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Global existence and decay of smooth solutions for the 3-D MHD-type equations without magnetic diffusion

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Abstract We study the large time behavior of a 3-D MHD (magneto-hydrodynamical)-type system without magnetic diffusion introduced by Lin and Zhang (2014). By using the elementary energy method and interpolation technique, we prove the global existence and decay estimate of smooth solution near the equilibrium state $(x_3, 0)$.

Keywords global existence, decay estimates, magneto-hydrodynamical equations, zero magnetic diffusion

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1 Introduction

In this paper, we consider the 3-D incompressible MHD-type equations without magnetic diffusion

$$\begin{cases} \partial_t\phi + u\cdot\nabla\phi = 0,\\ \partial_t u + u\cdot\nabla u - \Delta u + \nabla p = -\mathrm{div}\left(\nabla\phi\otimes\nabla\phi\right),\\ \mathrm{div}\, u = 0 \end{cases} \tag{1.1}$$

with the initial data (ϕ_0, u_0) in $\mathbb{R}^3$, where ϕ is a scalar function and u is the velocity. The system (1.1) is an analogy of the 2-D incompressible MHD equations without magnetic diffusion, which read as follows,

$$\begin{cases} \partial_t b + u\cdot\nabla b = b\cdot\nabla u,\\ \partial_t u + u\cdot\nabla u - \Delta u + \nabla\pi = b\cdot\nabla b,\\ \mathrm{div}\, u = \mathrm{div}\, b = 0. \end{cases} \tag{1.2}$$

To see this, we use $\mathrm{div}\, b = 0$ to obtain $b = (\partial_{x_2}\phi, -\partial_{x_1}\phi)$ for some potential function ϕ. Then in terms of ϕ, the system (1.2) can be rewritten as the 2-D version of (1.1), i.e.,

$$\begin{cases} \partial_t\phi + u\cdot\nabla\phi = 0,\\ \partial_t u + u\cdot\nabla u - \Delta u + \nabla p = -\mathrm{div}\left(\nabla\phi\otimes\nabla\phi\right),\\ \mathrm{div}\, u = 0, \end{cases} \tag{1.3}$$

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where $p=\pi-|\nabla\phi|^2$. The MHD equations without magnetic diffusion can be applied to plasmas when the plasmas are strongly collisional, or the resistivity due to these collisions are extremely small [2, 6, 12, 19].

In the last decades, much attention has been paid to the local/global existence of solutions to the 2-D or 3-D MHD equations with partial viscosity (see [3, 4, 7, 9] for 2-D case and [10, 11, 24] for 3-D case, and the references therein). It is well known that the 2-D MHD equations have the global smooth solution when the magnetic diffusion is included, while in the case without magnetic diffusion, the question of whether smooth solution of the 2-D MHD equations develops singularity in finite time is open [6, 19]. In a recent remarkable paper [13], Lin et al. proved the global existence of smooth solution of the system (1.3) around the trivial solution $(x_2,0)$. To capture the weak dissipation, the anisotropic Littlewood-Paley decomposition as well as anisotropic Besov space is used in [13]. More recently, Zhang [25] gave a simplified proof for the global existence and uniqueness by using the energy method. Wu et al. [23] further investigated the global well-posedness of this system with a velocity damping term. For the 3-D MHD-type equations (1.1), Lin and Zhang [14] established the global well-posedness for smooth initial data that is close to the nontrivial steady state $(x_3,0)$ by employing the anisotropic Littlewood-Paley analysis and the energy method. Then Lin and Zhang [15] simplified the proof by using the classical energy method.

On the other hand, whether there is certain dissipation or not is a very important problem for the inviscid MHD equations. It has been numerically showed that the energy is dissipated at a rate that is independent of the ohmic resistivity in the MHD systems (see [2]). In other words, the viscosity for the magnetic field equation can be zero and the system may still be dissipative. Recently, Ren et al. [17] confirmed this numerical observation for the 2-D MHD system (1.3) by providing an explicit time decay rate for various Sobolev norms of the solutions. The main tools used in [17] are the anisotropic Littlewood-Paley decomposition and the anisotropic Besov spaces. The main purpose of this paper is to establish the time decay estimates for the 3-D MHD-type system (1.1) by using the elementary energy method and the interpolation. As a by-product of our proof, we also obtain the global existence of small smooth solutions. We believe that the elementary energy-method techniques developed in this paper will be useful for various related problems.

Since we are considering the equations (1.1) close to the equilibrium state $(x_3,0)$, we may set $\psi=\phi-x_3$ and transform (1.1) into the following system for (ψ,u):

$$\begin{cases}\partial_t\psi+u\cdot\nabla\psi+u_3=0,\\ \partial_t u+u\cdot\nabla u-\Delta u+\nabla\partial_3\psi+\Delta\Psi+\nabla p=-\dfrac{1}{2}\nabla|\nabla\psi|^2-\Delta\psi\nabla\psi,\\ \operatorname{div}u=0,\\ (\psi,u)|_{t=0}=(\psi_0,u_0),\end{cases}\tag{1.4}$$

where $\Delta\Psi=(0,0,\Delta\psi)$. Furthermore, if we take divergence to the u-equation and solve the pressure p, we can transform (1.4) into the following form,

$$\begin{cases}\partial_t\psi+u_3=-u\cdot\nabla\psi,\\ \partial_t u_h-\Delta u_h-\partial_3\nabla_h\psi=-u\cdot\nabla u_h+f^h,\\ \partial_t u_3-\Delta u_3+\Delta_h\psi=-u\cdot\nabla u_3+f^v,\\ \operatorname{div}u=0,\\ (\psi,u)|_{t=0}=(\psi_0,u_0),\end{cases}\tag{1.5}$$

where $u_h=(u_1,u_2)^{\mathrm{T}}$, $\nabla_h=(\partial_1,\partial_2)^{\mathrm{T}}$ and

$$f^h=\frac{\nabla_h}{\Delta}(\partial_i u_j\partial_j u_i+\partial_i\partial_j(\partial_i\psi\partial_j\psi))-\partial_j(\nabla_h\psi\partial_j\psi),$$
$$f^v=\frac{\partial_3}{\Delta}(\partial_i u_j\partial_j u_i+\partial_i\partial_j(\partial_i\psi\partial_j\psi))-\partial_j(\partial_3\psi\partial_j\psi).$$

Hereafter, the indices i and j mean summation from $i,j=1$ to 3.

Our main result is stated as follows.

Theorem 1.1. *Assume that the initial data* (ψ_0, u_0) *satisfy* $(\nabla\psi_0, u_0) \in H^{14}(\mathbb{R}^3)$ *and* $\operatorname{div} u_0 = 0$. *Then there exist two small positive constants* δ *and* ε *such that if* $(\nabla\psi_0, u_0) \in H^{-s,4}(\mathbb{R}^3)$ *with* $s = 1-\delta$ *and*

$$\|\nabla\psi_0\|^2_{H^{14}(\mathbb{R}^3)} + \|u_0\|^2_{H^{14}(\mathbb{R}^3)} + \|\nabla\psi_0\|^2_{H^{-s,4}(\mathbb{R}^3)} + \|u_0\|^2_{H^{-s,4}(\mathbb{R}^3)} \leqslant C_0^2\varepsilon^2,$$

then the system (1.5) *admits a unique global solution* (ψ, u) *satisfying*

$$(\nabla\psi, u) \in C([0,+\infty); H^{14}(\mathbb{R}^3)) \cap C([0,+\infty); H^{-s,4}(\mathbb{R}^3)).$$

Moreover, it holds that

$$\|\nabla_h^\ell u\|_2 + \|\nabla\nabla_h^\ell\psi\|_2 + \|\nabla\nabla_h^\ell u\|_2 + \|\nabla^2\nabla_h^\ell\psi\|_2 \leqslant C\varepsilon(1+t)^{-\frac{s+\ell}{2}} \tag{1.6}$$

for any $t \in [0,+\infty)$ *and* $\ell = 0,1,2$. *Here* $H^{-s,k}(\mathbb{R}^3)$ *is the anisotropic Sobolev space with norm defined by*

$$\|u\|_{H^{-s,k}(\mathbb{R}^3)} = \||\xi_h|^{-s}(1+|\xi|^2)^{\frac{k}{2}}\hat{u}(\xi)\|_{L^2(\mathbb{R}^3)} < \infty,$$

where $\xi_h = (\xi_1, \xi_2)$.

Remark 1.2. Due to $s < 1$, we see that $(\nabla\psi_0, u_0) \in H^{-s,0}(\mathbb{R}^3)$ if $(\nabla\psi_0, u_0) \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$.

In Theorem 1.1, we obtained the anisotropic decay estimates for solutions of System (1.5). We can also establish the higher order decay estimates if the initial data has the more regularity. This anisotropic decay can be seen from the linearized version of (1.5):

$$\begin{cases} \partial_t\psi + u_3 = 0, \\ \partial_t u_h - \Delta u_h - \partial_3\nabla_h\psi = 0, \\ \partial_t u_3 - \Delta u_3 + \Delta_h\psi = 0, \\ \operatorname{div} u = 0. \end{cases}$$

By a simple calculation, we obtain

$$\begin{cases} \partial_{tt}\psi - \Delta_h\psi + \Delta u_3 = 0, \\ \partial_{tt}u_h - \Delta\partial_t u_h - \nabla_h(\nabla_h\cdot u_h) = 0, \\ \partial_{tt}u_3 - \Delta\partial_t u_3 - \Delta_h u_3 = 0, \end{cases}$$

which are damped wave equations with dissipation only on horizontal direction. A simple spectral analysis [17] reveals the anisotropy of the time decay rates on the spatial directions. This partial dissipation also implies that the solution has weak dissipation.

To prove Theorem 1.1, the main difficulty is that the system has a weak dissipation so that it is difficult to control the growth of nonlinear terms. To overcome this difficulty, we will use the anisotropic Sobolev space and the special structure of nonlinear terms. Our arguments are based on the elementary energy method and the interpolation.

The rest of this paper is organized as follows. In Section 2, we show the time decay estimates under the assumption that the solution is bounded in the anisotropic Sobolev spaces. Then in Section 3, we use the obtained decay estimates to prove that the solution is indeed bounded in the anisotropic Sobolev spaces with a refined bound. Thus, the theorem follows from a continuous argument. Finally in Section 4, we present some comments.

Notation. Throughout this paper, we set $x_h = (x_1, x_2)$ and use $\|\cdot\|_k$ to denote $\|\cdot\|_{L^k(\mathbb{R}^3)}$ for simplicity.

2 Proof of decay estimates

In this section, we will establish the decay estimates of solution under the assumption that the solution is bounded in some anisotropic Sobolev spaces, which will be closed in the next section. We begin with introducing the energy

$$D_\ell(t) := \|\nabla_h^\ell u\|_2^2 + \|\nabla\nabla_h^\ell\psi\|_2^2 + \|\nabla\nabla_h^\ell u\|_2^2 + \|\nabla^2\nabla_h^\ell\psi\|_2^2 + 2\epsilon_1\int \nabla_h^\ell u_3\nabla^2\nabla_h^\ell\psi,$$

$$H_\ell(t) := \|\nabla\nabla_h^\ell u\|_2^2 + \|\nabla^2\nabla_h^\ell u\|_2^2 + \epsilon_1\|\nabla\nabla_h^{\ell+1}\psi\|_2^2 - \epsilon_1\|\nabla\nabla_h^\ell u_3\|_2^2 - \epsilon_1\int \nabla^2\nabla_h^\ell u_3\nabla^2\nabla_h^\ell\psi$$

for $\ell = 0, 1, 2$. It follows from $\partial_3 u_3 = -\nabla_h \cdot u_h$ that we can fix a positive constant ϵ_1 such that

$$\begin{aligned} D_\ell(t) &\simeq (\|\nabla_h^\ell u\|_2^2 + \|\nabla\nabla_h^\ell\psi\|_2^2 + \|\nabla\nabla_h^\ell u\|_2^2 + \|\nabla^2\nabla_h^\ell\psi\|_2^2), \\ H_\ell(t) &\simeq (\|\nabla\nabla_h^\ell u\|_2^2 + \|\nabla^2\nabla_h^\ell u\|_2^2 + \|\nabla\nabla_h^{\ell+1}\psi\|_2^2) \end{aligned} \tag{2.1}$$

for $\ell = 0, 1, 2$.

Lemma 2.1 (See [16]). *There exists a constant $C > 0$ such that*

$$\|f\|_{L^\infty(\mathbb{R}^3)} \leqslant C\|\nabla f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}\|\nabla\nabla_h f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}.$$

We first deduce the following lower order energy decay estimates.

Proposition 2.2. *Assume that the solution (ψ, u) of (1.5) satisfies $\|u(t)\|_{H^2}^2 + \|\nabla\psi(t)\|_{H^3}^2 \leqslant c_0^2$ for any $t \in [0, T]$. If c_0 is suitably small, then for some $c > 0$, we have*

$$\frac{d}{dt}D_0(t) + cH_0(t) \leqslant 0 \quad \text{for any} \quad t \in [0, T].$$

Proof. By a standard energy estimate, we have

$$\frac{1}{2}\frac{d}{dt}(\|u\|_2^2 + \|\nabla\psi\|_2^2) + \|\nabla u\|_2^2 = 0.$$

Similarly, we also have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\nabla u\|_2^2 + \|\nabla^2\psi\|_2^2) + \|\nabla^2 u\|_2^2 \\ &\quad = \int (u\cdot\nabla u)\cdot\nabla^2 u - \int \nabla^2(u\cdot\nabla\psi)\cdot\nabla^2\psi - \int u_h\cdot\nabla^2 f^h - \int u_3\nabla^2 f^v \end{aligned}$$

and

$$\begin{aligned} &\frac{d}{dt}\int u_3\nabla^2\psi + \|\nabla\nabla_h\psi\|_2^2 - \|\nabla u_3\|_2^2 - \int \nabla^2 u_3\nabla^2\psi \\ &\quad = -\int (u\cdot\nabla u_3)\cdot\nabla^2\psi - \int (u\cdot\nabla\psi)\nabla^2 u_3 + \int f^v\nabla^2\psi. \end{aligned}$$

Then combining the above three equalities, we obtain

$$\begin{aligned} \frac{d}{dt}D_0(t) + H_0(t) &= \int (u\cdot\nabla u)\cdot\nabla^2 u - \int \nabla^2(u\cdot\nabla\psi)\cdot\nabla^2\psi - \int u_h\cdot\nabla^2 f^h - \int u_3\nabla^2 f^v \\ &\quad - \epsilon_1\int (u\cdot\nabla u_3)\cdot\nabla^2\psi - \epsilon_1\int (u\cdot\nabla\psi)\nabla^2 u_3 + \epsilon_1\int f^v\nabla^2\psi \\ &=: I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7. \end{aligned} \tag{2.2}$$

We now estimate $I_1, I_2, \ldots, I_7$ one by one. For I_1, it is clear that

$$I_1 \leqslant \|u\|_3\|\nabla u\|_6\|\nabla^2 u\|_2 \lesssim \|u\|_{H^1}\|\nabla^2 u\|_2^2 \leqslant Cc_0 H_0(t).$$

For the term I_2, we first use the integration by parts and the divergence free of u to deduce that

$$\begin{aligned} I_2 &= -\int \nabla_h^2(u\cdot\nabla\psi)\nabla_h^2\psi - 2\int \nabla_h^2(u\cdot\nabla\psi)\partial_3^2\psi - \int \partial_3^2(u\cdot\nabla\psi)\partial_3^2\psi \\ &= -\int (\nabla_h^2(u\cdot\nabla\psi) + 2\partial_3^2(u\cdot\nabla\psi))\nabla_h^2\psi - \int (\partial_3^2(u_h\cdot\nabla_h\psi) + \partial_3^2(u_3\partial_3\psi))\partial_3^2\psi \\ &= -\int (\nabla_h^2 u\cdot\nabla\psi + 2\nabla_h u\cdot\nabla\nabla_h\psi + 2\partial_3^2 u\cdot\nabla\psi + 4\partial_3 u\cdot\nabla\partial_3\psi + 2u\cdot\nabla\partial_3^2\psi)\nabla_h^2\psi \\ &\quad - \int (\partial_3^2 u_h\cdot\nabla_h\psi + 2\partial_3 u_h\cdot\nabla_h\partial_3\psi + \partial_3^2 u_3\partial_3\psi + 2\partial_3 u_3\partial_3^2\psi)\partial_3^2\psi. \end{aligned}$$

Notice that

$$\int \partial_3^2 u_3 \partial_3 \psi \partial_3^2 \psi = -\frac{1}{2} \int \nabla_h \cdot \partial_3 u_h \partial_3 (\partial_3 \psi)^2 = -\int \partial_3^2 u_h \cdot \nabla_h \partial_3 \psi \partial_3 \psi$$

and

$$\int \partial_3 u_3 \partial_3^2 \psi \partial_3^2 \psi = -\int \nabla_h \cdot u_h (\partial_3^2 \psi)^2 = -2 \int \partial_3 u_h \cdot \nabla_h \partial_3 \psi \partial_3^2 \psi - 2 \int u_h \cdot \nabla_h \partial_3 \psi \partial_3^3 \psi$$

by the divergence free of u and the integration by parts again. It then follows from Hölder's inequality and Sobolev's embedding that

$$\begin{aligned} I_2 &\lesssim (\|\nabla^2 u\|_2 \|\nabla \psi\|_\infty + \|\nabla u\|_6 \|\nabla^2 \psi\|_3 + \|u\|_6 \|\nabla^3 \psi\|_3) \|\nabla_h^2 \psi\|_2 + \|\nabla^2 u\|_2 \|\nabla_h \psi\|_6 \|\nabla^2 \psi\|_3 \\ &\quad + (\|\nabla u\|_6 \|\nabla^2 \psi\|_3 + \|\nabla^2 u\|_2 \|\nabla \psi\|_\infty + \|\nabla u\|_2 \|\nabla^2 \psi\|_\infty + \|u\|_6 \|\nabla^3 \psi\|_3) \|\nabla \nabla_h \psi\|_2 \\ &\lesssim \|\nabla \psi\|_{H^3} (\|\nabla u\|_2^2 + \|\nabla^2 u\|_2^2 + \|\nabla \nabla_h \psi\|_2^2) \\ &\leqslant C c_0 H_0(t). \end{aligned}$$

For I_3 and I_4, we have

$$\begin{aligned} I_3 &= -\int \partial_i \partial_j u_h \cdot \nabla_h (u_i u_j + \partial_i \psi \partial_j \psi) + \int \nabla^2 u_h \cdot \partial_j (\nabla_h \psi \partial_j \psi) \\ &\lesssim \|\nabla^2 u\|_2 (\|\nabla u\|_2 \|u\|_\infty + \|\nabla \nabla_h \psi\|_2 \|\nabla \psi\|_\infty + \|\nabla_h \psi\|_6 \|\nabla^2 \psi\|_3) \\ &\lesssim (\|u\|_{H^2} + \|\nabla \psi\|_{H^2}) (\|\nabla u\|_2^2 + \|\nabla^2 u\|_2^2 + \|\nabla \nabla_h \psi\|_2^2) \\ &\leqslant C c_0 H_0(t) \end{aligned}$$

and

$$\begin{aligned} I_4 &= -\int \partial_i \partial_j u_3 \partial_3 (u_i u_j) - \int \partial_3 \partial_j u_3 \nabla_h \cdot (\nabla_h \psi \partial_j \psi) + \int \nabla_h \partial_j u_3 \cdot \nabla_h (\partial_3 \psi \partial_j \psi) \\ &\leqslant \|\nabla^2 u\|_2 (\|\nabla u\|_2 \|u\|_\infty + \|\nabla \nabla_h \psi\|_2 \|\nabla \psi\|_\infty) \\ &\lesssim (\|u\|_{H^2} + \|\nabla \psi\|_{H^2}) (\|\nabla u\|_2^2 + \|\nabla^2 u\|_2^2 + \|\nabla \nabla_h \psi\|_2^2) \\ &\leqslant C c_0 H_0(t). \end{aligned}$$

Also, it is clear that

$$I_5 + I_6 \lesssim \|u\|_6 (\|\nabla u\|_2 \|\nabla^2 \psi\|_3 + \|\nabla \psi\|_3 \|\nabla^2 u\|_2) \lesssim \|\nabla \psi\|_{H^2} (\|\nabla u\|_2^2 + \|\nabla^2 u\|_2^2) \leqslant C c_0 H_0(t).$$

