(\alpha)^6| d\alpha\right)^{1/2} \\
&\quad \times \left(\sum_{\substack{1\leqslant b_1,b_2\leqslant B \\ (b_1b_2,B)=1 \\ b_1^3-b_2^3\equiv 0 \pmod B}} \int_0^1 |g(\alpha; b_2)^2 h(\alpha)^6| d\alpha\right)^{1/2} \\
&= \sum_{\substack{1\leqslant b\leqslant B \\ (b,B)=1}} \upsilon(b) \int_0^1 |g(\alpha; b)^2 h(\alpha)^6| d\alpha,
\end{aligned}$$

where

$$\upsilon(b) = \sum_{\substack{1\leqslant c\leqslant B \\ (c,B)=1 \\ c^3-b^3\equiv 0 \pmod B}} 1.$$

We complete the proof by observing that $\upsilon(b) \ll B^\varepsilon$ and

$$S^-(A,B) = \sum_{\substack{1\leqslant b\leqslant B\\(b,B)=1}} \int_0^1 |g(\alpha;b)^2 h(\alpha)^6|d\alpha. \qquad \square$$

Lemma 3.2. *Suppose that $B < P$. For any $\varepsilon > 0$, we have*

$$S(A,B) \ll B^{-1+\varepsilon}P^2Q^3 + P^{-1/2+\varepsilon}Q^{5+\varepsilon} + P^{1/2+\varepsilon}Q^{4+\varepsilon} + PQ^{7/2+\varepsilon} + B^{-1+\varepsilon}P^{3/2}Q^{7/2+\varepsilon}.$$

Proof. By changing variables $x_2 = x, x_1 = x + hB$, (3.1) is reduced to

$$A(3hx^2 + 3h^2Bx + h^3B^2) = y_1^3 + y_2^3 + y_3^3 - y_4^3 - y_5^3 - y_6^3. \tag{3.3}$$

By (2.1), the contribution from $h = 0$ to $S^-(A,B)$ is $O(PQ^{7/2+\varepsilon})$. We use $S^+(A,B)$ to denote the number of solutions to equation (3.3) with $1 \leqslant h \leqslant P/B$, $1 \leqslant x \leqslant P$ and $1 \leqslant y_1, y_2, y_3, y_4, y_5, y_6 \leqslant Q$. Then it follows from the above that

$$S^-(A,B) \leqslant 2S^+(A,B) + O(PQ^{7/2+\varepsilon}).$$

We complete the proof by applying Lemmas 2.5 and 3.1. $\square$

As a consequence of Lemma 3.2, we conclude the following.

Lemma 3.3. *Suppose that $AP^3 = BQ^3$ and $A \leqslant B \leqslant P^{1/2}Q^\rho$ with $0 \leqslant \rho \leqslant 1/2$. For any $\varepsilon > 0$, we have*

$$S(A,B) \ll B^{-1}P^{2+\varepsilon}Q^{3+\rho}.$$

Let $S^\#(A,B)$ denote the number of solutions to

$$A(p_1^3 - p_2^3) = B(p_3^3 + p_4^3 + p_5^3 - p_6^3 - p_7^3 - p_8^3), \tag{3.4}$$

where $1 \leqslant p_1, p_2 \leqslant P$ and $1 \leqslant p_3, p_4, p_5, p_6, p_7, p_8 \leqslant Q$.

Lemma 3.4. *Suppose that $AP^3 = BQ^3$ and $A \leqslant B \leqslant P^{1/2}Q^\rho$ with $0 \leqslant \rho \leqslant 1/2$. If $(A,B) = 1$, then we have*

$$S^\#(A,B) \ll B^{-1}P^{2+\varepsilon}Q^{3+\rho}.$$

Proof. Since $(A,B) = 1$, the equation (3.4) forces that $B \mid p_1^3 - p_2^3$. The contribution from $(p_1p_2, B) = 1$ is at most $S(A,B)$. Note that $(p_1p_2, B) > 1$ implies $p_1 \mid B$ and $p_2 \mid B$, and thus the contribution from $(p_1p_2, B) > 1$ is $O(B^\varepsilon Q^{7/2+\varepsilon})$. The desired estimate follows from Lemma 3.3 immediately. $\square$

