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Second order duality for multiobjective programming with cone constraints

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Abstract We focus on second order duality for a class of multiobjective programming problem subject to cone constraints. Four types of second order duality models are formulated. Weak and strong duality theorems are established in terms of the generalized convexity, respectively. Converse duality theorems, essential parts of duality theory, are presented under appropriate assumptions. Moreover, some deficiencies in the work of Ahmad and Agarwal (2010) are discussed.

Keywords multiobjective programming, cone constraints, second order duality

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1 Introduction

Duality plays an important role in studying the solutions of nonlinear programming problems. Many authors have formulated different duality models, for example, Wolfe dual [11] and Mond-Weir dual [15]. In 1996, Nanda and Das [16] proposed four types of duality models associated with the nonlinear programming problem with cone constraints. These results were motivated by the work of Bazaraa and Goode [3] and Hanson and Mond [9]. Later on, Chandra and Abha [4] identified some shortcomings in these duals presented by Nanda and Das [16] followed by the corrected versions given below:

$$
\begin{aligned}
(\mathrm{ND})_1 \quad & \max \ f(u)+y^{\mathrm{T}}g(u)-u^{\mathrm{T}}\nabla(f+y^{\mathrm{T}}g)(u)\\
& \ \text{s.t.} \ -\nabla(f+y^{\mathrm{T}}g)(u)\in C_1^*,\\
& \qquad\ y\in C_2,\\
(\mathrm{ND})_2 \quad & \max \ f(u)\\
& \ \text{s.t.} \ -\nabla(f+y^{\mathrm{T}}g)(u)\in C_1^*,\\
& \qquad\ y^{\mathrm{T}}g(u)-u^{\mathrm{T}}\nabla(f+y^{\mathrm{T}}g)(u)\geqslant 0,\\
& \qquad\ y\in C_2,\\
(\mathrm{ND})_3 \quad & \max \ f(u)-u^{\mathrm{T}}\nabla(f+y^{\mathrm{T}}g)(u)
\end{aligned}
$$

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$$\begin{aligned}
&\text{s.t.}\quad -\nabla(f+y^{\mathrm{T}}g)(u)\in C_1^*,\\
&\qquad\ \ y^{\mathrm{T}}g(u)\geqslant 0,\\
&\qquad\ \ y\in C_2,\\
(\mathrm{ND})_4\quad &\max\quad f(u)+y^{\mathrm{T}}g(u)\\
&\text{s.t.}\quad -\nabla(f+y^{\mathrm{T}}g)(u)\in C_1^*,\\
&\qquad\ \ u^{\mathrm{T}}\nabla(f+y^{\mathrm{T}}g)(u)\leqslant 0,\\
&\qquad\ \ y\in C_2.
\end{aligned}$$

Furthermore, they also established the corresponding weak and strong duality theorems.

From the weak duality result, it is known that the objective value of a feasible solution to the primal problem is not less than the corresponding dual one. This result provides a lower bound for the primal optimal value if a feasible dual solution is known. The strong duality theorem tells us that, whenever the primal problem has an optimal solution, the dual problem also has one and there is no duality gap. However, the essential, but the most difficult part of the duality theory, is on the converse duality theorem. It deals with the issues on how to obtain the primal solution from the dual solution and on conditions under which there is no gap between the primal problem and the dual problem. In order to handle such a matter between a nonlinear problem with cone constraints and its corresponding four duality models mentioned above, Yang et al. [21] established converse duality theorems, under suitable assumptions, such as nonsingularity, positive/negative definiteness.

The study of second order duality is attractive due to its computational advantage over first order duality. It provides a tighter bound for the value of the objective function when approximations are used (see [1, 12, 14, 17, 22]). For multiobjective programming, it appears naturally and frequently in various areas of our daily life. Thus, it is valuable to investigate multiobjective programming. Nevertheless, the results of second order duality for multiobjective programming are mostly on symmetric duality (see [6–8, 10, 13, 18–20, 23, 24]). In particular, Yang et al. [23, 24] studied second order symmetric dual programs and established duality theorems under F-convexity conditions. Following the work of Yang et al. [23, 24], Mishra and Lai [13] established second order symmetric dual results for multiobjective programs under the assumption of cone second-order pseudo-invexity. Gulati et al. [6, 7] obtained duality theorems for second order multiobjective symmetric dual problems under η-bonvexity/η-pseudobonvexity assumptions. Kailey et al. [10] studied second-order multiobjective mixed symmetric dual under η-bonvexity/η-pseudobonvexity. Gupta and Kailey [8] investigated second order symmetric dual programs under generalized cone-invexity.

To the best of our knowledge, there are only a very few works dealing with nonsymmetric type of second order duality for multiobjective programming with cone constraints. Furthermore, unlike linear programming, a majority of dual formulations in nonlinear programming do not possess the symmetry property. Therefore, in this paper, we discuss the second order nonsymmetric dual for a class of multiobjective programming with cone constraints. Based on the first order duality results of Chandra and Abha [4] and Yang et al. [21] and the second order duality theorems of Yang et al. [22] and Ahmad and Agarwal [2] for nonlinear programming with cone constraints, four types of second order duality models are introduced. Weak duality theorems are presented under the assumptions of F-pseudoconvexity and F-quasiconvexity, which are more general than invexity. Strong duality theorems are established by using the characterization of efficient solutions of Chankong and Haimes [5] and the generalized Fritz John type conditions in [3]. Most importantly, converse duality theorems, which play a crucial role in duality theory, are discussed under certain suitable assumptions for the primal problem and its four second order duality models, respectively. Furthermore, some deficiencies in the recent work on the second order converse duality results obtained by Ahmad and Agarwal [2] are discussed.

2 Preliminaries

Throughout this paper, denote $\mathbb{R}^n$ the n-dimensional Euclidean space with its non-negative orthant $\mathbb{R}^n_+$.

The following conventions will be used: For each $x, y \in \mathbb{R}^n$,

$$x < y \Leftrightarrow y - x \in \mathrm{int}\mathbb{R}^n_+,$$
$$x \leqslant y \Leftrightarrow y - x \in \mathbb{R}^n_+ \backslash \{0\},$$
$$x \leqq y \Leftrightarrow y - x \in \mathbb{R}^n_+,$$
$$x \nleqslant y \Leftrightarrow y - x \notin \mathbb{R}^n_+ \backslash \{0\}.$$

In the sequel, we need the following definitions.

Definition 2.1. A set $C \subseteq \mathbb{R}^n$ is called a cone if for each $x \in C$ and $\lambda \geqslant 0$, $\lambda x \in C$. In addition, if C is convex, then it is called a convex cone.

Definition 2.2. Let $C \subseteq \mathbb{R}^n$ be a cone. The set $C^* = \{z \in \mathbb{R}^n : x^{\mathrm{T}} z \leqslant 0 \text{ for all } x \in C\}$ is called the polar of cone C.

Definition 2.3 (See [9]). A functional $F : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is said to be sublinear in its third argument, if for all $(x, u) \in \mathbb{R}^n \times \mathbb{R}^n$,

(i) $F(x, u, a_1 + a_2) \leqslant F(x, u, a_1) + F(x, u, a_2)$, for all $a_1, a_2 \in \mathbb{R}^n$,

(ii) $F(x, u, \alpha a) = \alpha F(x, u, a)$, for all $\alpha \in \mathbb{R}_+$ and for all $a \in \mathbb{R}^n$.

For convenience, we denote $F_{x,u}(a) = F(x, u, a)$.

Now we consider a sublinear functional $F : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and a twice differentiable function $h : \mathbb{R}^n \to \mathbb{R}$. Furthermore, denote by $\nabla h(u)$ and $\nabla^2 h(u)$ the gradient and the Hessian matrix of the function h evaluated at u, respectively.

Definition 2.4 (See [22]). h is said to be second order F- pseudoconvex at $u \in \mathbb{R}^n$ if

$$(x, p) \in \mathbb{R}^n \times \mathbb{R}^n,$$
$$F_{x,u}[\nabla h(u) + \nabla^2 h(u) p] \geqslant 0 \Rightarrow h(x) \geqslant h(u) - \frac{1}{2} p^{\mathrm{T}} \nabla^2 h(u) p.$$

Definition 2.5 (See [22]). h is said to be second order F- quasiconvex at $u \in \mathbb{R}^n$ if

$$(x, p) \in \mathbb{R}^n \times \mathbb{R}^n,$$
$$h(x) \leqslant h(u) - \frac{1}{2} p^{\mathrm{T}} \nabla^2 h(u) p \Rightarrow F_{x,u}[\nabla h(u) + \nabla^2 h(u) p] \leqslant 0.$$

In this paper, we consider the following multiobjective programming problem with cone constraints:

$$\begin{aligned} \text{(MOP)} \qquad \min \quad & f(x) \\ \text{s.t.} \quad & g(x) \in C_2^*, \\ & x \in C_1, \end{aligned}$$

where $f = (f_1, f_2, \ldots, f_p) : \mathbb{R}^n \to \mathbb{R}^p$ and $g = (g_1, g_2, \ldots, g_m) : \mathbb{R}^n \to \mathbb{R}^m$ are vector-valued functions such that each component function is twice continuously differentiable. Let C_1 and C_2 be two closed convex cones with nonempty interiors in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Furthermore, let

$$S = \{x \in \mathbb{R}^n : g(x) \in C_2^*,\ x \in C_1\}$$

be the feasible set of (MOP).

For (MOP), we need the following notation: For each $x, y, u \in \mathbb{R}^n$, and $\alpha \in \mathbb{R}^p$,

$$\alpha^{\mathrm{T}} \nabla f(u) := \sum_{i=1}^{p} \alpha_i \nabla f_i(u),$$
$$\nabla f(u) x := [\nabla f_1(u)^{\mathrm{T}} x, \ldots, \nabla f_p(u)^{\mathrm{T}} x]^{\mathrm{T}},$$
$$x^{\mathrm{T}} \nabla^2 f(u) y := [x^{\mathrm{T}} \nabla^2 f_1(u) y, \ldots, x^{\mathrm{T}} \nabla^2 f_p(u) y]^{\mathrm{T}}.$$

The solution involved in this paper is defined in the sense of efficiency as given below:

Definition 2.6 (See [5]). A point $\bar{x} \in S$ is said to be an efficient solution of (MOP), if there exists no other $x \in S$ such that $f(x) \leqslant f(\bar{x})$.

We shall use the characterization of efficiency from [5, Theorem 4.11].

Lemma 2.7. *$\bar{x}$ is an efficient solution for* (MOP) *if and only if $\bar{x}$ solves*

$$P_k(\bar{x}) \begin{cases} \min & f_k(x) \\ \text{s.t.} & f_j(x) \leqslant f_j(\bar{x}) \quad \textit{for all } j \neq k, \\ & g(x) \in C_2^*, \\ & x \in C_1, \end{cases}$$

for all $k = 1, \dots, p$.

Motivated by the first order duals of Chandra and Abha [4] and the second order duals of Yang et al. [22] for nonlinear programming with cone constraints, we now introduce four types of second order nonsymmetric duality models for multiobjective programming problems with cone constraints (MOP) as follows:

$$\begin{aligned} (\mathrm{ND}')_1 \quad \max \quad & f(u) + \Big\{ y^{\mathrm{T}} g(u) - \frac{1}{2} p^{\mathrm{T}} \nabla^2 (\lambda^{\mathrm{T}} f + y^{\mathrm{T}} g)(u) p \\ & - u^{\mathrm{T}} [\nabla (\lambda^{\mathrm{T}} f + y^{\mathrm{T}} g)(u) + \nabla^2 (\lambda^{\mathrm{T}} f + y^{\mathrm{T}} g)(u) p] \Big\} e \\ \text{s.t.} \quad & - [\nabla (\lambda^{\mathrm{T}} f + y^{\mathrm{T}} g)(u) + \nabla^2 (\lambda^{\mathrm{T}} f + y^{\mathrm{T}} g)(u) p] \in C_1^*, \\ & y \in C_2, \\ & \lambda > 0, \quad \lambda^{\mathrm{T}} e = 1, \end{aligned}$$

$$\begin{aligned} (\mathrm{ND}')_2 \quad \max \quad & f(u) - \Big\{ \frac{1}{2} p^{\mathrm{T}} \nabla^2 (\lambda^{\mathrm{T}} f)(u) p \Big\} e \\ \text{s.t.} \quad & - [\nabla (\lambda^{\mathrm{T}} f + y^{\mathrm{T}} g)(u) + \nabla^2 (\lambda^{\mathrm{T}} f + y^{\mathrm{T}} g)(u) p] \in C_1^*, \\ & y^{\mathrm{T}} g(u) - \frac{1}{2} p^{\mathrm{T}} \nabla^2 y^{\mathrm{T}} g(u) p - u^{\mathrm{T}} [\nabla (\lambda^{\mathrm{T}} f + y^{\mathrm{T}} g)(u) \\ & + \nabla^2 (\lambda^{\mathrm{T}} f + y^{\mathrm{T}} g)(u) p] \geqslant 0, \\ & y \in C_2, \\ & \lambda > 0, \quad \lambda^{\mathrm{T}} e = 1, \end{aligned}$$

$$\begin{aligned} (\mathrm{ND}')_3 \quad \max \quad & f(u) - \Big\{ \frac{1}{2} p^{\mathrm{T}} \nabla^2 (\lambda^{\mathrm{T}} f)(u) p + u^{\mathrm{T}} [\nabla (\lambda^{\mathrm{T}} f + y^{\mathrm{T}} g)(u) \\ & + \nabla^2 (\lambda^{\mathrm{T}} f + y^{\mathrm{T}} g)(u) p] \Big\} e \\ \text{s.t.} \quad & - [\nabla (\lambda^{\mathrm{T}} f + y^{\mathrm{T}} g)(u) + \nabla^2 (\lambda^{\mathrm{T}} f + y^{\mathrm{T}} g)(u) p] \in C_1^*, \\ & y^{\mathrm{T}} g(u) - \frac{1}{2} p^{\mathrm{T}} \nabla^2 y^{\mathrm{T}} g(u) p \geqslant 0, \\ & y \in C_2, \\ & \lambda > 0, \quad \lambda^{\mathrm{T}} e = 1, \end{aligned}$$

$$\begin{aligned} (\mathrm{ND}')_4 \quad \max \quad & f(u) + \Big\{ y^{\mathrm{T}} g(u) - \frac{1}{2} p^{\mathrm{T}} \nabla^2 (\lambda^{\mathrm{T}} f + y^{\mathrm{T}} g)(u) p \Big\} e \\ \text{s.t.} \quad & - [\nabla (\lambda^{\mathrm{T}} f + y^{\mathrm{T}} g)(u) + \nabla^2 (\lambda^{\mathrm{T}} f + y^{\mathrm{T}} g)(u) p] \in C_1^*, \\ & u^{\mathrm{T}} [\nabla (\lambda^{\mathrm{T}} f + y^{\mathrm{T}} g)(u) + \nabla^2 (\lambda^{\mathrm{T}} f + y^{\mathrm{T}} g)(u) p] \leqslant 0, \\ & y \in C_2, \\ & \lambda > 0, \quad \lambda^{\mathrm{T}} e = 1. \end{aligned}$$

