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Strategically supported cooperation in dynamic games with coalition structures

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Abstract The problem of strategic stability of long-range cooperative agreements in dynamic games with coalition structures is investigated. Based on imputation distribution procedures, a general theoretical framework of the differential game with a coalition structure is proposed. A few assumptions about the deviation instant for a coalition are made concerning the behavior of a group of many individuals in certain dynamic environments. From these, the time-consistent cooperative agreement can be strategically supported by ε-Nash or strong ε-Nash equilibria. While in games in the extensive form with perfect information, it is somewhat surprising that without the assumptions of deviation instant for a coalition, Nash or strong Nash equilibria can be constructed.

Keywords cooperative game theory, coalition structure, strategic stability, imputation distribution procedure, deviation instant, ε-Nash equilibrium, strong ε-Nash equilibrium

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1 Introduction

Human behavior is dynamic, and cooperation runs through human behavior. It often happens that players agree to cooperate over a certain period. It also happens often that some cooperative agreements are abandoned before reaching the maturity. Therefore, it is important that cooperation remains stable on a time interval.

When we analyze the problem of stability of long-range cooperative agreements, there are three important aspects which must be taken into account, including time consistency, strategic stability and the irrational-behavior-proof condition.

Time consistency involves the property that as the cooperation develops, partners are guided by the same optimal principle at each instant of time and hence do not possess incentives to deviate from the previous cooperative behavior. The concept of time consistency and its implementation was initially

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proposed in [18, 21–23] and was developed in [19, 20, 26]. Some new results about time consistency could be found in [8, 17, 25, 30].

Strategic stability means that the outcome of cooperative agreements must be attained in some Nash equilibrium, which will guarantee the strategic support of the cooperation. The agreement will be developed in such a manner that at least individual deviations from the cooperation will not give any advantage to the deviator. Some results about strategic stability could be found in [3, 5, 6, 9–11, 15, 27].

The irrational-behavior-proof condition means that the partners involved in the cooperation must be sure that even in the worst scenario they will not lose compared with the noncooperative behavior. Since one cannot be sure that the partners will behave rationally on a long time interval, this aspect must be also taken into account. The concept of the irrational-behavior-proof condition was initially proposed in [29]. A further investigation could be found in [7].

Some results of dynamic games with coalition structures are given in [1, 2, 12, 13, 16, 24]. In [27], the existence of ε-Nash equilibrium in the regularized differential game without coalition structures was firstly proved. In this paper, we focus on *the problem of strategic stability* in dynamic games with coalition structures. We consider the general coalition setting, when not only the grand coalition, but also a coalition partition of players can be formed.

We build a general theoretical framework of the differential game with a coalition structure based on *imputation distribution procedures* (IDP). The notion of IDP is the basic ingredient in our theory. This notion may be interpreted as the instantaneous payoff of an individual at some moment, which prescribes the distribution of the total gain among the members of a group and yields the existence of Nash equilibrium.

To construct ε-Nash or strong ε-Nash equilibria in such games a few assumptions about the *deviation instant for a coalition* concerning the behavior of a group of many individuals in certain dynamic environments are made. It turns out that ε-Nash or strong ε-Nash equilibria exist in the differential game with a coalition structure, which guarantees the strategic support of cooperation.

We also consider the problem in games in the extensive form with perfect information, in which it is somewhat surprising that without the assumptions of deviation instant for a coalition, Nash or strong Nash equilibria can be constructed.

The paper is organized as follows. In Section 2, we define the basic concepts and set up standard terminologies and notation about the differential game with a coalition structure. In Section 3, we prove the existence of ε-Nash equilibrium in a regularized differential game with a coalition structure and the existence of strong ε-Nash equilibrium in a strictly regularized differential game with a coalition structure. In Section 4, we consider the problem in a discrete time case with perfect information.

2 Formal definitions and terminologies

In this section, we define the basic concepts of the differential game with a coalition structure and set up standard terminologies and notation.

Differential game, $\Gamma(x_0, T-t_0)$:

We consider an *n-person differential game* $\Gamma(x_0, T-t_0)$ with independent motions on the time interval $[t_0, T]$ (see [4]). Motion equations have the form

$$\dot{x_i} = f_i(x_i, u_i), \quad u_i \in U_i \subset \mathbb{R}^p, \quad x_i \in X \subset \mathbb{R}^q, \quad i = 1, 2, \ldots, n. \tag{2.1}$$

It is assumed that the system of differential equations (2.1) satisfies all conditions necessary for the existence, sustainability and uniqueness of the solution for any n-tuple of measurable controls $u_1(t), \ldots, u_n(t)$.

Let $N = \{1, 2, \ldots, n\}$ be the set of players. The payoff of player i is given by

$$H_i(x_0, T-t_0; u_1(\cdot), \ldots, u_n(\cdot)) = \int_{t_0}^{T} h_i(x(\tau))d\tau,$$

where $h_i(x)$ is a continuous function and $x(\tau)=(x_1(\tau),\dots,x_n(\tau))$ is the solution of (2.1) when open-loop controls $u_1(\tau),\dots,u_n(\tau)$ are used and $x(t_0)=(x_1(t_0),\dots,x_n(t_0))=x_0$.

Optimal cooperative trajectory, $\bar{x}(t)$:

Suppose that there exist an n-tuple of open-loop controls $\bar{u}(t)=(\bar{u}_1(t),\dots,\bar{u}_n(t))$ and a trajectory $\bar{x}(t), t\in[t_0,T]$, such that

$$\begin{aligned}&\max_{u_1(t),\dots,u_n(t)}\sum_{i=1}^{n}H_i(x_0,T-t_0;u_1(t),\dots,u_n(t))\\&=\sum_{i=1}^{n}H_i(x_0,T-t_0;\bar{u}_1(t),\dots,\bar{u}_n(t))=\sum_{i=1}^{n}\int_{t_0}^{T}h_i(\bar{x}(\tau))d\tau.\end{aligned}\tag{2.2}$$

We shall call a trajectory $\bar{x}(t)=(\bar{x}_1(t),\dots,\bar{x}_n(t))$ satisfying (2.2) an *optimal cooperative trajectory.*

Characteristic function:

The *characteristic function* in $\Gamma(x_0,T-t_0)$ is defined in a classical way:

$$\begin{aligned}&V(x_0,T-t_0;N)=\sum_{i=1}^{n}\int_{t_0}^{T}h_i(\bar{x}(\tau))d\tau,\\&V(x_0,T-t_0;\emptyset)=0,\\&V(x_0,T-t_0;S)=\mathrm{Val}\Gamma_{S,N\backslash S}(x_0,T-t_0),\end{aligned}\tag{2.3}$$

where $\mathrm{Val}\Gamma_{S,N\backslash S}(x_0,T-t_0)$ is the value of the zero-sum game between coalition S acting as player 1 and coalition $N\backslash S$ acting as player 2 where the payoff of S equals $\sum_{i\in S}H_i(x_0,T-t_0;u_1(t),\dots,u_n(t))$. In the special case when $S=\{i\}$, $V(x_0,T-t_0;\{i\})$ is the value of the zero-sum game between player i and coalition $N\backslash\{i\}$.

Imputation set, $L(x_0,T-t_0)$:

Define $L(x_0,T-t_0)$ as the *imputation set* of $\Gamma(x_0,T-t_0)$ (see [28]):

$$L(x_0,T-t_0)=\bigg\{\alpha=(\alpha_1,\dots,\alpha_n):\alpha_i\geqslant V(x_0,T-t_0;\{i\}),\sum_{i\in N}\alpha_i=V(x_0,T-t_0;N)\bigg\}.\tag{2.4}$$

Core, $C(x_0,T-t_0)$:

Define $C(x_0,T-t_0)$ as the *core* of $\Gamma(x_0,T-t_0)$:

$$\begin{aligned}&C(x_0,T-t_0)\\&\quad=\bigg\{\alpha=(\alpha_1,\dots,\alpha_n):\sum_{i\in S}\alpha_i\geqslant V(x_0,T-t_0;S),\forall\, S\subset N,\sum_{i\in N}\alpha_i=V(x_0,T-t_0;N)\bigg\}.\end{aligned}\tag{2.5}$$

Imputation distribution procedure, $\beta(\tau)$:

Let $\alpha\in L(x_0,T-t_0)$. Define the *imputation distribution procedure* (IDP) (see [19]) as a function

$$\beta(\tau)=(\beta_1(\tau),\dots,\beta_n(\tau)),\quad \tau\in[t_0,T],$$

such that

$$\alpha_i=\int_{t_0}^{T}\beta_i(\tau)d\tau.\tag{2.6}$$

Regularized game, $\Gamma_\alpha(x_0,T-t_0)$:

For every $\alpha\in L(x_0,T-t_0)$, we define the noncooperative game $\Gamma_\alpha(x_0,T-t_0)$ which differs from game $\Gamma(x_0,T-t_0)$ only by payoffs defined along the optimal cooperative trajectory $\bar{x}(\tau),\tau\in[t_0,T]$.