4 Proof of Theorem 1.1

For $X \geqslant 2$, we define

$$f_X(\alpha) = \sum_{p\leqslant X} e(p^3\alpha).$$

Lemma 4.1. *Suppose that there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a,q) = 1$, $1 \leqslant q \leqslant X$ such that $|\alpha - a/q| \leqslant (qX^2)^{-1}$. Then one has*

$$f_X(\alpha) \ll \frac{q^\varepsilon(\log X)^cX}{\sqrt{q(1 + X^3|\alpha - a/q|)}} + X^{11/12+\varepsilon},$$

where c is an absolute constant.

Proof. This follows from [8, Theorem 2] and [18, Lemma 8.5] $\square$

Lemma 4.2. *Suppose that $\delta < 1/18$. Let $\alpha \in \mathbb{R}$, $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with*

$$(a,q)=1, \quad 1 \leqslant q \leqslant N^{5\delta/6}, \quad |\alpha - a/q| \leqslant N^{5\delta/6}(qN)^{-1}.$$

Suppose that $X \gg N^{5/18}$ and $N = AX^3$. Then one has

$$f_X(A\alpha) \ll \frac{q^{\varepsilon}(\log X)X\sqrt{(q,A)}}{\sqrt{q(1+N|\alpha - a/q|)}}.$$

Proof. The desired conclusion follows from Lemma 4.1 by noting

$$\frac{q^{\varepsilon}(\log X)^c X\sqrt{(q,A)}}{\sqrt{q(1+N|\alpha - a/q|)}} \gg XN^{-5\delta/12} \gg X^{1-3\delta/2} \geqslant X^{11/12+\varepsilon}$$

provided that $\delta < 1/18$. □

Lemma 4.3. *Let $\delta < 1/12$. Suppose that $A_jX_j^3 = N$ and $A_jX_j^{2\delta} \leqslant \sqrt{N}/2$ for all $1 \leqslant j \leqslant 5$. Suppose that $|\alpha - a/q| \leqslant N^{5\delta/6}(qN)^{-1}$ for some $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with*

$$(a,q)=1 \quad \text{and} \quad N^{5\delta/6} < q \leqslant N^{1-5\delta/6}.$$

If $A_1, \ldots, A_5$ are pairwise coprime, then there exists $1 \leqslant j \leqslant 5$ such that

$$f_{X_j}(A_j\alpha) \ll X_j^{1-\delta+\varepsilon}.$$

Proof. Suppose that $f_{X_j}(A_j\alpha) \gg X_j^{1-\delta+\varepsilon}$ for all $1 \leqslant j \leqslant 5$. Then for $1 \leqslant j \leqslant 5$, we deduce from Lemma 4.1 that

$$\left|A_j\alpha - \frac{a_j}{q_j}\right| \leqslant \frac{X_j^{2\delta}}{q_jX_j^3} \quad \text{for some} \quad a_j \in \mathbb{Z}, \quad q_j \in \mathbb{N} \quad \text{with} \quad (a_j,q_j)=1, \quad q_j \leqslant X_j^{2\delta}.$$

Then for any $1 \leqslant i < j \leqslant 5$, one has

$$\begin{aligned} |a_iA_jq_j - a_jA_iq_i| &= A_iA_jq_iq_j\left|\frac{a_i}{A_iq_i} - \frac{a_j}{A_jq_j}\right| \leqslant A_iA_jq_iq_j\left(\left|\alpha - \frac{a_i}{A_iq_i}\right| + \left|\alpha - \frac{a_j}{A_jq_j}\right|\right) \\ &\leqslant A_iA_jq_iq_j\left(\frac{X_i^{2\delta}}{q_iN} + \frac{X_j^{2\delta}}{q_jN}\right) \leqslant \frac{2A_iA_jX_i^{2\delta}X_j^{2\delta}}{N} \leqslant \frac{1}{2}. \end{aligned}$$