Finally, for I_7, we can use

$$\begin{aligned} I_7 &= -\epsilon_1 \int \partial_3 \psi \partial_i u_j \partial_j u_i - \epsilon_1 \int \partial_j \psi \partial_i \partial_3 (\partial_i \psi \partial_j \psi) + \epsilon_1 \int \partial_j \psi \nabla^2 (\partial_3 \psi \partial_j \psi) \\ &= -\epsilon_1 \int \partial_3 \psi \partial_i u_j \partial_j u_i - \epsilon_1 \int \partial_j \psi \nabla_h \cdot \partial_3 (\nabla_h \psi \partial_j \psi) + \epsilon_1 \int \partial_j \psi \nabla_h^2 (\partial_3 \psi \partial_j \psi) \\ &= -\epsilon_1 \int \partial_3 \psi \partial_i u_j \partial_j u_i + \epsilon_1 \int \nabla_h \partial_j \psi \cdot \partial_3 (\nabla_h \psi \partial_j \psi) - \epsilon_1 \int \nabla_h \partial_j \psi \cdot \nabla_h (\partial_3 \psi \partial_j \psi) \end{aligned}$$

to deduce that

$$\begin{aligned} I_7 &\lesssim \|\nabla \psi\|_\infty \|\nabla u\|_2^2 + \|\nabla \nabla_h \psi\|_2 (\|\nabla \nabla_h \psi\|_2 \|\nabla \psi\|_\infty + \|\nabla_h \psi\|_6 \|\nabla^2 \psi\|_3) \\ &\lesssim \|\nabla \psi\|_{H^2} (\|\nabla u\|_2^2 + \|\nabla \nabla_h \psi\|_2^2) \\ &\leqslant C c_0 H_0(t). \end{aligned}$$

Summarily, we can substitute the estimates of $I_1, I_2, \ldots, I_7$ into (2.2) and then complete the proof of Proposition 2.2 by taking c_0 small suitably. □

Proposition 2.3. *Let $s = 1 - \delta$. Assume that the solution (ψ, u) of (1.5) satisfies*

$$\|u(t)\|_{H^3}^2 + \|\nabla \psi(t)\|_{H^8}^2 + \|u(t)\|_{\dot{H}^{-s,2}}^2 + \|\nabla \psi(t)\|_{\dot{H}^{-s,1}}^2 \leqslant c_0^2$$

for any $t \in [0, T]$. If δ and c_0 are suitably small, then for some $c > 0$, we have

$$\frac{d}{dt} D_1(t) + c H_1(t) \leqslant 0 \quad \textit{for any} \quad t \in [0, T].$$

Proof. It follows from the standard energy method that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\nabla_h u\|_2^2+\|\nabla\nabla_h\psi\|_2^2)+\|\nabla\nabla_h u\|_2^2\\&\quad=-\int\nabla\nabla_h\psi\cdot\nabla\nabla_h(u\cdot\nabla\psi)-\int\nabla_h u\cdot\nabla_h(u\cdot\nabla u)+\int\nabla_h u_h\cdot\nabla_h f^h+\int\nabla_h u_3\cdot\nabla_h f^v.\end{aligned}$$

Similarly, we also have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\nabla\nabla_h u\|_2^2+\|\nabla^2\nabla_h\psi\|_2^2)+\|\nabla^2\nabla_h u\|_2^2\\&\quad=-\int\nabla^2\nabla_h\psi\cdot\nabla^2\nabla_h(u\cdot\nabla\psi)-\int\nabla\nabla_h u\cdot\nabla\nabla_h(u\cdot\nabla u)\\&\qquad+\int\nabla\nabla_h u_h\cdot\nabla\nabla_h f^h+\int\nabla\nabla_h u_3\cdot\nabla\nabla_h f^v\end{aligned}$$

and

$$\begin{aligned}&-\frac{d}{dt}\int\nabla^2 u_3\nabla_h^2\psi+\|\nabla\nabla_h^2\psi\|_2^2-\|\nabla\nabla_h u_3\|_2^2-\int\nabla^2\nabla_h\psi\cdot\nabla^2\nabla_h u_3\\&\quad=-\int\nabla\nabla_h^2\psi\cdot\nabla(u\cdot\nabla u_3)+\int\nabla\nabla_h^2\psi\cdot\nabla f^v+\int\nabla^2 u_3\nabla_h^2(u\cdot\nabla\psi).\end{aligned}$$

Thus combining the above three equalities, we obtain

$$\begin{aligned}&\frac{d}{dt}D_1(t)+H_1(t)\\&\quad=-\int\nabla\nabla_h\psi\cdot\nabla\nabla_h(u\cdot\nabla\psi)-\int\nabla_h u\cdot\nabla_h(u\cdot\nabla u)+\int\nabla_h u_h\cdot\nabla_h f^h+\int\nabla_h u_3\cdot\nabla_h f^v\\&\qquad-\int\nabla^2\nabla_h\psi\cdot\nabla^2\nabla_h(u\cdot\nabla\psi)-\int\nabla\nabla_h u\cdot\nabla\nabla_h(u\cdot\nabla u)+\int\nabla\nabla_h u_h\cdot\nabla\nabla_h f^h\\&\qquad+\int\nabla\nabla_h u_3\cdot\nabla\nabla_h f^v-\epsilon_1\int\nabla\nabla_h^2\psi\cdot\nabla(u\cdot\nabla u_3)+\epsilon_1\int\nabla\nabla_h^2\psi\cdot\nabla f^v+\epsilon_1\int\nabla^2 u_3\nabla_h^2(u\cdot\nabla\psi)\\&\quad=:J_1+J_2+\cdots+J_{11}.\end{aligned}\tag{2.3}$$

We now estimate the terms $J_1, J_2, \dots, J_{11}$ one by one. For J_1, we first use the integration by parts and $\operatorname{div} u=0$ to rewrite it as

$$\begin{aligned}J_1&=-\int\nabla_h^2\psi\nabla_h^2(u\cdot\nabla\psi)-\int\nabla_h^2\psi\partial_3^2(u\cdot\nabla\psi)\\&=-\int\nabla_h^2\psi(\nabla_h^2u\cdot\nabla\psi+2\nabla_h u\cdot\nabla\nabla_h\psi)-\int\partial_3\nabla_h\psi\cdot(\partial_3\nabla_h u\cdot\nabla\psi+\nabla_h u\cdot\nabla\partial_3\psi+\partial_3 u\cdot\nabla\nabla_h\psi)\\&=-\int\nabla_h^2\psi(\nabla_h^2u\cdot\nabla\psi+2\nabla_h u\cdot\nabla\nabla_h\psi)-\int\partial_3\nabla_h\psi\cdot(\partial_3\nabla_h u_h\cdot\nabla_h\psi+\nabla_h u_h\cdot\nabla_h\partial_3\psi\\&\quad+\partial_3 u_h\cdot\nabla_h\nabla_h\psi)-\int\partial_3\nabla_h\psi\cdot(\partial_3\nabla_h u_3\partial_3\psi+\nabla_h u_3\partial_3^2\psi+\partial_3 u_3\partial_3\nabla_h\psi).\end{aligned}$$

Due to $\partial_3 u_3=-\nabla_h\cdot u_h$, we have

$$\begin{aligned}J_1&=-\int\nabla_h^2\psi(\nabla_h^2u\cdot\nabla\psi+2\nabla_h u\cdot\nabla\nabla_h\psi)-\int\partial_3\nabla_h\psi\cdot(\partial_3\nabla_h u_h\cdot\nabla_h\psi+\nabla_h u_h\cdot\nabla_h\partial_3\psi\\&\quad+\partial_3 u_h\cdot\nabla_h\nabla_h\psi)+\int\partial_3\nabla_h\psi\cdot(\nabla_h(\nabla_h\cdot u_h)\partial_3\psi-\nabla_h u_3\partial_3^2\psi+(\nabla_h\cdot u_h)\partial_3\nabla_h\psi)\\&=-\int(\nabla_h^2\psi\nabla_h^2u\cdot\nabla\psi-2\nabla_h\psi\cdot\nabla_h u\cdot\nabla\nabla_h^2\psi)-\int\partial_3\nabla_h\psi\cdot(\partial_3\nabla_h u_h\cdot\nabla_h\psi+\nabla_h u_h\cdot\nabla_h\partial_3\psi\\&\quad+\partial_3 u_h\cdot\nabla_h\nabla_h\psi)-\int(\partial_3\nabla_h^2\psi(\nabla_h\cdot u_h)\partial_3\psi+(\nabla_h\cdot u_h)|\partial_3\nabla_h\psi|^2)\\&\quad-\int\partial_3\nabla_h\psi\cdot(\nabla_h u_3\partial_3^2\psi-(\nabla_h\cdot u_h)\partial_3\nabla_h\psi).\end{aligned}$$

It then follows from Hölder's inequality and Sobolev's embedding that

$$\begin{aligned} J_1 &\lesssim \|\nabla_h^2\psi\|_6\|\nabla_h^2 u\|_2\|\nabla\psi\|_3 + \|\nabla\nabla_h^2\psi\|_2\|\nabla_h u\|_6\|\nabla\psi\|_3 + \|\nabla\nabla_h u\|_6\|\nabla\nabla_h\psi\|_{\frac{12}{5}}\|\nabla_h\psi\|_{\frac{12}{5}} \\ &\quad + \|\nabla_h u\|_6\|\nabla\nabla_h\psi\|_{\frac{12}{5}}^2 + \|\nabla\nabla_h\psi\|_{\frac{12}{5}}\|\nabla u\|_{\frac{12}{5}}\|\nabla_h^2\psi\|_6 + \|\nabla_h u\|_6\|\nabla\nabla_h\psi\|_{\frac{12}{5}}\|\nabla^2\psi\|_{\frac{12}{5}} \\ &\lesssim \|\nabla\nabla_h^2\psi\|_2\|\nabla\nabla_h u\|_2\|\nabla\psi\|_{H^1} + \|\nabla^2\nabla_h u\|_2\|\nabla\nabla_h\psi\|_{\frac{12}{5}}\|\nabla\psi\|_{\frac{12}{5}} + \|\nabla\nabla_h u\|_2\|\nabla\nabla_h\psi\|_{\frac{12}{5}}^2 \\ &\quad + \|\nabla\nabla_h^2\psi\|_2\|\nabla\nabla_h\psi\|_{\frac{12}{5}}\|\nabla u\|_{\frac{12}{5}} + \|\nabla\nabla_h u\|_2\|\nabla\nabla_h\psi\|_{\frac{12}{5}}\|\nabla\psi\|_{\frac{12}{5}}^{\frac{5}{6s+1}}\|\nabla\psi\|_{H^6}^{\frac{6s-4}{6s+1}} \end{aligned} \tag{2.4}$$

for $s \geqslant \frac{7}{8}$. Notice that

$$\begin{aligned} \|\nabla\nabla_h\psi\|_{\frac{12}{5}} &\lesssim \Big\|\|\nabla\nabla_h^2\psi\|_{L^2_{x_h}}^{\frac{6s+7}{6(s+2)}}\|\Lambda_h^{-s}\nabla\psi\|_{L^2_{x_h}}^{\frac{5}{6(s+2)}}\Big\|_{L_{x_3}^{\frac{12}{5}}} \\ &\lesssim \|\nabla\nabla_h^2\psi\|_2^{\frac{6s+7}{6(s+2)}}\Big\|\|\Lambda_h^{-s}\nabla\psi\|_{L^2_{x_h}}\Big\|_{L_{x_3}^{\frac{10}{3-s}}}^{\frac{5}{6(s+2)}} \\ &\lesssim \|\nabla\nabla_h^2\psi\|_2^{\frac{6s+7}{6(s+2)}}\|\nabla\psi\|_{H^{-s,1}}^{\frac{5}{6(s+2)}} \end{aligned} \tag{2.5}$$

by the interpolation and Sobolev's embedding. Similarly, we also have

$$\begin{aligned} \|\nabla\psi\|_{\frac{12}{5}} &\lesssim \Big\|\|\nabla\nabla_h^2\psi\|_{L^2_{x_h}}^{\frac{6s+1}{6(s+2)}}\|\Lambda_h^{-s}\nabla\psi\|_{L^2_{x_h}}^{\frac{11}{6(s+2)}}\Big\|_{L_{x_3}^{\frac{12}{5}}} \\ &\lesssim \|\nabla\nabla_h^2\psi\|_2^{\frac{6s+1}{6(s+2)}}\Big\|\|\Lambda_h^{-s}\nabla\psi\|_{L^2_{x_h}}\Big\|_{L_{x_3}^{\frac{22}{9-s}}}^{\frac{11}{6(s+2)}} \\ &\lesssim \|\nabla\nabla_h^2\psi\|_2^{\frac{6s+1}{6(s+2)}}\|\nabla\psi\|_{H^{-s,1}}^{\frac{11}{6(s+2)}} \end{aligned} \tag{2.6}$$

and

$$\|\nabla u\|_{\frac{12}{5}} \lesssim \|\nabla\nabla_h^2 u\|_2^{\frac{6s+1}{6(s+2)}}\|\nabla u\|_{H^{-s,1}}^{\frac{11}{6(s+2)}} \lesssim \|\nabla\nabla_h^2 u\|_2^{\frac{6s+1}{6(s+2)}}\|u\|_{H^{-s,2}}^{\frac{11}{6(s+2)}}. \tag{2.7}$$

Thus substituting (2.5)–(2.7) into (2.4), we can deduce that

$$J_1 \lesssim (\|\nabla\psi\|_{H^6} + \|\nabla\psi\|_{H^{-s,1}} + \|u\|_{H^{-s,2}})(\|\nabla\nabla_h^2\psi\|_2^2 + \|\nabla\nabla_h u\|_2^2 + \|\nabla^2\nabla_h u\|_2^2) \leqslant Cc_0 H_1(t).$$

The term J_2 can be directly estimated as follows:

$$J_2 = \int \nabla_h u \cdot \nabla\nabla_h u \cdot u \leqslant \|\nabla_h u\|_6\|\nabla\nabla_h u\|_2\|u\|_3 \lesssim \|u\|_{H^1}\|\nabla\nabla_h u\|_2^2 \leqslant Cc_0 H_1(t).$$

For J_3, we use the L^p boundedness of Riesz operator, the interpolation and Sobolev's embedding to obtain

$$\begin{aligned} J_3 &= -\int \nabla_h^2 u_h \cdot \frac{\partial_i\partial_j}{\Delta}\nabla_h(u_i u_j) + \int \nabla_h u_h \cdot \nabla_h\frac{\partial_i\partial_j}{\Delta}\nabla_h(\partial_i\psi\partial_j\psi) + \int \partial_j\nabla_h u_h \cdot \nabla_h(\nabla_h\psi\partial_j\psi) \\ &\leqslant \|\nabla_h^2 u\|_2\|\nabla_h(u\otimes u)\|_2 + \|\nabla_h u\|_6\|\nabla_h^2(\nabla\psi\otimes\nabla\psi)\|_{\frac{6}{5}} + \|\nabla\nabla_h u\|_2\|\nabla_h(\nabla_h\psi\otimes\nabla\psi)\|_2 \\ &\lesssim \|\nabla\nabla_h u\|_2(\|\nabla_h u\|_6\|u\|_3 + \|\nabla\nabla_h^2\psi\|_2\|\nabla\psi\|_3 + \|\nabla\nabla_h\psi\|_{\frac{12}{5}}^2 + \|\nabla_h^2\psi\|_6\|\nabla\psi\|_3 + \|\nabla\nabla_h\psi\nabla_h\psi\|_2) \\ &\lesssim \|\nabla\nabla_h u\|_2(\|\nabla\nabla_h u\|_2\|u\|_{H^1} + \|\nabla\nabla_h^2\psi\|_2\|\nabla\psi\|_{H^1} + \|\nabla\nabla_h\psi\|_{\frac{12}{5}}^2 + \|\nabla\nabla_h\psi\nabla_h\psi\|_2). \end{aligned}$$

Then we have $J_3 \lesssim (\|\nabla\psi\|_{H^3} + \|\nabla\psi\|_{H^{-s,1}} + \|u\|_{H^1})(\|\nabla\nabla_h^2\psi\|_2^2 + \|\nabla\nabla_h u\|_2^2) \leqslant Cc_0 H_1(t)$ by (2.5) and the fact that

$$\begin{aligned} \|\nabla\nabla_h\psi\nabla_h\psi\|_2 &\leqslant \Big\|\|\nabla\nabla_h\psi\|_{L^4_{x_h}}\|\nabla_h\psi\|_{L^4_{x_h}}\Big\|_{L^2_{x_3}} \lesssim \Big\|\|\nabla\nabla_h^2\psi\|_{L^2_{x_h}}^{\frac{2(s+1)}{s+2}}\|\Lambda_h^{-s}\nabla\psi\|_{L^2_{x_h}}^{\frac{2}{s+2}}\Big\|_{L^2_{x_3}} \\ &\lesssim \|\nabla\nabla_h^2\psi\|_2\|\nabla\nabla_h^2\psi\|_{L^\infty_{x_3}L^2_{x_h}}^{\frac{s}{s+2}}\|\Lambda_h^{-s}\nabla\psi\|_{L^\infty_{x_3}L^2_{x_h}}^{\frac{2}{s+2}} \\ &\lesssim \|\nabla\nabla_h^2\psi\|_2\|\nabla\psi\|_{H^3}^{\frac{s}{s+2}}\|\nabla\psi\|_{H^{-s,1}}^{\frac{2}{s+2}}. \end{aligned}$$

To estimate J_4, we can first rewrite it as

$$\begin{aligned}J_4 &= -\int \nabla_h \partial_3 u_3 \cdot \frac{\partial_i \partial_j}{\Delta} \nabla_h (u_i u_j) + \int \nabla_h u_3 \cdot \nabla_h \left(\frac{\partial_i \partial_3}{\Delta} \partial_j (\partial_i \psi \partial_j \psi) - \partial_j (\partial_3 \psi \partial_j \psi) \right) \\ &= -\int \nabla_h \partial_3 u_3 \cdot \frac{\partial_i \partial_j}{\Delta} \nabla_h (u_i u_j) + \int \nabla_h u_3 \cdot \nabla_h \left(\frac{\partial_3 \partial_j}{\Delta} \nabla_h \cdot (\nabla_h \psi \partial_j \psi) - \frac{\nabla_h^2}{\Delta} \partial_j (\partial_3 \psi \partial_j \psi) \right)\end{aligned}$$

and then take a similar procedure as J_3 to obtain

$$J_4 \lesssim \|\nabla \nabla_h u\|_2 \|\nabla_h (u \otimes u)\|_2 + \|\nabla_h u\|_6 \|\nabla_h^2 (\nabla \psi \otimes \nabla \psi)\|_{\frac{6}{5}} \leqslant C c_0 H_1(t).$$

For J_5, we first rewrite it as

$$\begin{aligned}J_5 = &-\int (\nabla_h^2 \nabla_h \psi \cdot (\nabla_h^2 \nabla_h u \cdot \nabla \psi + 3\nabla_h^2 u \cdot \nabla \nabla_h \psi + 3\nabla_h u \cdot \nabla \nabla_h^2 \psi) \\ &+ 2\nabla_h^2 \partial_3 \psi (\partial_3 \nabla_h^2 u \cdot \nabla \psi + \nabla_h^2 u \cdot \nabla \partial_3 \psi + 2\partial_3 \nabla_h u \cdot \nabla \nabla_h \psi + 2\nabla_h u \cdot \nabla \nabla_h \partial_3 \psi + \partial_3 u \cdot \nabla \nabla_h^2 \psi)) \\ &- \int \partial_3^2 \nabla_h \psi \cdot \partial_3^2 \nabla_h (u \cdot \nabla \psi) \\ =:&\, J_5^1 + J_5^2.\end{aligned}$$

Then it is clear that

$$\begin{aligned}J_5^1 &\lesssim \|\nabla \nabla_h^2 \psi\|_2 (\|\nabla^2 \nabla_h u\|_2 \|\nabla \psi\|_\infty + \|\nabla \nabla_h u\|_6 \|\nabla^2 \psi\|_3 + \|\nabla_h u\|_6 \|\nabla^3 \psi\|_3 + \|\nabla \nabla_h^2 \psi\|_2 \|\nabla u\|_\infty) \\ &\lesssim (\|u\|_{H^3} + \|\nabla \psi\|_{H^3})(\|\nabla \nabla_h^2 \psi\|_2^2 + \|\nabla^2 \nabla_h u\|_2^2 + \|\nabla \nabla_h u\|_2^2).\end{aligned} \tag{2.8}$$