Denote the payoff function in $\Gamma_\alpha(x_0,T-t_0)$ by $H_i^\alpha(x_0,T-t_0;u_1(t),\dots,u_n(t))$ and the corresponding trajectory by $x(\tau)$. Then

$$H_i^\alpha(x_0,T-t_0;u_1(t),\dots,u_n(t))=H_i(x_0,T-t_0;u_1(t),\dots,u_n(t)),$$

if there does not exist $\tau \in (t_0, T]$ such that $x(\tau) = \bar{x}(\tau)$.

Let $t = \inf\{t' : x(\tau) \neq \bar{x}(\tau), \tau \in (t', T]\}$. Then

$$\begin{aligned}&H_i^{\alpha}(x_0, T - t_0; u_1(t), \dots, u_n(t))\\&\quad= \int_{t_0}^{t} \beta_i(\tau)d\tau + H_i(\bar{x}(t), T - t; u_1(t), \dots, u_n(t))\\&\quad= \int_{t_0}^{t} \beta_i(\tau)d\tau + \int_{t}^{T} h_i(x(\tau))d\tau,\end{aligned}$$

where $h_i(x(\tau))$ is the instantaneous payoff of player i from the state $\bar{x}(t)$.

In a special case, when $x(\tau) = \bar{x}(\tau), \tau \in (t_0, T]$, we have

$$H_i^{\alpha}(x_0, T - t_0; \bar{u}_1(t), \dots, \bar{u}_n(t)) = \int_{t_0}^{T} \beta_i(\tau)d\tau = \alpha_i.$$

Consider the current subgames $\Gamma(\bar{x}(t), T - t)$, imputation sets $L(\bar{x}(t), T - t)$ and cores $C(\bar{x}(t), T - t)$. Let $\alpha(t) \in L(\bar{x}(t), T - t)$. Suppose that $\alpha(t)$ can be selected as a differentiable function of $t, t \in [t_0, T]$. Game $\Gamma_{\alpha}(x_0, T - t_0)$ is called a *regularized game* of $\Gamma(x_0, T - t_0)$ (*α-regularization*) if IDP β is defined in such a way that

$$\alpha_i(t) = \int_{t}^{T} \beta_i(\tau)d\tau,$$

or

$$\beta_i(t) = -\alpha_i'(t). \tag{2.7}$$

In particular, if $\alpha(t) \in C(\bar{x}(t), T - t)$, $\Gamma_{\alpha}(x_0, T - t_0)$ is called a *strictly regularized game* of $\Gamma(x_0, T - t_0)$.

Time consistency:

From (2.7), we get

$$\alpha_i = \int_{t_0}^{t} \beta_i(\tau)d\tau + \alpha_i(t). \tag{2.8}$$

Now suppose that $M(x_0, T - t_0) \subset L(x_0, T - t_0)$ is some optimality principle in the cooperative version of game $\Gamma(x_0, T - t_0)$, and $M(\bar{x}(t), T - t) \subset L(\bar{x}(t), T - t)$ is the same optimality principle defined in the subgame $L(\bar{x}(t), T - t)$ with initial conditions on the optimal trajectory. M can be the core, the stable set, the Shapley value, the nucleolus etc. If $\alpha \in M(x_0, T - t_0)$ and $\alpha(t) \in M(\bar{x}(t), T - t)$ condition (2.8) gives us the *time consistency* of the chosen imputation α or the chosen optimality principle in game $\Gamma(x_0, T - t_0)$.

Differential game with a coalition structure, $\Gamma^{\mathcal{P}}(x_0, T - t_0)$:

Let $\mathcal{P} = \{S_1, \dots, S_m\}$ be a partition of the player set N such that $S_i \cap S_j = \emptyset, i \neq j, \bigcup_{i=1}^{m} S_i = N, |S_i| = n_i, \sum_{i=1}^{m} n_i = n$.

Suppose that each player i from N is playing in the interests of coalition $S_k \in \mathcal{P}$ which he belongs to, trying to maximize the sum of payoffs of coalition members, i.e.,

$$\max_{u_i, i \in S_k} \sum_{i \in S_k} H_i(x_0, T - t_0; u_1(t), \dots, u_n(t)). \tag{2.9}$$

Define $u_{S_k} = \{u_i, i \in S_k\}$ as the strategy of coalition S_k and $x_{S_k} = \{x_i, i \in S_k\}$ as the trajectory of coalition S_k. Write

$$H_{S_k}(x_0, T - t_0; u_{S_1}(t), \dots, u_{S_m}(t)) = \sum_{i \in S_k} H_i(x_0, T - t_0; u_1(t), \dots, u_n(t))$$

as the payoff of coalition S_k.

Suppose that coalitions in $\mathcal{P}$ are playing cooperatively with objective (2.2) and state dynamics (2.1). We call the above game the *cooperative differential game with a coalition structure* denoted by $\Gamma^{\mathcal{P}}(x_0, T-t_0)$.

Suppose that there exist an n-tuple of open-loop controls $\bar{u}(t) = (\bar{u}_1(t), \dots, \bar{u}_n(t))$ and a trajectory $\bar{x}(t) = (\bar{x}_1(t), \dots, \bar{x}_n(t)), t \in [t_0, T]$ satisfying (2.2). Then the trajectory $\bar{x}(t)$ is an optimal cooperative trajectory of $\Gamma(x_0, T-t_0)$. We define $\bar{x}(t)$ as an *optimal cooperative trajectory* of $\Gamma^{\mathcal{P}}(x_0, T-t_0)$ at the same time.

The *characteristic function* in $\Gamma^{\mathcal{P}}(x_0, T-t_0)$ is defined by

$$\begin{aligned}
&V(x_0, T-t_0; \mathcal{P}) = \sum_{i=1}^{n} \int_{t_0}^{T} h_i(\bar{x}(\tau)) d\tau, \\
&V(x_0, T-t_0; \emptyset) = 0, \\
&V(x_0, T-t_0; \mathcal{S}) = \mathrm{Val}\Gamma_{\mathcal{S}, \mathcal{P} \setminus \mathcal{S}}(x_0, T-t_0),
\end{aligned} \tag{2.10}$$

where $\mathrm{Val}\Gamma_{\mathcal{S}, \mathcal{P} \setminus \mathcal{S}}(x_0, T-t_0)$ is the value of the zero-sum game played between coalition $\mathcal{S}$ acting as player 1 and coalition $\mathcal{P} \setminus \mathcal{S}$ acting as player 2 where the payoff of coalition $\mathcal{S}$ equals

$$\sum_{S_k \in \mathcal{S}} H_{S_k}(x_0, T-t_0; u_{S_1}(t), \dots, u_{S_m}(t)).$$

Define $L^{\mathcal{P}}(x_0, T-t_0)$ as the *imputation set* in $\Gamma^{\mathcal{P}}(x_0, T-t_0)$:

$$\begin{aligned}
L^{\mathcal{P}}(x_0, T-t_0) = \bigg\{ & \alpha = (\alpha_{S_1}, \dots, \alpha_{S_m}) : \\
& \alpha_{S_k} \geqslant V(x_0, T-t_0; \{S_k\}), \sum_{S_k \in \mathcal{P}} \alpha_{S_k} = V(x_0, T-t_0; \mathcal{P}) \bigg\}.
\end{aligned} \tag{2.11}$$