Thus there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a,q)=1$ such that $\frac{a_j}{A_jq_j} = \frac{a}{q}$ for all $1 \leqslant j \leqslant 5$. Therefore $q = q_1A_1' = \cdots = q_5A_5'$, where $A_j' \mid A_j$. On recalling that $A_1, \ldots, A_5$ are pairwise coprime, we have $q = q'A_1'A_2'A_3'A_4'A_5'$ for some $q' \in \mathbb{N}$. Without loss of generality, we assume that $A_1' \leqslant A_2' \leqslant \cdots \leqslant A_5'$, and thus $A_1' \leqslant (q'A_1'A_2'A_3'A_4')^{1/4} = q_5^{1/4} \leqslant X_5^{\delta/2}$. It follows that $q = q_1A_1' \leqslant X_1^{2\delta}X_5^{\delta/2} \leqslant N^{5\delta/6}$ and

$$\left|\alpha - \frac{a}{q}\right| = \left|\alpha - \frac{a_1}{A_1q_1}\right| \leqslant \frac{X_1^{2\delta}}{q_1N} \leqslant \frac{X_1^{5\delta/2}}{qN} \leqslant \frac{N^{5\delta/6}}{qN}.$$

We complete the proof. □

Without loss of generality, we assume that $0 < |a_1| < |a_2| < \cdots < |a_9|$. Suppose that $P_1 = \mathcal{K}(|a_9|^2 + |b|^{1/3})$, where $\mathcal{K} > 1$ is a sufficiently large constant. Then we define $P_j = (|a_1|/|a_j|)^{1/3}P_1$ for $2 \leqslant j \leqslant 9$. We write $N = |a_1|P_1^3$ and

$$\mathcal{P} = \prod_{j=1}^{9} P_j.$$

We define

$$r(n) = \sum_{\substack{a_1p_1^3 + \cdots + a_9p_9^3 = n \\ 1 \leqslant p_j \leqslant P_j (1 \leqslant j \leqslant 9)}} 1.$$

We introduce the singular series

$$\mathfrak{S}(a_1,\dots,a_9,n)=\sum_{q=1}^{\infty}\frac{1}{\phi^9(q)}\sum_{\substack{a=1\\(a,q)=1}}^{q}\bigg(\prod_{j=1}^{9}C(q,a_ja)\bigg)e(-an/q),$$

where

$$C(q,a)=\sum_{\substack{x=1\\(x,q)=1}}^{q}e(ax^3/q).$$

The singular integral is given by

$$\mathfrak{I}(n)=\int_{-\infty}^{\infty}\bigg(\prod_{j=1}^{9}\int_1^{P_j}\frac{e(a_jx^3\beta)}{\log x}dx\bigg)e(-n\beta)d\beta.$$

We point out that

$$\mathfrak{S}(a_1,\dots,a_9,n)\gg 1\quad\text{if}\quad a_1+\cdots+a_9\equiv n\pmod 2,$$

and

$$\mathfrak{I}(n)\gg\mathcal{P}(\log N)^{-9}N^{-1}\quad\text{if}\quad a_1,\dots,a_9\quad\text{are not of the same sign.}$$

Therefore Theorem 1.1 follows from

$$r(n)=\mathfrak{S}(a_1,\dots,a_9,n)\mathfrak{I}(n)+O(\mathcal{P}N^{-1}\mathcal{L}^{-1/100}),$$

where $\mathcal{L}=(\log N)^{\mathcal{C}}$ with $\mathcal{C}$ a sufficiently large constant. We define

$$f_j(\alpha)=f_{P_j}(a_j\alpha)\quad(1\leqslant j\leqslant 9),$$

and write

$$\mathcal{F}(\alpha)=\prod_{j=1}^{9}f_j(\alpha).$$

By orthogonality, we have

$$r(n)=\int_0^1\mathcal{F}(\alpha)e(-n\alpha)d\alpha.$$

Suppose that $1/20<\delta<1/18$. Let

$$\mathfrak{M}(X)=\bigcup_{q\leqslant X}\bigcup_{\substack{a=1\\(a,q)=1}}^{q}\bigg[\frac{a}{q}-\frac{X}{qN},\frac{a}{q}+\frac{X}{qN}\bigg].$$