Remark 2.8. When $f:\mathbb{R}^n\to\mathbb{R}$, the second order duality models (i.e., $(\mathrm{ND}')_1$, $(\mathrm{ND}')_2$, $(\mathrm{ND}')_3$ and $(\mathrm{ND}')_4$) for (MOP) reduce to the second order duality models introduced by Yang et al. [22] for nonlinear programming with cone constraints. In addition, if $p=0$, the above second order duality models reduce to the first order duality models (i.e., $(\mathrm{ND})_1$, $(\mathrm{ND})_2$, $(\mathrm{ND})_3$ and $(\mathrm{ND})_4$) proposed by Chandra and Abha [4], respectively.

3 Main results

In this section, we establish weak, strong and converse duality theorems between (MOP) and $(\mathrm{ND}')_1$, $(\mathrm{ND}')_2$, $(\mathrm{ND}')_3$ and $(\mathrm{ND}')_4$, respectively.

First, we present weak duality results, which give the relationships between the objective values of feasible solutions to the primal problem (MOP) and those to the respective duality models $(\mathrm{ND}')_1$, $(\mathrm{ND}')_2$, $(\mathrm{ND}')_3$ and $(\mathrm{ND}')_4$.

3.1 Weak duality

The following theorems state that the objective value of any feasible solution of (MOP) is not less than those of $(\mathrm{ND}')_1$, $(\mathrm{ND}')_2$, $(\mathrm{ND}')_3$ and $(\mathrm{ND}')_4$, respectively, under some appropriate conditions, such as second order F-pseudoconvexity and second order F-quasiconvexity.

Theorem 3.1 (Weak duality for (MOP) and $(\mathrm{ND}')_1$)**.** *Let x and (u,y,λ,p) be feasible for* (MOP) *and* $(\mathrm{ND}')_1$*, respectively. Assume that there exists a sublinear functional $F:\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ such that $\lambda^{\mathrm{T}}f(\cdot)+y^{\mathrm{T}}g(\cdot)+(\cdot)^{\mathrm{T}}v$ is second order F-pseudoconvex at u with*

$$v=-[\nabla(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)+\nabla^2(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)p].$$

Then

$$\begin{aligned}f(x)\nleqslant f(u)+\bigg\{&y^{\mathrm{T}}g(u)-\frac{1}{2}p^{\mathrm{T}}\nabla^2(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)p\\&-u^{\mathrm{T}}[\nabla(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)+\nabla^2(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)p]\bigg\}e.\end{aligned}$$

Proof. Assume that

$$\begin{aligned}f(x)\leqslant f(u)+\bigg\{&y^{\mathrm{T}}g(u)-\frac{1}{2}p^{\mathrm{T}}\nabla^2(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)p\\&-u^{\mathrm{T}}[\nabla(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)+\nabla^2(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)p]\bigg\}e.\end{aligned}$$

Then, from the constraints of (MOP) and $(\mathrm{ND}')_1$, we have

$$\begin{aligned}\lambda^{\mathrm{T}}f(x)<\lambda^{\mathrm{T}}f(u)+y^{\mathrm{T}}g(u)-\frac{1}{2}p^{\mathrm{T}}\nabla^2(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)p&\\-u^{\mathrm{T}}[\nabla(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)+\nabla^2(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)p],&\qquad(3.1)\\-x^{\mathrm{T}}[\nabla(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)+\nabla^2(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)p]\leqslant 0,&\qquad(3.2)\\y^{\mathrm{T}}g(x)\leqslant 0.&\qquad(3.3)\end{aligned}$$

Putting (3.1)–(3.3) together, we get

$$\begin{aligned}&\lambda^{\mathrm{T}}f(x)+y^{\mathrm{T}}g(x)-x^{\mathrm{T}}[\nabla(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)+\nabla^2(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)p]\\&\quad<\lambda^{\mathrm{T}}f(u)+y^{\mathrm{T}}g(u)-u^{\mathrm{T}}[\nabla(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)+\nabla^2(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)p]\\&\qquad-\frac{1}{2}p^{\mathrm{T}}\nabla^2(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)p.\end{aligned}$$

Since $\lambda^{\mathrm{T}} f(\cdot) + y^{\mathrm{T}} g(\cdot) + (\cdot)^{\mathrm{T}} v$ is second order F-pseudoconvex at u with

$$v = -[\nabla(\lambda^{\mathrm{T}} f + y^{\mathrm{T}} g)(u) + \nabla^2(\lambda^{\mathrm{T}} f + y^{\mathrm{T}} g)(u)p],$$

we obtain $F_{x,u}(0) < 0$, which contradicts $F_{x,u}(0) = 0$. Hence,

$$\begin{aligned} f(x) \nleqslant f(u) + \bigg\{ & y^{\mathrm{T}} g(u) - \frac{1}{2} p^{\mathrm{T}} \nabla^2(\lambda^{\mathrm{T}} f + y^{\mathrm{T}} g)(u)p \\ & - u^{\mathrm{T}}[\nabla(\lambda^{\mathrm{T}} f + y^{\mathrm{T}} g)(u) + \nabla^2(\lambda^{\mathrm{T}} f + y^{\mathrm{T}} g)(u)p] \bigg\} e. \end{aligned}$$ □

Remark 3.2. Theorem 3.1 reduces to Theorem 1 presented by Yang et al. [22], when $f : \mathbb{R}^n \to \mathbb{R}$. Furthermore, let $p = 0$ and $F_{x,u}(a) = \eta(x,u)^{\mathrm{T}} a$. Then, the second order F-pseudoconvexity reduces to psedoinvexity, and Theorem 3.1 reduces to Theorem 1 established by Chandra and Abha [4].

Theorem 3.3 (Weak duality for (MOP) and $(\mathrm{ND}')_2$)**.** *Let x and (u, y, λ, p) be feasible for* (MOP) *and* $(\mathrm{ND}')_2$*, respectively. Suppose that there exists a sublinear functional $F : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that $\lambda^{\mathrm{T}} f(\cdot)$ is second order F-pseudoconvex at u and $y^{\mathrm{T}} g(\cdot) + (\cdot)^{\mathrm{T}} v$ is second order F-quasiconvex at u with*

$$v = -[\nabla(\lambda^{\mathrm{T}} f + y^{\mathrm{T}} g)(u) + \nabla^2(\lambda^{\mathrm{T}} f + y^{\mathrm{T}} g)(u)p].$$

Then

$$f(x) \nleqslant f(u) - \left[\frac{1}{2} p^{\mathrm{T}} \nabla^2(\lambda^{\mathrm{T}} f)(u)p\right] e.$$

Proof. Suppose that

$$f(x) \leqslant f(u) - \left[\frac{1}{2} p^{\mathrm{T}} \nabla^2(\lambda^{\mathrm{T}} f)(u)p\right] e.$$

Since $\lambda > 0$ and $\lambda^{\mathrm{T}} e = 1$,

$$\lambda^{\mathrm{T}} f(x) < \lambda^{\mathrm{T}} f(u) - \frac{1}{2} p^{\mathrm{T}} \nabla^2(\lambda^{\mathrm{T}} f)(u)p.$$

By virtue of the second order F-pseudoconvexity of $\lambda^{\mathrm{T}} f(\cdot)$ at u, we get

$$F_{x,u}(\nabla(\lambda^{\mathrm{T}} f)(u) + \nabla^2(\lambda^{\mathrm{T}} f)(u)p) < 0. \tag{3.4}$$

It follows from the sublinearity of F and

$$v + [\nabla(\lambda^{\mathrm{T}} f + y^{\mathrm{T}} g)(u) + \nabla^2(\lambda^{\mathrm{T}} f + y^{\mathrm{T}} g)(u)p] = 0$$

that

$$F_{x,u}(v + [\nabla(y^{\mathrm{T}} g)(u) + \nabla^2(y^{\mathrm{T}} g)(u)p]) + F_{x,u}(\nabla(\lambda^{\mathrm{T}} f)(u) + \nabla^2(\lambda^{\mathrm{T}} f)(u)p) \geqslant 0,$$

which combined with (3.4) yields

$$F_{x,u}(v + [\nabla(y^{\mathrm{T}} g)(u) + \nabla^2(y^{\mathrm{T}} g)(u)p]) > 0.$$

Then, by the second order F-quasiconvexity of $y^{\mathrm{T}} g(\cdot) + (\cdot)^{\mathrm{T}} v$ at u with

$$v = -[\nabla(\lambda^{\mathrm{T}} f + y^{\mathrm{T}} g)(u) + \nabla^2(\lambda^{\mathrm{T}} f + y^{\mathrm{T}} g)(u)p],$$

we have

$$\begin{aligned} & y^{\mathrm{T}} g(x) - x^{\mathrm{T}}[\nabla(\lambda^{\mathrm{T}} f + y^{\mathrm{T}} g)(u) + \nabla^2(\lambda^{\mathrm{T}} f + y^{\mathrm{T}} g)(u)p] \\ & \quad > y^{\mathrm{T}} g(u) - u^{\mathrm{T}}[\nabla(\lambda^{\mathrm{T}} f + y^{\mathrm{T}} g)(u) + \nabla^2(\lambda^{\mathrm{T}} f + y^{\mathrm{T}} g)(u)p] - \frac{1}{2} p^{\mathrm{T}} \nabla^2(y^{\mathrm{T}} g)(u)p. \end{aligned} \tag{3.5}$$

Note that

$$0 \geqslant y^{\mathrm{T}} g(x) - x^{\mathrm{T}}[\nabla(\lambda^{\mathrm{T}} f + y^{\mathrm{T}} g)(u) + \nabla^2(\lambda^{\mathrm{T}} f + y^{\mathrm{T}} g)(u)p],$$

which together with (3.5) implies that

$$y^{\mathrm{T}}g(u)-\frac{1}{2}p^{\mathrm{T}}\nabla^2(y^{\mathrm{T}}g)(u)p-u^{\mathrm{T}}[\nabla(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)+\nabla^2(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)p]<0.$$

This contradicts the feasibility of (u,λ,y,p) for $(\mathrm{ND}')_2$. Therefore,

$$f(x)\nleq f(u)-\left[\frac{1}{2}p^{\mathrm{T}}\nabla^2(\lambda^{\mathrm{T}}f)(u)p\right]e. \qquad \square$$

Remark 3.4. Similar to Theorem 3.1, Theorem 3.3 reduces to Theorem 2 given by Yang et al. [22], when $f:\mathbb{R}^n\to\mathbb{R}$. Furthermore, if we choose $p=0$ and $F_{x,u}(a)=\eta(x,u)^{\mathrm{T}}a$, then it is easy to see that the second order F-pseudoconvexity and the second order F-quasiconvexity reduce to pseudoinvexity and quasiinvexity, respectively, and Theorem 3.3 reduces to Theorem 2 established by Chandra and Abha [4].