Define $C^{\mathcal{P}}(x_0, T-t_0)$ as the *core* in $\Gamma^{\mathcal{P}}(x_0, T-t_0)$:

$$\begin{aligned}
C^{\mathcal{P}}(x_0, T-t_0) = \bigg\{ & \alpha = (\alpha_{S_1}, \dots, \alpha_{S_m}) : \\
& \sum_{S_k \in \mathcal{S}} \alpha_{S_k} \geqslant V(x_0, T-t_0; \mathcal{S}), \forall \mathcal{S} \subset \mathcal{P}, \sum_{S_k \in \mathcal{P}} \alpha_{S_k} = V(x_0, T-t_0; \mathcal{P}) \bigg\}.
\end{aligned} \tag{2.12}$$

Let $\alpha \in L^{\mathcal{P}}(x_0, T-t_0)$. Define the *imputation distribution procedure* (IDP) of $\Gamma^{\mathcal{P}}(x_0, T-t_0)$ as a function $\beta(\tau) = (\beta_{S_1}(\tau), \dots, \beta_{S_m}(\tau)), \tau \in [t_0, T]$, such that

$$\alpha_{S_k} = \int_{t_0}^{T} \beta_{S_k}(\tau) d\tau, \quad S_k \in \mathcal{P}. \tag{2.13}$$

Regularized game with a coalition structure, $\Gamma_{\alpha}^{\mathcal{P}}(x_0, T-t_0)$:

For every $\alpha \in L^{\mathcal{P}}(x_0, T-t_0)$, we define the noncooperative game $\Gamma_{\alpha}^{\mathcal{P}}(x_0, T-t_0)$ which differs from game $\Gamma^{\mathcal{P}}(x_0, T-t_0)$ only by payoffs defined along the optimal cooperative trajectory $\bar{x}(\tau), \tau \in [t_0, T]$.

Denote the payoff function in game $\Gamma_{\alpha}^{\mathcal{P}}(x_0, T-t_0)$ by $H_{S_k}^{\alpha}(x_0, T-t_0; u_{S_1}(t), \dots, u_{S_m}(t))$ and the corresponding trajectory by $x(\tau)$. Then

$$H_{S_k}^{\alpha}(x_0, T-t_0; u_{S_1}(t), \dots, u_{S_m}(t)) = H_{S_k}(x_0, T-t_0; u_{S_1}(t), \dots, u_{S_m}(t)),$$

if there does not exist $\tau \in (t_0, T]$ such that $x(\tau) = \bar{x}(\tau)$ for $\tau \in (t_0, T]$.

Let $t = \inf\{t' : x(\tau) \neq \bar{x}(\tau), \tau \in (t', T]\}$. Then

$$\begin{aligned}
&H_{S_k}^{\alpha}(x_0, T-t_0; u_{S_1}(t), \dots, u_{S_m}(t)) \\
&\quad = \int_{t_0}^{t} \beta_{S_k}(\tau) d\tau + H_{S_k}(\bar{x}(t), T-t; u_{S_1}(t), \dots, u_{S_m}(t))
\end{aligned}$$

$$= \int_{t_0}^{t} \beta_{S_k}(\tau)d\tau + \int_{t}^{T} h_{S_k}(x(\tau))d\tau,$$

where $h_{S_k}(x(\tau)) = \sum_{i\in S_k} h_i(x(\tau))$.

In a special case, when $x(\tau) = \bar{x}(\tau), \tau \in (t_0, T]$, we have

$$H^{\alpha}_{S_k}(x_0, T - t_0; u_{S_1}(t), \ldots, u_{S_m}(t)) = \int_{t_0}^{T} \beta_{S_k}(\tau)d\tau = \alpha_{S_k}.$$

Consider subgames $\Gamma^{\mathcal{P}}(\bar{x}(t), T - t)$, imputation sets $L^{\mathcal{P}}(\bar{x}(t), T - t)$ and cores $C^{\mathcal{P}}(\bar{x}(t), T - t)$. Let $\alpha(t) \in L^{\mathcal{P}}(\bar{x}(t), T - t)$. Suppose that $\alpha(t)$ can be selected as a differentiable function of $t, t \in [t_0, T]$. Game $\Gamma^{\mathcal{P}}_{\alpha}(x_0, T - t_0)$ is called the *regularized game* of $\Gamma^{\mathcal{P}}(x_0, T - t_0)$ (*α-regularization*) if IDP β is defined in such a way that

$$\alpha_{S_k}(t) = \int_{t}^{T} \beta_{S_k}(\tau)d\tau,$$

or

$$\beta_{S_k}(t) = -\alpha'_{S_k}(t). \tag{2.14}$$

In particular, if $\alpha(t) \in C^{\mathcal{P}}(\bar{x}(t), T - t)$, $\Gamma^{\mathcal{P}}_{\alpha}(x_0, T - t_0)$ is called the *strictly regularized game* of $\Gamma^{\mathcal{P}}(x_0, T - t_0)$.

From (2.14), we get

$$\alpha_{S_k} = \int_{t_0}^{t} \beta_{S_k}(\tau)d\tau + \alpha_{S_k}(t), \quad S_k \in \mathcal{P}. \tag{2.15}$$

Now suppose that $M^{\mathcal{P}}(x_0, T - t_0) \subset L^{\mathcal{P}}(x_0, T - t_0)$ is some optimality principle in the cooperative version of game $\Gamma^{\mathcal{P}}(x_0, T - t_0)$, and $M^{\mathcal{P}}(\bar{x}(t), T - t) \subset L^{\mathcal{P}}(\bar{x}(t), T - t)$ is the same optimality principle defined in the subgame $L^{\mathcal{P}}(\bar{x}(t), T - t)$ with initial conditions on the optimal trajectory. If $\alpha \in M^{\mathcal{P}}(x_0, T - t_0)$ and $\alpha(t) \in M^{\mathcal{P}}(\bar{x}(t), T - t)$, condition (2.15) gives us the *time consistency* of the chosen imputation α or the chosen optimality principle in game $\Gamma^{\mathcal{P}}(x_0, T - t_0)$.

ε-Nash and strong ε-Nash equilibria of $\Gamma^{\mathcal{P}}_{\alpha}(x_0, T - t_0)$:

In the differential game with a coalition structure, the deviation of strategies of different members in a coalition happens possibly in different time. And the trajectory realized by the deviation of strategies of some members possibly has no changing, which cannot be regarded as the actual deviation. To define ε-Nash and strong ε-Nash equilibria of $\Gamma^{\mathcal{P}}_{\alpha}(x_0, T - t_0)$, we shall define the deviation instant for a coalition.

In $\Gamma(x_0, T - t_0)$, we say that for player $i \in N$ strategy $u_i(\cdot)$ is *essentially different* from strategy $\bar{u}_i(\cdot)$ under n-tuple $\bar{u}(\cdot)$, if the trajectory $x_i(\cdot)$ under n-tuple $\bar{u}(\cdot) \,\|\, u_i(\cdot)$ is different from the trajectory $\bar{x}_i(\cdot)$ under $\bar{u}(\cdot)$, i.e., there is $t \in (t_0, T]$ such that $x_i(t) \neq \bar{x}_i(t)$. If strategies $u_i(\cdot)$ and $\bar{u}_i(\cdot)$ are essentially different, we define $\bar{t}_i(\bar{u}(\cdot) \,\|\, u_i(\cdot)) = \inf\{t' : x_i(\tau) \neq \bar{x}_i(\tau), \tau \in (t', T]\}$ as the *deviation instant* between strategies $u_i(\cdot)$ and $\bar{u}_i(\cdot)$.

We say that coalition $S_k \in \mathcal{P}$ has *the same deviation instant* under n-tuple $\bar{u}(\cdot)$ if $\bar{t}_i(\bar{u}(\cdot) \,\|\, u_i(\cdot))$ is the same for every $i \in S_k$. We shall write $\bar{t}(\bar{u}(\cdot) \,\|\, u_{S_k}(\cdot))$ to denote $\bar{t}_i(\bar{u}(\cdot) \,\|\, u_i(\cdot))$ if S_k has the same deviation instant. We say that $\mathcal{S} \subset \mathcal{P}$ has *the same deviation instant* if $\bar{t}(\bar{u}(\cdot) \,\|\, u_{S_k}(\cdot))$ is the same for every $S_k \in \mathcal{S}$.