However, the estimate for J_5^2 is very subtle. We first notice that

$$\begin{aligned}J_5^2 = &-\int \partial_3^2 \nabla_h \psi \cdot (\partial_3^2 \nabla_h u_h \cdot \nabla_h \psi + 2\partial_3 \nabla_h u_h \cdot \nabla_h \partial_3 \psi + \nabla_h u_h \cdot \nabla_h \partial_3^2 \psi + \partial_3^2 u_h \cdot \nabla_h \nabla_h \psi \\ &+ 2\partial_3 u_h \cdot \nabla_h \nabla_h \partial_3 \psi + \partial_3^2 \nabla_h u_3 \partial_3 \psi + 2\partial_3 \nabla_h u_3 \partial_3^2 \psi + \nabla_h u_3 \partial_3^3 \psi + \partial_3^2 u_3 \partial_3 \nabla_h \psi + 2\partial_3 u_3 \partial_3^2 \nabla_h \psi).\end{aligned}$$

Then we can use $\partial_3 u_3 = -\nabla_h \cdot u_h$ and the integration by parts to obtain

$$\begin{aligned}J_5^2 &\lesssim (\|\nabla^2 \nabla_h u\|_2 \|\nabla^2 \nabla_h \psi\|_3 \|\nabla_h \psi\|_6 + \|\nabla \nabla_h u\|_6 \|\nabla^2 \nabla_h \psi\|_3 \|\nabla \nabla_h \psi\|_2 + \|\nabla_h u\|_6 \|\nabla^2 \nabla_h \psi\|_{\frac{12}{5}}^2 \\ &\quad + \|\nabla_h^2 \psi\|_6 \|\nabla^2 \nabla_h \psi\|_3 \|\nabla^2 u\|_2 + \|\nabla \nabla_h^2 \psi\|_2 \|\nabla^2 \nabla_h \psi\|_3 \|\nabla u\|_6 + \|\nabla^2 \nabla_h u\|_2 \|\nabla \nabla_h^2 \psi\|_2 \|\nabla \psi\|_\infty \\ &\quad + \|\nabla \nabla_h^2 \psi\|_2 \|\nabla \nabla_h u\|_6 \|\nabla^2 \psi\|_3 + \|\nabla_h u\|_6 \|\nabla \nabla_h^2 \psi\|_2 \|\nabla^3 \psi\|_3) + \left| \int \partial_3^2 \nabla_h \psi \cdot \nabla_h u_3 \partial_3^3 \psi \right| \\ &\lesssim (\|\nabla^2 \nabla_h u\|_2 \|\nabla^2 \nabla_h \psi\|_3 \|\nabla \nabla_h \psi\|_2 + \|\nabla \nabla_h u\|_2 \|\nabla^2 \nabla_h \psi\|_{\frac{12}{5}}^2 + \|\nabla \nabla_h^2 \psi\|_2 \|\nabla^2 \nabla_h \psi\|_3 \|\nabla^2 u\|_2 \\ &\quad + \|\nabla^2 \nabla_h u\|_2 \|\nabla \nabla_h^2 \psi\|_2 \|\nabla \psi\|_{H^2} + \|\nabla \nabla_h u\|_2 \|\nabla \nabla_h^2 \psi\|_2 \|\nabla \psi\|_{H^3}) + \left| \int \partial_3^2 \nabla_h \psi \cdot \nabla_h u_3 \partial_3^3 \psi \right|.\end{aligned} \tag{2.9}$$

It follows from the interpolation and Sobolev's embedding that

$$\begin{aligned}\|\nabla \nabla_h \psi\|_2 &\lesssim \| \|\nabla \nabla_h^2 \psi\|_{L^2_{x_h}}^{\frac{s+1}{s+2}} \|\Lambda_h^{-s} \nabla \psi\|_{L^2_{x_h}}^{\frac{1}{s+2}} \|_{L^2_{x_3}} \\ &\lesssim \|\nabla \nabla_h^2 \psi\|_2^{\frac{s+1}{s+2}} \|\Lambda_h^{-s} \nabla \psi\|_2^{\frac{1}{s+2}} \lesssim \|\nabla \nabla_h^2 \psi\|_2^{\frac{s+1}{s+2}} \|\nabla \psi\|_{H^{-s,0}}^{\frac{1}{s+2}},\end{aligned} \tag{2.10}$$

which implies that

$$\begin{aligned}\|\nabla^2 \nabla_h \psi\|_3 &\lesssim \|\nabla \nabla_h \psi\|_2^{\frac{s+2}{(s+1)^2}} \|\nabla^{\frac{5s^2+8s+1}{2(s^2+s-1)}} \nabla_h \psi\|_2^{\frac{s^2+s-1}{(s+1)^2}} \\ &\lesssim \|\nabla \nabla_h \psi\|_2^{\frac{s+2}{(s+1)^2}} \|\nabla \psi\|_{H^8}^{\frac{s^2+s-1}{(s+1)^2}} \\ &\lesssim \|\nabla \nabla_h^2 \psi\|_2^{\frac{1}{s+1}} \|\nabla \psi\|_{H^{-s,0}}^{\frac{1}{(s+1)^2}} \|\nabla \psi\|_{H^8}^{\frac{s^2+s-1}{(s+1)^2}}\end{aligned} \tag{2.11}$$

for $s=1-\delta$ with small δ, and that

$$\|\nabla^2\nabla_h\psi\|_{\frac{12}{5}} \lesssim \|\nabla\nabla_h\psi\|_2^{\frac{s+2}{2(s+1)}}\|\nabla^{\frac{7s+5}{2s}}\nabla_h\psi\|_2^{\frac{s}{2(s+1)}} \lesssim \|\nabla\nabla_h^2\psi\|_2^{\frac{1}{2}}\|\nabla\psi\|_{H^{-s,0}}^{\frac{1}{2(s+1)}}\|\nabla\psi\|_{H^7}^{\frac{s}{2(s+1)}}. \tag{2.12}$$

Similarly, we also have

$$\|\nabla^2 u\|_2 \lesssim \|\|\nabla^2\nabla_h u\|_{L^2_{x_h}}^{\frac{s}{s+1}}\|\Lambda_h^{-s}\nabla^2 u\|_{L^2_{x_h}}^{\frac{1}{s+1}}\|_{L^2_{x_3}} \lesssim \|\nabla^2\nabla_h u\|_2^{\frac{s}{s+1}}\|\Lambda_h^{-s}\nabla^2 u\|_2^{\frac{1}{s+1}} \lesssim \|\nabla^2\nabla_h u\|_2^{\frac{s}{s+1}}\|u\|_{H^{-s,2}}^{\frac{1}{s+1}}.$$

Substituting the above estimates into (2.9), we obtain

$$\begin{aligned} J_5^2 &\lesssim (\|\nabla\psi\|_{H^8}+\|\nabla\psi\|_{H^{-s,0}}+\|u\|_{H^{-s,2}})(\|\nabla\nabla_h^2\psi\|_2^2+\|\nabla\nabla_h u\|_2^2+\|\nabla^2\nabla_h u\|_2^2)\\ &\quad+\left|\int \partial_3^2\nabla_h\psi\cdot\nabla_h u_3\partial_3^3\psi\right|. \end{aligned} \tag{2.13}$$

It remains to estimate the integral term in (2.13). For this purpose, we first use the integration by parts to obtain

$$\begin{aligned} \int \partial_3^2\nabla_h\psi\cdot\nabla_h u_3\partial_3^3\psi &= -\int u_3\partial_3^2\nabla_h^2\psi\partial_3^3\psi-\int u_3\partial_3^2\nabla_h\psi\cdot\partial_3^3\nabla_h\psi\\ &= \int \partial_3 u_3\partial_3\nabla_h^2\psi\partial_3^3\psi+\int u_3\partial_3\nabla_h^2\psi\partial_3^4\psi-\frac{1}{2}\int u_3\partial_3|\partial_3^2\nabla_h\psi|^2\\ &= -\int(\nabla_h\cdot u_h)\partial_3\nabla_h^2\psi\partial_3^3\psi+\int u_3\partial_3\nabla_h^2\psi\partial_3^4\psi-\frac{1}{2}\int(\nabla_h\cdot u_h)|\partial_3^2\nabla_h\psi|^2. \end{aligned}$$

It then follows from Lemma 2.1 and (2.12) that

$$\begin{aligned} &\left|\int \partial_3^2\nabla_h\psi\cdot\nabla_h u_3\partial_3^3\psi\right|\\ &\quad\leqslant \|\nabla_h u\|_6\|\nabla\nabla_h^2\psi\|_2\|\nabla^3\psi\|_3+\|u_3\|_\infty\|\nabla\nabla_h^2\psi\|_2\|\nabla^4\psi\|_2+\|\nabla_h u\|_6\|\nabla^2\nabla_h\psi\|_{\frac{12}{5}}^2\\ &\quad\lesssim (\|\nabla\psi\|_{H^7}+\|\nabla\psi\|_{H^{-s,0}})\|\nabla\nabla_h u\|_2\|\nabla\nabla_h^2\psi\|_2+\|\nabla u_3\|_2^{\frac{1}{2}}\|\nabla\nabla_h u_3\|_2^{\frac{1}{2}}\|\nabla\nabla_h^2\psi\|_2\|\nabla^4\psi\|_2, \end{aligned}$$

which together with $\|\nabla u_3\|_2 \lesssim \|\nabla_h u\|_2 \lesssim \|\|\nabla_h^2 u\|_{L^2_{x_h}}^{\frac{s+1}{s+2}}\|\Lambda_h^{-s}u\|_{L^2_{x_h}}^{\frac{1}{s+2}}\|_{L^2_{x_3}} \lesssim \|\nabla\nabla_h u\|_2^{\frac{s+1}{s+2}}\|\Lambda_h^{-s}u\|_2^{\frac{1}{s+2}}$ and

$$\begin{aligned} \|\nabla^4\psi\|_2 &\lesssim \|\nabla\psi\|_2^{\frac{5}{8}}\|\nabla^8\nabla\psi\|_2^{\frac{3}{8}}\\ &\lesssim \|\|\nabla\nabla_h^2\psi\|_{L^2_{x_h}}^{\frac{s}{s+2}}\|\Lambda_h^{-s}\nabla\psi\|_{L^2_{x_h}}^{\frac{2}{s+2}}\|_{L^2_{x_3}}^{\frac{5}{8}}\|\nabla\psi\|_{H^8}^{\frac{3}{8}}\\ &\lesssim \|\nabla\nabla_h^2\psi\|_2^{\frac{5s}{8(s+2)}}\|\Lambda_h^{-s}\nabla\psi\|_2^{\frac{5}{4(s+2)}}\|\nabla\psi\|_{H^8}^{\frac{3}{8}} \end{aligned}$$

yields that

$$\left|\int \partial_3^2\nabla_h\psi\cdot\nabla_h u_3\partial_3^3\psi\right| \lesssim (\|\nabla\psi\|_{H^8}+\|\nabla\psi\|_{H^{-s,0}}+\|u\|_{H^{-s,0}})(\|\nabla\nabla_h u\|_2^2+\|\nabla\nabla_h^2\psi\|_2^2). \tag{2.14}$$

Substituting (2.14) into (2.13) and using (2.8), we have

$$\begin{aligned} J_5 &\lesssim (\|\nabla\psi\|_{H^8}+\|u\|_{H^3}+\|\nabla\psi\|_{H^{-s,0}}+\|u\|_{H^{-s,2}})(\|\nabla\nabla_h^2\psi\|_2^2+\|\nabla\nabla_h u\|_2^2+\|\nabla^2\nabla_h u\|_2^2)\\ &\leqslant Cc_0H_1(t). \end{aligned} \tag{2.15}$$

For J_6, a direct calculation yields that

$$J_6 \lesssim \|\nabla\nabla_h u\|_2(\|\nabla\nabla_h u\|_2\|\nabla u\|_\infty+\|\nabla_h u\|_6\|\nabla^2 u\|_3) \lesssim \|u\|_{H^3}\|\nabla\nabla_h u\|_2^2 \leqslant Cc_0H_1(t).$$

For J_7, we first use the integration by parts, Hölder's inequality and Sobolev's embedding to have

$$\begin{aligned} J_7 &= -\int \partial_i\partial_j\nabla_h u_h\cdot\nabla_h\nabla_h(u_iu_j+\partial_i\psi\partial_j\psi)+\int \nabla^2\nabla_h u_h\cdot\nabla_h\partial_j(\nabla_h\psi\partial_j\psi)\\ &\lesssim \|\nabla^2\nabla_h u\|_2(\|\nabla_h^2(u\otimes u)\|_2+\|\nabla_h^2(\nabla\psi\otimes\nabla\psi)\|_2+\|\nabla\nabla_h(\nabla_h\psi\otimes\nabla\psi)\|_2)\\ &\lesssim \|\nabla^2\nabla_h u\|_2(\|\nabla_h^2 u\|_2\|u\|_\infty+\|\nabla_h u\|_6\|\nabla_h u\|_3+\|\nabla\nabla_h^2\psi\|_2\|\nabla\psi\|_\infty+\|\nabla\nabla_h\psi\|_4^2\\ &\quad+\|\nabla_h^2\psi\|_6\|\nabla^2\psi\|_3+\|\nabla_h\psi\|_6\|\nabla^2\nabla_h\psi\|_3)\\ &\lesssim \|\nabla^2\nabla_h u\|_2(\|\nabla\nabla_h u\|_2\|u\|_{H^2}+\|\nabla\nabla_h^2\psi\|_2\|\nabla\psi\|_{H^2}+\|\nabla\nabla_h\psi\|_4^2+\|\nabla\nabla_h\psi\|_2\|\nabla^2\nabla_h\psi\|_3). \end{aligned}$$

Then by (2.10) and the fact that $\|\nabla\nabla_h\psi\|_4^2 \lesssim \|\nabla\nabla_h\psi\|_2^{\frac{s+2}{s+1}}\|\nabla^{\frac{5s+3}{2s}}\nabla_h\psi\|_2^{\frac{s}{s+1}} \lesssim \|\nabla\nabla_h\psi\|_2^{\frac{s+2}{s+1}}\|\nabla\psi\|_{H^5}^{\frac{s}{s+1}}$, we obtain

$$J_7 \lesssim (\|\nabla\psi\|_{H^8} + \|u\|_{H^2} + \|\nabla\psi\|_{H^{-s,0}})(\|\nabla\nabla_h^2\psi\|_2^2 + \|\nabla\nabla_h u\|_2^2 + \|\nabla^2\nabla_h u\|_2^2) \leqslant Cc_0H_1(t).$$

Similarly, for J_8, we have

$$\begin{aligned}
J_8 &= -\int \partial_i\partial_j\nabla_h u_3\cdot\nabla_h\partial_3(u_iu_j) - \int \nabla_h u_3\cdot\nabla_h\partial_3\partial_j\nabla_h\cdot(\nabla_h\psi\partial_j\psi) + \int \nabla_h u_3\cdot\nabla_h\nabla_h^2\partial_j(\partial_3\psi\partial_j\psi)\\
&= -\int \partial_i\partial_j\nabla_h u_3\cdot\nabla_h\partial_3(u_iu_j) - \int \partial_3\partial_j\nabla_h u_3\cdot\nabla_h\nabla_h\cdot(\nabla_h\psi\partial_j\psi) + \int \partial_j\nabla_h^2 u_3\nabla_h^2(\partial_3\psi\partial_j\psi)\\
&\lesssim \|\nabla^2\nabla_h u\|_2(\|\nabla\nabla_h u\|_2\|u\|_\infty + \|\nabla_h u\|_6\|\nabla u\|_3 + \|\nabla\nabla_h^2\psi\|_2\|\nabla\psi\|_\infty + \|\nabla\nabla_h\psi\|_4^2)\\
&\leqslant Cc_0H_1(t).
\end{aligned}$$

For J_9, a direct calculation gives that

$$\begin{aligned}
J_9 &= -\epsilon_1\int \nabla\nabla_h^2\psi\cdot\nabla(u_h\cdot\nabla_h u_3 + u_3\partial_3 u_3) = -\epsilon_1\int \nabla\nabla_h^2\psi\cdot\nabla(u_h\cdot\nabla_h u_3 - u_3\nabla_h\cdot u_h)\\
&\lesssim \|\nabla\nabla_h^2\psi\|_2(\|\nabla\nabla_h u\|_2\|u\|_\infty + \|\nabla_h u\|_6\|\nabla u\|_3)\\
&\lesssim \|u\|_{H^2}(\|\nabla\nabla_h u\|_2^2 + \|\nabla\nabla_h^2\psi\|_2^2)\\
&\leqslant Cc_0H_1(t).
\end{aligned}$$

For J_{10}, we can take a similar procedure as J_7 to obtain

$$\begin{aligned}
J_{10} &= -\epsilon_1\int \nabla_h^2\psi\partial_3\partial_i\partial_j(u_iu_j) - \epsilon_1\int \nabla_h^2\psi\partial_3\partial_j\nabla_h\cdot(\nabla_h\psi\partial_j\psi) + \epsilon_1\int \nabla_h^2\psi\partial_j\nabla_h^2(\partial_3\psi\partial_j\psi)\\
&= \epsilon_1\int \partial_3\nabla_h^2\psi\partial_i\partial_j(u_iu_j) + \epsilon_1\int \partial_3\nabla_h^2\psi\partial_j\nabla_h\cdot(\nabla_h\psi\partial_j\psi) - \epsilon_1\int \partial_j\nabla_h^2\psi\nabla_h^2(\partial_3\psi\partial_j\psi)\\
&\lesssim \|\nabla\nabla_h^2\psi\|_2(\|\nabla(u\nabla_h u)\|_2 + \|\partial_3^2u_3^2\|_2 + \|\nabla\nabla_h(\nabla_h\psi\nabla\psi)\|_2 + \|\nabla_h^2(\nabla\psi\nabla\psi)\|_2)\\
&\leqslant Cc_0H_1(t).
\end{aligned}$$

Finally, for J_{11}, we have

$$\begin{aligned}
J_{11} &= -\epsilon_1\int \nabla^2\nabla_h u_3\cdot(\nabla_h(u_h\cdot\nabla_h\psi) + \nabla_h u_3\partial_3\psi) - \epsilon_1\int \nabla^2\nabla_h u_3\cdot u_3\partial_3\nabla_h\psi\\
&\lesssim \|\nabla^2\nabla_h u\|_2(\|\nabla_h u\|_6\|\nabla\psi\|_3 + \|u\|_3\|\nabla_h^2\psi\|_6)\\
&\lesssim (\|\nabla\psi\|_{H^1} + \|u\|_{H^1})(\|\nabla\nabla_h u\|_2^2 + \|\nabla^2\nabla_h u\|_2^2 + \|\nabla\nabla_h^2\psi\|_2^2)\\
&\leqslant Cc_0H_1(t).
\end{aligned}$$

Here we used the fact that

$$\begin{aligned}
&-\int \nabla^2\nabla_h u_3\cdot u_3\partial_3\nabla_h\psi\\
&\quad= \int \nabla^2(\nabla_h\cdot u_h)\nabla_h u_3\cdot\nabla_h\psi + \int \nabla^2(\nabla_h\cdot u_h)u_3\nabla_h^2\psi - \int \nabla^2\nabla_h u_3(\nabla_h\cdot u_h)\nabla_h\psi.
\end{aligned}$$

Summarily, collecting the estimates for $J_1, J_2, \dots, J_{11}$, we can conclude that

$$\frac{d}{dt}D_1(t) + H_1(t) \leqslant Cc_0H_1(t).$$

This completes the proof of Proposition 2.3 by taking c_0 small suitably. □

Proposition 2.4. *Let $s = 1-\delta$. Assume that the solution (ψ, u) of (1.5) satisfies*

$$\|u(t)\|_{H^{14}}^2 + \|\nabla\psi(t)\|_{H^{14}}^2 + \|u(t)\|_{H^{-s,3}}^2 + \|\nabla\psi(t)\|_{H^{-s,4}}^2 \leqslant c_0^2$$

for any $t \in [0,T]$. If δ and c_0 are suitably small, then for some $c > 0$, we have

$$\frac{d}{dt}D_2(t) + cH_2(t) \leqslant 0 \quad \textit{for any} \quad t \in [0,T].$$