Theorem 3.5 (Weak duality for (MOP) and $(\mathrm{ND}')_3$)**.** *Let x and (u,y,λ,p) be feasible for* (MOP) *and* $(\mathrm{ND}')_3$*, respectively. Suppose that there exists a sublinear functional $F:\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ such that $\lambda^{\mathrm{T}}f(\cdot)+(\cdot)^{\mathrm{T}}v$ is second order F-pseudoconvex at u for $v=-[\nabla(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)+\nabla^2(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)p]$ and $y^{\mathrm{T}}g(\cdot)$ is second order F-quasiconvex at u. Then*

$$f(x)\nleq f(u)-\left\{\frac{1}{2}p^{\mathrm{T}}\nabla^2(\lambda^{\mathrm{T}}f)(u)p+u^{\mathrm{T}}[\nabla(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)\right.$$
$$\left.+\nabla^2(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)p]\right\}e.$$

Proof. Suppose that

$$f(x)\leqslant f(u)-\left\{\frac{1}{2}p^{\mathrm{T}}\nabla^2(\lambda^{\mathrm{T}}f)(u)p+u^{\mathrm{T}}[\nabla(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)\right.$$
$$\left.+\nabla^2(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)p]\right\}e.$$

Then, by the constraints of (MOP) and $(\mathrm{ND}')_3$,

$$\lambda^{\mathrm{T}}f(x)<\lambda^{\mathrm{T}}f(u)-\left\{\frac{1}{2}p^{\mathrm{T}}\nabla^2(\lambda^{\mathrm{T}}f)(u)p+u^{\mathrm{T}}[\nabla(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)\right.$$
$$\left.+\nabla^2(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)p]\right\},$$

and $-x^{\mathrm{T}}[\nabla(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)+\nabla^2(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)p]\leqslant 0$.

It follows from the above two inequalities that

$$\lambda^{\mathrm{T}}f(x)-x^{\mathrm{T}}[\nabla(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)+\nabla^2(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)p]$$
$$<\lambda^{\mathrm{T}}f(u)-\frac{1}{2}p^{\mathrm{T}}\nabla^2(\lambda^{\mathrm{T}}f)(u)p-u^{\mathrm{T}}[\nabla(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)$$
$$+\nabla^2(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)p].$$

As $\lambda^{\mathrm{T}}f(\cdot)+(\cdot)^{\mathrm{T}}v$ is second order F-pseudoconvex at u for $v=-[\nabla(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)+\nabla^2(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)p]$, we get $F_{x,u}(\nabla(\lambda^{\mathrm{T}}f)(u)+v+\nabla^2(\lambda^{\mathrm{T}}f)(u)p)<0$, which together with the sublinearity of F and $v+[\nabla(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)+\nabla^2(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)p]=0$ implies

$$F_{x,u}(\nabla(y^{\mathrm{T}}g)(u)+\nabla^2(y^{\mathrm{T}}g)(u)p)>0.$$

Owing to the second order F-quasiconvexity of $y^{\mathrm{T}}g(\cdot)$ at u, we have $y^{\mathrm{T}}g(x)>y^{\mathrm{T}}g(u)-\frac{1}{2}p^{\mathrm{T}}\nabla^2(y^{\mathrm{T}}g)(u)p$. Note that $y^{\mathrm{T}}g(x)\leqslant 0$, thus $y^{\mathrm{T}}g(u)-\frac{1}{2}p^{\mathrm{T}}\nabla^2(y^{\mathrm{T}}g)(u)p<0$, which contradicts the constraints of $(ND')_3$. So,

$$f(x)\nleq f(u)-\left\{\frac{1}{2}p^{\mathrm{T}}\nabla^2(\lambda^{\mathrm{T}}f)(u)p+u^{\mathrm{T}}[\nabla(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)+\nabla^2(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)p]\right\}e. \qquad \square$$

Remark 3.6. As mentioned in Remark 3.4, Theorem 3.5 reduces to Theorem 3 obtained by Yang et al. [22], when $f:\mathbb{R}^n\to\mathbb{R}$. In addition, if

$$p=0 \quad \text{and} \quad F_{x,u}(a)=\eta(x,u)^{\mathrm{T}}a,$$

then Theorem 3.5 reduces to Theorem 3 established by Chandra and Abha [4].

Theorem 3.7 (Weak duality for (MOP) and $(\mathrm{ND}')_4$)**.** *Let x and (u,y,λ,p) be feasible for* (MOP) *and* $(\mathrm{ND}')_4$*, respectively. Suppose that there exists a sublinear functional $F:\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ satisfying*

$$F_{x,u}(a)+a^{\mathrm{T}}u\leqslant 0, \quad \textit{for all} \quad a\in C_1^*, \tag{A}$$

and $\lambda^{\mathrm{T}}f(\cdot)+y^{\mathrm{T}}g(\cdot)$ is second order F-pseudoconvex at u. Then

$$f(x)\nleq f(u)+\left\{y^{\mathrm{T}}g(u)-\frac{1}{2}p^{\mathrm{T}}\nabla^2(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)p\right\}e.$$

Proof. Suppose the conclusion is not true. Then

$$\lambda^{\mathrm{T}}f(x)<\lambda^{\mathrm{T}}f(u)+y^{\mathrm{T}}g(u)-\frac{1}{2}p^{\mathrm{T}}\nabla^2(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)p.$$

Since $y^{\mathrm{T}}g(x)\leqslant 0$, we get

$$\lambda^{\mathrm{T}}f(x)+y^{\mathrm{T}}g(x)<\lambda^{\mathrm{T}}f(u)+y^{\mathrm{T}}g(u)-\frac{1}{2}p^{\mathrm{T}}\nabla^2(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)p.$$

It is easy to see that

$$F_{x,u}(\nabla(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)+\nabla^2(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)p)<0, \tag{3.6}$$

for $\lambda^{\mathrm{T}}f(\cdot)+y^{\mathrm{T}}g(\cdot)$ is second order F-pseudoconvex at u.

On the other hand, taking into account the condition of (A) and the constraints of $(\mathrm{ND}')_4$, we have

$$\begin{aligned}&F_{x,u}(-[\nabla(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)+\nabla^2(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)p])\\&\quad\leqslant u^{\mathrm{T}}[\nabla(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)+\nabla^2(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)p]\leqslant 0.\end{aligned}$$

Hence, it follows from the sublinearity of F that

$$F_{x,u}(\nabla(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)+\nabla^2(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)p)\geqslant 0,$$

contradicting (3.6). Thus,

$$f(x)\nleq f(u)+\left\{y^{\mathrm{T}}g(u)-\frac{1}{2}p^{\mathrm{T}}\nabla^2(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)p\right\}e.$$

□

Remark 3.8. Similar to the above three theorems, Theorem 3.7 reduces to Theorem 4 presented by Yang et al. [22], when $f:\mathbb{R}^n\to\mathbb{R}$. Furthermore, when

$$p=0 \quad \text{and} \quad F_{x,u}(a)=\eta(x,u)^{\mathrm{T}}a,$$

the condition

$$\text{“}F_{x,u}(a)+a^{\mathrm{T}}u\leqslant 0, \quad \text{for all} \quad a\in C_1^*\text{”}$$

in Theorem 3.7 becomes

$$\text{“}\eta(x,u)+u\in C_1\text{”}$$

in the first order weak duality theorem (i.e., Theorem 4) of Chandra and Abha [4], and Theorem 3.7 reduces to Theorem 4 established by Chandra and Abha [4].

3.2 Strong duality

Based on the above weak duality theorems, we now present strong duality theorems for efficient solutions, which are focused on how to obtain efficient solutions of four second order duals $(\mathrm{ND}')_1$, $(\mathrm{ND}')_2$, $(\mathrm{ND}')_3$ and $(\mathrm{ND}')_4$ from the ones of the primal programming (MOP), respectively. These results are established in terms of the characterization of efficient solutions (see Lemma 2.7) and the generalized Fritz John conditions [3].

Theorem 3.9 (Strong duality for (MOP) and $(\mathrm{ND}')_1$)**.** *Let $\bar{x}$ be an efficient solution of* (MOP) *at which a suitable constraint qualification* [11] *is satisfied. Then there exist $\bar{\lambda} > 0$ and $\bar{y} \in C_2$ such that $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p} = 0)$ is feasible for* $(\mathrm{ND}')_1$ *and the objective values of* (MOP) *and* $(\mathrm{ND}')_1$ *are equal. Furthermore, if hypotheses of Theorem* 3.1 *are satisfied, then $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p} = 0)$ is an efficient solution to* $(\mathrm{ND}')_1$.

Proof. Since $\bar{x}$ is an efficient solution of (MOP) at which a suitable constraint qualification is satisfied, by Lemma 2.7 and the generalized Fritz John conditions [3], there exist $\bar{\lambda} \in \mathbb{R}^p$ with $\bar{\lambda} > 0$ and $\bar{y} \in C_2$ such that

$$[\nabla(\bar{\lambda}^{\mathrm{T}} f + \bar{y}^{\mathrm{T}} g)(\bar{x})]^{\mathrm{T}}(x - \bar{x}) \geqslant 0, \quad \forall\, x \in C_1, \tag{3.7}$$

and

$$\bar{y}^{\mathrm{T}} g(\bar{x}) = 0. \tag{3.8}$$

Since C_1 is a convex cone, taking $x + \bar{x}$ for each $x \in C_1$ into (3.7), we get

$$[\nabla(\bar{\lambda}^{\mathrm{T}} f + \bar{y}^{\mathrm{T}} g)(\bar{x})]^{\mathrm{T}} x \geqslant 0, \quad \forall\, x \in C_1,$$

which yields

$$-\nabla(\bar{\lambda}^{\mathrm{T}} f + \bar{y}^{\mathrm{T}} g)(\bar{x}) \in C_1^*,$$

i.e., $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p} = 0)$ is feasible for $(\mathrm{ND}')_1$.

Substituting $x = 0$ and $x = 2\bar{x}$ in (3.7), respectively, we obtain

$$\bar{x}^{\mathrm{T}} \nabla(\bar{\lambda}^{\mathrm{T}} f + \bar{y}^{\mathrm{T}} g)(\bar{x}) = 0. \tag{3.9}$$

Consequently, it follows from (3.8), (3.9) and $\bar{p} = 0$ that

$$\begin{aligned} f(\bar{x}) = f(\bar{x}) + \Big\{ & \bar{y}^{\mathrm{T}} g(\bar{x}) - \frac{1}{2}\bar{p}^{\mathrm{T}} \nabla^2(\bar{\lambda}^{\mathrm{T}} f + \bar{y}^{\mathrm{T}} g)(\bar{x})\bar{p} \\ & - \bar{x}^{\mathrm{T}}[\nabla(\bar{\lambda}^{\mathrm{T}} f + \bar{y}^{\mathrm{T}} g)(\bar{x}) + \nabla^2(\bar{\lambda}^{\mathrm{T}} f + \bar{y}^{\mathrm{T}} g)(\bar{x})\bar{p}] \Big\} e, \end{aligned}$$

which together with the weak duality theorem (see Theorem 3.1) implies that $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p} = 0)$ is an efficient solution to $(ND')_1$. □

Remark 3.10. (i) The constraint "$\lambda^{\mathrm{T}} e = 1$" is not essential for $(\mathrm{ND}')_1$. For example, by taking

$$\bar{\lambda} := \frac{\bar{\lambda}}{\bar{\lambda}^{\mathrm{T}} e} \quad \text{and} \quad \bar{y} := \frac{\bar{y}}{\bar{\lambda}^{\mathrm{T}} e}$$

in the proof of Theorem 3.9, we obtain all the constraints of $(\mathrm{ND}')_1$.

(ii) Similar to the proof of Theorem 3.9, strong duality theorems between (MOP) and $(\mathrm{ND}')_i$ $(i = 2, 3, 4)$ can also be established, respectively.

(iii) In the same way, when choosing $f : \mathbb{R}^n \to \mathbb{R}$, it follows that the strong duality theorems between (MOP) and the four second order duality models (i.e., $(\mathrm{ND}')_1$, $(\mathrm{ND}')_2$, $(\mathrm{ND}')_3$ and $(\mathrm{ND}')_4$) reduce to the corresponding strong duality theorems given by Yang et al. [22], respectively. Moreover, if $p = 0$ and $F_{x,u}(a) = \eta(x, u)^{\mathrm{T}} a$, then the above results reduce to those established by Chandra and Abha [4].

3.3 Converse duality

In what follows, we move our attention to converse duality theorems between the primal problem (MOP) and four second order duals $(\mathrm{ND}')_1$, $(\mathrm{ND}')_2$, $(\mathrm{ND}')_3$ and $(\mathrm{ND}')_4$ under appropriate assumptions, respectively. These results are about the issue that how to get efficient solutions of the primal programming (MOP) from those of second order duals $(\mathrm{ND}')_1$, $(\mathrm{ND}')_2$, $(\mathrm{ND}')_3$ and $(\mathrm{ND}')_4$, respectively.

As pointed out in Remark 3.10(i), the constraint

$$\text{“}\lambda^{\mathrm{T}}e = 1\text{”}$$

is not essential to the four second order duals (i.e., $(\mathrm{ND}')_1$, $(\mathrm{ND}')_2$, $(\mathrm{ND}')_3$ and $(\mathrm{ND}')_4$). So, in the following converse duality theorems, we do not consider this constraint “$\lambda^{\mathrm{T}}e = 1$” unless otherwise stated.

Theorem 3.11 (Converse duality for (MOP) and $(\mathrm{ND}')_1$). *Let $(\bar{u}, \bar{y}, \bar{\lambda}, \bar{p})$ be an efficient solution to* $(\mathrm{ND}')_1$. *Suppose that*

(i) *the $n \times n$ Hessian matrix $\nabla^2(\bar{\lambda}^{\mathrm{T}} f + \bar{y}^{\mathrm{T}} g)(\bar{u})$ is nonsingular, and,*

(ii) $\bar{p}^{\mathrm{T}}\nabla(\bar{\lambda}^{\mathrm{T}} f)(\bar{u}) + \frac{1}{2}\bar{p}^{\mathrm{T}}\nabla^2(\bar{\lambda}^{\mathrm{T}} f)(\bar{u})\bar{p} = 0 \Rightarrow \bar{p} = 0.$

Then $\bar{u}$ is feasible for (MOP), *and the objective values of* (MOP) *and* $(\mathrm{ND}')_1$ *are equal.*

In addition, if the assumptions of weak duality theorem (*see Theorem* 3.1) *are satisfied for all feasible solutions of* (MOP) *and* $(\mathrm{ND}')_1$, *then $\bar{u}$ is an efficient solution of* (MOP).