Suppose that every $S_k \in \mathcal{P}$ has the same deviation instant. An m-tuple $u^*(\cdot) = (u^*_{S_1}(\cdot), \ldots, u^*_{s_m}(\cdot))$ is an *ε-Nash equilibrium* of $\Gamma^{\mathcal{P}}_{\alpha}(x_0, T - t_0)$ if and only if

$$H^{\alpha}_{S_k}(x_0, T - t_0; u^*(\cdot)) \geqslant H^{\alpha}_{S_k}(x_0, T - t_0; u^*(\cdot) \,\|\, u_{S_k}(\cdot)) - \varepsilon, \tag{2.16}$$

for all $S_k \in \mathcal{P}$ and all u_{S_k}.

Suppose that every $\mathcal{S} \subset \mathcal{P}$ has the same deviation instant. An m-tuple $u^*(\cdot) = (u^*_{S_1}(\cdot), \ldots, u^*_{S_m}(\cdot))$ is a *strong ε-Nash equilibrium* of $\Gamma^{\mathcal{P}}_{\alpha}(x_0, T - t_0)$ if and only if

$$\sum_{S_k \in \mathcal{S}} H^{\alpha}_{S_k}(x_0, T - t_0; u^*(\cdot)) \geqslant \sum_{S_k \in \mathcal{S}} H^{\alpha}_{S_k}(x_0, T - t_0; u^*(\cdot) \,\|\, u_{\mathcal{S}}(\cdot)) - \varepsilon, \tag{2.17}$$

for all $\mathcal{S} \subset \mathcal{P}$ and all $u_{\mathcal{S}} = \{u_{S_k}, S_k \in \mathcal{S}\}$.

3 Existence of ε-Nash and strong ε-Nash equilibria in differential games with coalition structures

Theorem 3.1. *Suppose that every $S_k \in \mathcal{P}$ has the same deviation instant. For every $\varepsilon > 0$, the regularized game $\Gamma^{\mathcal{P}}_{\alpha}(x_0, T-t_0)$ has an ε-Nash equilibrium with payoff α.*

Proof. The proof is based on the construction of ε-Nash equilibrium in piecewise open-loop (POL) strategies with memory.

Remind the definition of POL strategies with memory in a differential game. Denote by $\hat{x}(t)$ any admissible trajectory of the system (2.1) on the time interval $[t_0, t], t \in [t_0, T]$. The strategy $u_{S_i}(\cdot)$ of player S_i is called POL if it consists of the pair (a, σ), where σ is a partition of time interval $[t_0, T]$: $t_0 < t_1 < \cdots < t_s = T, t_{k+1} - t_k = \delta > 0, k = 0, 1, 2, \ldots, s-1$, and a is a map which corresponds an open-loop control $u_{S_i}(t), t \in [t_k, t_{k+1})$ for each point $(\hat{x}(t_k), t_k), t_k \in \sigma$.

Consider POL strategies $\bar{u}(\cdot) = (\bar{a}, \sigma)$, where $\bar{a}$ maps each point $(\bar{x}(t_k), t_k)$ on the optimal cooperative trajectory to an open-loop control $\bar{u}_{S_i}(t), t \in [t_k, t_{k+1})$ satisfying (2.2) and $\bar{a}$ is arbitrary at other points.

Consider a family of zero-sum games $\Gamma^{\mathcal{P}}_{\{S_i\}, \mathcal{P}\backslash\{S_i\}}(x, T-t)$ from the initial position x and duration $T-t$ between coalition $\mathcal{S}$ consisting of a single player S_i and coalition $\mathcal{P}\backslash\{S_i\}$. The payoff of coalition $\{S_i\}$ is equal to $H_{S_i}(x, T-t; u_{S_1}(t), \ldots, u_{S_m}(t))$ and the payoff of coalition $\mathcal{P}\backslash\{S_i\}$ is equal to $(-H_{S_i})$. Let $\hat{u}(x, t; \cdot)$ be the $\frac{\varepsilon}{2}$-optimal POL strategy of player $\mathcal{P}\backslash\{S_i\}$ in $\Gamma^{\mathcal{P}}_{\{S_i\}, \mathcal{P}\backslash\{S_i\}}(x, T-t)$. Note that $\hat{u}(x, t; \cdot) = \{u_{S_j}, S_j \in \mathcal{P}\backslash\{S_i\}\}$.

Let $\hat{x}(t) = \{\hat{x}_{S_1}(t), \ldots, \hat{x}_{S_m}(t)\}$ be the segment of an admissible trajectory satisfying (2.1) on time interval $[t_0, t], t \in [t_0, T]$. For each $S_i \in \mathcal{P}$ define $\bar{t}(S_i) = \inf\{t' : \hat{x}_{S_i}(\tau) \neq \bar{x}_{S_i}(\tau), \tau \in (t', T]\}$ and $\bar{\bar{t}}(S_j) = \min_{S_i} \bar{t}(S_i) = \bar{t}(S_j)$. $\bar{\bar{t}}(S_j)$ lies in one of the intervals $[t_k, t_{k+1}), k = 0, 1, \ldots, s-1$, and $\bar{\bar{t}}(S_j) - t_0$ is the length of the time interval starting from t_0 on which $\hat{x}(t)$ coincides with the cooperative trajectory $\bar{x}(t)$.

Define the following strategies of coalition $S_i \in \mathcal{P}$:

$$
u^*_{S_i}(\cdot) = \begin{cases} \bar{u}_{S_i}(t), & \text{for } (\hat{x}(t_k), t_k) \text{ on the optimal cooperative trajectory,} \\ \hat{u}_{S_i}(\hat{x}(t_{k+1}), t_{k+1}; \cdot), & S_i\text{-th component of the } \frac{\varepsilon}{2}\text{-optimal POL strategy of coalition} \\ & \mathcal{P}\backslash\{S_j\} \text{ in game } \Gamma^{\mathcal{P}}_{\{S_j\}, \mathcal{P}\backslash\{S_j\}}(\hat{x}(t_{k+1}), T-t_{k+1}), \\ & \text{if } t_k \leqslant \bar{\bar{t}}(S_j) < t_{k+1}, \\ \text{arbitrary}, & \text{for all the other positions.} \end{cases}
$$

We shall show that $u^*(\cdot) = (u^*_{S_1}(\cdot), \ldots, u^*_{S_m}(\cdot))$ is an ε-Nash equilibrium in $\Gamma^{\mathcal{P}}_{\alpha}(x_0, T-t_0)$. We have to show that

$$
H^{\alpha}_{S_i}(x_0, T-t_0; u^*(\cdot)) \geqslant H^{\alpha}_{S_i}(x_0, T-t_0; u^*(\cdot) \parallel u_{S_i}(\cdot)) - \varepsilon, \tag{3.1}
$$

for all $S_i \in \mathcal{P}$ and all u_{S_i}.

It is easy to see that when the m-tuple $u^*(\cdot)$ is played, the game develops along the optimal trajectory $\bar{x}(t)$. If under $u^*(\cdot) \parallel u_{S_i}(\cdot)$ the trajectory $\bar{x}(t)$ is also realized then (3.1) will be true.

Now suppose that under $u^*(\cdot) \parallel u_{S_i}(\cdot)$ the trajectory $x(t)$ is different from $\bar{x}(t)$. Suppose $\bar{t}(S_i) \in [t_k, t_{k+1})$. Since the motion of coalitions are independent, we get $x_{S_j}(t_{k+1}) = \bar{x}_{S_j}(t_{k+1})$ for $S_j \in \mathcal{P}\backslash\{S_i\}$. From the definition of $u^*(\cdot)$ it follows that the coalitions in $\mathcal{P}\backslash\{S_i\}$ will use their strategies $\hat{u}_{S_j}(x(t_{k+1}), t_{k+1}; \cdot)$ and coalition S_i starting from position $(x(t_{k+1}), t_{k+1})$ will get no more than

$$
V(x(t_{k+1}), T-t_{k+1}; \{S_i\}) + \frac{\varepsilon}{2},
$$

where $V(x(t_{k+1}), T-t_{k+1}; \{S_i\})$ is the value of game $\Gamma^{\mathcal{P}}_{\{S_i\}, \mathcal{P}\backslash\{S_i\}}(x(t_{k+1}), T-t_{k+1})$.