Proof. Similar to Proposition 2.3, we can use the standard energy method to deduce that

$$
\begin{aligned}
&\frac{d}{dt}D_2(t)+H_2(t)\\
&=-\int\nabla\nabla_h^2\psi\cdot\nabla\nabla_h^2(u\cdot\nabla\psi)-\int\nabla_h^2u\cdot\nabla_h^2(u\cdot\nabla u)+\int\nabla_h^2u_h\cdot\nabla_h^2f^h+\int\nabla_h^2u_3\cdot\nabla_h^2f^v\\
&\quad-\int\nabla^2\nabla_h^2\psi\nabla^2\nabla_h^2(u\cdot\nabla\psi)-\int\nabla\nabla_h^2u\cdot\nabla\nabla_h^2(u\cdot\nabla u)+\int\nabla\nabla_h^2u_h\cdot\nabla\nabla_h^2f^h\\
&\quad+\int\nabla\nabla_h^2u_3\cdot\nabla\nabla_h^2f^v+\epsilon_1\int\nabla\nabla_h^2\psi\cdot\nabla\nabla_h^2(u\cdot\nabla u_3)-\epsilon_1\int\nabla\nabla_h^2\psi\cdot\nabla\nabla_h^2f^v\\
&\quad-\epsilon_1\int\nabla^2\nabla_h^2u_3\nabla_h^2(u\cdot\nabla\psi)\\
&=:K_1+K_2+\cdots+K_{11}.
\end{aligned}\tag{2.16}
$$

We need to estimate each term on the right-hand side of (2.16). For K_1, we first rewrite it as

$$K_1=-\int\nabla_h^3\psi\cdot\nabla_h^3(u\cdot\nabla\psi)-\int\partial_3\nabla_h^2\psi\partial_3\nabla_h^2(u\cdot\nabla\psi)=:K_1^1+K_1^2.$$

By the integration by parts, we obtain

$$K_1^1=-\int\nabla_h^3\psi\cdot\nabla_h^3u_j\partial_j\psi+3\int\partial_j\nabla_h^3\psi\cdot\nabla_h^2u_j\nabla_h\psi+3\int\partial_j\nabla_h^3\psi\cdot\nabla_hu_j\nabla_h^2\psi.$$

Thus we have

$$
\begin{aligned}
K_1^1&\lesssim(\|\nabla_h^3\psi\|_6\|\nabla_h^3u\|_2+\|\nabla\nabla_h^3\psi\|_2\|\nabla_h^2u\|_6)\|\nabla\psi\|_3+\|\nabla\nabla_h^3\psi\|_2\|\nabla_hu\nabla_h^2\psi\|_2\\
&\lesssim(\|\nabla\psi\|_{H^4}+\|u\|_{H^{-s,1}}+\|\nabla\psi\|_{H^{-s,1}})(\|\nabla\nabla_h^2u\|_2^2+\|\nabla\nabla_h^3\psi\|_2^2)\\
&\leqslant Cc_0H_2(t).
\end{aligned}\tag{2.17}
$$

Here we used the fact that

$$
\begin{aligned}
\|\nabla_hu\nabla_h^2\psi\|_2&\leqslant\|\|\nabla_hu\|_{L^4_{x_h}}\|\nabla_h^2\psi\|_{L^4_{x_h}}\|_{L^2_{x_3}}\\
&\lesssim\|\|\nabla_h^3u\|_{L^2_{x_h}}^{\frac{2s+3}{2(s+3)}}\|\Lambda_h^{-s}u\|_{L^2_{x_h}}^{\frac{3}{2(s+3)}}\|\nabla_h^4\psi\|_{L^2_{x_h}}^{\frac{2s+3}{2(s+3)}}\|\Lambda_h^{-s}\nabla_h\psi\|_{L^2_{x_h}}^{\frac{3}{2(s+3)}}\|_{L^2_{x_3}}\\
&\lesssim(\|\nabla\psi\|_{H^4}+\|u\|_{H^{-s,1}}+\|\nabla\psi\|_{H^{-s,1}})(\|\nabla\nabla_h^2u\|_2+\|\nabla\nabla_h^3\psi\|_2).
\end{aligned}
$$

The estimate for K_1^2 is more subtle. By using $u\cdot\nabla=u_h\cdot\nabla_h+u_3\partial_3$ and the divergence free of u, we have

$$
\begin{aligned}
K_1^2=&-\int\partial_3\nabla_h^2\psi(\partial_3\nabla_h^2u_h\cdot\nabla_h\psi+\nabla_h^2u_h\cdot\nabla_h\partial_3\psi+2\partial_3\nabla_hu_h\cdot\nabla_h\nabla_h\psi+2\nabla_hu_h\cdot\nabla_h\nabla_h\partial_3\psi\\
&+\partial_3u_h\cdot\nabla_h\nabla_h^2\psi)+\int\partial_3\nabla_h^2\psi(\nabla_h^2(\nabla_h\cdot u_h)\partial_3\psi-\nabla_h^2u_3\partial_3^2\psi+2\nabla_h(\nabla_h\cdot u_h)\partial_3\nabla_h\psi\\
&-2\nabla_hu_3\cdot\nabla_h\partial_3^2\psi+(\nabla_h\cdot u_h)\partial_3\nabla_h^2\psi).
\end{aligned}
$$

It then follows from Sobolev's embedding that

$$
\begin{aligned}
K_1^2&\lesssim\|\nabla\nabla_h^2u\|_2\|\nabla\nabla_h^2\psi\nabla_h\psi\|_2+\|\nabla_h^2u\|_6\|\nabla\nabla_h^2\psi\|_{\frac{12}{5}}\|\nabla\nabla_h\psi\|_{\frac{12}{5}}+\|\nabla\nabla_h^3\psi\|_2\|\nabla\nabla_hu\nabla_h\psi\|_2\\
&\quad+\|\nabla_hu\|_6\|\nabla\nabla_h^2\psi\|_{\frac{12}{5}}^2+\|\nabla_h^3\psi\|_6\|\nabla\nabla_h^2\psi\|_{\frac{12}{5}}\|\nabla u\|_{\frac{12}{5}}+\|\nabla\nabla_h^3\psi\|_2\|\nabla_h^2u\|_6\|\nabla\psi\|_3\\
&\quad+\|\nabla\nabla_h^3\psi\|_2\|\nabla_hu_3\nabla^2\psi\|_2+\|\nabla_hu_3\|_6\|\nabla\nabla_h^2\psi\|_{\frac{12}{5}}\|\nabla^2\nabla_h\psi\|_{\frac{12}{5}}\\
&=:R_1+R_2+\cdots+R_8.
\end{aligned}\tag{2.18}
$$

We estimate the terms on the right-hand side of (2.18) one by one. Let $s=1-\delta$ with $\delta>0$ suitably small. Since

$$\|\nabla\nabla_h^2\psi\nabla_h\psi\|_2\leqslant\|\|\nabla\nabla_h^2\psi\|_{L^4_{x_h}}\|\nabla_h\psi\|_{L^4_{x_h}}\|_{L^2_{x_3}}$$

$$\begin{aligned}
&\lesssim \| \|\nabla\nabla_h^3\psi\|_{L^2_{x_h}}^{\frac{2s+3}{s+3}} \|\Lambda_h^{-s}\nabla\psi\|_{L^2_{x_h}}^{\frac{3}{s+3}} \|_{L^2_{x_3}} \\
&\lesssim \|\nabla\nabla_h^3\psi\|_2 \|\nabla\nabla_h^3\psi\|_{L^\infty_{x_3}L^2_{x_h}}^{\frac{s}{s+3}} \|\Lambda_h^{-s}\nabla\psi\|_{L^\infty_{x_3}L^2_{x_h}}^{\frac{3}{s+3}}, \qquad (2.19)
\end{aligned}$$

we have

$$R_1 \lesssim (\|\nabla\psi\|_{H^4} + \|\nabla\psi\|_{H^{-s,1}})\|\nabla\nabla_h^2 u\|_2\|\nabla\nabla_h^3\psi\|_2 \leqslant Cc_0H_2(t).$$

Similarly, we have

$$\begin{aligned}
R_3 &\leqslant \|\nabla\nabla_h^3\psi\|_2 \| \|\nabla\nabla_h u\|_{L^4_{x_h}} \|\nabla_h\psi\|_{L^4_{x_h}} \|_{L^2_{x_3}} \\
&\lesssim \|\nabla\nabla_h^3\psi\|_2 \| \|\nabla\nabla_h^2 u\|_{L^2_{x_h}}^{\frac{2s+3}{2(s+2)}} \|\Lambda_h^{-s}\nabla u\|_{L^2_{x_h}}^{\frac{1}{2(s+2)}} \|\nabla\nabla_h^3\psi\|_{L^2_{x_h}}^{\frac{2s+1}{2(s+3)}} \|\Lambda_h^{-s}\nabla\psi\|_{L^2_{x_h}}^{\frac{5}{2(s+3)}} \|_{L^2_{x_3}} \\
&\lesssim (\|\nabla\psi\|_{H^4} + \|u\|_{H^{-s,2}} + \|\nabla\psi\|_{H^{-s,1}})(\|\nabla\nabla_h^2 u\|_2^2 + \|\nabla\nabla_h^3\psi\|_2^2) \\
&\leqslant Cc_0H_2(t).
\end{aligned}$$

To estimate R_2, we use Sobolev's embedding and the interpolation to obtain

$$\begin{aligned}
\|\nabla\nabla_h^2\psi\|_{\frac{12}{5}} &\lesssim \| \|\nabla\nabla_h^3\psi\|_{L^2_{x_h}}^{\frac{6s+13}{6(s+3)}} \|\Lambda_h^{-s}\nabla\psi\|_{L^2_{x_h}}^{\frac{5}{6(s+3)}} \|_{L^{\frac{12}{5}}_{x_3}} \\
&\lesssim \|\nabla\nabla_h^3\psi\|_2^{\frac{6s+13}{6(s+3)}} \| \|\Lambda_h^{-s}\nabla\psi\|_{L^2_{x_h}} \|_{L^{\frac{10}{2-s}}_{x_3}}^{\frac{5}{6(s+3)}} \\
&\lesssim \|\nabla\nabla_h^3\psi\|_2^{\frac{6s+13}{6(s+3)}} \|\nabla\psi\|_{H^{-s,1}}^{\frac{5}{6(s+3)}} \qquad (2.20)
\end{aligned}$$

and

$$\begin{aligned}
\|\nabla\nabla_h\psi\|_{\frac{12}{5}} &\lesssim \| \|\nabla\nabla_h^3\psi\|_{L^2_{x_h}}^{\frac{6s+7}{6(s+3)}} \|\Lambda_h^{-s}\nabla\psi\|_{L^2_{x_h}}^{\frac{11}{6(s+3)}} \|_{L^{\frac{12}{5}}_{x_3}} \\
&\lesssim \|\nabla\nabla_h^3\psi\|_2^{\frac{6s+7}{6(s+3)}} \| \|\Lambda_h^{-s}\nabla\psi\|_{L^2_{x_h}} \|_{L^{\frac{22}{8-s}}_{x_3}}^{\frac{11}{6(s+3)}} \\
&\lesssim \|\nabla\nabla_h^3\psi\|_2^{\frac{6s+7}{6(s+3)}} \|\nabla\psi\|_{H^{-s,1}}^{\frac{11}{6(s+3)}}. \qquad (2.21)
\end{aligned}$$

Thus we have

$$R_2 \lesssim (\|\nabla\psi\|_{H^3} + \|\nabla\psi\|_{H^{-s,1}})\|\nabla\nabla_h^2 u\|_2\|\nabla\nabla_h^3\psi\|_2 \leqslant Cc_0H_2(t).$$

Similarly, it is easy to see that

$$\|\nabla_h u\|_6 \lesssim \|\nabla\nabla_h u\|_2 \lesssim \|\nabla\nabla_h^2 u\|_2^{\frac{s+1}{s+2}} \|\Lambda_h^{-s}\nabla u\|_2^{\frac{1}{s+2}} \lesssim \|\nabla\nabla_h^2 u\|_2^{\frac{s+1}{s+2}} \|u\|_{H^{-s,1}}^{\frac{1}{s+2}}, \qquad (2.22)$$

which together with (2.20) yields that

$$\begin{aligned}
R_4 &\lesssim \|\nabla\nabla_h^2 u\|_2^{\frac{s+1}{s+2}} \|u\|_{H^{-s,1}}^{\frac{1}{s+2}} \|\nabla\nabla_h^3\psi\|_2^{\frac{6s+13}{3(s+3)}} \|\nabla\psi\|_{H^{-s,1}}^{\frac{5}{3(s+3)}} \\
&\lesssim (\|\nabla\psi\|_{H^3} + \|\nabla\psi\|_{H^{-s,1}} + \|u\|_{H^{-s,1}})(\|\nabla\nabla_h^2 u\|_2^2 + \|\nabla\nabla_h^3\psi\|_2^2) \\
&\leqslant Cc_0H_2(t).
\end{aligned}$$

For R_5, it follows from Sobolev's embedding and the interpolation that

$$\begin{aligned}
\|\nabla u\|_{\frac{12}{5}} &\lesssim \| \|\nabla\nabla_h^3 u\|_{L^2_{x_h}}^{\frac{6s+1}{6(s+3)}} \|\Lambda_h^{-s}\nabla u\|_{L^2_{x_h}}^{\frac{17}{6(s+3)}} \|_{L^{\frac{12}{5}}_{x_3}} \\
&\lesssim \|\nabla\nabla_h^3 u\|_2^{\frac{6s+1}{6(s+3)}} \| \|\Lambda_h^{-s}\nabla u\|_{L^2_{x_h}} \|_{L^{\frac{34}{14-s}}_{x_3}}^{\frac{17}{6(s+3)}} \\
&\lesssim \|\nabla^2\nabla_h^2 u\|_2^{\frac{6s+1}{6(s+3)}} \|u\|_{H^{-s,2}}^{\frac{17}{6(s+3)}}.
\end{aligned}$$

Then by (2.20), we have

$$\begin{aligned}R_5 &\lesssim \|\nabla\nabla_h^3\psi\|_2^{1+\frac{6s+13}{6(s+3)}}\|\nabla\psi\|_{H^{-s,1}}^{\frac{5}{6(s+3)}}\|\nabla^2\nabla_h^2u\|_2^{\frac{6s+1}{6(s+3)}}\|u\|_{H^{-s,2}}^{\frac{17}{6(s+3)}}\\ &\lesssim (\|u\|_{H^4}+\|\nabla\psi\|_{H^{-s,1}}+\|u\|_{H^{-s,2}})(\|\nabla^2\nabla_h^2u\|_2^2+\|\nabla\nabla_h^3\psi\|_2^2)\\ &\leqslant Cc_0H_2(t).\end{aligned}$$

It is clear that

$$R_6 \lesssim \|\nabla\nabla_h^3\psi\|_2\|\nabla\nabla_h^2u\|_2\|\nabla\psi\|_{H^1} \lesssim \|\nabla\psi\|_{H^1}(\|\nabla\nabla_h^2u\|_2^2+\|\nabla\nabla_h^3\psi\|_2^2) \leqslant Cc_0H_2(t).$$

To estimate R_7, we first deduce that

$$\|\nabla_h^2u\|_2 \lesssim \|\|\nabla_h^3u\|_{L^2_{x_h}}^{\frac{s+2}{s+3}}\|\Lambda_h^{-s}u\|_{L^2_{x_h}}^{\frac{1}{s+3}}\|_{L^2_{x_3}} \lesssim \|\nabla\nabla_h^2u\|_2^{\frac{s+2}{s+3}}\|\Lambda_h^{-s}u\|_2^{\frac{1}{s+3}} = \|\nabla\nabla_h^2u\|_2^{\frac{s+2}{s+3}}\|u\|_{H^{-s,0}}^{\frac{1}{s+3}}. \tag{2.23}$$

Then by Lemma 2.1 we obtain

$$\|\nabla_hu_3\|_\infty \lesssim \|\nabla\nabla_hu_3\|_2^{\frac12}\|\nabla\nabla_h^2u_3\|_2^{\frac12} \lesssim \|\nabla_h^2u\|_2^{\frac12}\|\nabla\nabla_h^2u\|_2^{\frac12} \lesssim \|\nabla\nabla_h^2u\|_2^{\frac12+\frac{s+2}{2(s+3)}}\|u\|_{H^{-s,0}}^{\frac{1}{2(s+3)}}.$$

Thus we have

$$\begin{aligned}R_7 &\lesssim \|\nabla\nabla_h^3\psi\|_2\|\nabla\nabla_h^2u\|_2^{\frac12+\frac{s+2}{2(s+3)}}\|u\|_{H^{-s,0}}^{\frac{1}{2(s+3)}}\|\nabla\psi\|_2^{\frac{1}{2s}}\|\nabla^{\frac{2s}{2s-1}}\nabla\psi\|_2^{1-\frac{1}{2s}}\\ &\lesssim \|\nabla\nabla_h^3\psi\|_2\|\nabla\nabla_h^2u\|_2^{\frac12+\frac{s+2}{2(s+3)}}\|u\|_{H^{-s,0}}^{\frac{1}{2(s+3)}}\|\nabla\nabla_h^3\psi\|_2^{\frac{1}{2(s+3)}}\|\nabla\psi\|_{H^{-s,0}}^{\frac{3}{2s(s+3)}}\|\nabla\psi\|_{H^4}^{1-\frac{1}{2s}}\\ &\lesssim (\|\nabla\psi\|_{H^4}+\|\nabla\psi\|_{H^{-s,0}}+\|u\|_{H^{-s,0}})(\|\nabla\nabla_h^2u\|_2^2+\|\nabla\nabla_h^3\psi\|_2^2)\\ &\leqslant Cc_0H_2(t).\end{aligned}$$

Here we used the fact that

$$\|\nabla\psi\|_2 \lesssim \|\|\nabla\nabla_h^3\psi\|_{L^2_{x_h}}^{\frac{s}{s+3}}\|\Lambda_h^{-s}\nabla\psi\|_{L^2_{x_h}}^{\frac{3}{s+3}}\|_{L^2_{x_3}} \lesssim \|\nabla\nabla_h^3\psi\|_2^{\frac{s}{s+3}}\|\Lambda_h^{-s}\nabla\psi\|_2^{\frac{3}{s+3}}. \tag{2.24}$$

Finally, we estimate R_8. Indeed, we have

$$\|\nabla^2\nabla_h\psi\|_{\frac{12}{5}} \lesssim \|\nabla\nabla_h\psi\|_{\frac{12}{5}}^{\frac{11}{6s+7}}\|\nabla^{\frac{6s+7}{6s-4}}\nabla\nabla_h\psi\|_{\frac{12}{5}}^{\frac{6s-4}{6s+7}} \lesssim \|\nabla\nabla_h\psi\|_{\frac{12}{5}}^{\frac{11}{6s+7}}\|\nabla\psi\|_{H^9}^{\frac{6s-4}{6s+7}},$$

which together with (2.23), (2.20) and (2.21) gives that

$$\begin{aligned}R_8 &\lesssim \|\nabla_h^2u\|_2\|\nabla\nabla_h^2\psi\|_{\frac{12}{5}}\|\nabla\nabla_h\psi\|_{\frac{12}{5}}^{\frac{11}{6s+7}}\|\nabla\psi\|_{H^9}^{\frac{6s-4}{6s+7}}\\ &\lesssim \|\nabla\nabla_h^2u\|_2^{\frac{s+2}{s+3}}\|u\|_{H^{-s,0}}^{\frac{1}{s+3}}\|\nabla\nabla_h^3\psi\|_2^{\frac{6s+13}{6(s+3)}}\|\nabla\psi\|_{H^{-s,1}}^{\frac{5}{6(s+3)}}\|\nabla\nabla_h^3\psi\|_2^{\frac{11}{6(s+3)}}\|\nabla\psi\|_{H^{-s,1}}^{\frac{121}{6(s+3)(6s+7)}}\|\nabla\psi\|_{H^9}^{\frac{6s-4}{6s+7}}\\ &\lesssim (\|\nabla\psi\|_{H^9}+\|\nabla\psi\|_{H^{-s,1}}+\|u\|_{H^{-s,0}})(\|\nabla\nabla_h^2u\|_2^2+\|\nabla\nabla_h^3\psi\|_2^2)\\ &\leqslant Cc_0H_2(t).\end{aligned}$$