Proof. Let

$$L = \alpha^{\mathrm{T}}\left\{f(u) + \left\{y^{\mathrm{T}}g(u) - \frac{1}{2}p^{\mathrm{T}}\nabla^2(\lambda^{\mathrm{T}}f + y^{\mathrm{T}}g)(u)p - u^{\mathrm{T}}[\nabla(\lambda^{\mathrm{T}}f + y^{\mathrm{T}}g)(u) + \nabla^2(\lambda^{\mathrm{T}}f + y^{\mathrm{T}}g)(u)p]\right\}e\right\}$$
$$+ \beta^{\mathrm{T}}[\nabla(\lambda^{\mathrm{T}}f + y^{\mathrm{T}}g)(u) + \nabla^2(\lambda^{\mathrm{T}}f + y^{\mathrm{T}}g)(u)p] - \delta^{\mathrm{T}}y + \eta^{\mathrm{T}}\lambda.$$

Since $(\bar{u}, \bar{\lambda}, \bar{y}, \bar{p})$ is an efficient solution for $(\mathrm{ND}')_1$, it follows from Lemma 2.7 and the generalized Fritz John type necessary conditions [3] that there exist $\alpha \in \mathbb{R}^p_+, \beta \in C_1, \eta \in \mathbb{R}^p_+$ and $\delta \in C_2^*$ such that

$$\frac{\partial L}{\partial u}\bigg|_{(\bar{u},\bar{y},\bar{\lambda},\bar{p})} = (\alpha - \alpha^{\mathrm{T}}e\bar{\lambda})^{\mathrm{T}}\nabla f(\bar{u}) + \left(\beta - \frac{1}{2}\alpha^{\mathrm{T}}e\bar{p} - \alpha^{\mathrm{T}}e\bar{u}\right)^{\mathrm{T}}\nabla(\nabla^2(\bar{\lambda}^{\mathrm{T}}f + \bar{y}^{\mathrm{T}}g)(\bar{u})\bar{p})$$
$$+ (\beta - \alpha^{\mathrm{T}}e\bar{p} - \alpha^{\mathrm{T}}e\bar{u})^{\mathrm{T}}\nabla^2(\bar{\lambda}^{\mathrm{T}}f + \bar{y}^{\mathrm{T}}g)(\bar{u}) = 0, \tag{3.10}$$

$$\frac{\partial L}{\partial \lambda}\bigg|_{(\bar{u},\bar{y},\bar{\lambda},\bar{p})} = \left(\beta - \frac{1}{2}\alpha^{\mathrm{T}}e\bar{p} - \alpha^{\mathrm{T}}e\bar{u}\right)^{\mathrm{T}}\nabla^2 f(\bar{u})\bar{p} + \nabla f(\bar{u})(\beta - \alpha^{\mathrm{T}}e\bar{u}) + \eta = 0, \tag{3.11}$$

$$\frac{\partial L}{\partial y}\bigg|_{(\bar{u},\bar{y},\bar{\lambda},\bar{p})} = \alpha^{\mathrm{T}}eg(\bar{u}) + \left(\beta - \frac{1}{2}\alpha^{\mathrm{T}}e\bar{p} - \alpha^{\mathrm{T}}e\bar{u}\right)^{\mathrm{T}}\nabla^2 g(\bar{u})\bar{p} + \nabla g(\bar{u})(\beta - \alpha^{\mathrm{T}}e\bar{u}) - \delta = 0, \tag{3.12}$$

$$\frac{\partial L}{\partial p}\bigg|_{(\bar{u},\bar{y},\bar{\lambda},\bar{p})} = (\beta - \alpha^{\mathrm{T}}e\bar{p} - \alpha^{\mathrm{T}}e\bar{u})^{\mathrm{T}}\nabla^2(\bar{\lambda}^{\mathrm{T}}f + \bar{y}^{\mathrm{T}}g)(\bar{u}) = 0, \tag{3.13}$$

$$\beta^{\mathrm{T}}[\nabla(\bar{\lambda}^{\mathrm{T}}f + \bar{y}^{\mathrm{T}}g)(\bar{u}) + \nabla^2(\bar{\lambda}^{\mathrm{T}}f + \bar{y}^{\mathrm{T}}g)(\bar{u})\bar{p}] = 0, \tag{3.14}$$

$$\delta^{\mathrm{T}}\bar{y} = 0, \tag{3.15}$$

$$\eta^{\mathrm{T}}\bar{\lambda} = 0, \tag{3.16}$$

$$(\alpha, \beta, \eta, \delta) \neq 0. \tag{3.17}$$

Since $\bar{\lambda} > 0$ and $\eta \in \mathbb{R}^n_+$, it is clear from (3.16) that

$$\eta = 0. \tag{3.18}$$

Note that $\nabla^2(\bar{\lambda}^{\mathrm{T}}f + \bar{y}^{\mathrm{T}}g)(\bar{u})$ is nonsingular. Thus, by (3.13), we have

$$\beta = \alpha^{\mathrm{T}}e\bar{p} + \alpha^{\mathrm{T}}e\bar{u}. \tag{3.19}$$

Now, we claim that $\alpha \neq 0$. Otherwise, by (3.19) and (3.12), we obtain

$$\beta = 0, \quad \delta = 0,$$

i.e.,

$$(\alpha, \beta, \eta, \delta) = 0,$$

contradicting (3.17). Therefore,

$$\alpha \geqslant 0, \quad \text{and} \quad \alpha^{\mathrm{T}} e > 0.$$

Multiplying (3.11) by $\bar{\lambda}$, we get from (3.19) that

$$\alpha^{\mathrm{T}} e\left[\bar{p}^{\mathrm{T}} \nabla(\bar{\lambda}^{\mathrm{T}} f)(\bar{u}) + \frac{1}{2}\bar{p}^{\mathrm{T}} \nabla^2(\bar{\lambda}^{\mathrm{T}} f)(\bar{u})\bar{p}\right] = 0.$$

Since $\alpha^{\mathrm{T}} e > 0$,

$$\bar{p}^{\mathrm{T}} \nabla(\bar{\lambda}^{\mathrm{T}} f)(\bar{u}) + \frac{1}{2}\bar{p}^{\mathrm{T}} \nabla^2(\bar{\lambda}^{\mathrm{T}} f)(\bar{u})\bar{p} = 0.$$

By Assumption (ii), we obtain

$$\bar{p} = 0. \tag{3.20}$$

Thus, (3.19) reduces to

$$\beta = (\alpha^{\mathrm{T}} e)\bar{u}.$$

As $\alpha^{\mathrm{T}} e > 0$ and $\beta \in C_1$, it follows that

$$\bar{u} = \frac{\beta}{\alpha^{\mathrm{T}} e} \in C_1. \tag{3.21}$$

Substituting $\bar{p} = 0$ and $\beta = (\alpha^{\mathrm{T}} e)\bar{u}$ into (3.12), we have

$$(\alpha^{\mathrm{T}} e)g(\bar{u}) = \delta \in C_2^*.$$

For $\alpha^{\mathrm{T}} e > 0$, we get

$$g(\bar{u}) = \frac{\delta}{\alpha^{\mathrm{T}} e} \in C_2^*, \tag{3.22}$$

which together with (3.21) implies that $\bar{u}$ is feasible for (MOP).

On the other hand, it follows directly from substituting (3.21) into (3.14) that

$$\bar{u}^{\mathrm{T}}[\nabla(\bar{\lambda}^{\mathrm{T}} f + \bar{y}^{\mathrm{T}} g)(\bar{u}) + \nabla^2(\bar{\lambda}^{\mathrm{T}} f + \bar{y}^{\mathrm{T}} g)(\bar{u})\bar{p}] = 0. \tag{3.23}$$

Multiplying (3.22) by $\bar{y}$, it is clear from (3.15) that

$$\bar{y}^{\mathrm{T}} g(\bar{u}) = \frac{1}{\alpha^{\mathrm{T}} e}\delta^{\mathrm{T}} \bar{y} = 0, \tag{3.24}$$

as $\alpha^{\mathrm{T}} e > 0$.

Putting (3.20), (3.23) and (3.24) together, we arrive at

$$\begin{aligned} f(\bar{u}) = f(\bar{u}) + \Big\{ &\bar{y}^{\mathrm{T}} g(\bar{u}) - \frac{1}{2}\bar{p}^{\mathrm{T}} \nabla^2(\bar{\lambda}^{\mathrm{T}} f + \bar{y}^{\mathrm{T}} g)(\bar{u})\bar{p} \\ &- \bar{u}^{\mathrm{T}}[\nabla(\bar{\lambda}^{\mathrm{T}} f + \bar{y}^{\mathrm{T}} g)(\bar{u}) + \nabla^2(\bar{\lambda}^{\mathrm{T}} f + \bar{y}^{\mathrm{T}} g)(\bar{u})\bar{p}] \Big\} e. \end{aligned}$$

This implies that the objective values of (MOP) and $(\mathrm{ND}')_1$ are equal. The efficiency of $\bar{u}$ for (MOP) follows from the weak duality theorem (see Theorem 3.1). □

This converse duality result between (MOP) and $(\mathrm{ND}')_1$ reveals the fact that under assumptions of the weak duality theorem (see Theorem 3.1) between (MOP) and $(\mathrm{ND}')_1$, when $(\bar{u}, \bar{y}, \bar{\lambda}, \bar{p})$ is an efficient solution of $(\mathrm{ND}')_1$, the Hessian matrix $\nabla^2(\bar{\lambda}^{\mathrm{T}} f + \bar{y}^{\mathrm{T}} g)(\bar{u})$ is nonsingular, and

$$\bar{p}^{\mathrm{T}}\nabla(\bar{\lambda}^{\mathrm{T}}f)(\bar{u})+\frac{1}{2}\bar{p}^{\mathrm{T}}\nabla^2(\bar{\lambda}^{\mathrm{T}}f)(\bar{u})\bar{p}=0\Rightarrow\bar{p}=0,$$

we can conclude that $\bar{u}$ is an efficient solution of (MOP).

Remark 3.12. (i) Note that the condition

$$“\bar{p}^{\mathrm{T}}\nabla(\bar{\lambda}^{\mathrm{T}}f)(\bar{u})+\frac{1}{2}\bar{p}^{\mathrm{T}}\nabla^2(\bar{\lambda}^{\mathrm{T}}f)(\bar{u})\bar{p}=0\Rightarrow\bar{p}=0”$$

in Theorem 3.11 is different from the assumption

$$“\bar{p}^{\mathrm{T}}\nabla(\nabla^2 f(\bar{u})\bar{p}+\nabla^2\bar{y}^{\mathrm{T}}g(\bar{u})\bar{p})=0\Rightarrow\bar{p}=0”$$

in [2, Theorem 1]. In fact, the primal and the second order duality models discussed in [2] are single-objective, but our models are multi-objective. There are some essential differences between scalar and multiobjective programming. This is why the conditions are not the same. Furthermore, the assumption in [2],

$$“\bar{p}^{\mathrm{T}}\nabla(\nabla^2 f(\bar{u})\bar{p}+\nabla^2\bar{y}^{\mathrm{T}}g(\bar{u})\bar{p})=0\Rightarrow\bar{p}=0”$$

requires the third derivative, but our condition

$$“\bar{p}^{\mathrm{T}}\nabla(\bar{\lambda}^{\mathrm{T}}f)(\bar{u})+\frac{1}{2}\bar{p}^{\mathrm{T}}\nabla^2(\bar{\lambda}^{\mathrm{T}}f)(\bar{u})\bar{p}=0\Rightarrow\bar{p}=0”$$

only needs the second derivative. In this sense, our condition is superior to the one in [2].

(ii) Furthermore, if

$$p=0 \quad \text{and} \quad F_{x,u}(a)=\eta(x,u)^{\mathrm{T}}a,$$

then the concept of the second order F-pseudoconvexity reduces to pseudo-invexity, and Theorem 3.11 completely reduces to Theorem 1 established by Yang et al. [21].

Before presenting the second order converse duality theorems of $(ND')_2$ and $(ND')_3$, we should point out that there is some drawback in Assumption (ii) of Theorems 2 and 3 established by Ahmad and Agarwal [2]: “the vectors $\{[\nabla^2 f(\bar{u})]_j,\ [\nabla^2(\bar{y}^{\mathrm{T}}g)(\bar{u})]_j,\ j=1,\ldots,n\}$ are linearly independent, where $[\nabla^2 f(\bar{u})]_j$ is the j-th row of $\nabla^2 f(\bar{u})$ and $[\nabla^2(\bar{y}^{\mathrm{T}}g)(\bar{u})]_j$ is the j-th row of $\nabla^2(\bar{y}^{\mathrm{T}}g)(\bar{u})$.” Note that both of the matrices $\nabla^2 f(\bar{u})$ and $\nabla^2(\bar{y}^{\mathrm{T}}g)(\bar{u})$ are $n\times n$, and the number of n-dimensional vectors $\{[\nabla^2 f(\bar{u})]_j,\ [\nabla^2(\bar{y}^{\mathrm{T}}g)(\bar{u})]_j,\ j=1,\ldots,n\}$ is $2n$. Consequently, the vectors $\{[\nabla^2 f(\bar{u})]_j,\ [\nabla^2(\bar{y}^{\mathrm{T}}g)(\bar{u})]_j,\ j=1,\ldots,n\}$ are linearly dependent. Therefore, this assumption is unreasonable.