By choosing $\delta = t_{k+1} - t_k$ sufficiently small, one can achieve that integral $\int_{t_k}^{t_{k+1}} h_{S_i}(x(\tau))d\tau$ will be small (less than $\frac{\varepsilon}{4}$). Then the total payoff $H^{\alpha}_{S_i}(x_0, T-t_0; u^*(\cdot) \parallel u_{S_i}(\cdot))$ of coalition S_i in game $\Gamma^{\mathcal{P}}_{\alpha}(x_0, T-t_0)$ when the m-tuple of strategies $u^*(\cdot) \parallel u_{S_i}(\cdot)$ is played cannot exceed the amount

$$
\int_{t_0}^{t_k} \beta_{S_i}(\tau)d\tau + \int_{t_k}^{t_{k+1}} h_{S_i}(x(\tau))d\tau + V(x(t_{k+1}), T-t_{k+1}; \{S_i\}) + \frac{\varepsilon}{2}
$$

$$\leqslant \int_{t_0}^{t_k} \beta_{S_i}(\tau)d\tau + V(x(t_{k+1}), T - t_{k+1}; \{S_i\}) + \frac{3\varepsilon}{4}. \tag{3.2}$$

When the m-tuple $u^*(\cdot)$ is played, the payoff $H^{\alpha}_{S_i}(x_0, T - t_0; u^*(\cdot))$ of coalition S_i is equal to

$$\alpha_{S_i} = \int_{t_0}^{T} \beta_{S_i}(\tau)d\tau = \int_{t_0}^{t_k} \beta_{S_i}(\tau)d\tau + \alpha_{S_i}(t_k).$$

But $\alpha_{S_i}(t_k) \in L^{\mathcal{P}}(\bar{x}(t_k), T - t_k)$, then we get

$$\alpha_{S_i}(t_k) \geqslant V(\bar{x}(t_k), T - t_k; \{S_i\}).$$

From the continuity of function V and trajectory $x(t)$ by appropriate choice of $\delta = t_{k+1} - t_k$ the following inequality can be guaranteed:

$$V(\bar{x}(t_k), T - t_k; \{S_i\}) \geqslant V(x(t_{k+1}), T - t_{k+1}; \{S_i\}) - \frac{\varepsilon}{4}.$$

So $H^{\alpha}_{S_i}(x_0, T - t_0; u^*(\cdot))$ will be no less than

$$\int_{t_0}^{t_k} \beta_{S_i}(\tau)d\tau + V(x(t_{k+1}), T - t_{k+1}; \{S_i\}) - \frac{\varepsilon}{4}. \tag{3.3}$$

Combining (3.2) and (3.3), we finish the proof of Theorem 3.1. □

This means that the cooperative solution (any imputation) can be strategically supported in a regularized game $\Gamma^{\mathcal{P}}_{\alpha}(x_0, T - t_0)$ by a specially constructed ε-Nash equilibrium.

Theorem 3.2. *Suppose that every $\mathcal{S} \subset \mathcal{P}$ has the same deviation instant. For every $\varepsilon > 0$, the strictly regularized game $\Gamma^{\mathcal{P}}_{\alpha}(x_0, T - t_0)$ has a strong ε-Nash equilibrium with payoff α.*

Proof. The proof is based on the construction of strong ε-Nash equilibrium in piecewise open-loop (POL) strategies with memory.

Consider POL strategies $\bar{u}(\cdot) = (\bar{a}, \sigma)$, where $\bar{a}$ maps each point $(\bar{x}(t_k), t_k)$ on the optimal cooperative trajectory to an open-loop control $\bar{u}_{S_i}(t), t \in [t_k, t_{k+1}), S_i \in \mathcal{P}$, satisfying (2.2) and $\bar{a}$ is arbitrary at other points.

Consider a family of zero-sum games $\Gamma^{\mathcal{P}}_{\mathcal{S},\mathcal{P}\backslash\mathcal{S}}(x, T - t)$ from the initial position x and duration $T - t$ between coalition $\mathcal{S}$ and coalition $\mathcal{P}\backslash\mathcal{S}$ where the payoff of coalition $\mathcal{S}$ equals $\sum_{S_i \in \mathcal{S}} H_{S_i}(x, T - t; u_{S_1}(t), \dots, u_{S_m}(t))$. Let $\hat{u}_{\mathcal{P}\backslash\mathcal{S}}(x, t; \cdot)$ be the $\frac{\varepsilon}{2}$-optimal POL strategy of coalition $\mathcal{P}\backslash\mathcal{S}$ in $\Gamma^{\mathcal{P}}_{\mathcal{S},\mathcal{P}\backslash\mathcal{S}}(x, T - t)$. Note that $\hat{u}_{\mathcal{P}\backslash\mathcal{S}}(x, t; \cdot) = \{u_{S_j}, S_j \in \mathcal{P}\backslash\mathcal{S}\}$.

Let $\hat{x}(t) = \{\hat{x}_{S_1}(t), \dots, \hat{x}_{S_m}(t)\}$ be the segment of an admissible trajectory satisfying (2.1) on time interval $[t_0, t], t \in [t_0, T]$. Since every $\mathcal{S} \subset \mathcal{P}$ has the same deviation instant, for every $\mathcal{S} \subset \mathcal{P}$ we can define $\bar{t}(\mathcal{S}) = \bar{t}(S_i) = \inf\{t' : \hat{x}_{S_i}(\tau) \neq \bar{x}_{S_i}(\tau), \tau \in (t', T]\}, S_i \in \mathcal{S}$ and $\bar{\bar{t}}(\mathcal{S}) = \min_{\mathcal{S}} \bar{t}(\mathcal{S})$. $\bar{\bar{t}}(\mathcal{S})$ lies in one of the intervals $[t_k, t_{k+1}), k = 0, 1, 2, \dots, s - 1$. And $\bar{\bar{t}}(\mathcal{S}) - t_0$ is the length of the time interval starting from t_0 on which $\hat{x}(t)$ coincides with the cooperative trajectory $\bar{x}(t)$.

Define the following strategies of coalition $S_i \in \mathcal{P}$:

$$u^*_{S_i}(\cdot) = \begin{cases} \bar{u}_{S_i}(t), & \text{for } (\hat{x}(t_k), t_k) \text{ on the optimal cooperative trajectory,} \\ \hat{u}_{S_i}(\hat{x}(t_{k+1}), t_{k+1}; \cdot), & S_i\text{-th component of the } \frac{\varepsilon}{2}\text{-optimal POL strategy of} \\ & \text{coalition } \mathcal{P}\backslash\mathcal{S} \text{ in game } \Gamma^{\mathcal{P}}_{\mathcal{S},\mathcal{P}\backslash\mathcal{S}}(\hat{x}(t_{k+1}), T - t_{k+1}), \\ & \text{if } t_k \leqslant \bar{\bar{t}}(\mathcal{S}) < t_{k+1}, \\ \text{arbitrary}, & \text{for all the other positions.} \end{cases}$$

We shall show that $u^*(\cdot) = (u^*_{S_1}(\cdot), \dots, u^*_{S_m}(\cdot))$ is a strong ε-Nash equilibrium in $\Gamma^{\mathcal{P}}_{\alpha}(x_0, T - t_0)$. We have to show that

$$\sum_{S_i \in \mathcal{S}} H^{\alpha}_{S_i}(x_0, T - t_0; u^*(\cdot)) \geqslant \sum_{S_i \in \mathcal{S}} H^{\alpha}_{S_i}(x_0, T - t_0; u^*(\cdot) \parallel u_{\mathcal{S}}(\cdot)) - \varepsilon, \tag{3.4}$$

for all $\mathcal{S} \subset \mathcal{P}$ and all $u_{\mathcal{S}} = \{u_{S_i}, S_i \in \mathcal{S}\}$.

It is easy to see that when the m-tuple $u^*(\cdot)$ is played, the game develops along the optimal trajectory $\bar{x}(t)$. If under $u^*(\cdot) \parallel u_{\mathcal{S}}(\cdot)$ the trajectory $\bar{x}(t)$ is also realized then (3.4) will be true.