Substituting the estimates of $R_1,\dots,R_8$ into (2.18) and using (2.17), we deduce that

$$K_1 \leqslant K_1^1+K_1^2 \leqslant Cc_0H_2(t).$$

For K_2, it follows from the integration by parts and the divergence free of u that

$$K_2 = -\int \nabla_h^2u\cdot(\nabla_h^2u\cdot\nabla u+2\nabla_hu\cdot\nabla\nabla_hu) = \int \nabla_h^2\partial_ku_j\nabla_h^2u_ku_j + 2\int \nabla_h^2\partial_ku_j\nabla_hu_k\cdot\nabla_hu_j.$$

Then we deduce that

$$K_2 \lesssim \|\nabla\nabla_h^2u\|_2(\|\nabla_h^2u\|_6\|u\|_3+\|\nabla_hu\|_4^2) \lesssim (\|u\|_{H^4}+\|u\|_{H^{-s,1}})\|\nabla\nabla_h^2u\|_2^2 \leqslant Cc_0H_2(t).$$

Here we used the fact that

$$\begin{aligned}\|\nabla_h u\|_4^2 &\lesssim \|\|\nabla_h^3 u\|_{L^2_{x_h}}^{\frac{2s+3}{s+3}}\|\Lambda_h^{-s}u\|_{L^2_{x_h}}^{\frac{3}{s+3}}\|_{L^2_{x_3}}\\ &\lesssim \|\nabla_h^3 u\|_2\|\nabla_h^3 u\|_{L^\infty_{x_3}L^2_{x_h}}^{\frac{s}{s+3}}\|\Lambda_h^{-s}u\|_{L^\infty_{x_3}L^2_{x_h}}^{\frac{3}{s+3}}\\ &\lesssim (\|u\|_{H^4}+\|u\|_{H^{-s,1}})\|\nabla\nabla_h^2 u\|_2. \end{aligned} \tag{2.25}$$

For K_3, we have

$$\begin{aligned}K_3 &= -\int(\nabla_h^3\cdot u_h)\frac{\partial_i\partial_j}{\Delta}\nabla_h^2(u_iu_j)+\int\nabla_h^2 u_h\cdot\frac{\partial_i\partial_j}{\Delta}\nabla_h^3(\partial_i\psi\partial_j\psi)+\int\partial_j\nabla_h^2 u_h\cdot\nabla_h^2(\nabla_h\psi\partial_j\psi)\\ &\leqslant \|\nabla_h^3 u\|_2\|\nabla_h^2(u\otimes u)\|_2+\|\nabla_h^2 u\|_6\|\nabla_h^3(\nabla\psi\otimes\nabla\psi)\|_{\frac{6}{5}}+\|\nabla\nabla_h^2 u\|_2\|\nabla_h^2(\nabla_h\psi\otimes\nabla\psi)\|_2\\ &\lesssim \|\nabla\nabla_h^2 u\|_2(\|u\|_3\|\nabla_h^2 u\|_6+\|\nabla_h u\|_4^2+\|\nabla\nabla_h^3\psi\|_2\|\nabla\psi\|_3+\|\nabla\nabla_h^2\psi\|_{\frac{12}{5}}\|\nabla\nabla_h\psi\|_{\frac{12}{5}}\\ &\quad+\|\nabla_h^3\psi\|_6\|\nabla\psi\|_3+\|\nabla_h^2\psi\nabla\nabla_h\psi\|_2+\|\nabla_h\psi\nabla\nabla_h^2\psi\|_2).\end{aligned}$$

Similar to (2.19), we have

$$\|\nabla_h^2\psi\nabla\nabla_h\psi\|_2\lesssim\|\|\nabla\nabla_h^3\psi\|_{L^2_{x_h}}^{\frac{2s+3}{s+3}}\|\Lambda_h^{-s}\nabla\psi\|_{L^2_{x_h}}^{\frac{3}{s+3}}\|_{L^2_{x_3}}\lesssim\|\nabla\nabla_h^3\psi\|_2\|\nabla\nabla_h^3\psi\|_{L^\infty_{x_3}L^2_{x_h}}^{\frac{s}{s+3}}\|\Lambda_h^{-s}\nabla\psi\|_{L^\infty_{x_3}L^2_{x_h}}^{\frac{3}{s+3}},$$

which together with (2.25), (2.19), (2.20) and (2.21) yields that

$$K_3\lesssim(\|u\|_{H^4}+\|\nabla\psi\|_{H^4}+\|u\|_{H^{-s,1}}+\|\nabla\psi\|_{H^{-s,1}})(\|\nabla\nabla_h^2 u\|_2^2+\|\nabla\nabla_h^3\psi\|_2^2)\leqslant Cc_0H_2(t).$$

The estimate for K_4 can be similarly obtained. Precisely, we have

$$\begin{aligned}K_4 &= -\int\nabla_h^2\partial_3 u_3\frac{\partial_i\partial_j}{\Delta}\nabla_h^2(u_iu_j)+\int\nabla_h^2 u_3\nabla_h^2\left(\frac{\partial_i\partial_3}{\Delta}\partial_j(\partial_i\psi\partial_j\psi)-\partial_j(\partial_3\psi\partial_j\psi)\right)\\ &= -\int\nabla_h^2\partial_3 u_3\frac{\partial_i\partial_j}{\Delta}\nabla_h^2(u_iu_j)+\int\nabla_h^2 u_3\nabla_h^2\left(\frac{\partial_3\partial_j}{\Delta}\nabla_h\cdot(\nabla_h\psi\partial_j\psi)-\frac{\partial_j}{\Delta}\nabla_h^2(\partial_3\psi\partial_j\psi)\right)\\ &\lesssim\|\nabla_h^3 u\|_2\|\nabla_h^2(u\otimes u)\|_2+\|\nabla_h^2 u\|_6\|\nabla_h^3(\nabla\psi\otimes\nabla\psi)\|_{\frac{6}{5}}\\ &\leqslant Cc_0H_2(t).\end{aligned}$$

For the term K_5, we need the more subtle estimates. We first rewrite it as

$$\begin{aligned}K_5 &= -\int(\nabla_h^2\nabla_h^2\psi\nabla_h^2\nabla_h^2(u\cdot\nabla\psi)+2\nabla_h^2\nabla_h^2\psi\partial_3^2\nabla_h^2(u\cdot\nabla\psi))-\int\partial_3^2\nabla_h^2\psi\partial_3^2\nabla_h^2(u\cdot\nabla\psi)\\ &=: K_5^1+K_5^2.\end{aligned}$$

Then for K_5^1, we have

$$\begin{aligned}K_5^1 &= -\int(\nabla_h^2\nabla_h^2\psi\nabla_h^2\nabla_h^2(u\cdot\nabla\psi)-2\partial_3\nabla_h^3\psi\cdot\partial_3\nabla_h^3(u\cdot\nabla\psi))\\ &\lesssim\|\nabla\nabla_h^3\psi\|_2\|\nabla\nabla_h^3(u\cdot\nabla\psi)-u\cdot\nabla\nabla\nabla_h^3\psi\|_2\\ &\lesssim\|\nabla\nabla_h^3\psi\|_2(\|\nabla\nabla_h^3 u\|_2\|\nabla\psi\|_\infty+\|\nabla_h^3 u\|_6\|\nabla^2\psi\|_3+\|\nabla\nabla_h^2 u\|_2\|\nabla\nabla_h\psi\|_\infty\\ &\quad+\|\nabla_h^2 u\|_6\|\nabla^2\nabla_h\psi\|_3+\|\nabla\nabla_h u\nabla\nabla_h^2\psi\|_2+\|\nabla_h u\|_6\|\nabla^2\nabla_h^2\psi\|_3+\|\nabla u\|_\infty\|\nabla\nabla_h^3\psi\|_2).\end{aligned} \tag{2.26}$$

By the interpolation, we obtain

$$\begin{aligned}\|\nabla\nabla_h u\nabla\nabla_h^2\psi\|_2 &\lesssim\|\|\nabla\nabla_h^2 u\|_{L^2_{x_h}}^{\frac{2s+3}{2(s+2)}}\|\Lambda_h^{-s}\nabla u\|_{L^2_{x_h}}^{\frac{1}{2(s+2)}}\|\nabla\nabla_h^3\psi\|_{L^2_{x_h}}^{\frac{2s+5}{2(s+3)}}\|\Lambda_h^{-s}\nabla\psi\|_{L^2_{x_h}}^{\frac{1}{2(s+3)}}\|_{L^2_{x_3}}\\ &\lesssim(\|\nabla\nabla_h^2 u\|_2+\|\nabla\nabla_h^3\psi\|_2)(\|\nabla\psi\|_{H^4}+\|u\|_{H^{-s,2}}+\|\nabla\psi\|_{H^{-s,1}}).\end{aligned}$$

Similarly, we also have

$$\|\nabla^2\nabla_h^2\psi\|_3\lesssim\|\nabla\nabla_h^2\psi\|_{\frac{12}{5}}^{\frac{2}{3}}\|\nabla^5\nabla_h^2\psi\|_2^{\frac{1}{3}}.$$

Thus by (2.22), (2.20) and (2.26), we deduce that

$$\begin{aligned} K_5^1 &\lesssim (\|u\|_{H^3} + \|\nabla\psi\|_{H^6} + \|u\|_{H^{-s,1}} + \|\nabla\psi\|_{H^{-s,1}})(\|\nabla\nabla_h^2 u\|_2^2 + \|\nabla^2\nabla_h^2 u\|_2^2 + \|\nabla\nabla_h^3\psi\|_2^2) \\ &\leqslant Cc_0 H_2(t). \end{aligned}$$

For K_5^2, we further rewrite it as

$$\begin{aligned} K_5^2 = -\int \partial_3\nabla_h^3\psi \cdot (&\partial_3^3\nabla_h u_h \cdot \nabla_h\psi + 3\partial_3^2\nabla_h u_h \cdot \nabla_h\partial_3\psi + 3\partial_3\nabla_h u_h \cdot \nabla_h\partial_3^2\psi + \nabla_h u_h \cdot \nabla_h\partial_3^3\psi \\ &+ \partial_3^3 u_h \cdot \nabla_h\nabla_h\psi + 3\partial_3^2 u_h \cdot \nabla_h\nabla_h\partial_3\psi + 3\partial_3 u_h \cdot \nabla_h\nabla_h\partial_3^2\psi + u_h \cdot \nabla_h\nabla_h\partial_3^3\psi + \partial_3^3\nabla_h u_3\partial_3\psi \\ &+ 3\partial_3^2\nabla_h u_3\partial_3^2\psi + 3\partial_3\nabla_h u_3\partial_3^3\psi + \nabla_h u_3\partial_3^4\psi + \partial_3^3 u_3\nabla_h\partial_3\psi + 3\partial_3^2 u_3\nabla_h\partial_3^2\psi \\ &+ 3\partial_3 u_3\nabla_h\partial_3^3\psi + u_3\nabla_h\partial_3^4\psi). \end{aligned}$$

It then follows from Hölder's inequality that

$$\begin{aligned} K_5^2 \leqslant C\|\nabla\nabla_h^3\psi\|_2(&\|\nabla^3\nabla_h u\nabla_h\psi\|_2 + \|\nabla^2\nabla_h u\nabla\nabla_h\psi\|_2 + \|\nabla\nabla_h u\nabla^2\nabla_h\psi\|_2 + \|\nabla_h u\nabla^3\nabla_h\psi\|_2 \\ &+ \|\nabla^3 u\nabla_h^2\psi\|_2 + \|\nabla^2 u\nabla\nabla_h^2\psi\|_2 + \|\nabla u\nabla^2\nabla_h^2\psi\|_2 + \|u\nabla^3\nabla_h^2\psi\|_2 + \|\nabla^3\nabla_h u_3\nabla\psi\|_2 \\ &+ \|\nabla^2\nabla_h u_3\nabla^2\psi\|_2 + \|\nabla\nabla_h u_3\nabla^3\psi\|_2 + \|\nabla_h u_3\nabla^4\psi\|_2 + \|\nabla^3 u_3\nabla\nabla_h\psi\|_2 \\ &+ \|\nabla^2 u_3\nabla^2\nabla_h\psi\|_2 + \|\nabla u_3\nabla^3\nabla_h\psi\|_2 + \|u_3\nabla^4\nabla_h\psi\|_2) \\ =:& C\|\nabla\nabla_h^3\psi\|_2(S_1 + S_2 + \cdots + S_{16}). \end{aligned} \tag{2.27}$$

We now estimate $S_1, \ldots, S_{16}$ one by one. The following estimates will be frequently used

$$\begin{aligned} \|\nabla^2\nabla_h u\|_2 &\lesssim \|\|\nabla^2\nabla_h^2 u\|_{L^2_{x_h}}^{\frac{s+1}{s+2}}\|\Lambda_h^{-s}\nabla^2 u\|_{L^2_{x_h}}^{\frac{1}{s+2}}\|_{L^2_{x_3}} \\ &\lesssim \|\nabla^2\nabla_h^2 u\|_2^{\frac{s+1}{s+2}}\|u\|_{H^{-s,2}}^{\frac{1}{s+2}} \\ &\lesssim \|u\|_{H^{-s,2}}^{\frac{1}{s+2}}\|\nabla^2\nabla_h^2 u\|_2^{\frac{s+1}{s+2}} \end{aligned} \tag{2.28}$$

and

$$\|\nabla\nabla_h^2\psi\|_2 \lesssim \|\|\nabla\nabla_h^3\psi\|_{L^2_{x_h}}^{\frac{s+2}{s+3}}\|\Lambda_h^{-s}\nabla\psi\|_{L^2_{x_h}}^{\frac{1}{s+3}}\|_{L^2_{x_3}} \lesssim \|\nabla\nabla_h^3\psi\|_2^{\frac{s+2}{s+3}}\|\Lambda_h^{-s}\nabla\psi\|_2^{\frac{1}{s+3}} \tag{2.29}$$

as well as

$$\|\nabla\nabla_h\psi\|_2 \lesssim \|\|\nabla\nabla_h^3\psi\|_{L^2_{x_h}}^{\frac{s+1}{s+3}}\|\Lambda_h^{-s}\nabla\psi\|_{L^2_{x_h}}^{\frac{2}{s+3}}\|_{L^2_{x_3}} \lesssim \|\nabla\nabla_h^3\psi\|_2^{\frac{s+1}{s+3}}\|\Lambda_h^{-s}\nabla\psi\|_2^{\frac{2}{s+3}}. \tag{2.30}$$

For S_1, we use the interpolation and Sobolev's embedding to obtain

$$S_1 \leqslant \|\nabla^3\nabla_h u\|_2\|\nabla_h\psi\|_\infty \lesssim \|\nabla^2\nabla_h u\|_2^{\frac{3(s+2)}{2(s+1)(s+3)}}\|\nabla^{\frac{6(s^2+3s+1)}{2s^2+5s}}\nabla_h u\|_2^{\frac{2s^2+5s}{2(s+1)(s+3)}}\|\nabla\nabla_h\psi\|_2^{\frac{1}{2}}\|\nabla\nabla_h^2\psi\|_2^{\frac{1}{2}},$$

which together with (2.28)–(2.30) gives that

$$S_1 \lesssim (\|u\|_{H^6} + \|\nabla\psi\|_{H^3} + \|u\|_{H^{-s,2}} + \|\nabla\psi\|_{H^{-s,0}})(\|\nabla^2\nabla_h^2 u\|_2 + \|\nabla\nabla_h^3\psi\|_2). \tag{2.31}$$

Similarly, for S_2, we have

$$\begin{aligned} S_2 &\lesssim \|\|\nabla^2\nabla_h^2 u\|_{L^2_{x_h}}^{\frac{2s+3}{2(s+2)}}\|\Lambda_h^{-s}\nabla^2 u\|_{L^2_{x_h}}^{\frac{1}{2(s+2)}}\|\nabla\nabla_h^3\psi\|_{L^2_{x_h}}^{\frac{2s+3}{2(s+3)}}\|\Lambda_h^{-s}\nabla\psi\|_{L^2_{x_h}}^{\frac{3}{2(s+3)}}\|_{L^2_{x_3}} \\ &\lesssim (\|\nabla\psi\|_{H^4} + \|u\|_{H^{-s,3}} + \|\nabla\psi\|_{H^{-s,1}})(\|\nabla^2\nabla_h^2 u\|_2 + \|\nabla\nabla_h^3\psi\|_2). \end{aligned} \tag{2.32}$$

Noticing that

$$S_3 \leqslant \|\nabla\nabla_h u\|_\infty\|\nabla^2\nabla_h\psi\|_2 \lesssim \|\nabla^2\nabla_h u\|_2^{\frac{1}{2}}\|\nabla^2\nabla_h^2 u\|_2^{\frac{1}{2}}\|\nabla\nabla_h\psi\|_2^{\frac{s+3}{2(s+2)}}\|\nabla^{\frac{3s+5}{s+1}}\nabla_h\psi\|_2^{\frac{s+1}{2(s+2)}},$$

we can use (2.28) and (2.30) to obtain

$$S_3 \lesssim (\|\nabla\psi\|_{H^5} + \|u\|_{H^{-s,2}} + \|\nabla\psi\|_{H^{-s,0}})(\|\nabla^2\nabla_h^2 u\|_2 + \|\nabla\nabla_h^3\psi\|_2). \tag{2.33}$$

Similarly, we have

$$S_4 \leqslant \|\nabla_h u\|_\infty \|\nabla^3 \nabla_h \psi\|_2 \leqslant C\|\nabla\nabla_h u\|_2^{\frac{1}{2}} \|\nabla\nabla_h^2 u\|_2^{\frac{1}{2}} \|\nabla\nabla_h \psi\|_2^{\frac{s+3}{2(s+2)}} \|\nabla^{\frac{5s+9}{s+1}} \nabla_h \psi\|_2^{\frac{s+1}{2(s+2)}}.$$

Thus by (2.22) and (2.30), we obtain

$$S_4 \leqslant C(\|\nabla\psi\|_{H^8} + \|u\|_{H^{-s,1}} + \|\nabla\psi\|_{H^{-s,0}})(\|\nabla\nabla_h^2 u\|_2 + \|\nabla\nabla_h^3 \psi\|_2). \tag{2.34}$$

To estimate S_5, S_6 and S_7, we will use the following inequality:

$$\begin{aligned} \|\nabla^2 u\|_2 &\lesssim \| \|\nabla^2 \nabla_h^2 u\|_{L^2_{x_h}}^{\frac{s}{s+2}} \|\Lambda_h^{-s} \nabla^2 u\|_{L^2_{x_h}}^{\frac{2}{s+2}} \|_{L^2_{x_3}} \\ &\lesssim \|\nabla^2 \nabla_h^2 u\|_2^{\frac{s}{s+2}} \|\Lambda_h^{-s} \nabla^2 u\|_2^{\frac{2}{s+2}} \\ &\lesssim \|\nabla^2 \nabla_h^2 u\|_2^{\frac{s}{s+2}} \|u\|_{H^{-s,2}}^{\frac{2}{s+2}}. \end{aligned} \tag{2.35}$$

Then for S_5, we have

$$\begin{aligned} S_5 &\leqslant \|\nabla^3 u\|_2 \|\nabla_h^2 \psi\|_\infty \\ &\lesssim \|\nabla^2 u\|_2^{\frac{s+2}{2s(s+3)}} \|\nabla^{\frac{2(3s^2+8s-2)}{2s^2+5s-2}} u\|_2^{\frac{2s^2+5s-2}{2s(s+3)}} \|\nabla\nabla_h^2 \psi\|_2^{\frac{1}{2}} \|\nabla\nabla_h^3 \psi\|_2^{\frac{1}{2}} \\ &\lesssim (\|u\|_{H^4} + \|\nabla\psi\|_{H^{-s,0}})(\|\nabla^2 \nabla_h^2 u\|_2 + \|\nabla\nabla_h^3 \psi\|_2) \end{aligned} \tag{2.36}$$

by (2.29). Similarly, for S_6 and S_7, we can use (2.35), (2.28), (2.29) and the interpolation to deduce that

$$\begin{aligned} S_6 &\leqslant \|\nabla^2 u\|_\infty \|\nabla\nabla_h^2 \psi\|_2 \lesssim \|\nabla^3 u\|_2^{\frac{1}{2}} \|\nabla^3 \nabla_h u\|_2^{\frac{1}{2}} \|\nabla\nabla_h^2 \psi\|_2 \\ &\lesssim (\|u\|_{H^6} + \|u\|_{H^{-s,2}} + \|\nabla\psi\|_{H^{-s,0}})(\|\nabla^2 \nabla_h^2 u\|_2 + \|\nabla\nabla_h^3 \psi\|_2) \end{aligned} \tag{2.37}$$

and

$$\begin{aligned} S_7 &\lesssim \|\nabla^2 u\|_2^{\frac{1}{2}} \|\nabla^2 \nabla_h u\|_2^{\frac{1}{2}} \|\nabla\nabla_h^2 \psi\|_2^{\frac{3(s+3)}{2(s+2)^2}} \|\nabla^{\frac{4s^2+13s+7}{2s^2+5s-1}} \nabla_h^2 \psi\|_2^{\frac{2s^2+5s-1}{2(s+2)^2}} \\ &\lesssim (\|\nabla\psi\|_{H^6} + \|u\|_{H^{-s,2}} + \|\nabla\psi\|_{H^{-s,0}})(\|\nabla^2 \nabla_h^2 u\|_2 + \|\nabla\nabla_h^3 \psi\|_2). \end{aligned} \tag{2.38}$$