We define

$$\mathfrak{M}=\mathfrak{M}(\mathcal{L}),\quad\mathfrak{n}=\mathfrak{M}(N^{5\delta/6})\setminus\mathfrak{M}(\mathcal{L}),\quad\mathfrak{m}=\bigg[\frac{N^{5\delta/6}}{N},1+\frac{N^{5\delta/6}}{N}\bigg]\setminus\mathfrak{M}(N^{5\delta/6}).$$

We have

$$r(n)=\int_{\mathfrak{M}}\mathcal{F}(\alpha)e(-n\alpha)d\alpha+\int_{\mathfrak{n}}\mathcal{F}(\alpha)e(-n\alpha)d\alpha+\int_{\mathfrak{m}}\mathcal{F}(\alpha)e(-n\alpha)d\alpha.$$

The standard argument in the Waring-Goldbach problem leads to

$$\int_{\mathfrak{M}}F(\alpha)e(-n\alpha)d\alpha=\mathfrak{S}(a_1,\dots,a_9,n)\mathfrak{I}(n)+O(\mathcal{P}N^{-1}\mathcal{L}^{-1/100}).\tag{4.1}$$

Therefore it suffices to consider the contribution from the integration over $\mathfrak{m} \cup \mathfrak{n}$.

Note that $N = |a_1|P_1^3 = \cdots = |a_9|P_9^3$. For all $1 \leqslant j \leqslant 9$, we easily check

$$P_j = (|a_1|/|a_j|)^{1/3} P_1 \geqslant |a_1|^{1/3} P_1^{5/6} \geqslant (|a_1|P_1^3)^{5/18} = N^{5/18}. \tag{4.2}$$

Thus by Lemma 4.3, for $\alpha \in \mathfrak{m}$, there exists $2 \leqslant j \leqslant 6$ such that $f_j(\alpha) \ll P_j^{1-\delta+\varepsilon}$. We write $\{2,3,4,5,6\} \setminus \{j\} = \{k_1, k_2, k_3, k\}$. Then

$$\int_{\mathfrak{m}} |F(\alpha)| d\alpha \ll P_j^{1-\delta+\varepsilon} \left(\prod_{i=1}^{3} \int_0^1 |f_1(\alpha)^2 f_{k_i}(\alpha)^6| d\alpha \right)^{1/6} \left(\prod_{i=7}^{9} \int_0^1 |f_k(\alpha)^2 f_i(\alpha)^6| d\alpha \right)^{1/6}.$$

For $1 \leqslant i \leqslant 3$, by Lemma 3.4 with $\rho = 0$, we have

$$\int_0^1 |f_1(\alpha)^2 f_{k_i}(\alpha)^6| d\alpha \ll N^{-1+\varepsilon} P_1^2 P_{k_i}^6.$$

For $7 \leqslant i \leqslant 9$, by (4.2), one has

$$P_k^{1/2} P_i^{1/10} \geqslant P_1^{1/2} \geqslant |a_9| > |a_i|.$$

Then we apply Lemma 3.4 with $\rho = 1/10$ to conclude that

$$\int_0^1 |f_k(\alpha)^2 f_i(\alpha)^6| d\alpha \ll N^{-1+\varepsilon} P_k^2 P_i^{6+1/10} \leqslant N^{-1+\varepsilon} P_k^2 P_i^6 P_j^{1/10} \quad (7 \leqslant i \leqslant 9).$$

It follows from the above that

$$\int_{\mathfrak{m}} |F(\alpha)| d\alpha \ll P_j^{1/20-\delta+\varepsilon} N^{-1} \mathcal{P}. \tag{4.3}$$