Now suppose that under $u^*(\cdot) \parallel u_{\mathcal{S}}(\cdot)$ the trajectory $x(t)$ is different from $\bar{x}(t)$. Suppose $\bar{t}(\mathcal{S}) \in [t_k, t_{k+1})$. Since the motion of coalitions are independent, we get $x_{S_j}(t_{k+1}) = \bar{x}_{S_j}(t_{k+1})$ for $S_j \in \mathcal{P}\backslash\mathcal{S}$.

From the definition of $u^*(\cdot)$ it follows that coalitions in $\mathcal{P}\backslash\mathcal{S}$ will use their strategies $\hat{u}_{S_j}(x(t_{k+1}), t_{k+1}; \cdot)$ and coalition $\mathcal{S}$ starting from the position $(x(t_{k+1}), t_{k+1})$ will get no more than

$$V(x(t_{k+1}), T - t_{k+1}; \mathcal{S}) + \frac{\varepsilon}{2},$$

where $V(x(t_{k+1}), T - t_{k+1}; \mathcal{S})$ is the value of the game $\Gamma^{\mathcal{P}}_{\mathcal{S},\mathcal{P}\backslash\mathcal{S}}(x(t_{k+1}), T - t_{k+1})$.

By choosing $\delta = t_{k+1} - t_k$ sufficiently small, one can achieve that the integral $\int_{t_k}^{t_{k+1}} \sum_{S_i \in \mathcal{S}} h_{S_i}(x(\tau))d\tau$ will be small (less than $\frac{\varepsilon}{4}$). Then the total payoff $\sum_{S_i \in \mathcal{S}} H^{\alpha}_{S_i}(x_0, T - t_0; u^*(\cdot) \parallel u_{S_i}(\cdot))$ of coalition $\mathcal{S}$ in game $\Gamma^{\mathcal{P}}_{\alpha}(x_0, T - t_0)$ when the m-tuple of strategies $u^*(\cdot) \parallel u_{\mathcal{S}}(\cdot)$ is played cannot exceed the amount

$$\begin{aligned} &\sum_{S_i \in \mathcal{S}} \int_{t_0}^{t_k} \beta_{S_i}(\tau)d\tau + \sum_{S_i \in \mathcal{S}} \int_{t_k}^{t_{k+1}} h_{S_i}(x(\tau))d\tau + V(x(t_{k+1}), T - t_{k+1}; \mathcal{S}) + \frac{\varepsilon}{2} \\ &\quad \leqslant \sum_{S_i \in \mathcal{S}} \int_{t_0}^{t_k} \beta_{S_i}(\tau)d\tau + V(x(t_{k+1}), T - t_{k+1}; \mathcal{S}) + \frac{3\varepsilon}{4}. \end{aligned} \tag{3.5}$$

When the m-tuple $u^*(\cdot)$ is played the payoff $\sum_{S_i \in \mathcal{S}} H^{\alpha}_{S_i}(x_0, T - t_0; u^*(\cdot))$ of coalition $\mathcal{S}$ is equal to

$$\sum_{S_i \in \mathcal{S}} \alpha_{S_i} = \sum_{S_i \in \mathcal{S}} \int_{t_0}^{T} \beta_{S_i}(\tau)d\tau = \sum_{S_i \in \mathcal{S}} \int_{t_0}^{t_k} \beta_{S_i}(\tau)d\tau + \sum_{S_i \in \mathcal{S}} \alpha_{S_i}(t_k).$$

But $\alpha_{S_i}(t_k) \in C^{\mathcal{P}}(\bar{x}(t_k), T - t_k)$, then we get

$$\sum_{S_i \in \mathcal{S}} \alpha_{S_i}(t_k) \geqslant V(\bar{x}(t_k), T - t_k; \mathcal{S}).$$

From the continuity of function V and trajectory $x(t)$ by appropriate choice of $\delta = t_{k+1} - t_k$ the following inequality can be guaranteed:

$$V(\bar{x}(t_k), T - t_k; \mathcal{S}) \geqslant V(x(t_{k+1}), T - t_{k+1}; \mathcal{S}) - \frac{\varepsilon}{4}.$$

So $\sum_{S_i \in \mathcal{S}} H^{\alpha}_{S_i}(x_0, T - t_0; u^*(\cdot))$ will be no less than

$$\sum_{S_i \in \mathcal{S}} \int_{t_0}^{t_k} \beta_{S_i}(\tau)d\tau + V(x(t_{k+1}), T - t_{k+1}; \mathcal{S}) - \frac{\varepsilon}{4}. \tag{3.6}$$

Combining (3.5) and (3.6), we finish the proof of Theorem 3.2. □

4 Discrete time case with perfect information

In what follows we consider the problem of strategic stability in the game with a coalition structure in the extensive form with perfect information. We need to set up standard terminology and notation. But in this case, the assumptions of deviation instant for a coalition disappear.

A game tree is a finite oriented treelike graph K with the root x_0. Let x be some vertex (position). We denote by $K(x)$ a subtree K with the root in x. We denote by $Z(x)$ the immediate successors of x. A vertex y, directly following after x, is called an alternative in x ($y \in Z(x)$). The player who makes a decision in x will be denoted by $i(x)$.

Let $N = \{1, 2, \ldots, n\}$ be the set of all players in the game. *A game in the extensive form with perfect information* (see [14]), $G(x_0)$ is a graph tree $K(x_0)$, with the following additional properties:

• The set of vertices is split up into $n+1$ subsets $P_1, P_2, \dots, P_{n+1}$, which form a partition of the set of all vertices of the graph tree K. The vertices in P_i are called personal positions of player i, $i = 1, 2, \dots, n$; vertices in P_{n+1} are called terminal positions.

• In each vertex x the system of real numbers $h(x) = (h_1(x), \dots, h_n(x))$ is defined, where $h_i(x)$ is interpreted as the stage payoff of player i in the vertex x.

A strategy of player i is a mapping $F_i(\cdot)$, which associates to each position $x \in P_i$ a unique alternative $y \in Z(x)$. Denote by $H_i(x, ; u_1(\cdot), \dots, u_n(\cdot))$ the payoff function of player $i \in N$ in the subgame $G(x)$ starting from the position x. $H_i(x, ; u_1(\cdot), \dots, u_n(\cdot)) = \sum_{k=1}^{\nu} h_i(x'_k)$, where $(x'_1, x'_2, \dots, x'_\nu)$ is the path realized in the subgame $G(x)$, when the n-tuple of strategies $(u_1(\cdot), \dots, u_n(\cdot))$ is played, $x'_1 = x$.

Denote by $\bar{u}(\cdot) = (\bar{u}_1(\cdot), \dots, \bar{u}_n(\cdot))$ the n-tuple of strategies and the path $\bar{x} = (\bar{x}_0, \bar{x}_1, \dots, \bar{x}_l), \bar{x}_l \in P_{n+1}$ such that

$$\max_{u_1(\cdot), \dots, u_n(\cdot)} \sum_{i=1}^{n} H_i(x_0; u_1(\cdot), \dots, u_n(\cdot)) = \sum_{i=1}^{n} H_i(x_0; \bar{u}_1(\cdot), \dots, \bar{u}_n(\cdot)) = \sum_{i=1}^{n} \sum_{k=0}^{l} h_i(\bar{x}_k). \tag{4.1}$$

A path $\bar{x} = (\bar{x}_0, \bar{x}_1, \dots, \bar{x}_l)$ satisfying (4.1) we shall call an *optimal cooperative path.*

Define the *characteristic function* in $G(x_0)$ in a classical way:

$$\begin{aligned} V(x_0; N) &= \sum_{i=1}^{n} \sum_{k=0}^{l} h_i(\bar{x}_k), \\ V(x_0; \emptyset) &= 0, \\ V(x_0; S) &= \mathrm{Val} G_{S, N\backslash S}(x_0), \end{aligned} \tag{4.2}$$

where $\mathrm{Val} G_{S,N\backslash S}(x_0)$ is the value of the zero-sum game played between coalition S acting as player 1 and coalition $N\backslash S$ acting as player 2 where the payoff of coalition S equals $\sum_{i \in S} H_i(x_0; u_1(\cdot), \dots, u_n(\cdot))$. In the special case when $S = \{i\}$, $V(x_0; \{i\})$ is the value of zero-sum game between player i and coalition $N\backslash\{i\}$.