In order to overcome this deficiency, we impose some restriction on the second order duality models $(\mathrm{ND}')_2$ and $(\mathrm{ND}')_3$, given in the following theorem.

Theorem 3.13 (Converse duality for (MOP) and $(\mathrm{ND}')_2$)**.** *Let $(\bar{u},\bar{y},\bar{\lambda},\bar{p})$ be an efficient solution to $(\mathrm{ND}')_2$. Suppose that*

(i) $\nabla(\bar{\lambda}^{\mathrm{T}}f+\bar{y}^{\mathrm{T}}g)(\bar{u})=0$,

(ii) *the $n\times n$ Hessian matrix $\nabla^2(\bar{\lambda}^{\mathrm{T}}f)(\bar{u})$ is positive or negative definite,*

(iii) *the $n\times n$ Hessian matrix $\nabla^2(\bar{y}^{\mathrm{T}}g)(\bar{u})$ is positive definite and $\bar{y}^{\mathrm{T}}g(\bar{u})\geqslant 0$, or the $n\times n$ Hessian matrix $\nabla^2(\bar{y}^{\mathrm{T}}g)(\bar{u})$ is negative definite and $\bar{y}^{\mathrm{T}}g(\bar{u})\leqslant 0$,*

(iv) *the $n\times n$ Hessian matrix $\nabla^2(\bar{\lambda}^{\mathrm{T}}f)(\bar{u})+\nabla^2(\bar{y}^{\mathrm{T}}g)(\bar{u})$ is nonsingular, and,*

(v) *the vectors $\{\nabla f_i(\bar{u}),\ i=1,\ldots,p\}$ are linearly independent, where $\nabla f_i(\bar{u})$ is the i-th row of $\nabla f(\bar{u})$. Then $\bar{u}$ is feasible for* (MOP)*, and the objective values of* (MOP) *and* $(\mathrm{ND}')_2$ *are equal.*

Furthermore, if the hypotheses of weak duality theorem (*see Theorem* 3.3) *are satisfied for all feasible solutions of* (MOP) *and* $(\mathrm{ND}')_2$, *then $\bar{u}$ is an efficient solution to* (MOP).

Proof. Let

$$L=\alpha^{\mathrm{T}}\left[f(u)-\frac{1}{2}p^{\mathrm{T}}\nabla^2(\lambda^{\mathrm{T}}f)(u)pe\right]+\beta^{\mathrm{T}}[\nabla(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)+\nabla^2(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)p]$$

$$+\gamma\left\{y^{\mathrm{T}}g(u)-\frac{1}{2}p^{\mathrm{T}}\nabla^2y^{\mathrm{T}}g(u)p-u^{\mathrm{T}}[\nabla(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)+\nabla^2(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)p]\right\}-\delta^{\mathrm{T}}y+\eta^{\mathrm{T}}\lambda.$$

Since $(\bar{u},\bar{y},\bar{\lambda},\bar{p})$ is an efficient solution for $(\mathrm{ND}')_2$, it follows from Lemma 2.7 and the generalized Fritz John type necessary conditions [3] that there exist $\alpha\in\mathbb{R}^p_+,\beta\in C_1,\gamma\in\mathbb{R}_+,\delta\in C_2^*$ and $\eta\in\mathbb{R}^p_+$ such that

$$\begin{aligned}\left.\frac{\partial L}{\partial u}\right|_{(\bar{u},\bar{y},\bar{\lambda},\bar{p})}&=(\alpha-\gamma\bar{\lambda})^{\mathrm{T}}\nabla f(\bar{u})+(\beta-\gamma\bar{u}-\gamma\bar{p})^{\mathrm{T}}\nabla^2(\bar{\lambda}^{\mathrm{T}}f+\bar{y}^{\mathrm{T}}g)(\bar{u})\\&\quad+\left(\beta-\frac{1}{2}\alpha^{\mathrm{T}}e\bar{p}-\gamma\bar{u}\right)^{\mathrm{T}}\nabla(\nabla^2(\bar{\lambda}^{\mathrm{T}}f)(\bar{u})\bar{p})\\&\quad+\left(\beta-\frac{1}{2}\gamma\bar{p}-\gamma\bar{u}\right)^{\mathrm{T}}\nabla(\nabla^2(\bar{y}^{\mathrm{T}}g)(\bar{u})\bar{p})=0,\end{aligned}\tag{3.25}$$

$$\left.\frac{\partial L}{\partial \lambda}\right|_{(\bar{u},\bar{y},\bar{\lambda},\bar{p})}=\nabla f(\bar{u})(\beta-\gamma\bar{u})+\left(\beta-\frac{1}{2}\alpha^{\mathrm{T}}e\bar{p}-\gamma\bar{u}\right)^{\mathrm{T}}\nabla^2 f(\bar{u})\bar{p}+\eta=0,\tag{3.26}$$

$$\left.\frac{\partial L}{\partial y}\right|_{(\bar{u},\bar{y},\bar{\lambda},\bar{p})}=\gamma g(\bar{u})+\nabla g(\bar{u})(\beta-\gamma\bar{u})+\left(\beta-\frac{1}{2}\gamma\bar{p}-\gamma\bar{u}\right)^{\mathrm{T}}\nabla^2 g(\bar{u})\bar{p}-\delta=0,\tag{3.27}$$

$$\left.\frac{\partial L}{\partial p}\right|_{(\bar{u},\bar{y},\bar{\lambda},\bar{p})}=(\beta-\alpha^{\mathrm{T}}e\bar{p}-\gamma\bar{u})^{\mathrm{T}}\nabla^2(\bar{\lambda}^{\mathrm{T}}f)(\bar{u})+(\beta-\gamma\bar{p}-\gamma\bar{u})^{\mathrm{T}}\nabla^2(\bar{y}^{\mathrm{T}}g)(\bar{u})=0,\tag{3.28}$$

$$\beta^{\mathrm{T}}[\nabla(\bar{\lambda}^{\mathrm{T}}f+\bar{y}^{\mathrm{T}}g)(\bar{u})+\nabla^2(\bar{\lambda}^{\mathrm{T}}f+\bar{y}^{\mathrm{T}}g)(\bar{u})\bar{p}]=0,\tag{3.29}$$

$$\gamma\left\{\bar{y}^{\mathrm{T}}g(\bar{u})-\frac{1}{2}\bar{p}^{\mathrm{T}}\nabla^2\bar{y}^{\mathrm{T}}g(\bar{u})\bar{p}-\bar{u}^{\mathrm{T}}[\nabla(\bar{\lambda}^{\mathrm{T}}f+\bar{y}^{\mathrm{T}}g)(\bar{u})+\nabla^2(\bar{\lambda}^{\mathrm{T}}f+\bar{y}^{\mathrm{T}}g)(\bar{u})\bar{p}]\right\}=0,\tag{3.30}$$

$$\delta^{\mathrm{T}}\bar{y}=0,\tag{3.31}$$

$$\eta^{\mathrm{T}}\bar{\lambda}=0,\tag{3.32}$$

$$(\alpha,\beta,\gamma,\eta)\neq 0.\tag{3.33}$$

Since $\bar{\lambda}>0$ and $\eta\in\mathbb{R}^n_+$, it is clear from (3.32) that

$$\eta=0.\tag{3.34}$$

Multiplying (3.27) by $\bar{y}$ and combining with (3.30) and (3.31), we have

$$\beta^{\mathrm{T}}[\nabla(\bar{y}^{\mathrm{T}}g)(\bar{u})+\nabla^2(\bar{y}^{\mathrm{T}}g)(\bar{u})\bar{p}]+\gamma\bar{u}^{\mathrm{T}}[\nabla(\bar{\lambda}^{\mathrm{T}}f)(\bar{u})+\nabla^2(\bar{\lambda}^{\mathrm{T}}f)(\bar{u})\bar{p}]=0.\tag{3.35}$$

Subtracting (3.29) from (3.35), we obtain

$$(\beta-\gamma\bar{u})^{\mathrm{T}}[\nabla(\bar{\lambda}^{\mathrm{T}}f)(\bar{u})+\nabla^2(\bar{\lambda}^{\mathrm{T}}f)(\bar{u})\bar{p}]=0.\tag{3.36}$$

Multiplying (3.26) by $\bar{\lambda}$ and combining with (3.34) and (3.36), we get

$$\frac{1}{2}(\alpha^{\mathrm{T}}e)\bar{p}^{\mathrm{T}}\nabla^2(\bar{\lambda}^{\mathrm{T}}f)(\bar{u})\bar{p}=0.$$

Since $\nabla^2(\bar{\lambda}^{\mathrm{T}}f)(\bar{u})$ is positive or negative definite,

$$(\alpha^{\mathrm{T}}e)\cdot\bar{p}=0.\tag{3.37}$$

First, we show $\gamma>0$. If not, then $\gamma=0$. It follows from (3.37) that (3.28) implies

$$\beta^{\mathrm{T}}[\nabla^2(\bar{\lambda}^{\mathrm{T}}f)(\bar{u})+\nabla^2(\bar{y}^{\mathrm{T}}g)(\bar{u})]=0.$$

Since $\nabla^2(\bar{\lambda}^{\mathrm{T}}f)(\bar{u})+\nabla^2(\bar{y}^{\mathrm{T}}g)(\bar{u})$ is nonsingular, $\beta=0$. Thus, from (3.25), we have

$$\sum_{i=1}^{p}\alpha_i\nabla f_i(\bar{u})=0,$$

because $\beta = 0$, $\gamma = 0$ and $(\alpha^{\mathrm{T}} e) \cdot \bar{p} = 0$. However, the vectors $\{\nabla f_i(\bar{u}),\ i = 1, \ldots, p\}$ are linearly independent and the above equation yields

$$\alpha = 0,$$

contradicting (3.33). Hence, $\gamma > 0$.

Now, we claim that $\alpha \neq 0$. Suppose that $\alpha = 0$. As $\nabla(\bar{\lambda}^{\mathrm{T}} f + \bar{y}^{\mathrm{T}} g)(\bar{u}) = 0$, it follows from (3.29) that

$$\beta^{\mathrm{T}}[\nabla^2(\bar{\lambda}^{\mathrm{T}} f)(\bar{u}) + \nabla^2(\bar{y}^{\mathrm{T}} g)(\bar{u})]\bar{p} = 0,$$

which together with (3.28) yields that

$$\gamma \bar{u}^{\mathrm{T}}[\nabla^2(\bar{\lambda}^{\mathrm{T}} f)(\bar{u})\bar{p} + \nabla^2(\bar{y}^{\mathrm{T}} g)(\bar{u})]\bar{p} + \gamma \bar{p}^{\mathrm{T}} \nabla^2(\bar{y}^{\mathrm{T}} g)(\bar{u})\bar{p} = 0, \tag{3.38}$$

since $(\alpha^{\mathrm{T}} e) \cdot \bar{p} = 0$. Taking (3.38) and $\nabla(\bar{\lambda}^{\mathrm{T}} f + \bar{y}^{\mathrm{T}} g)(\bar{u}) = 0$ into (3.30), we obtain that

$$\gamma \left\{ \bar{y}^{\mathrm{T}} g(\bar{u}) + \frac{1}{2} \bar{p}^{\mathrm{T}} \nabla^2(\bar{y}^{\mathrm{T}} g)(\bar{u})\bar{p} \right\} = 0. \tag{3.39}$$

As $\gamma \neq 0$, (3.39) yields

$$\bar{y}^{\mathrm{T}} g(\bar{u}) + \frac{1}{2} \bar{p}^{\mathrm{T}} \nabla^2 \bar{y}^{\mathrm{T}} g(\bar{u})\bar{p} = 0. \tag{3.40}$$

By Assumption (iii), $\bar{p} = 0$. Thus, (3.28) reduces to

$$(\beta - \gamma \bar{u})^{\mathrm{T}} \nabla^2(\bar{\lambda}^{\mathrm{T}} f + \bar{y}^{\mathrm{T}} g)(\bar{u}) = 0.$$

Since $\nabla^2(\bar{\lambda}^{\mathrm{T}} f)(\bar{u}) + \nabla^2(\bar{y}^{\mathrm{T}} g)(\bar{u})$ is nonsingular, $\beta = \gamma \bar{u}$, which together with $\alpha = 0$, $\bar{p} = 0$ and $\gamma \neq 0$ shows that (3.25) reduces to $\nabla(\bar{\lambda}^{\mathrm{T}} f)(\bar{u}) = 0$. Noting that the vectors $\{\nabla f_i(\bar{u}),\ i = 1, \ldots, p\}$ are linearly independent, we get $\bar{\lambda} = 0$, contradicting $\bar{\lambda} > 0$. Thus, $\alpha \neq 0$, and $\alpha^{\mathrm{T}} e > 0$.