We introduce the function $\Psi(\alpha)$ on $\mathfrak{M}(N^{5\delta/6})$ by taking

$$\Psi(\alpha) = \frac{1}{q(1+N|\beta|)}$$

when $\alpha = a/q + \beta$ with $1 \leqslant a \leqslant q \leqslant N^{5\delta/6}$, $(a,q) = 1$ and $|\beta| \leqslant N^{5\delta/6}(qN)^{-1}$. We have by Lemma 4.2 that

$$F(\alpha) \ll \mathcal{P} (\log N)^c \Psi(\alpha)^{3.9} \quad \text{for} \quad \alpha \in \mathfrak{M}(N^{5\delta/6}).$$

In particular, one has

$$F(\alpha) \ll \mathcal{P} (\log N)^c \mathcal{L}^{-3.9} \quad \text{for} \quad \alpha \in \mathfrak{n}.$$

Therefore,

$$\begin{aligned} \int_{\mathfrak{n}} |F(\alpha)| d\alpha &\ll \mathcal{P}^{\frac{10}{39}} (\log N)^{\frac{10c}{39}} \mathcal{L}^{-1} \int_{\mathfrak{n}} |F(\alpha)|^{\frac{29}{39}} d\alpha \\ &\ll \mathcal{P} (\log N)^c \mathcal{L}^{-1} \int_{\mathfrak{M}(N^{5\delta/6})} \Psi(\alpha)^{2.9} d\alpha. \end{aligned}$$

It follows that

$$\int_{\mathfrak{n}} |F(\alpha)| d\alpha \ll \mathcal{P} (\log N)^c \mathcal{L}^{-1} N^{-1}. \tag{4.4}$$

We complete the proof of Theorem 1.1 by combining (4.1), (4.3) and (4.4).

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References

1 Baker A. On some Diophantine inequalities involving primes. J Reine Angew Math, 1967, 228: 166–181
2 Choi S K K, Kumchev A V. Mean values of Dirichlet polynomials and applications to linear equations with prime variables. Acta Arith, 2006, 132: 125–142
3 Choi S K K, Liu J Y. Small primes solutions to quadratic equations. Canad J Math, 2002, 54: 71–91
4 Choi S K K, Liu J Y. Small primes solutions to quadratic equations II. Proc Amer Math Soc, 2005, 133: 945–951
5 Davenport H. Sums of three positive cubes. J Lond Math Soc, 1950, 25: 339–343
6 Harman G, Kumchev A V. On sums of squares of primes. Math Proc Cambridge Philos Soc, 2006, 140: 1–13
7 Hua L K. Some results in additive number theory. Q J Math, 1938, 9: 68–80
8 Kumchev A V. On Weyl sums over primes and almost primes. Michigan Math J, 2006, 54: 243–268
9 Leung D. Small prime solutions to cubic Diophantine equations. ArXiv:0610739, 2006
10 Li H Z. Small prime solutions of linear ternary equations. Acta Arith, 2001, 98: 293–309
11 Liu J Y, Tsang K M. Small prime solutions of ternary linear equations. Acta Arith, 2005, 118: 79–100
12 Liu M C, Tsang K M. Small prime solutions of linear equations. In: Théorie des Nombres. Berlin: Walter de Gruyter, 1991, 595–624
13 Liu M C, Tsang K M. Small prime solutions of some additive equations. Monatsh Math, 1991, 111: 147–169
14 Liu Z X. Small prime solutions to cubic Diophantine equations. Canad Math Bull, 2013, 56: 785–794
15 Vaughan R C. Sum of three cubes. Bull Lond Math Soc, 1985, 17: 17–20
16 Vaughan R C. The Hardy-Littlewood Method. 2nd ed. Cambridge: Cambridge University Press, 1997
17 Vinogradov I M. Representation of an odd number as a sum of three primes. Dokl Akad Nauk SSSR, 1937, 15: 291–294
18 Zhao L L. On the Waring-Goldbach problem for fourth and sixth powers. Proc Lond Math Soc, 2014, 108: 1593–1622