Define $L(x_0)$ as the *imputation set* in game $G(x_0)$:

$$L(x_0) = \left\{ \alpha = (\alpha_1, \dots, \alpha_n) : \alpha_i \geqslant V(x_0; \{i\}), \sum_{i \in N} \alpha_i = V(x_0; N) \right\}. \tag{4.3}$$

Define $C(x_0)$ as the *core* in game $G(x_0)$:

$$C(x_0) = \left\{ \alpha = (\alpha_1, \dots, \alpha_n) : \sum_{i \in S} \alpha_i \geqslant V(x_0; S), \forall\, S \subset N, \sum_{i \in N} \alpha_i = V(x_0; N) \right\}. \tag{4.4}$$

Let $\alpha \in L(x_0)$. Define the *imputation distribution procedure* (IDP) as a function $\beta(k) = (\beta_1(k), \dots, \beta_n(k)), k = 0, 1, \dots, l$, such that

$$\alpha_i = \sum_{k=0}^{l} \beta_i(k). \tag{4.5}$$

For every $\alpha \in L(x_0)$, we define the noncooperative game $G_\alpha(x_0)$, which differs from game $G(x_0)$ only by payoffs defined along the optimal cooperative path $\bar{x} = (\bar{x}_0, \bar{x}_1, \dots, \bar{x}_l)$. Suppose under the strategy profile $(u_1(\cdot), \dots, u_n(\cdot))$, the path $(x_0, \dots, x_{l'})$ is realized. Denote the payoff function in game $G_\alpha(x_0)$ by

$$H_i^\alpha(x_0; u_1(\cdot), \dots, u_n(\cdot)) = \sum_{k=0}^{r-1} \beta_i(k) + \sum_{k=r}^{l'} h_i(x_k),$$

where $r = \min\{k : x_k \neq \bar{x}_k, k = 0, 1, \dots, l'\}$.

Consider subgames $G(\bar{x}_k)$ and imputation sets $L(\bar{x}_k)$. Let $\alpha(k) \in L(\bar{x}_k)$. Game $G_\alpha(x_0)$ is called a *regularized game* of $G(x_0)$ (*α-regularization*) if IDP β is defined in such a way that

$$\alpha_i(k) = \sum_{j=k}^{l} \beta_i(j),$$

or

$$\beta_i(k) = \alpha_i(k) - \alpha_i(k+1), \quad k = 0, 1, \ldots, l-1, \quad \beta_i(l) = \alpha_i(l), \quad \alpha_i(0) = \alpha_i. \tag{4.6}$$

In particular, if $\alpha(k) \in C(\bar{x}_k)$, $G_\alpha(x_0)$ is called a *strictly regularized game* of $G(x_0)$.

From (4.6), we get

$$\alpha_i = \sum_{j=0}^{k-1} \beta_i(j) + \alpha_i(k). \tag{4.7}$$

Now suppose that $M(x_0) \subset L(x_0)$ is some optimality principle in the cooperative version of game $G(x_0)$, and $M(\bar{x}_k) \subset L(\bar{x}_k)$ is the same optimality principle defined in the subgame $L(\bar{x}_k)$ with initial conditions on the optimal path. If $\alpha \in M(x_0)$ and $\alpha(k) \in M(\bar{x}_k)$, (4.7) gives us the *time consistency* of the chosen imputation α or the chosen optimality principle in game $G(x_0)$.

Let $\mathcal{P} = \{S_1, \ldots, S_m\}$ be a partition of the player set N. Suppose that each player $i \in N$ is playing in the interests of coalition $S_k \in \mathcal{P}$ which he belongs to, trying to maximize the sum of payoffs of coalition members, i.e.,

$$\max_{u_i, i \in S_k} \sum_{i \in S_k} H_i(x_0; u_1(\cdot), \ldots, u_n(\cdot)). \tag{4.8}$$

Define $u_{S_k} = \{u_i, i \in S_k\}$ as the strategy of coalition S_k. Write

$$H_{S_k}(x_0; u_{S_1}(\cdot), \ldots, u_{S_m}(\cdot)) = \sum_{i \in S_k} H_i(x_0; u_1(\cdot), \ldots, u_n(\cdot))$$

as the payoff of coalition S_k.

Suppose that coalitions in $\mathcal{P}$ are playing cooperatively with objective (4.1). We call the above game as the *cooperative game in the extensive form with a coalition structure*, denoted by $G^{\mathcal{P}}(x_0)$.

Suppose that there exist an n-tuple of strategies $\bar{u}(\cdot) = (\bar{u}_1(\cdot), \ldots, \bar{u}_n(\cdot))$ and the path $\bar{x} = (\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_l), \bar{x}_l \in P_{n+1}$ satisfying (4.1). Then the path $\bar{x}$ is an optimal cooperative path of $G(x_0)$. We define $\bar{x}$ as an *optimal cooperative path* of $G^{\mathcal{P}}(x_0)$ at the same time.

The *characteristic function* in $G^{\mathcal{P}}(x_0, T - t_0)$ is defined by:

$$\begin{aligned} V(x_0; \mathcal{P}) &= \sum_{i=1}^{n} \sum_{k=0}^{l} h_i(\bar{x}_k), \\ V(x_0; \emptyset) &= 0, \\ V(x_0; \mathcal{S}) &= \mathrm{Val} G_{\mathcal{S}, \mathcal{P} \backslash \mathcal{S}}(x_0), \end{aligned} \tag{4.9}$$

where $\mathrm{Val} G_{\mathcal{S}, \mathcal{P} \backslash \mathcal{S}}(x_0)$ is the value of the zero-sum game played between coalition $\mathcal{S}$ acting as player 1 and coalition $\mathcal{P} \backslash \mathcal{S}$ acting as player 2 where the payoff of coalition $\mathcal{S}$ equals $\sum_{S_k \in \mathcal{S}} H_{S_k}(x_0; u_{S_1}(\cdot), \ldots, u_{S_m}(\cdot))$.

Define $L^{\mathcal{P}}(x_0)$ as the *imputation set* in game $G^{\mathcal{P}}(x_0)$:

$$L^{\mathcal{P}}(x_0) = \left\{ \alpha = (\alpha_{S_1}, \ldots, \alpha_{S_m}) : \alpha_{S_k} \geqslant V(x_0; \{S_k\}), \sum_{S_k \in \mathcal{P}} \alpha_{S_k} = V(x_0; \mathcal{P}) \right\}. \tag{4.10}$$

Define $C^{\mathcal{P}}(x_0)$ as the *core* in game $G^{\mathcal{P}}(x_0)$:

$$C^{\mathcal{P}}(x_0) = \left\{ \alpha = (\alpha_{S_1}, \ldots, \alpha_{S_m}) : \sum_{S_k \in \mathcal{S}} \alpha_{S_k} \geqslant V(x_0; \mathcal{S}), \forall \mathcal{S} \subset \mathcal{P}, \sum_{S_k \in \mathcal{P}} \alpha_{S_k} = V(x_0; \mathcal{P}) \right\}. \tag{4.11}$$

Let $\alpha \in L^{\mathcal{P}}(x_0)$. Define the *imputation distribution procedure* (IDP) of $G^{\mathcal{P}}(x_0)$ as a function $\beta(j) = (\beta_{S_1}(j), \dots, \beta_{S_m}(j)), j = 0, 1, \dots, l$, such that

$$\alpha_{S_k} = \sum_{j=0}^{l} \beta_{S_k}(j), \quad S_k \in \mathcal{P}. \tag{4.12}$$

For every $\alpha \in L(x_0)$, we define the noncooperative game $G_\alpha^{\mathcal{P}}(x_0)$, which differs from game $G^{\mathcal{P}}(x_0)$ only by payoffs defined along the optimal cooperative path $\bar{x} = (\bar{x}_0, \bar{x}_1, \dots, \bar{x}_l)$.