For S_8, it follows from (2.35), (2.22) and (2.29) that

$$\begin{aligned} S_8 &\lesssim \|\nabla u\|_2^{\frac{1}{2}} \|\nabla\nabla_h u\|_2^{\frac{1}{2}} \|\nabla^3 \nabla_h^2 \psi\|_2 \\ &\lesssim \|\nabla^2 u\|_2^{\frac{1}{2}} \|\nabla^2 \nabla_h u\|_2^{\frac{1}{2}} \|\nabla\nabla_h^2 \psi\|_2^{\frac{9-s^2}{2(s+2)^2}} \|\nabla^{\frac{4(s+2)^2}{3s^2+8s-1}} \nabla_h^2 \psi\|_2^{\frac{3s^2+8s-1}{2(s+2)^2}} \\ &\lesssim (\|\nabla\psi\|_{H^{10}} + \|u\|_{H^{-s,2}} + \|\nabla\psi\|_{H^{-s,0}})(\|\nabla\nabla_h^2 u\|_2 + \|\nabla^2 \nabla_h^2 u\|_2 + \|\nabla\nabla_h^3 \psi\|_2). \end{aligned} \tag{2.39}$$

For S_9, S_{10} and S_{11}, it is easy to see that

$$\begin{aligned} S_9 + S_{10} + S_{11} &\leqslant \|\nabla^3 \nabla_h u_3\|_2 \|\nabla\psi\|_\infty + \|\nabla^2 \nabla_h u_3\|_6 \|\nabla^2 \psi\|_3 + \|\nabla\nabla_h u_3\|_6 \|\nabla^3 \psi\|_2 \\ &\lesssim \|\nabla\psi\|_{H^2}(\|\nabla^2 \nabla_h^2 u\|_2 + \|\nabla\nabla_h^2 u\|_2) \end{aligned} \tag{2.40}$$

by $\partial_3 u_3 = -\nabla_h \cdot u_h$ and Sobolev's embedding. Similarly, for S_{12}, we first notice that

$$S_{12} \lesssim \|\nabla\nabla_h u_3\|_2^{\frac{1}{2}} \|\nabla\nabla_h^2 u_3\|_2^{\frac{1}{2}} \|\nabla\psi\|_2^{\frac{1}{2s}} \|\nabla^{\frac{6s}{2s-1}} \nabla\psi\|_2^{1-\frac{1}{2s}} \lesssim \|\nabla_h^2 u\|_2^{\frac{1}{2}} \|\nabla\nabla_h^2 u\|_2^{\frac{1}{2}} \|\nabla\psi\|_2^{\frac{1}{2s}} \|\nabla\psi\|_{H^7}^{1-\frac{1}{2s}}$$

and then use (2.23) and (2.24) to obtain

$$S_{12} \lesssim (\|\nabla\psi\|_{H^7} + \|u\|_{H^{-s,0}} + \|\nabla\psi\|_{H^{-s,0}})(\|\nabla\nabla_h^2 u\|_2 + \|\nabla\nabla_h^3 \psi\|_2). \tag{2.41}$$

The estimates for S_{13}, S_{14} and S_{15} are exactly similar to those of S_2, S_3 and S_4 by $\partial_3 u_3 = -\nabla_h \cdot u_h$. Finally, since

$$\|\nabla u_3\|_2 \lesssim \|\nabla_h u\|_2 \lesssim \|\nabla_h^3 u\|_2^{\frac{s+1}{s+3}} \|\Lambda_h^{-s} u\|_2^{\frac{2}{s+3}},$$

we can use the interpolation, (2.23) and (2.30) to obtain

$$\begin{aligned} S_{16} &\lesssim \|\nabla u_3\|_2^{\frac{1}{2}} \|\nabla\nabla_h u_3\|_2^{\frac{1}{2}} \|\nabla\nabla_h^3\psi\|_2^{\frac{3}{2(s+1)}} \|\nabla^{\frac{8s+5}{2s-1}}\nabla_h\psi\|_2^{\frac{2s-1}{2(s+1)}} \\ &\lesssim (\|\nabla\psi\|_{H^{14}} + \|u\|_{H^{-s,0}} + \|\nabla\psi\|_{H^{-s,0}})(\|\nabla\nabla_h^2 u\|_2 + \|\nabla\nabla_h^3\psi\|_2). \end{aligned} \tag{2.42}$$

Substituting the estimates of $S_1, S_2, \dots, S_{16}$ into (2.27), we deduce that $K_5 \leqslant K_5^1 + K_5^2 \leqslant Cc_0 H_2(t)$.

For K_6, we have

$$\begin{aligned} K_6 &= \int \nabla^2\nabla_h^2 u \cdot (\nabla_h^2 u \cdot \nabla u + 2\nabla_h u \cdot \nabla\nabla_h u + u \cdot \nabla\nabla_h^2 u) \\ &\lesssim \|\nabla^2\nabla_h^2 u\|_2(\|\nabla_h^2 u\|_6\|\nabla u\|_3 + \|\nabla\nabla_h u\|_2^{\frac{1}{2}}\|\nabla\nabla_h^2 u\|_2^{\frac{1}{2}}\|\nabla\nabla_h u\|_2 + \|u\|_\infty\|\nabla\nabla_h^2 u\|_2) \\ &\lesssim (\|u\|_{H^2} + \|u\|_{H^{-s,2}})(\|\nabla\nabla_h^2 u\|_2^2 + \|\nabla^2\nabla_h^2 u\|_2^2) \\ &\leqslant Cc_0 H_2(t) \end{aligned}$$

by (2.22) and (2.28).

To estimate K_7, we first notice that $\|\nabla^2\nabla_h\psi\|_2 \lesssim \|\nabla\nabla_h\psi\|_2^{\frac{3}{4}}\|\nabla^5\nabla_h\psi\|_2^{\frac{1}{4}}$ and $\|\nabla^2\nabla_h^2\psi\|_2 \lesssim \|\nabla\nabla_h^2\psi\|_2^{\frac{3}{4}}$ $\|\nabla^5\nabla_h^2\psi\|_2^{\frac{1}{4}}$. Then by (2.29) and (2.30), we obtain

$$\begin{aligned} K_7 &= -\int \nabla^2\nabla_h^2 u_h \cdot \left(\frac{\partial_i\partial_j}{\Delta}\nabla_h^3(u_iu_j + \partial_i\psi\partial_j\psi) + \nabla_h^2\partial_j(\nabla_h\psi\partial_j\psi)\right) \\ &\lesssim \|\nabla^2\nabla_h^2 u\|_2(\|\nabla_h^3(u\otimes u)\|_2 + \|\nabla_h^3(\nabla\psi\otimes\nabla\psi)\|_2 + \|\nabla\nabla_h^2(\nabla_h\psi\otimes\nabla\psi)\|_2) \\ &\lesssim \|\nabla^2\nabla_h^2 u\|_2(\|\nabla_h^3 u\|_2\|u\|_\infty + \|\nabla_h^2 u\|_6\|\nabla_h u\|_3 + \|\nabla\nabla_h^3\psi\|_2\|\nabla\psi\|_\infty \\ &\quad + \|\nabla\nabla_h^2\psi\|_2\|\nabla^2\nabla_h\psi\|_2^{\frac{1}{2}}\|\nabla^2\nabla_h^2\psi\|_2^{\frac{1}{2}} + \|\nabla_h^3\psi\|_6\|\nabla^2\psi\|_3 + \|\nabla\nabla_h^2\psi\|_2^{\frac{1}{2}}\|\nabla\nabla_h^3\psi\|_2^{\frac{1}{2}}\|\nabla^2\nabla_h\psi\|_2 \\ &\quad + \|\nabla\nabla_h\psi\|_2^{\frac{1}{2}}\|\nabla\nabla_h^2\psi\|_2^{\frac{1}{2}}\|\nabla^2\nabla_h^2\psi\|_2) \\ &\lesssim (\|u\|_{H^2} + \|\nabla\psi\|_{H^6} + \|u\|_{H^{-s,2}} + \|\nabla\psi\|_{H^{-s,0}})(\|\nabla\nabla_h^2 u\|_2^2 + \|\nabla^2\nabla_h^2 u\|_2^2 + \|\nabla\nabla_h^3\psi\|_2^2) \\ &\leqslant Cc_0 H_2(t). \end{aligned}$$

Similarly, for K_8, we have

$$\begin{aligned} K_8 &= -\int \partial_i\partial_j\nabla_h\partial_3 u_3 \cdot \nabla_h^3(u_iu_j) - \int \partial_3\partial_j\nabla_h^2 u_3\nabla_h^3 \cdot (\nabla_h\psi\partial_j\psi) + \int \partial_j\nabla_h^3 u_3 \cdot \nabla_h^3(\partial_3\psi\partial_j\psi) \\ &\lesssim \|\nabla^2\nabla_h^2 u\|_2(\|\nabla_h^3(u\otimes u)\|_2 + \|\nabla_h^3(\nabla\psi\otimes\nabla\psi)\|_2) \\ &\leqslant Cc_0 H_2(t). \end{aligned}$$

For K_9, it is easy to see that

$$\begin{aligned} K_9 &= -\epsilon_1\int \nabla\nabla_h^3\psi \cdot \nabla\nabla_h(u_h\cdot\nabla_h u_3 + u_3\partial_3 u_3) = -\epsilon_1\int \nabla\nabla_h^3\psi \cdot \nabla\nabla_h(u_h\cdot\nabla_h u_3 - u_3\nabla_h\cdot u_h) \\ &\lesssim \|\nabla\nabla_h^3\psi\|_2(\|\nabla\nabla_h u\|_2\|\nabla\nabla_h u\|_2^{\frac{1}{2}}\|\nabla\nabla_h^2 u\|_2^{\frac{1}{2}} + \|\nabla u\|_3\|\nabla_h^2 u\|_6 + \|u\|_\infty\|\nabla\nabla_h^2 u\|_2) \\ &\lesssim (\|u\|_{H^2} + \|u\|_{H^{-s,1}})(\|\nabla\nabla_h^2 u\|_2^2 + \|\nabla^2\nabla_h^2 u\|_2^2 + \|\nabla\nabla_h^3\psi\|_2^2) \\ &\leqslant Cc_0 H_2(t). \end{aligned}$$

Similar to K_6 and K_7, we also have

$$\begin{aligned} K_{10} &= \epsilon_1\int \partial_3\nabla_h^3\psi \cdot \nabla_h\partial_i\partial_j(u_iu_j) + \epsilon_1\int \partial_j\nabla_h^3\psi \cdot \partial_3\nabla_h\nabla_h \cdot (\nabla_h\psi\partial_j\psi) - \epsilon_1\int \partial_j\nabla_h^3\psi \cdot \nabla_h^3(\partial_3\psi\partial_j\psi) \\ &\lesssim \|\nabla\nabla_h^3\psi\|_2(\|\nabla\nabla_h^2(uu)\|_2 + \|\nabla\nabla_h(u\nabla_h u)\|_2 + \|\nabla\nabla_h^2(\nabla_h\psi\otimes\nabla\psi)\|_2 + \|\nabla_h^3(\nabla\psi\otimes\nabla\psi)\|_2) \\ &\leqslant Cc_0 H_2(t). \end{aligned}$$

Finally, for K_{11}, we have

$$\begin{aligned}K_{11} &= \epsilon_1 \int \nabla^2\nabla_h u_3 \cdot \nabla_h^3(u\cdot\nabla\psi)\\ &\lesssim \|\nabla\nabla_h^2 u\|_2(\|\nabla_h^3 u\|_2\|\nabla\psi\|_\infty + \|\nabla_h^2 u\|_6\|\nabla\nabla_h\psi\|_3 + \|\nabla_h u\|_\infty\|\nabla\nabla_h^2\psi\|_2 + \|u\|_\infty\|\nabla\nabla_h^3\psi\|_2)\\ &\lesssim \|\nabla\nabla_h^2 u\|_2(\|\nabla\nabla_h^2 u\|_2\|\nabla\psi\|_{H^2} + \|\nabla\nabla_h u\|_2^{\frac12}\|\nabla\nabla_h^2 u\|_2^{\frac12}\|\nabla\nabla_h^2\psi\|_2 + \|u\|_{H^2}\|\nabla\nabla_h^3\psi\|_2)\\ &\lesssim (\|u\|_{H^2} + \|\nabla\psi\|_{H^2} + \|\nabla\psi\|_{H^{-s,0}})(\|\nabla\nabla_h^2 u\|_2^2 + \|\nabla\nabla_h^3\psi\|_2^2)\\ &\leqslant Cc_0 H_2(t).\end{aligned}$$

Substituting the estimates for $K_1, K_2, \dots, K_{11}$ into (2.16), we conclude that

$$\frac{d}{dt}D_2(t) + H_2(t) \leqslant Cc_0 H_2(t).$$

This completes the proof of Proposition 2.4 by taking c_0 small suitably. □

Now we are in position to complete the decay estimates of the solution.

Proposition 2.5. *Assume that the solution (ψ, u) of (1.5) satisfies*

$$\|\nabla\psi(t)\|_{H^{14}(\mathbb{R}^3)}^2 + \|u(t)\|_{H^{14}(\mathbb{R}^3)}^2 + \|\nabla\psi(t)\|_{H^{-s,4}(\mathbb{R}^3)}^2 + \|u(t)\|_{H^{-s,3}(\mathbb{R}^3)}^2 \leqslant 4C_0^2\varepsilon^2$$

for any $t \in [0,T]$. If ε is suitably small, then it holds that

$$\begin{aligned}&\|\nabla_h^\ell u\|_2 + \|\nabla\nabla_h^\ell\psi\|_2 + \|\nabla\nabla_h^\ell u\|_2 + \|\nabla^2\nabla_h^\ell\psi\|_2 \leqslant C\varepsilon(1+t)^{-\frac{s+\ell}{2}},\\ &\int_0^t (\|\nabla\nabla_h^\ell u\|_2^2 + \|\nabla^2\nabla_h^\ell u\|_2^2 + \|\nabla\nabla_h^{\ell+1}\psi\|_2^2)d\tau \leqslant C\varepsilon^2,\end{aligned} \tag{2.43}$$

for any $t \in [0,T]$ and $\ell = 0,1,2$, where C is a constant independent of ε, t.

Proof. We first choose ε small enough such that $4C_0^2\varepsilon^2 \leqslant c_0^2$. Then the estimates $(2.43)_2$ can be obtained by using a direct integral in Propositions 2.2–2.4.

To establish the decay estimates $(2.43)_1$, we first use the interpolation to obtain

$$\|\nabla_h^\ell u\|_2^2 \lesssim \|\Lambda_h^{-s}u\|_2^{\frac{2}{s+\ell+1}}\|\nabla_h^{\ell+1}u\|_2^{\frac{2(s+\ell)}{s+\ell+1}} \leqslant C\varepsilon^{\frac{2}{s+\ell+1}}H_\ell(t)^{\frac{s+\ell}{s+\ell+1}}.$$

Similarly, we have

$$\|\nabla\nabla_h^\ell u\|_2^2 \lesssim \|\Lambda_h^{-s}\nabla u\|_2^{\frac{2}{s+\ell+1}}\|\nabla\nabla_h^{\ell+1}u\|_2^{\frac{2(s+\ell)}{s+\ell+1}} \leqslant C\varepsilon^{\frac{2}{s+\ell+1}}H_\ell(t)^{\frac{s+\ell}{s+\ell+1}}$$

and

$$\|\nabla\nabla_h^\ell\psi\|_2^2 \lesssim \|\Lambda_h^{-s}\nabla\psi\|_2^{\frac{2}{s+\ell+1}}\|\nabla_h^{\ell+1}\nabla\psi\|_2^{\frac{2(s+\ell)}{s+\ell+1}} \leqslant C\varepsilon^{\frac{2}{s+\ell+1}}H_\ell(t)^{\frac{s+\ell}{s+\ell+1}}.$$

Also, we can use the Littlewood-Paley decomposition to deduce that

$$\begin{aligned}\|\nabla^2\nabla_h^\ell\psi\|_2^2 &\lesssim \|\Lambda_h^{-s}\nabla^{s+\ell+2}\psi\|_{L^2}^{\frac{2}{s+\ell+1}}\|\nabla_h^{\ell+1}\nabla\psi\|_{L^2}^{\frac{2(s+\ell)}{s+\ell+1}}\\ &\lesssim \|\nabla\psi\|_{H^{-s,s+\ell+1}}^{\frac{2}{s+\ell+1}}\|\nabla_h^{\ell+1}\nabla\psi\|_{L^2}^{\frac{2(s+\ell)}{s+\ell+1}}\\ &\leqslant C\varepsilon^{\frac{2}{s+\ell+1}}H_\ell(t)^{\frac{s+\ell}{s+\ell+1}}.\end{aligned}$$

Then we have

$$D_\ell(t) \leqslant C\varepsilon^{\frac{2}{s+\ell+1}}H_\ell(t)^{\frac{s+\ell}{s+\ell+1}},$$

which along with Propositions 2.2–2.4 gives

$$\frac{d}{dt}D_\ell(t) + c\varepsilon^{-\frac{2}{s+\ell}}D_\ell(t)^{1+\frac{1}{s+\ell}} \leqslant 0.$$

This implies the desired decay estimates $(2.43)_1$ by (2.1). □

3 Proof of global existence

In this section, we will establish the global existence by using the decay estimates (2.43) obtained in last section, which in turn closes the decay estimates (2.43) by using a continuous argument.

Proposition 3.1. *Let $s = 1-\delta$. Assume that the solution (ψ, u) of Equations (1.5) satisfies*

$$\mathcal{E}_s(t) := \|\nabla\psi(t)\|_{H^{14}}^2 + \|u(t)\|_{H^{14}}^2 + \|\nabla\psi(t)\|_{H^{-s,4}}^2 + \|u(t)\|_{H^{-s,4}}^2 \leqslant 4C_0^2\varepsilon^2$$

for any $t \in [0,T]$. If ε and δ are suitably small, then there exists a positive constant $\kappa_0 > 1$ such that

$$\frac{d}{dt}\mathcal{E}_s(t) \leqslant \widehat{C}_0\varepsilon\mathcal{E}_s(t)(1+t)^{-\kappa_0} \tag{3.1}$$

for any $t \in [0,T]$, where $\widehat{C}_0$ is a constant independent of ε and t.