We may now claim that $\bar{p} = 0$. As $\alpha^{\mathrm{T}} e > 0$, (3.37) implies

$$\bar{p} = 0.$$

As $\bar{p} = 0$, (3.28) reduces to

$$(\beta - \gamma \bar{u})^{\mathrm{T}} [\nabla^2(\bar{\lambda}^{\mathrm{T}} f)(\bar{u}) + \nabla^2(\bar{y}^{\mathrm{T}} g)(\bar{u})] = 0.$$

Since $\nabla^2(\bar{\lambda}^{\mathrm{T}} f)(\bar{u}) + \nabla^2(\bar{y}^{\mathrm{T}} g)(\bar{u})$ is nonsingular,

$$\beta = \gamma \bar{u}, \tag{3.41}$$

which implies

$$\bar{u} = \frac{1}{\gamma} \beta \in C_1, \tag{3.42}$$

as $\gamma > 0$.

From (3.27) and (3.41) along with $\gamma > 0$ and $\bar{p} = 0$, we get

$$g(\bar{u}) = \frac{1}{\gamma} \delta \in C_2^*. \tag{3.43}$$

Consequently, it follows from (3.42) and (3.43) that $\bar{u}$ is feasible for (MOP). Furthermore, from $\bar{p} = 0$, we have

$$f(\bar{u}) = f(\bar{u}) - \left[\frac{1}{2} \bar{p}^{\mathrm{T}} \nabla^2(\bar{\lambda}^{\mathrm{T}} f)(\bar{u})\bar{p} \right] e,$$

i.e., the objective values of (MOP) and $(\mathrm{ND}')_2$ are equal. The efficiency of $\bar{u}$ for (MOP) follows from the weak duality theorem (see Theorem 3.3). □

This converse duality result between (MOP) and $(\mathrm{ND}')_2$ reveals the fact that under mild assumptions, such as positive or negative definiteness, nonsingularity and linearly independent property, feasible solutions of (MOP) can be derived from efficient solutions of the second order duality model $(\mathrm{ND}')_2$, and the values of objective functions for both problems are equal. In addition, if the conditions of the weak duality theorem (see Theorem 3.3) between (MOP) and $(\mathrm{ND}')_2$ hold, these feasible solutions are efficient solutions to (MOP) .

Remark 3.14. When $f:\mathbb{R}^n\to\mathbb{R}$, Condition (v) in such a second order converse duality theorem between (MOP) and $(\mathrm{ND}')_2$ (see Theorem 3.13)

$$\text{“the vectors } \{\nabla f_i(\bar u),\ i=1,\dots,p\} \text{ are linearly independent”}$$

reduces to

$$\text{“}\nabla f(\bar u)\neq 0\text{”}.$$

However, even when we assume that $p=0$ and $F_{x,u}(a)=\eta(x,u)^{\mathrm{T}}a$, Theorem 3.13 could not completely reduce to Theorem 2 given by Yang et al. [21], some other conditions are needed in Theorem 3.13 to ensure $\alpha\neq 0$. This is because both of the primal and second order duality models (see (MOP), $(\mathrm{ND}')_2$) are multiobjective programming, and there are more parameters in the case of second order duality models.

As mentioned above, there exists some shortcoming in Assumption (ii) of Theorem 3 established by Ahmad and Agarwal [2]. Accordingly, like Theorem 3.13, some restriction on the duality model $(\mathrm{ND}')_3$ is imposed to obtain the second order converse duality theorem between (MOP) and $(\mathrm{ND}')_3$.

Theorem 3.15 (Converse duality for (MOP) and $(\mathrm{ND}')_3$)**.** *Let $(\bar u,\bar y,\bar\lambda,\bar p)$ be an efficient solution for $(\mathrm{ND}')_3$. Suppose that*

(i) $\nabla\bar y^{\mathrm{T}}g(\bar u)\neq 0$,

(ii) $\nabla(\bar\lambda^{\mathrm{T}}f+\bar y^{\mathrm{T}}g)(\bar u)=0$,

(iii) $\nabla^2(\bar y^{\mathrm{T}}g)(\bar u)$ *is positive definite and* $\bar y^{\mathrm{T}}g(\bar u)\leqslant 0$, *or,* $\nabla^2(\bar y^{\mathrm{T}}g)(\bar u)$ *is negative definite and* $\bar y^{\mathrm{T}}g(\bar u)\geqslant 0$,

(iv) $\nabla^2(\bar\lambda^{\mathrm{T}}f)(\bar u)$ *is positive or negative definite,*

(v) *the $n\times n$ Hessian matrix* $\nabla^2(\bar\lambda^{\mathrm{T}}f)(\bar u)+\nabla^2(\bar y^{\mathrm{T}}g)(\bar u)$ *is nonsingular, and,*

(vi) *the vectors* $\{\nabla f_i(\bar u),\ i=1,\dots,p\}$ *are linearly independent, where* $\nabla f_i(\bar u)$ *is the i-th row of* $\nabla f(\bar u)$.

Then $\bar u$ is feasible for (MOP)*, and the objective values of* (MOP) *and* $(\mathrm{ND}')_3$ *are equal.*

In addition, if the hypotheses of weak duality theorem (*see Theorem* 3.5) *are satisfied for all feasible solutions of* (MOP) *and* $(\mathrm{ND}')_3$*, then $\bar u$ is an efficient solution of* (MOP).

Proof. Let

$$\begin{aligned}L=\alpha^{\mathrm{T}}\bigg\{f(u)-\bigg\{\frac12p^{\mathrm{T}}\nabla^2(\lambda^{\mathrm{T}}f)(u)p+u^{\mathrm{T}}[\nabla(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)+\nabla^2(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)p]\bigg\}e\bigg\}\\ +\beta^{\mathrm{T}}[\nabla(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)+\nabla^2(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)p]+\gamma\bigg[y^{\mathrm{T}}g(u)-\frac12p^{\mathrm{T}}\nabla^2(y^{\mathrm{T}}g)(u)p\bigg]+\eta^{\mathrm{T}}\lambda.\end{aligned}$$

Since $(\bar u,\bar y,\bar\lambda,\bar p)$ is an efficient solution for $(\mathrm{ND}')_3$, it follows from Lemma 2.7 and the generalized Fritz John conditions [3] that there exist $\alpha\in\mathbb{R}^p_+$, $\beta\in C_1$, $\gamma\in\mathbb{R}_+$ and $\eta\in\mathbb{R}^p_+$ such that

$$\begin{aligned}\frac{\partial L}{\partial u}\bigg|_{(\bar u,\bar y,\bar\lambda,\bar p)}&=(\alpha-\alpha^{\mathrm{T}}e\bar\lambda)^{\mathrm{T}}\nabla f(\bar u)\\&\quad+(\gamma-\alpha^{\mathrm{T}}e)\nabla(\bar y^{\mathrm{T}}g)(\bar u)+\bigg(\beta-\frac12\alpha^{\mathrm{T}}e\bar p-\alpha^{\mathrm{T}}e\bar u\bigg)^{\mathrm{T}}\nabla(\nabla^2(\bar\lambda^{\mathrm{T}}f)(\bar u)\bar p)\\&\quad+\bigg(\beta-\frac12\gamma\bar p-\alpha^{\mathrm{T}}e\bar u\bigg)^{\mathrm{T}}\nabla(\nabla^2(\bar y^{\mathrm{T}}g)(\bar u)\bar p)+(\beta-\alpha^{\mathrm{T}}e\bar u-\alpha^{\mathrm{T}}e\bar p)^{\mathrm{T}}\nabla^2(\bar\lambda^{\mathrm{T}}f+\bar y^{\mathrm{T}}g)(\bar u)\\&=0,\end{aligned}\tag{3.44}$$

$$\frac{\partial L}{\partial\lambda}\bigg|_{(\bar u,\bar y,\bar\lambda,\bar p)}=\nabla f(\bar u)(\beta-\alpha^{\mathrm{T}}e\bar u)+\bigg(\beta-\frac12\alpha^{\mathrm{T}}e\bar p-\alpha^{\mathrm{T}}e\bar u\bigg)^{\mathrm{T}}\nabla^2f(\bar u)\bar p+\eta=0,\tag{3.45}$$

$$(y-\bar{y})^{\mathrm{T}}\frac{\partial L}{\partial y}\bigg|_{(\bar{u},\bar{y},\bar{\lambda},\bar{p})}=\bigg[\gamma g(\bar{u})+\bigg(\beta-\frac{1}{2}\gamma\bar{p}-\alpha^{\mathrm{T}}e\bar{u}\bigg)^{\mathrm{T}}\nabla^2 g(\bar{u})\bar{p}$$
$$+\nabla g(\bar{u})(\beta-\alpha^{\mathrm{T}}e\bar{u})\bigg]^{\mathrm{T}}(y-\bar{y})\leqslant 0,\quad \forall\, y\in C_2, \tag{3.46}$$

$$\frac{\partial L}{\partial p}\bigg|_{(\bar{u},\bar{y},\bar{\lambda},\bar{p})}=(\beta-\alpha^{\mathrm{T}}e\bar{p}-\alpha^{\mathrm{T}}e\bar{u})^{\mathrm{T}}\nabla^2(\bar{\lambda}^{\mathrm{T}}f)(\bar{u})+(\beta-\gamma\bar{p}-\alpha^{\mathrm{T}}e\bar{u})^{\mathrm{T}}\nabla^2(\bar{y}^{\mathrm{T}}g)(\bar{u})=0, \tag{3.47}$$

$$\beta^{\mathrm{T}}[\nabla(\bar{\lambda}^{\mathrm{T}}f+\bar{y}^{\mathrm{T}}g)(\bar{u})+\nabla^2(\bar{\lambda}^{\mathrm{T}}f+\bar{y}^{\mathrm{T}}g)(\bar{u})\bar{p}]=0, \tag{3.48}$$

$$\gamma\bigg[\bar{y}^{\mathrm{T}}g(\bar{u})-\frac{1}{2}\bar{p}^{\mathrm{T}}\nabla^2(\bar{y}^{\mathrm{T}}g)(\bar{u})\bar{p}\bigg]=0, \tag{3.49}$$

$$\eta^{\mathrm{T}}\bar{\lambda}=0, \tag{3.50}$$

$$(\alpha,\beta,\gamma,\eta)\neq 0. \tag{3.51}$$

Since $\bar{\lambda}>0$ and $\eta\in\mathbb{R}^n_+$, it is clear from (3.50) that

$$\eta=0.$$

As C_2 is a convex cone, it follows from (3.46) that

$$\gamma(\bar{y}^{\mathrm{T}}g(\bar{u}))+(\beta-\alpha^{\mathrm{T}}e\bar{u})^{\mathrm{T}}\nabla(\bar{y}^{\mathrm{T}}g)(\bar{u})+\bigg(\beta-\frac{1}{2}\gamma\bar{p}-\alpha^{\mathrm{T}}e\bar{u}\bigg)^{\mathrm{T}}\nabla^2(\bar{y}^{\mathrm{T}}g)(\bar{u})\bar{p}=0, \tag{3.52}$$

which together with (3.49) implies

$$(\beta-\alpha^{\mathrm{T}}e\bar{u})^{\mathrm{T}}[\nabla(\bar{y}^{\mathrm{T}}g)(\bar{u})+\nabla^2(\bar{y}^{\mathrm{T}}g)(\bar{u})\bar{p}]=0. \tag{3.53}$$

First, we show that $\alpha\neq 0$. Suppose that $\alpha=0$. Then, (3.45), (3.47) and (3.53), respectively reduce to

$$\nabla f(\bar{u})\beta+\beta^{\mathrm{T}}\nabla^2 f(\bar{u})\bar{p}=0, \tag{3.54}$$

$$\beta^{\mathrm{T}}\nabla^2(\bar{\lambda}^{\mathrm{T}}f)(\bar{u})+(\beta-\gamma\bar{p})^{\mathrm{T}}\nabla^2(\bar{y}^{\mathrm{T}}g)(\bar{u})=0, \tag{3.55}$$

$$\beta^{\mathrm{T}}[\nabla(\bar{y}^{\mathrm{T}}g)(\bar{u})+\nabla^2(\bar{y}^{\mathrm{T}}g)(\bar{u})\bar{p}]=0. \tag{3.56}$$

Multiplying (3.54) by $\bar{\lambda}$ and combining with (3.56), we get

$$\beta^{\mathrm{T}}[\nabla(\bar{\lambda}^{\mathrm{T}}f+\bar{y}^{\mathrm{T}}g)(\bar{u})+\nabla^2(\bar{\lambda}^{\mathrm{T}}f+\bar{y}^{\mathrm{T}}g)(\bar{u})\bar{p}]=0, \tag{3.57}$$

which together with (3.55) yields

$$\beta^{\mathrm{T}}\nabla(\bar{\lambda}^{\mathrm{T}}f+\bar{y}^{\mathrm{T}}g)(\bar{u})=-\gamma\bar{p}^{\mathrm{T}}\nabla^2(\bar{y}^{\mathrm{T}}g)(\bar{u})\bar{p}. \tag{3.58}$$