Suppose under the strategy profile $(u_{S_1}(\cdot), \dots, u_{S_m}(\cdot))$ the path $(x_0, \dots, x_{l'})$ is realized. Denote the payoff function in game $G_\alpha^{\mathcal{P}}(x_0)$ by

$$H_{S_k}^\alpha(x_0; u_{S_1}(\cdot), \dots, u_{S_m}(\cdot)) = \sum_{j=0}^{r-1} \beta_{S_k}(j) + \sum_{j=r}^{l'} h_{S_k}(x_j),$$

where $r = \min\{j : x_j \neq \bar{x}_j, j = 0, 1, \dots, l'\}$ and $h_{S_k}(x_j) = \sum_{i \in S_k} h_i(x_j)$.

Consider subgames $G^{\mathcal{P}}(\bar{x}_k)$ and imputation sets $L^{\mathcal{P}}(\bar{x}_k)$. Let $\alpha(k) \in L^{\mathcal{P}}(\bar{x}_k)$. Game $G_\alpha^{\mathcal{P}}(x_0)$ is called a *regularized game* of $G^{\mathcal{P}}(x_0)$ (α*-regularization*) if IDP β is defined in such a way that

$$\alpha_{S_i}(k) = \sum_{j=k}^{l} \beta_{S_i}(j),$$

or

$$\beta_{S_i}(k) = \alpha_{S_i}(k) - \alpha_{S_i}(k+1), \quad k = 0, 1, \dots, l-1, \quad \beta_{S_i}(l) = \alpha_{S_i}(l), \quad \alpha_{S_i}(0) = \alpha_{S_i}. \tag{4.13}$$

In particular, if $\alpha(k) \in C^{\mathcal{P}}(\bar{x}_k)$, $G_\alpha^{\mathcal{P}}(x_0)$ is called a *strictly regularized game* of $G^{\mathcal{P}}(x_0)$.

From (4.13), we get

$$\alpha_{S_i} = \sum_{j=0}^{k-1} \beta_{S_i}(j) + \alpha_{S_i}(k), \quad S_i \in \mathcal{P}. \tag{4.14}$$

Now suppose that $M^{\mathcal{P}}(x_0) \subset L^{\mathcal{P}}(x_0)$ is some optimality principle in the cooperative version of game $G^{\mathcal{P}}(x_0)$, and $M^{\mathcal{P}}(\bar{x}_k) \subset L^{\mathcal{P}}(\bar{x}_k)$ is the same optimality principle defined in the subgame $L^{\mathcal{P}}(\bar{x}_k)$ with initial conditions on the optimal path. If $\alpha \in M^{\mathcal{P}}(x_0)$ and $\alpha(k) \in M^{\mathcal{P}}(\bar{x}_k)$, (4.14) gives us the *time consistency* of the chosen imputation α or the chosen optimality principle in game $G^{\mathcal{P}}(x_0)$.

In the game in extensive form with a coalition structure, the deviation of strategies of different members in a coalition can bring the changing of the path directly. To define Nash or strong Nash equilibria of $G_\alpha^{\mathcal{P}}(x_0)$, we do not need the assumptions of deviation instant for a coalition.

An m-tuple $u^*(\cdot) = (u_{S_1}^*(\cdot), \dots, u_{s_m}^*(\cdot))$ is a *Nash equilibrium* of $G_\alpha^{\mathcal{P}}(x_0)$ if and only if

$$H_{S_k}^\alpha(x_0; u^*(\cdot)) \geqslant H_{S_k}^\alpha(x_0; u^*(\cdot) \parallel u_{S_k}(\cdot)), \tag{4.15}$$

for all $S_k \in \mathcal{P}$ and all u_{S_k}.

An m-tuple $u^*(\cdot) = (u_{S_1}^*(\cdot), \dots, u_{S_m}^*(\cdot))$ is a *strong Nash equilibrium* of $G_\alpha^{\mathcal{P}}(x_0)$ if and only if

$$\sum_{S_k \in \mathcal{S}} H_{S_k}^\alpha(x_0; u^*(\cdot)) \geqslant \sum_{S_k \in \mathcal{S}} H_{S_k}^\alpha(x_0; u^*(\cdot) \parallel u_{\mathcal{S}}(\cdot)), \tag{4.16}$$

for all $\mathcal{S} \subset \mathcal{P}$ and all $u_{\mathcal{S}} = \{u_{S_k}, S_k \in \mathcal{S}\}$.

Theorem 4.1. *The regularized game $G_\alpha^{\mathcal{P}}(x_0)$ has a Nash equilibrium with payoff α.*

Proof. Since $\alpha(k) \in L^{\mathcal{P}}(\bar{x}_k)$ in game $G^{\mathcal{P}}(\bar{x}_k)$, along the optimal cooperative path we have

$$\alpha_{S_i}(k) \geqslant V(\bar{x}_k; \{S_i\}), \quad S_i \in \mathcal{P}, \quad k = 0, 1, \dots, l.$$

At the same time $\alpha_{S_i}(k) = \sum_{j=k}^{l} \beta_{S_i}(j)$, and we get

$$\sum_{j=k}^{l} \beta_{S_i}(j) \geqslant V(\bar{x}_k; \{S_i\}). \tag{4.17}$$

But $\sum_{j=k}^{l} \beta_{S_i}(j)$ is the payoff of coalition S_i in the subgame $G_\alpha^{\mathcal{P}}(\bar{x}_k)$ along the optimal cooperative path, and from (4.17) using the arguments similar to those in the proof of Theorem 3.1, one can construct a Nash equilibrium with payoff α, which leads to the optimal cooperative path $\bar{x}$. □

Theorem 4.2. *The strictly regularized game $G_\alpha^{\mathcal{P}}(x_0)$ has a strong Nash equilibrium with payoff α.*

Proof. Since $\alpha(k) \in C^{\mathcal{P}}(\bar{x}_k)$ in game $G^{\mathcal{P}}(\bar{x}_k)$, along the optimal cooperative path we have

$$\sum_{S_i \in \mathcal{S}} \alpha_{S_i}(k) \geqslant V(\bar{x}_k; \mathcal{S}), \quad \forall\, \mathcal{S} \subset \mathcal{P}, \quad k = 0, 1, \ldots, l.$$

At the same time $\sum_{S_i \in \mathcal{S}} \alpha_{S_i}(k) = \sum_{S_i \in \mathcal{S}} \sum_{j=k}^{l} \beta_{S_i}(j)$, and we get

$$\sum_{S_i \in \mathcal{S}} \sum_{j=k}^{l} \beta_{S_i}(j) \geqslant V(\bar{x}_k; \mathcal{S}). \tag{4.18}$$

But $\sum_{S_i \in \mathcal{S}} \sum_{j=k}^{l} \beta_{S_i}(j)$ is the payoff of coalition $\mathcal{S}$ in the subgame $G_\alpha^{\mathcal{P}}(\bar{x}_k)$ along the optimal cooperative path, and from (4.18) using the arguments similar to those in the proof of Theorem 3.2, one can construct a strong Nash equilibrium with payoff α, which leads to the optimal cooperative path $\bar{x}$. □

It should be noticed that if every $\mathcal{S} \subset \mathcal{P}$ has the same deviation instant, a strong ε-Nash equilibrium is also an ε-Nash equilibrium in the strictly regularized game $\Gamma_\alpha^{\mathcal{P}}(x_0, T - t_0)$. So the existence of strong ε-Nash equilibrium implies the existence of ε-Nash equilibrium in $\Gamma_\alpha^{\mathcal{P}}(x_0, T - t_0)$. And if every $S \subset N$ has the same deviation instant, we can easily construct a strong ε-Nash equilibrium in the strictly regularized game $\Gamma_\alpha^{\mathcal{P}}(x_0, T - t_0)$ from a strong ε-Nash equilibrium in the strictly regularized game $\Gamma_\alpha(x_0, T - t_0)$ (see [27]). So the existence of strong ε-Nash equilibrium in the strictly regularized game $\Gamma_\alpha(x_0, T - t_0)$ implies the existence of strong ε-Nash equilibrium in the strictly regularized game $\Gamma_\alpha^{\mathcal{P}}(x_0, T - t_0)$. While in the discrete time case with perfect information, without the assumptions of deviation instant for a coalition we can get the similar conclusions.

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