Proof. The following energy conservation has been established in the proof of Proposition 2.2:

$$\frac{1}{2}\frac{d}{dt}(\|\nabla\psi(t)\|_2^2 + \|u(t)\|_2^2) + \|\nabla u(t)\|_2^2 = 0.$$

We now establish the energy estimates related to the higher order or negative derivatives. It is convenient to use the vorticity equation when we consider the higher derivatives of u. For this purpose, we apply ∇^2 to $(1.4)_1$ and take the curl of $(1.4)_2$–$(1.4)_3$ to obtain

$$\begin{cases} \partial_t\nabla^2\psi + \nabla^2 u_3 = -\nabla^2(u\cdot\nabla\psi), \\ \partial_t\omega + u\cdot\nabla\omega - \Delta\omega + \mathcal{W} = \omega\cdot\nabla u - \operatorname{curl}(\Delta\psi\nabla\psi), \end{cases} \tag{3.2}$$

where $\omega = \operatorname{curl} u$ and $\mathcal{W} = (\Delta\partial_2\psi, -\Delta\partial_1\psi, 0)^{\mathrm{T}}$. Since the lower order energy estimates are easier than the higher order ones, we only pay our attention to the highest order energy estimates. For this purpose, we apply ∇^{13} to (3.2), multiply the resulting equations by $\nabla^{15}\psi$ and $\nabla^{13}\omega$, respectively, and then integrate over $\mathbb{R}^3$ to obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\nabla^{15}\psi(t)\|_2^2 + \|\nabla^{13}\omega(t)\|_2^2) + \|\nabla^{14}\omega\|_2^2 + \int\nabla^{15}\psi\cdot\nabla^{15}u_3 + \int\nabla^{14}\omega_h\cdot\nabla^{14}\nabla_h^{\perp}\psi \\ &\quad = -\int\nabla^{15}\psi\cdot\nabla^{15}(u\cdot\nabla\psi) - \int\nabla^{14}\omega\cdot\nabla^{12}(\omega\cdot\nabla u) + \int\nabla^{14}\omega\cdot\nabla^{12}(u\cdot\nabla\omega) \\ &\qquad + \int\nabla^{14}\omega\cdot\nabla^{12}\operatorname{curl}(\Delta\psi\nabla\psi) \\ &\quad =: L_1 + L_2 + L_3 + L_4, \end{aligned} \tag{3.3}$$

where $\nabla_h^{\perp} = (-\partial_2, \partial_1)^{\mathrm{T}}$ and $\omega_h = (\omega_1, \omega_2)$. For L_1, we use $u\cdot\nabla = u_h\cdot\nabla_h + u_3\partial_3$ and Leibniz's formula to obtain

$$\begin{aligned} L_1 &= -\int\left(\tilde{\nabla}_h^{15}\psi\cdot\tilde{\nabla}_h^{15}(u\cdot\nabla\psi) + \sum_{j=0}^{14}C_{15}^j\partial_3^{15}\psi\partial_3^{15-j}u_h\cdot\nabla_h\partial_3^j\psi\right) - \sum_{j=0}^{14}C_{15}^j\int\partial_3^{15}\psi\partial_3^{15-j}u_3\partial_3^{j+1}\psi \\ &=: L_1^1 + L_1^2, \end{aligned}$$

where $\tilde{\nabla}_h^{15}$ denotes ∇^{15} containing ∂_1 or ∂_2. It then follows from Hölder's inequality and Sobolev's embedding that

$$\begin{aligned} L_1^1 &\lesssim \|\nabla^{14}\nabla_h\psi\|_2\left(\|\nabla^{15}u\|_2\|\nabla\psi\|_\infty + \sum_{j=1}^{13}\|\nabla^{15-j}u\|_6\|\nabla^{j+1}\psi\|_3 + \|\nabla u\|_\infty\|\nabla^{15}\psi\|_2\right) \\ &\quad + \|\nabla^{15}\psi\|_2\left(\|\nabla^{15}u\|_2\|\nabla_h\psi\|_\infty + \sum_{j=1}^{13}\|\nabla^{15-j}u\|_6\|\nabla^j\nabla_h\psi\|_3 + \|\nabla u\|_\infty\|\nabla^{14}\nabla_h\psi\|_2\right) \end{aligned}$$

$$\lesssim \|\nabla\psi\|_{H^{14}}(\|\nabla\nabla_h\psi\|_{H^{13}}^2 + \|\nabla u\|_{H^{14}}^2), \tag{3.4}$$

where we used Lemma 2.1 to deal with the term $\|\nabla_h\psi\|_\infty$ in the last inequality. By the divergence free condition of u and integration by parts, we also have

$$\begin{aligned} L_1^2 &= \sum_{j=0}^{13} C_{15}^j \int \partial_3^{15}\psi\partial_3^{14-j}(\nabla_h\cdot u_h)\partial_3^{j+1}\psi - C_{15}^{14}\int \partial_3^{15}\psi\partial_3 u_3\partial_3^{15}\psi \\ &= \sum_{j=0}^{13} C_{15}^j \int \partial_3^{14}\nabla_h\psi\cdot(\partial_3^{15-j}u_h\partial_3^{j+1}\psi + \partial_3^{14-j}u_h\partial_3^{j+2}\psi) - \sum_{j=0}^{13} C_{15}^j\int \partial_3^{15}\psi\partial_3^{14-j}u_h\cdot\partial_3^{j+1}\nabla_h\psi \\ &\quad - C_{15}^{14}\int \partial_3^{15}\psi\partial_3 u_3\partial_3^{15}\psi. \end{aligned}$$

It follows from Hölder's inequality and Sobolev's embedding that

$$\begin{aligned} L_1^2 &\lesssim \|\nabla^{14}\nabla_h\psi\|_2\bigg(\|\nabla^{15}u\|_2\|\nabla\psi\|_\infty + \sum_{j=0}^{12}\|\nabla^{14-j}u\|_6\|\nabla^{j+2}\psi\|_3 + \|\nabla u\|_\infty\|\nabla^{15}\psi\|_2\bigg) \\ &\quad + \|\nabla^{15}\psi\|_2\bigg(\sum_{j=0}^{12}\|\nabla^{14-j}u\|_6\|\nabla^{j+1}\nabla_h\psi\|_3 + \|\nabla u\|_\infty\|\nabla^{14}\nabla_h\psi\|_2\bigg) + \|\partial_3 u_3\|_\infty\|\nabla^{15}\psi\|_2^2 \\ &\lesssim \|\nabla\psi\|_{H^{14}}(\|\nabla\nabla_h\psi\|_{H^13}^2 + \|\nabla u\|_{H^{14}}^2) + \|\nabla\psi\|_{H^{14}}^2\|\partial_3 u_3\|_\infty. \end{aligned} \tag{3.5}$$

For the terms L_2 and L_3, it is easy to see that

$$L_2 = -\sum_{j=0}^{12} C_{12}^j\int \nabla^{14}\omega\cdot(\nabla^{12-j}\omega\cdot\nabla\nabla^j u) \lesssim \sum_{j=0}^{12}\|\nabla^{14}\omega\|_2\|\nabla^{12-j}\omega\|_6\|\nabla\nabla^j u\|_3 \lesssim \|u\|_{H^{14}}\|\nabla u\|_{H^{14}}^2 \tag{3.6}$$

and

$$L_3 = \sum_{j=0}^{12} C_{12}^j\int \nabla^{14}\omega\cdot\nabla^{12-j}u\cdot\nabla\nabla^j\omega \lesssim \sum_{j=0}^{12}\|\nabla^{14}\omega\|_2\|\nabla^{12-j}u\|_3\|\nabla\nabla^j\omega\|_6 \lesssim \|u\|_{H^{14}}\|\nabla u\|_{H^{14}}^2. \tag{3.7}$$

For the term L_4, we first rewrite it as

$$\begin{aligned} L_4 &= \int \nabla^{14}\omega_1\nabla^{12}(\partial_3\psi\Delta\partial_2\psi - \partial_2\psi\Delta\partial_3\psi) + \int \nabla^{14}\omega_2\nabla^{12}(\partial_1\psi\Delta\partial_3\psi - \partial_3\psi\Delta\partial_1\psi) \\ &\quad + \int \nabla^{14}\omega_3\nabla^{12}(\partial_2\psi\Delta\partial_1\psi - \partial_1\psi\Delta\partial_2\psi) \\ &=: L_4^1 + L_4^2 + L_4^3. \end{aligned} \tag{3.8}$$

It then follows from Hölder's inequality, Sobolev's embedding and Lemma 2.1 that

$$\begin{aligned} L_4^1 &= \sum_{j=0}^{12} C_{12}^j\int \nabla^{14}\omega_1(\nabla^{12-j}\partial_3\psi\nabla^{j+2}\partial_2\psi - \nabla^{12-j}\partial_2\psi\nabla^{j+2}\partial_3\psi) \\ &\lesssim \|\nabla^{14}\omega\|_2\bigg(\sum_{j=0}^{11}\|\nabla^{12-j}\partial_3\psi\|_3\|\nabla^{j+2}\partial_2\psi\|_6 + \|\partial_3\psi\|_\infty\|\nabla^{14}\partial_2\psi\|_2 \\ &\quad + \sum_{j=0}^{11}\|\nabla^{12-j}\partial_2\psi\|_6\|\nabla^{j+2}\partial_3\psi\|_3 + \|\partial_2\psi\|_\infty\|\nabla^{14}\partial_3\psi\|_2\bigg) \\ &\lesssim \|\nabla\psi\|_{H^{14}}(\|\nabla\nabla_h\psi\|_{H^{13}}^2 + \|\nabla u\|_{H^{14}}^2). \end{aligned} \tag{3.9}$$

The terms L_4^2 and L_4^3 can be similarly dealt with. Summing up (3.3)–(3.9), we deduce that

$$\frac{1}{2}\frac{d}{dt}(\|\nabla^{15}\psi(t)\|_2^2 + \|\nabla^{14}u(t)\|_2^2) + \|\nabla^{15}u\|_2^2$$

$$\leqslant C(\|\nabla\psi\|_{H^{14}} + \|u\|_{H^{14}})(\|\nabla\nabla_h\psi\|_{H^{13}}^2 + \|\nabla u\|_{H^{14}}^2) + C\|\nabla\psi\|_{H^{14}}^2\|\partial_3 u_3\|_\infty. \tag{3.10}$$

Here we used the fact that $\int \nabla^{15}\psi\cdot\nabla^{15}u_3 + \int \nabla^{14}\omega_h\cdot\nabla^{14}\nabla_h^\perp\psi = 0$. Then (3.10) together with the lower order energy estimates yields that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\nabla\psi(t)\|_{H^{14}}^2 + \|u(t)\|_{H^{14}}^2) + \|\nabla u\|_{H^{14}}^2\\ &\quad\leqslant C(\|\nabla\psi\|_{H^{14}} + \|u\|_{H^{14}})(\|\nabla\nabla_h\psi\|_{H^{13}}^2 + \|\nabla u\|_{H^{14}}^2) + C\|\nabla\psi\|_{H^{14}}^2\|\partial_3 u_3\|_\infty.\end{aligned} \tag{3.11}$$

To close this estimate, we need to investigate the dissipation of $\nabla\nabla_h\psi$. In this case, it is convenient to consider the equation of the horizontal direction component of the vorticity. For this purpose, we use $(1.5)_1$ and $(3.2)_2$ to deduce that

$$\begin{cases}\partial_t\nabla_h^\perp\psi + \nabla_h^\perp u_3 = -\nabla_h^\perp(u\cdot\nabla\psi),\\ \partial_t\omega_h - \Delta\omega_h - \nabla_h^\perp\Delta\psi = -u\cdot\nabla\omega_h + \omega\cdot\nabla u_h - \mathcal{R},\end{cases} \tag{3.12}$$

where $\mathcal{R} = (\partial_3\psi\Delta\partial_2\psi - \partial_2\psi\Delta\partial_3\psi, \partial_1\psi\Delta\partial_3\psi - \partial_3\psi\Delta\partial_1\psi)^{\mathrm{T}}$. Similar to the estimate for u, we only investigate the highest order derivative case. Applying ∇^{13} to (3.12), and multiplying the resulting equations by $\nabla^{13}\omega_h$ and $\nabla^{13}\nabla_h^\perp\psi$ respectively, we have

$$\begin{aligned}&\frac{d}{dt}\int\nabla^{13}\omega_h\cdot\nabla^{13}\nabla_h^\perp\psi + \|\nabla^{14}\nabla_h\psi\|_2^2 + \int\nabla^{14}\omega_h\cdot\nabla^{14}\nabla_h^\perp\psi + \int\nabla^{13}\omega_h\cdot\nabla^{13}\nabla_h^\perp u_3\\ &\quad= \int\nabla^{14}\omega_h\cdot\nabla^{12}\nabla_h^\perp(u\cdot\nabla\psi) + \int\nabla^{14}\nabla_h^\perp\psi\cdot\nabla^{12}(u\cdot\nabla\omega_h)\\ &\qquad- \int\nabla^{14}\nabla_h^\perp\psi\cdot\nabla^{12}(\omega\cdot\nabla u_h) + \int\nabla^{14}\nabla_h^\perp\psi\cdot\nabla^{12}\mathcal{R}\\ &\quad=: M_1 + M_2 + M_3 + M_4.\end{aligned} \tag{3.13}$$

We shall estimate each term on the right-hand side of (3.13). For the first term M_1, we use Hölder's inequality and Sobolev's embedding to obtain

$$\begin{aligned}M_1 &= \sum_{j=0}^{12}C_{12}^j\int\nabla^{14}\omega_h\cdot(\nabla^{12-j}\nabla_h^\perp u\cdot\nabla\nabla^j\psi + \nabla^{12-j}u\cdot\nabla\nabla^j\nabla_h^\perp\psi)\\ &\lesssim \|\nabla^{14}\omega\|_2\sum_{j=0}^{12}(\|\nabla^{13-j}u\|_6\|\nabla^{j+1}\psi\|_3 + \|\nabla^{12-j}u\|_3\|\nabla^{j+1}\nabla_h\psi\|_6)\\ &\lesssim (\|\nabla\psi\|_{H^{13}} + \|u\|_{H^{13}})(\|\nabla\nabla_h\psi\|_{H^{13}}^2 + \|\nabla u\|_{H^{14}}^2).\end{aligned} \tag{3.14}$$

Similarly, for the terms M_2 and M_3, we have

$$\begin{aligned}M_2 &= \sum_{j=0}^{12}C_{12}^j\int\nabla^{14}\nabla_h^\perp\psi\cdot(\nabla^{12-j}u\cdot\nabla\nabla^j\omega_h)\\ &\lesssim \|\nabla^{14}\nabla_h\psi\|_2\sum_{j=0}^{12}\|\nabla^{12-j}u\|_3\|\nabla^{j+1}\omega\|_6\\ &\lesssim \|u\|_{H^{13}}(\|\nabla\nabla_h\psi\|_{H^{13}}^2 + \|\nabla u\|_{H^{13}}^2)\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}M_3 &= -\sum_{j=0}^{12}C_{12}^j\int\nabla^{14}\nabla_h^\perp\psi\cdot(\nabla^{12-j}\omega\cdot\nabla\nabla^j u_h)\\ &\lesssim \|\nabla^{14}\nabla_h\psi\|_2\sum_{j=0}^{12}\|\nabla^{12-j}\omega\|_6\|\nabla^{j+1}u\|_3\end{aligned}$$

$$\lesssim \|u\|_{H^{14}}(\|\nabla\nabla_h\psi\|_{H^{13}}^2+\|\nabla u\|_{H^{13}}^2). \tag{3.16}$$

Finally, for the last term M_4, we use the expression of $\mathcal{R}$ to rewrite it as

$$\begin{aligned}M_4&=-\int\nabla^{14}\partial_2\psi\nabla^{12}(\partial_3\psi\Delta\partial_2\psi-\partial_2\psi\Delta\partial_3\psi)+\int\nabla^{14}\partial_1\psi\nabla^{12}(\partial_1\psi\Delta\partial_3\psi-\partial_3\psi\Delta\partial_1\psi)\\&=:M_4^1+M_4^2.\end{aligned}$$

It then follows from Hölder's inequality, Sobolev's embedding and Lemma 2.1 that

$$\begin{aligned}M_4^1&=-\sum_{j=0}^{12}C_{12}^j\int\nabla^{14}\partial_2\psi(\nabla^{12-j}\partial_3\psi\nabla^{j+2}\partial_2\psi-\nabla^{12-j}\partial_2\psi\nabla^{j+2}\partial_3\psi)\\&\lesssim\|\nabla^{14}\partial_2\psi\|_2\bigg(\sum_{j=0}^{11}(\|\nabla^{12-j}\partial_3\psi\|_3\|\nabla^{j+2}\nabla_h\psi\|_6+\|\nabla^{12-j}\partial_2\psi\|_6\|\nabla^{j+2}\partial_3\psi\|_3)\\&\quad+\|\partial_3\psi\|_\infty\|\nabla^{14}\nabla_h\psi\|_2+\|\partial_2\psi\|_\infty\|\nabla^{14}\partial_3\psi\|_2\bigg)\\&\lesssim\|\nabla\psi\|_{H^{14}}\|\nabla\nabla_h\psi\|_{H^{13}}^2.\end{aligned}\tag{3.17}$$

Similarly, we can obtain the same estimate for M_4^2. Thus, summing up the estimates (3.13)–(3.17), we obtain

$$\begin{aligned}&\frac{d}{dt}\int\nabla^{13}\omega_h\cdot\nabla^{13}\nabla_h^\perp\psi+\|\nabla^{14}\nabla_h\psi\|_2^2+\int\nabla^{14}\omega_h\cdot\nabla^{14}\nabla_h^\perp\psi+\int\nabla^{13}\omega_h\cdot\nabla^{13}\nabla_h^\perp u_3\\&\quad\leqslant C(\|\nabla\psi\|_{H^{14}}+\|u\|_{H^{14}})(\|\nabla\nabla_h\psi\|_{H^{13}}^2+\|\nabla u\|_{H^{14}}^2),\end{aligned}$$

which together with the lower estimates gives that

$$\begin{aligned}&\frac{d}{dt}\langle\omega_h,\nabla_h^\perp\psi\rangle_{H^{13}}+\|\nabla\nabla_h\psi\|_{H^{13}}^2+\langle\nabla\omega_h,\nabla\nabla_h^\perp\psi\rangle_{H^{13}}+\langle\omega_h,\nabla_h^\perp u_3\rangle_{H^{13}}\\&\quad\leqslant C(\|\nabla\psi\|_{H^{14}}+\|u\|_{H^{14}})(\|\nabla\nabla_h\psi\|_{H^{13}}^2+\|\nabla u\|_{H^{14}}^2).\end{aligned}\tag{3.18}$$

Combining (3.11) with (3.18), we can take ε small enough and use Lemma 2.1 to deduce that

$$\frac{d}{dt}(\|\nabla\psi\|_{H^{14}}^2+\|u\|_{H^{14}}^2)+c(\|\nabla\nabla_h\psi\|_{H^{13}}^2+\|\nabla u\|_{H^{14}}^2)\leqslant C\|\nabla\psi\|_{H^{14}}^2\|\nabla\nabla_h u\|_2^{\frac12}\|\nabla\nabla_h^2u\|_2^{\frac12}.$$

Thus the decay estimates $(2.43)_1$ give that

$$\frac{d}{dt}(\|\nabla\psi\|_{H^{14}}^2+\|u\|_{H^{14}}^2)+c(\|\nabla\nabla_h\psi\|_{H^{13}}^2+\|\nabla u\|_{H^{14}}^2)\leqslant C\varepsilon\mathcal{E}_s(t)(1+t)^{-\frac{2s+3}{4}}. \tag{3.19}$$

We now turn to deal with the energy estimates related to negative derivatives. We first apply Λ_h^{-s} to (1.5) and obtain

$$\begin{cases}\partial_t\Lambda_h^{-s}\nabla\psi+\Lambda_h^{-s}\nabla u_3=-\Lambda_h^{-s}\nabla(u\cdot\nabla\psi),\\ \partial_t\Lambda_h^{-s}u_h-\Lambda_h^{-s}\Delta u_h-\Lambda_h^{-s}\partial_3\nabla_h\psi=-\Lambda_h^{-s}(u\cdot\nabla u_h)+\Lambda_h^{-s}f^h,\\ \partial_t\Lambda_h^{-s}u_3-\Lambda_h^{-s}\Delta u_3+\Lambda_h^{-s}\Delta_h\psi=-\Lambda_h^{-s}(u\cdot\nabla u_3)+\Lambda_h^{-s}f^v.\end{cases}\tag{3.20}$$

Similar to (3.11), we still focus on the highest order derivative case. For this purpose, we apply ∇^4 to (3.20), multiply the resulted equations by $\Lambda_h^{-s}\nabla^5\psi$, $\Lambda_h^{-s}\nabla^4u_h$ and $\Lambda_h^{-s}\nabla^4u_3$ respectively, and then integrate them on $\mathbb{R}^3$ to deduce that

$$\begin{aligned}&\frac12\frac{d}{dt}(\|\Lambda_h^{-s}\nabla^5\psi(t)\|_2^2+\|\Lambda_h^{-s}\nabla^4u(t)\|_2^2)+\|\Lambda_h^{-s}\nabla^5u(t)\|_2^2\\&\quad=-\int\Lambda_h^{-s}\nabla^5\psi\cdot\Lambda_h^{-s}\nabla^5(u\cdot\nabla\psi)-\int\Lambda_h^{-s}\nabla^4u\cdot\Lambda_h^{-s}\nabla^4(u\cdot\nabla u)\end{aligned}$$

$$
\begin{aligned}
&+ \int \Lambda_h^{-s}\nabla^4 u_h \cdot \Lambda_h^{-s}\nabla^4 f^h + \int \Lambda_h^{-s}\nabla^4 u_3 \cdot \Lambda_h^{-s}\nabla^4 f^v \\
&=: N_1 + N_2 + N_3 + N_4. \qquad (3.21)
\end{aligned}
$$