By Assumption (ii), (3.58) implies

$$\gamma\bar{p}^{\mathrm{T}}\nabla^2(\bar{y}^{\mathrm{T}}g)(\bar{u})\bar{p}=0.$$

As $\nabla^2(\bar{y}^{\mathrm{T}}g)(\bar{u})$ is positive or negative definite,

$$\gamma\bar{p}=0.$$

Substituting it into (3.55), we have

$$\beta=0,$$

since the $n\times n$ Hessian matrix $\nabla^2(\bar{\lambda}^{\mathrm{T}}f)(\bar{u})+\nabla^2(\bar{y}^{\mathrm{T}}g)(\bar{u})$ is nonsingular. Letting $\alpha=0$, $\gamma\bar{p}=0$ and $\beta=0$ in (3.44), we have

$$\gamma\nabla(\bar{y}^{\mathrm{T}}g)(\bar{u})=0.$$

By Assumption (i), $\gamma=0$. So $(\alpha,\beta,\gamma,\eta)=0$, contradicting (3.51). Therefore,

$$\alpha\geqslant 0\quad\text{and}\quad \alpha^{\mathrm{T}}e>0.$$

We now claim that $\gamma \neq 0$. If not, (3.47) reduces to

$$(\beta - \alpha^{\mathrm{T}} e\bar{u})^{\mathrm{T}} \nabla^2 (\bar{\lambda}^{\mathrm{T}} f + \bar{y}^{\mathrm{T}} g)(\bar{u}) = \alpha^{\mathrm{T}} e\bar{p}^{\mathrm{T}} \nabla^2 (\bar{\lambda}^{\mathrm{T}} f)(\bar{u}). \tag{3.59}$$

It follows from (3.45), (3.53) and (3.59) that

$$-\frac{1}{2}\alpha^{\mathrm{T}} e\bar{p}^{\mathrm{T}} \nabla^2 (\bar{\lambda}^{\mathrm{T}} f)(\bar{u})\bar{p} = (\beta - \alpha^{\mathrm{T}} e\bar{u})^{\mathrm{T}} \nabla (\bar{\lambda}^{\mathrm{T}} f + \bar{y}^{\mathrm{T}} g)(\bar{u})\bar{p}.$$

Since $\nabla(\bar{\lambda}^{\mathrm{T}} f + \bar{y}^{\mathrm{T}} g)(\bar{u}) = 0$ and $\alpha^{\mathrm{T}} e > 0$,

$$\bar{p}^{\mathrm{T}} \nabla^2 (\bar{\lambda}^{\mathrm{T}} f)(\bar{u})\bar{p} = 0,$$

which yields

$$\bar{p} = 0,$$

for $\nabla^2(\bar{\lambda}^{\mathrm{T}} f)(\bar{u})$ being positive or negative definite. Then, it is clear from (3.59) that

$$(\beta - \alpha^{\mathrm{T}} e\bar{u})^{\mathrm{T}} \nabla^2 (\bar{\lambda}^{\mathrm{T}} f + \bar{y}^{\mathrm{T}} g)(\bar{u}) = 0.$$

Because the $n \times n$ Hessian matrix $\nabla^2(\bar{\lambda}^{\mathrm{T}} f)(\bar{u}) + \nabla^2(\bar{y}^{\mathrm{T}} g)(\bar{u})$ is nonsingular,

$$\beta = \alpha^{\mathrm{T}} e\bar{u}.$$

Substituting $\bar{p} = 0$ and $\beta = \alpha^{\mathrm{T}} e\bar{u}$ into (3.44), we have

$$\nabla(\alpha^{\mathrm{T}} f)(\bar{u}) - \alpha^{\mathrm{T}} e \nabla(\bar{\lambda}^{\mathrm{T}} f + \bar{y}^{\mathrm{T}} g)(\bar{u}) = \nabla(\alpha^{\mathrm{T}} f)(\bar{u}) = 0.$$

Note that the vectors $\{\nabla f_i(\bar{u}),\ i = 1,\ldots,p,\ \nabla(\bar{y}^{\mathrm{T}} g)(\bar{u})\}$ are linearly independent, so $\alpha = 0$. This contradicts $\alpha \geqslant 0$. So $\gamma > 0$.

Then (3.49) gives

$$\bar{y}^{\mathrm{T}} g(\bar{u}) - \frac{1}{2}\bar{p}^{\mathrm{T}} \nabla^2 (\bar{y}^{\mathrm{T}} g)(\bar{u})\bar{p} = 0.$$

By Assumption (iii),

$$\bar{p} = 0.$$

Consequently, (3.47) reduces to

$$(\beta - \alpha^{\mathrm{T}} e\bar{u})^{\mathrm{T}} [\nabla^2 (\bar{\lambda}^{\mathrm{T}} f)(\bar{u}) + \nabla^2 (\bar{y}^{\mathrm{T}} g)(\bar{u})] = 0,$$

which yields

$$\beta = (\alpha^{\mathrm{T}} e)\bar{u}, \tag{3.60}$$

due to the nonsingularity of $\nabla^2(\bar{\lambda}^{\mathrm{T}} f + \bar{y}^{\mathrm{T}} g)(\bar{u})$.

Clearly, (3.60) implies that

$$\bar{u} = \frac{1}{\alpha^{\mathrm{T}} e}\beta \in C_1, \tag{3.61}$$

since $\alpha^{\mathrm{T}} e > 0$.

Combining with (3.46), (3.60) and $\bar{p} = 0$, we have

$$g(\bar{u}) \in C_2^*, \tag{3.62}$$

as $\gamma > 0$ and C_2 is a convex cone.

Accordingly, it follows from (3.61) and (3.62) that $\bar{u}$ is feasible for (MOP). In addition, by (3.48) and (3.60) along with $\alpha^{\mathrm{T}} e > 0$ and $\bar{p} = 0$, we have

$$\begin{aligned} f(\bar{u}) = f(\bar{u}) - \bigg\{ &\frac{1}{2}\bar{p}^{\mathrm{T}} \nabla^2 (\bar{\lambda}^{\mathrm{T}} f)(\bar{u})\bar{p} + \bar{u}^{\mathrm{T}} [\nabla(\bar{\lambda}^{\mathrm{T}} f + \bar{y}^{\mathrm{T}} g)(\bar{u}) \\ &+ \nabla^2 (\bar{\lambda}^{\mathrm{T}} f + \bar{y}^{\mathrm{T}} g)(\bar{u})\bar{p}] \bigg\} e, \end{aligned}$$

i.e., the objective values of (MOP) and $(\mathrm{ND}')_3$ are equal. The efficiency of $\bar{u}$ for (MOP) follows from the weak duality theorem (see Theorem 3.5). □

Similar to Theorem 3.13, this converse duality result between (MOP) and $(\mathrm{ND}')_3$ indicates that under suitable assumptions, feasible solutions of (MOP) can be derived from efficient solutions of second order duality model $(\mathrm{ND}')_3$, and the values of objective functions for both problems are equal. In addition, if the conditions of weak duality theorem (see Theorem 3.5) between (MOP) and $(\mathrm{ND}')_3$ hold, then these feasible solutions are efficient solutions to (MOP).

Remark 3.16. (i) We should point out that Condition (iii) in this converse duality theorem

"the vectors $\{\nabla f_i(\bar{u}),\ i=1,\dots,p\}$ are linearly independent"

indicates that the number p cannot be more than n.

(ii) Such a second order converse duality theorem between (MOP) and $(\mathrm{ND}')_3$ (see Theorem 3.15) is still stronger than that in Theorem 3 given by Yang et al. [21], even under the situation that $f:\mathbb{R}^n\to\mathbb{R}$, $p=0$ and $F_{x,u}(a)=\eta(x,u)^{\mathrm{T}}a$. Note that both of the primal and dual problems (i.e., (MOP) and $(\mathrm{ND}')_3$) are multiobjective programming, and there are more parameters in the case of second order duality models. This is why we need to strengthen conditions for this second order converse duality theorem.

Theorem 3.17 (Converse duality for (MOP) and $(\mathrm{ND}')_4$)**.** *Let $(\bar{u},\bar{y},\bar{\lambda},\bar{p})$ be an efficient solution to $(\mathrm{ND}')_4$. Suppose that*

(i) *either* (a) *$\nabla^2(\bar{\lambda}^{\mathrm{T}}f+\bar{y}^{\mathrm{T}}g)(\bar{u})$ is positive definite and $\bar{p}^{\mathrm{T}}[\nabla(\bar{\lambda}^{\mathrm{T}}f+\bar{y}^{\mathrm{T}}g)(\bar{u})]\geqslant 0$, or,* (b) *$\nabla^2(\bar{\lambda}^{\mathrm{T}}f+\bar{y}^{\mathrm{T}}g)(\bar{u})$ is negative definite and $\bar{p}^{\mathrm{T}}[\nabla(\bar{\lambda}^{\mathrm{T}}f+\bar{y}^{\mathrm{T}}g)(\bar{u})]\leqslant 0$,*

(ii) *the vectors $\{\nabla f_i(\bar{u}),\ i=1,\dots,p,\ \nabla(\bar{y}^{\mathrm{T}}g)(\bar{u})\}$ are linearly independent, where $\nabla f_i(\bar{u})$ is the i-th row of $\nabla f(\bar{u})$.*

Then $\bar{u}$ is feasible for (MOP)*, and the objective values of* (MOP) *and* $(\mathrm{ND}')_4$ *are equal.*

Moreover, if the hypotheses of the weak duality theorem (*see Theorem* 3.7) *are satisfied for all feasible solutions of* (MOP) *and* $(\mathrm{ND}')_4$*, then $\bar{u}$ is an efficient solution of* (MOP).

Proof. Let

$$L=\alpha^{\mathrm{T}}\left\{f(u)+\left[y^{\mathrm{T}}g(u)-\frac{1}{2}p^{\mathrm{T}}\nabla^2(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)p\right]e\right\}+\beta^{\mathrm{T}}[\nabla(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)+\nabla^2(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)p]$$
$$-\gamma u^{\mathrm{T}}[\nabla(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)+\nabla^2(\lambda^{\mathrm{T}}f+y^{\mathrm{T}}g)(u)p]+\eta^{\mathrm{T}}\lambda.$$

Since $(\bar{u},\bar{\lambda},\bar{y},\bar{p})$ is an efficient solution to $(\mathrm{ND}')_4$, it follows from Lemma 2.7 and the generalized Fritz John conditions [3] that there exist $\alpha\in\mathbb{R}^p_+,\beta\in C_1,\gamma\in\mathbb{R}_+$ and $\eta\in\mathbb{R}^p_+$ such that

$$\left.\frac{\partial L}{\partial u}\right|_{(\bar{u},\bar{y},\bar{\lambda},\bar{p})}=(\alpha-\gamma\bar{\lambda})^{\mathrm{T}}\nabla f(\bar{u})+(\alpha^{\mathrm{T}}e-\gamma)\nabla(\bar{y}^{\mathrm{T}}g)(\bar{u})+\left(\beta-\frac{1}{2}\alpha^{\mathrm{T}}e\bar{p}-\gamma\bar{u}\right)^{\mathrm{T}}\nabla(\nabla^2(\bar{\lambda}^{\mathrm{T}}f+\bar{y}^{\mathrm{T}}g)(\bar{u})\bar{p})$$
$$+(\beta-\gamma\bar{p}-\gamma\bar{u})^{\mathrm{T}}\nabla^2(\bar{\lambda}^{\mathrm{T}}f+\bar{y}^{\mathrm{T}}g)(\bar{u})=0, \tag{3.63}$$

$$\left.\frac{\partial L}{\partial \lambda}\right|_{(\bar{u},\bar{y},\bar{\lambda},\bar{p})}=\nabla f(\bar{u})(\beta-\gamma\bar{u})+\left(\beta-\frac{1}{2}\alpha^{\mathrm{T}}e\bar{p}-\gamma\bar{u}\right)^{\mathrm{T}}\nabla^2 f(\bar{u})\bar{p}+\eta=0, \tag{3.64}$$

$$(y-\bar{y})^{\mathrm{T}}\left.\frac{\partial L}{\partial y}\right|_{(\bar{u},\bar{y},\bar{\lambda},\bar{p})}=\left[\alpha^{\mathrm{T}}eg(\bar{u})+\left(\beta-\frac{1}{2}\alpha^{\mathrm{T}}e\bar{p}-\gamma\bar{u}\right)^{\mathrm{T}}\nabla^2 g(\bar{u})\bar{p}\right.$$
$$\left.+\nabla g(\bar{u})(\beta-\gamma\bar{u})\right]^{\mathrm{T}}(y-\bar{y})\leqslant 0,\quad \forall\, y\in C_2, \tag{3.65}$$

$$\left.\frac{\partial L}{\partial p}\right|_{(\bar{u},\bar{y},\bar{\lambda},\bar{p})}=(\beta-\alpha^{\mathrm{T}}e\bar{p}-\gamma\bar{u})^{\mathrm{T}}\nabla^2(\bar{\lambda}^{\mathrm{T}}f+\bar{y}^{\mathrm{T}}g)(\bar{u})=0, \tag{3.66}$$

$$\beta^{\mathrm{T}}[\nabla(\bar{\lambda}^{\mathrm{T}}f+\bar{y}^{\mathrm{T}}g)(\bar{u})+\nabla^2(\bar{\lambda}^{\mathrm{T}}f+\bar{y}^{\mathrm{T}}g)(\bar{u})\bar{p}]=0, \tag{3.67}$$

$$\gamma\bar{u}^{\mathrm{T}}[\nabla(\bar{\lambda}^{\mathrm{T}}f+\bar{y}^{\mathrm{T}}g)(\bar{u})+\nabla^2(\bar{\lambda}^{\mathrm{T}}f+\bar{y}^{\mathrm{T}}g)(\bar{u})\bar{p}]=0, \tag{3.68}$$

$$\eta^{\mathrm{T}}\bar{\lambda}=0, \tag{3.69}$$

$$(\alpha,\beta,\gamma,\eta)\neq 0. \tag{3.70}$$

Since $\bar{\lambda} > 0$ and $\eta \in \mathbb{R}^n_+$, it is clear from (3.69) that

$$\eta = 0.$$

By Assumption (i), $\nabla^2(\bar{\lambda}^{\mathrm{T}} f + \bar{y}^{\mathrm{T}} g)(\bar{u})$ is clearly nonsingular. Thus, by (3.66), we get

$$\beta = \alpha^{\mathrm{T}} e\bar{p} + \gamma\bar{u}. \tag{3.71}$$

We can claim that $\alpha \neq 0$. Indeed, if $\alpha = 0$, it follows from (3.71) that

$$\beta = \gamma\bar{u},$$

which along with (3.63) implies

$$\gamma[\nabla(\bar{\lambda}^{\mathrm{T}} f)(\bar{u}) + \nabla(\bar{y}^{\mathrm{T}} g)(\bar{u}) + \bar{p}^{\mathrm{T}}\nabla^2(\bar{\lambda}^{\mathrm{T}} f + \bar{y}^{\mathrm{T}} g)(\bar{u})] = 0.$$

Now, consider the following two cases: Case 1. $\gamma = 0$. Then $\beta = 0$, i.e., $(\alpha, \beta, \gamma, \eta) = 0$, contracting (3.70); Case 2. $\gamma \neq 0$. Then

$$\nabla(\bar{\lambda}^{\mathrm{T}} f)(\bar{u}) + \nabla(\bar{y}^{\mathrm{T}} g)(\bar{u}) + \nabla^2(\bar{\lambda}^{\mathrm{T}} f + \bar{y}^{\mathrm{T}} g)(\bar{u})\bar{p} = 0. \tag{3.72}$$

Multiplying (3.72) by $\bar{p}$, we get

$$\bar{p}^{\mathrm{T}}[\nabla(\bar{\lambda}^{\mathrm{T}} f)(\bar{u}) + \nabla(\bar{y}^{\mathrm{T}} g)(\bar{u})] + \bar{p}^{\mathrm{T}}\nabla^2(\bar{\lambda}^{\mathrm{T}} f + \bar{y}^{\mathrm{T}} g)(\bar{u})\bar{p} = 0.$$

By Assumption (i), we obtain $\bar{p} = 0$, which together with (3.72) yields

$$\sum_{i=1}^{p} \bar{\lambda}_i \nabla f_i(\bar{u}) + \nabla(\bar{y}^{\mathrm{T}} g)(\bar{u}) = 0.$$

This contradicts Assumption (ii). Hence,

$$\alpha \geqslant 0 \quad \text{and} \quad \alpha^{\mathrm{T}} e > 0.$$

Now, we should show $\bar{p} = 0$. It follows from (3.67) and (3.68) that

$$(\beta - \gamma\bar{u})^{\mathrm{T}}[\nabla(\bar{\lambda}^{\mathrm{T}} f + \bar{y}^{\mathrm{T}} g)(\bar{u}) + \nabla^2(\bar{\lambda}^{\mathrm{T}} f + \bar{y}^{\mathrm{T}} g)(\bar{u})\bar{p}] = 0,$$

which combined with (3.71) and $\alpha^{\mathrm{T}} e > 0$ yields

$$\bar{p}^{\mathrm{T}}[\nabla(\bar{\lambda}^{\mathrm{T}} f + \bar{y}^{\mathrm{T}} g)(\bar{u}) + \nabla^2(\bar{\lambda}^{\mathrm{T}} f + \bar{y}^{\mathrm{T}} g)(\bar{u})\bar{p}] = 0.$$

By Assumption (i),

$$\bar{p} = 0.$$

Accordingly, (3.71) reduces to

$$\beta = \gamma\bar{u}, \tag{3.73}$$

which together with (3.63) and $\bar{p} = 0$ yields

$$\sum_{i=1}^{p} (\alpha_i - \gamma\bar{\lambda}_i)\nabla f_i(\bar{u}) + (\alpha^{\mathrm{T}} e - \gamma)\nabla(\bar{y}^{\mathrm{T}} g)(\bar{u}) = 0.$$

Since the vectors $\{\nabla f_i(\bar{u}),\ i = 1, \ldots, p,\ \nabla(\bar{y}^{\mathrm{T}} g)(\bar{u})\}$ are linearly independent,

$$\gamma = \alpha^{\mathrm{T}} e > 0.$$

Therefore, (3.73) shows

$$\bar{u} = \frac{1}{\gamma}\beta \in C_1. \tag{3.74}$$

Substituting (3.73) and $\bar{p}=0$ in (3.65), we obtain

$$(\alpha^{\mathrm{T}}e)g(\bar{u})^{\mathrm{T}}(y-\bar{y})\leqslant 0,\quad \forall\, y\in C_2, \tag{3.75}$$

which implies

$$g(\bar{u})\in C_2^*, \tag{3.76}$$

and

$$\bar{y}^{\mathrm{T}}g(\bar{u})=0, \tag{3.77}$$

since $\alpha^{\mathrm{T}}e>0$ and C_2 is a convex cone.

Consequently, it follows form (3.74) and (3.76) that $\bar{u}$ is feasible for (MOP). From (3.77) and $\bar{p}=0$, we have

$$f(\bar{u})=f(\bar{u})+\left\{(\bar{y}^{\mathrm{T}}g)(\bar{u})-\frac{1}{2}\bar{p}^{\mathrm{T}}\nabla^2(\bar{\lambda}^{\mathrm{T}}f+\bar{y}^{\mathrm{T}}g)(\bar{u})\bar{p}\right\}e,$$

i.e., the objective values of (MOP) and $(\mathrm{ND}')_4$ are equal. The efficiency of $\bar{u}$ for (MOP) follows from the weak duality theorem (see Theorem 3.7). □

Remark 3.18. (i) Also, as mentioned in Remark 3.16(i), Theorem 3.17(ii),

"the vectors $\{\nabla f_i(\bar{u}),\ i=1,\ldots,p,\ \nabla(\bar{y}^{\mathrm{T}}g)(\bar{u})\}$ are linearly independent"

indicates that the number p cannot be more than n.

(ii) Note that in the second order duality theorem (see [2, Theorem 4]) given by Ahmad and Agarwal [2] for nonlinear programming, the condition

"$\nabla f(\bar{u})+\nabla(\bar{y}^{\mathrm{T}}g)(\bar{u})+\nabla^2 f(\bar{u})\bar{p}+\nabla^2(\bar{y}^{\mathrm{T}}g)(\bar{u})\bar{p}\neq 0$"

is essentially

"$\nabla f(\bar{u})+\nabla(\bar{y}^{\mathrm{T}}g)(\bar{u})\neq 0$",

since $\bar{p}=0$. Therefore, the linearly independent property in Theorem 3.17(ii) is stronger than the above condition, even for the case when $f:\mathbb{R}^n\to\mathbb{R}$. The main reason is that both of the primal and dual problems (i.e., (MOP) and $(\mathrm{ND}')_4$) are multiobjective programming.

(iii) In addition, in spite of $p=0$ and $F_{xu}(a)=\eta(x,u)^{\mathrm{T}}a$, the linearly independent property in Assumption (ii) of such a converse duality theorem (see Theorem 3.17) between (MOP) and $(\mathrm{ND}')_4$ is also stronger than the condition

"$\nabla f(\bar{u})+\nabla(\bar{y}^{\mathrm{T}}g)(\bar{u})\neq 0$"

used in the first order duality theorem (see Theorem 4) established by Yang et al. [21] for nonlinear programming. This is because that both of the primal and dual problems (i.e., (MOP) and $(\mathrm{ND}')_4$) are multiobjective programming, and there are more parameters in the second order duality model $(\mathrm{ND}')_4$.

4 Conclusion

Due to the computational advantage of the second order duality over the first order duality as well as the importance of multiobjective programming in practical applications, this paper is devoted to second order duals for a multiobjective programming with cone constraints. Based on the first order duals of Chandra and Abha [4] and the second order duals of Yang et al. [22] for nonlinear programming problems with cone constraints, four types of second order duality models are derived.

(i) Under the assumptions of second order F-pseudoconvexity and second order F-quasiconvexity, weak duality theorems are presented between (MOP) and $(\mathrm{ND}')_1$, $(\mathrm{ND}')_2$, $(\mathrm{ND}')_3$ and $(\mathrm{ND}')_4$, respectively.

(ii) Strong duality theorems are established by using the characterization of efficient solutions [5] and the generalized Fritz John conditions [3].

(iii) Converse duality theorems, the essential parts of duality theory, are discussed under certain suitable assumptions for the primal problem and four second order duality models, respectively. Note that there are some stronger conditions than those for the first order converse duality theorems of Yang et al. [21] and second order converse duality results of Ahmad and Agarwal [2] for nonlinear programming with cone constraints. These are due to the differences between scalar and multiobjective programming.

(iv) Meanwhile, we point out that there are some deficiencies in the second order converse duality theorems (see Theorems 2 and 3) of Ahmad and Agarwal [2]: "the vectors $\{[\nabla^2 f(\bar{u})]_j,\ [\nabla^2(\bar{y}^{\mathrm{T}} g)(\bar{u})]_j,\ j=1,\dots,n\}$ are linearly independent, where $[\nabla^2 f(\bar{u})]_j$ is the j-th row of $\nabla^2 f(\bar{u})$ and $[\nabla^2(\bar{y}^{\mathrm{T}} g)(\bar{u})]_j$ is the j-th row of $\nabla^2(\bar{y}^{\mathrm{T}} g)(\bar{u})$." In order to overcome this drawback, we impose some more restrictive conditions in the second order converse duality theorems (see Theorems 3.13 and 3.15).

Taking into account that the conditions imposed in Theorems 3.13 and 3.15 are quite strong, it is worthwhile to relax these conditions by weaker conditions in the second order converse duality Theorems 3.13 and 3.15.

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References

1 Aghezzaf B. Second order mixed type duality in multiobjective programming problems. J Math Anal Appl, 2003, 285: 97–106
2 Ahmad I, Agarwal R P. Second order converse duality for nonlinear programming. J Nonlinear Sci Appl, 2010, 3: 234–244
3 Bazaraa M S, Goode J J. On symmetric duality in nonlinear programming. Oper Res, 1973, 21: 1–9
4 Chandra S, Abha. A note on pseudo-invexity and duality in nonlinear programming. Eur J Oper Res, 2000, 122: 161–165
5 Chankong V, Haimes Y Y. Multiobjective Decision Making: Theory and Methodology. New York: North-Holland, 1983
6 Gulati T R, Mehndiratta G. Nondifferentiable multiobjective Mond-Weir type second-order symmetric duality over cones. Optim Lett, 2010, 4: 293–309
7 Gulati T R, Saini H, Gupta S K. Second-order multiobjective symmetric duality with cone constraints. Eur J Oper Res, 2010, 205: 247–252
8 Gupta S K, Kailey N. Second-order multiobjective symmetric duality involving cone-bonvex functions. J Global Optim, 2013, 55: 125–140
9 Hanson M A, Mond B. Further generalization of convexity in mathematical programming. J Inform Optim Sci, 1982, 3: 25–32
10 Kailey N, Gupta S K, Dangar D. Mixed second-order multiobjective symmetric duality with cone constraints. Nonlinear Anal Real, 2011, 12: 3373–3383
11 Mangasarian O L. Nonlinear Programming. New York: McGraw-Hill, 1969
12 Mangasarian O L. Second and higher order duality in nonlinear programming problem. J Math Anal Appl, 1975, 51: 607–620
13 Mishra S K, Lai K K. Second order symmetric duality in multiobjective programming involving generalized cone-invex functions. Eur J Oper Res, 2007, 178: 20–26
14 Mond B. Second order duality for nonlinear programming problem. Opsearch, 1974, 11: 90–99
15 Mond B, Weir T. Generalized concavity and duality. In: Generalized Concavity in Optimization and Economics. New York: Academic Press, 1981, 263–279
16 Nanda S, Das L N. Pseudo-invexity and duality in nonlinear programming. Eur J Oper Res, 1996, 88: 572–577
17 Pini R, Singh C. A survey of recent [1985–1995] advances in generalized convexity with applications to duality theory and optimality conditions. Optimization, 1997, 39: 311–360
18 Yang X M. On second order symmetric duality in nondifferentiable multiobjective programming. J Ind Manag Optim, 2009, 5: 697–703
19 Yang X M, Yang J, Lee H W J. Strong duality theorem for multiobjective higher order nondifferentiable symmetric dual programs. J Ind Manag Optim, 2013, 9: 525–530

20 Yang X M, Yang X Q, Teo K L. Non-differentiable second order symmetric duality in mathematical programming with F-convexity. Eur J Oper Res, 2003, 144: 554–559
21 Yang X M, Yang X Q, Teo K L. Converse duality in nonlinear programming with cone constraints. Eur J Oper Res, 2006, 170: 350–354
22 Yang X M, Yang X Q, Teo K L, et al. Second order duality for nonlinear programming. Indian J Pure Appl Math, 2004, 35: 699–708
23 Yang X M, Yang X Q, Teo K L, et al. Second order symmetric duality in non-differentiable multiobjective programming with F-convexity. Eur J Oper Res, 2005, 164: 406–416
24 Yang X M, Yang X Q, Teo K L, et al. Multiobjective second-order symmetric duality with F-convexity. Eur J Oper Res, 2005, 165: 585–591