We shall estimate each term on the right-hand side of (3.21). For the term N_1, we have

$$
\begin{aligned}
N_1 &\lesssim \|\Lambda_h^{-s}\nabla^5\psi\|_2 \|\|\nabla^5(u_h\cdot\nabla_h\psi + u_3\partial_3\psi)\|_{L_{x_h}^{\frac{2}{s+1}}}\|_{L_{x_3}^2} \\
&\lesssim \sum_{j=0}^5 \|\Lambda_h^{-s}\nabla^5\psi\|_2 (\|\|\nabla^{5-j}u_h\nabla^j\nabla_h\psi\|_{L_{x_h}^{\frac{2}{s+1}}}\|_{L_{x_3}^2} + \|\|\nabla^{5-j}u_3\nabla^j\partial_3\psi\|_{L_{x_h}^{\frac{2}{s+1}}}\|_{L_{x_3}^2}). \qquad (3.22)
\end{aligned}
$$

Here we used the Hardy-Littlewood-Sobolev inequality $\|\Lambda_h^{-s}f\|_{L_{x_h}^2} \leqslant C\|f\|_{L_{x_h}^{\frac{2}{s+1}}}$ in the second inequality. We only need to consider the cases $j=0$ and $j=5$, since the other cases can be similarly dealt with. For the term related to u_h with $j=0$, it follows from Hölder's inequality that

$$
\|\|\nabla^5 u_h\nabla_h\psi\|_{L_{x_h}^{\frac{2}{s+1}}}\|_{L_{x_3}^2} \leqslant \|\|\nabla^5 u_h\|_{L_{x_h}^2}\|\nabla_h\psi\|_{L_{x_h}^{\frac{2}{s}}}\|_{L_{x_3}^2} \leqslant \|\nabla^5 u\|_2 \|\|\nabla_h\psi\|_{L_{x_h}^{\frac{2}{s}}}\|_{L_{x_3}^\infty}.
$$

Noticing that

$$
\|\nabla^5 u\|_2 \lesssim \|\nabla u\|_2^{\frac{5}{9}}\|\nabla^{10}u\|_2^{\frac{4}{9}} \lesssim \|\|\Lambda_h^{-s}\nabla u\|_{L_{x_h}^2}^{\frac{2}{s+2}}\|\nabla\nabla_h^2 u\|_{L_{x_h}^2}^{\frac{s}{s+2}}\|_{L_{x_3}^2}^{\frac{5}{9}}\|u\|_{H^{10}}^{\frac{4}{9}} \lesssim \|u\|_{H^{-s,1}}^{\frac{10}{9(s+2)}}\|\nabla\nabla_h^2 u\|_2^{\frac{5s}{9(s+2)}}\|u\|_{H^{10}}^{\frac{4}{9}}
$$

and

$$
\begin{aligned}
\|\|\nabla_h\psi\|_{L_{x_h}^{\frac{2}{s}}}\|_{L_{x_3}^\infty} &\lesssim \|\|\nabla_h\psi\|_{L_{x_3}^2}^{\frac12}\|\partial_3\nabla_h\psi\|_{L_{x_3}^2}^{\frac12}\|_{L_{x_h}^{\frac{2}{s}}} \lesssim \|\|\nabla_h\psi\|_{L_{x_h}^{\frac{2}{s}}}\|_{L_{x_3}^2}^{\frac12}\|\|\nabla\nabla_h\psi\|_{L_{x_h}^{\frac{2}{s}}}\|_{L_{x_3}^2}^{\frac12} \\
&\lesssim \|\Lambda_h^{-s}\nabla_h\psi\|_2^{\frac{s}{2(s+1)}}\|\nabla_h^2\psi\|_2^{\frac{1}{2(s+1)}}\|\Lambda_h^{-s}\nabla\psi\|_2^{\frac{s}{2(s+2)}}\|\nabla\nabla_h^2\psi\|_2^{\frac{1}{s+2}} \\
&\lesssim \|\nabla\psi\|_{H^{-s,1}}^{\frac{2s^2+3s}{2(s+1)(s+2)}}\|\nabla\nabla_h\psi\|_2^{\frac{1}{2(s+1)}}\|\nabla\nabla_h^2\psi\|_2^{\frac{1}{s+2}}
\end{aligned}
$$

by the interpolation and Sobolev's embedding, we have

$$
\begin{aligned}
&\|\|\nabla^5 u_h\nabla_h\psi\|_{L_{x_h}^{\frac{2}{s+1}}}\|_{L_{x_3}^2} \\
&\quad\leqslant C\|\nabla\psi\|_{H^{-s,1}}^{\frac{2s^2+3s}{2(s+1)(s+2)}}\|u\|_{H^{-s,1}}^{\frac{10}{9(s+2)}}\|u\|_{H^{10}}^{\frac{4}{9}}\|\nabla\nabla_h\psi\|_2^{\frac{1}{2(s+1)}}\|\nabla\nabla_h^2\psi\|_2^{\frac{1}{s+2}}\|\nabla\nabla_h^2 u\|_2^{\frac{5s}{9(s+2)}}. \qquad (3.23)
\end{aligned}
$$

Similarly, for the term related to u_h with $j=5$, since

$$
\begin{aligned}
\|\|u\|_{L_{x_h}^{\frac{2}{s}}}\|_{L_{x_3}^\infty} &\lesssim \|\|u\|_{L_{x_h}^{\frac{2}{s}}}\|_{L_{x_3}^2}^{\frac12}\|\|\nabla u\|_{L_{x_h}^{\frac{2}{s}}}\|_{L_{x_3}^2}^{\frac12} \\
&\lesssim \|\Lambda_h^{-s}u\|_2^{\frac{s}{2(s+1)}}\|\nabla_h u\|_2^{\frac{1}{2(s+1)}}\|\Lambda_h^{-s}\nabla u\|_2^{\frac{s}{2(s+1)}}\|\nabla\nabla_h u\|_2^{\frac{1}{2(s+1)}} \\
&\lesssim \|u\|_{H^{-s,1}}^{\frac{s}{s+1}}\|\nabla_h u\|_{H^1}^{\frac{1}{(s+1)}}
\end{aligned}
$$

and

$$
\begin{aligned}
\|\nabla^5\nabla_h\psi\|_2 &\lesssim \|\nabla^2\nabla_h\psi\|_2^{\frac47}\|\nabla^9\nabla_h\psi\|_2^{\frac37} \\
&\lesssim \|\|\Lambda_h^{-s}\nabla^2\psi\|_{L_{x_h}^2}^{\frac{1}{s+2}}\|\nabla^2\nabla_h^2\psi\|_{L_{x_h}^2}^{\frac{s+1}{s+2}}\|_{L_{x_3}^2}^{\frac47}\|\nabla\psi\|_{H^9}^{\frac37} \\
&\lesssim \|\nabla\psi\|_{H^{-s,1}}^{\frac{4}{7(s+2)}}\|\nabla^2\nabla_h^2\psi\|_2^{\frac{4(s+1)}{7(s+2)}}\|\nabla\psi\|_{H^9}^{\frac37},
\end{aligned}
$$

we have

$$
\|\|u_h\nabla^5\nabla_h\psi\|_{L_{x_h}^{\frac{2}{s+1}}}\|_{L_{x_3}^2} \lesssim \|\|u_h\|_{L_{x_h}^{\frac{2}{s}}}\|\nabla^5\nabla_h\psi\|_{L_{x_h}^2}\|_{L_{x_3}^2}
$$

$$
\begin{aligned}
&\lesssim \| \|u\|_{L^{\frac{2}{s}}_{x_h}} \|_{L^\infty_{x_3}} \|\nabla^5 \nabla_h \psi\|_2 \\
&\lesssim \|u\|^{\frac{s}{s+1}}_{H^{-s,1}} \|\nabla\psi\|^{\frac{4}{7(s+2)}}_{H^{-s,1}} \|\nabla\psi\|^{\frac{3}{7}}_{H^9} \|\nabla_h u\|^{\frac{1}{s+1}}_{H^1} \|\nabla^2\nabla_h^2\psi\|_2^{\frac{4(s+1)}{7(s+2)}}.
\end{aligned} \tag{3.24}
$$

For the terms related to u_3, we take similar procedure and use the divergence free of u to obtain

$$
\begin{aligned}
&\| \|\nabla^5 u_3 \partial_3 \psi\|_{L^{\frac{2}{s+1}}_{x_h}} \|_{L^2_{x_3}} \\
&\lesssim \|\nabla^4 \nabla_h u\|_2 \| \|\nabla\psi\|_{L^{\frac{2}{s}}_{x_h}} \|_{L^\infty_{x_3}} \\
&\lesssim \|\Lambda_h^{-s}\nabla^4 u\|_2^{\frac{1}{s+2}} \|\nabla^4\nabla_h^2 u\|_2^{\frac{s+1}{s+2}} \| \|\Lambda_h^{-s}\nabla\psi\|^{\frac{s}{s+1}}_{L^2_{x_h}} \|\nabla\nabla_h\psi\|^{\frac{1}{s+1}}_{L^2_{x_h}} \|_{L^\infty_{x_3}} \\
&\lesssim \|u\|^{\frac{1}{s+2}}_{H^{-s,4}} \|\nabla\nabla_h^2 u\|_2^{\frac{4(s+1)}{7(s+2)}} \|\nabla^8\nabla_h^2 u\|_2^{\frac{3(s+1)}{7(s+2)}} \|\nabla\psi\|^{\frac{s}{s+1}}_{H^{-s,1}} \|\nabla\nabla_h\psi\|_2^{\frac{1}{2(s+1)}} \|\nabla^2\nabla_h\psi\|_2^{\frac{1}{2(s+1)}} \\
&\lesssim \|u\|^{\frac{1}{s+2}}_{H^{-s,4}} \|u\|^{\frac{3(s+1)}{7(s+2)}}_{H^{10}} \|\nabla\psi\|^{\frac{s}{s+1}}_{H^{-s,1}} \|\nabla\nabla_h^2 u\|_2^{\frac{4(s+1)}{7(s+2)}} \|\nabla\nabla_h\psi\|_2^{\frac{1}{2(s+1)}} \|\nabla^2\nabla_h\psi\|_2^{\frac{1}{2(s+1)}}
\end{aligned} \tag{3.25}
$$

for $j=0$, and

$$
\begin{aligned}
&\| \|u_3 \nabla^5 \partial_3 \psi\|_{L^{\frac{2}{s+1}}_{x_h}} \|_{L^2_{x_3}} \\
&\lesssim \| \|u_3\|_{L^{\frac{2}{s}}_{x_h}} \|_{L^\infty_{x_3}} \|\nabla^6\psi\|_2 \\
&\lesssim \| \|u_3\|^{\frac{1}{2}}_{L^2_{x_3}} \|\partial_3 u_3\|^{\frac{1}{2}}_{L^2_{x_3}} \|_{L^{\frac{2}{s}}_{x_h}} \|\nabla^2\psi\|_2^{\frac{5}{9}} \|\nabla^{11}\psi\|_2^{\frac{4}{9}} \\
&\lesssim \| \|u_3\|_{L^{\frac{2}{s}}_{x_h}} \|^{\frac{1}{2}}_{L^2_{x_3}} \| \|\partial_3 u_3\|_{L^{\frac{2}{s}}_{x_h}} \|^{\frac{1}{2}}_{L^2_{x_3}} \|\Lambda_h^{-s}\nabla^2\psi\|_2^{\frac{10}{9(s+2)}} \|\nabla^2\nabla_h^2\psi\|_2^{\frac{5s}{9(s+2)}} \|\nabla\psi\|^{\frac{4}{9}}_{H^{10}} \\
&\lesssim \|u\|^{\frac{2s+1}{2(s+2)}}_{H^{-s,1}} \|\nabla\psi\|^{\frac{10}{9(s+2)}}_{H^{-s,1}} \|\nabla\psi\|^{\frac{4}{9}}_{H^{10}} \|\nabla_h^2 u\|_2^{\frac{3}{2(s+2)}} \|\nabla^2\nabla_h^2\psi\|_2^{\frac{5s}{9(s+2)}}
\end{aligned} \tag{3.26}
$$

for $j=5$. In the last inequality, we have used the facts $\|u_3\|_{L^{\frac{2}{s}}_{x_h}} \lesssim \|\Lambda_h^{-s}u\|_2^{\frac{s+1}{s+2}} \|\nabla_h^2 u\|_2^{\frac{1}{s+2}}$ and $\|\partial_3 u_3\|_{L^{\frac{2}{s}}_{x_h}} \lesssim \|\Lambda_h^{-s}u\|_2^{\frac{s}{s+2}} \|\nabla_h^2 u\|_2^{\frac{2}{s+2}}$. Thus by (3.22)–(3.26) and the decay estimates $(2.43)_1$, we can deduce that

$$
N_1 \leqslant C\varepsilon \mathcal{E}_s(t)(1+t)^{-\kappa} \tag{3.27}
$$

for some $\kappa > 1$ and any $t \in [0,T]$, provided that the positive constant $\delta = 1-s$ is small enough. For the terms N_2 and N_3, we have

$$
\begin{aligned}
N_2 &\leqslant \|\Lambda_h^{-s}\nabla^5 u\|_2 \|\Lambda_h^{-s}\nabla^3(u\cdot\nabla u)\|_2 \\
&\lesssim \|\Lambda_h^{-s}\nabla^5 u\|_2 \sum_{j=0}^{3} \| \|\nabla^{3-j}u\nabla^{j+1}u\|_{L^{\frac{2}{s+1}}_{x_h}} \|_{L^2_{x_3}} \\
&\lesssim \|\Lambda_h^{-s}\nabla^5 u\|_2 \sum_{j=0}^{3} \| \|\nabla^{3-j}u\|_{L^{\frac{2}{s}}_{x_h}} \|_{L^\infty_{x_3}} \|\nabla^{j+1}u\|_2 \\
&\lesssim \|u\|_{H^5} (\|\Lambda_h^{-s}\nabla^5 u\|_2^2 + \|\nabla u\|^2_{H^3}),
\end{aligned} \tag{3.28}
$$

and

$$
\begin{aligned}
N_3 &\lesssim \|\Lambda_h^{-s}\nabla^5 u\|_2 \left\| \Lambda_h^{-s}\nabla^3 \left(\frac{\partial_i\partial_j}{\Delta} \nabla_h (u_i u_j + \partial_i\psi\partial_j\psi) - \partial_j(\nabla_h\psi\partial_j\psi) \right) \right\|_2 \\
&\lesssim \|\Lambda_h^{-s}\nabla^5 u\|_2 (\|\Lambda_h^{-s}\nabla^3(u\nabla u)\|_2 + \|\Lambda_h^{-s}\nabla^3(\nabla\psi\nabla\nabla_h\psi)\|_2 + \|\Lambda_h^{-s}\nabla^4(\nabla_h\psi\nabla\psi)\|_2),
\end{aligned} \tag{3.29}
$$

which can be estimated as N_1 and N_2. For the term N_4, we have

$$
N_4 = \int \Lambda_h^{-s}\nabla^5 u_3 \cdot \Lambda_h^{-s}\nabla^3 \left(\frac{\partial_i\partial_j}{\Delta} \partial_3(u_i u_j) + \frac{\partial_3\partial_j}{\Delta} \nabla_h \cdot (\partial_j\psi\nabla_h\psi) - \frac{\partial_j\nabla_h}{\Delta} \cdot \nabla_h(\partial_3\psi\partial_j\psi) \right)
$$

$$\lesssim \|\Lambda_h^{-s}\nabla^5 u\|_2(\|\Lambda_h^{-s}\nabla^3(u\nabla u)\|_2 + \|\Lambda_h^{-s}\nabla^3(\nabla\psi\nabla\nabla_h\psi)\|_2), \tag{3.30}$$

which can be estimated as N_3. Substituting the estimates (3.27)–(3.30) into (3.21) and taking similar procedure for the lower order derivatives, we can deduce that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\nabla\psi(t)\|^2_{H^{-s,4}} + \|u(t)\|^2_{H^{-s,4}}) + \|\nabla u(t)\|^2_{H^{-s,4}}\\ &\quad\leqslant C(\|u\|_{H^5} + \|\nabla\psi\|_{H^5})(\|\nabla u\|^2_{H^{-s,4}} + \|\nabla u\|^2_{H^3} + \|\nabla\nabla_h\psi\|^2_{H^4}) + C\varepsilon\mathcal{E}_s(t)(1+t)^{-\kappa}\\ &\quad\leqslant 2CC_0\varepsilon(\|\nabla u\|^2_{H^{-s,4}} + \|\nabla u\|^2_{H^3} + \|\nabla\nabla_h\psi\|^2_{H^4}) + C\varepsilon\mathcal{E}_s(t)(1+t)^{-\kappa}\end{aligned} \tag{3.31}$$

for some $\kappa > 1$ and any $t \in [0, T]$.

By taking ε suitably small such that $4CC_0\varepsilon \leqslant \min\{c, 1\}$ and then using (3.19) and (3.31), we can obtain (3.1) for some $\widehat{C}_0$ and $\kappa_0 > 1$. This completes the proof of Proposition 3.1. □

Proof of Theorem 1.1. By using the standard energy method, it can be proved that there exists $T > 0$ such that the system (1.5) admits a unique solution (ψ, u) on $[0, T]$ satisfying

$$(\nabla\psi, u) \in C([0, T]; H^{14}(\mathbb{R}^3) \cap H^{-s,4}(\mathbb{R}^3))$$

(we may refer to [5, 8, 21] for details). Let us further take $\varepsilon > 0$ small enough such that $2\widehat{C}_0\varepsilon \leqslant \kappa_0 - 1$ and then define

$$T^* = \sup\{t \mid \mathcal{E}_s(t) \leqslant 4C_0^2\varepsilon^2\}.$$

If $T^* < +\infty$, we can use Proposition 3.1 to deduce that

$$\mathcal{E}_s(t) \leqslant \mathcal{E}_s(0) + \widehat{C}_0\varepsilon\int_0^t \mathcal{E}_s(\tau)(1+\tau)^{-\kappa_0}d\tau \leqslant \mathcal{E}_s(0) + \frac{\widehat{C}_0\varepsilon}{\kappa_0 - 1}\sup_{\tau\in[0,T^*)}\mathcal{E}_s(\tau) \leqslant \mathcal{E}_s(0) + \frac{1}{2}\sup_{\tau\in[0,T^*)}\mathcal{E}_s(\tau)$$

for all $t \in [0, T^*)$. Then it holds that $\mathcal{E}_s(t) \leqslant 2\mathcal{E}_s(0) \leqslant 2C_0^2\varepsilon^2$ for any $t \in [0, T^*)$, which contradicts the definition of T^* and therefore $T^* = +\infty$. Thus we have closed the *a priori* assumption used in Proposition 2.5 and established the global existence and decay estimates. This completes the proof of Theorem 1.1. □

4 Several comments on the MHD equation without magnetic diffusion

In the 2-D case, as mentioned in the Introduction, the system (1.1) is exactly the same as the incompressible zero-magnetic-diffusion MHD system (1.2). In this paper, we use the elementary energy method and the interpolation to establish the time decay estimates for the 3-D MHD-type system (1.1). This elementary energy-method technique is useful for various related problems (see [22]). Recently, for the 3-D incompressible MHD system without magnetic diffusion, Xu and Zhang [24] wrote the system in the Lagrangian formulation and proved the global existence of smooth solution for the small initial data around the trivial state $(e_1, 0)$ under the assumption of an admissible condition, which is removed by Abidi and Zhang [1] by introducing a new Lagrangian formulation of the system.

Considering that the fluids and the magnetic fields are often posed in a perfectly conducting container with boundary, Ren et al. [18] proved that the initial boundary value problem of the 2-D MHD system without magnetic diffusion in a strip domain has a unique global strong solution around the equilibrium state $(\bar{b}, \bar{u}) = (e_1, 0)$ for both the non-slip boundary condition and Navier slip boundary condition on the velocity. In the 3-D case, Tan and Wang [20] also established the global existence and convergence rate of solutions for initial data around the steady state $(\bar{b}, \bar{u}) = (e_3, 0)$.

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