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Acyclic colorings of graphs with bounded degree

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Abstract A k-coloring (not necessarily proper) of vertices of a graph is called *acyclic*, if for every pair of distinct colors i and j the subgraph induced by the edges whose endpoints have colors i and j is acyclic. We consider some generalized acyclic k-colorings, namely, we require that each color class induces an acyclic or bounded degree graph. Mainly we focus on graphs with maximum degree 5. We prove that any such graph has an acyclic 5-coloring such that each color class induces an acyclic graph with maximum degree at most 4. We prove that the problem of deciding whether a graph G has an acyclic 2-coloring in which each color class induces a graph with maximum degree at most 3 is NP-complete, even for graphs with maximum degree 5. We also give a linear-time algorithm for an acyclic t-improper coloring of any graph with maximum degree d assuming that the number of colors is large enough.

Keywords acyclic coloring, bounded degree graph, computational complexity

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1 Introduction

We consider only finite, simple graphs. We use standard notation. For a graph G, we denote its vertex set and edge set by $V(G)$ and $E(G)$, respectively. Let $v \in V(G)$. By $N_G(v)$ (or $N(v)$) we denote the set of the neighbours of v in G. The cardinality of $N_G(v)$ is called the *degree* of v, denoted by $d_G(v)$ (or $d(v)$). The maximum and minimum vertex degrees in G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. For undefined concepts, we refer the reader to [22].

A k*-coloring* of a graph G is a mapping c from the set of vertices of G to the set $\{1, 2, \ldots, k\}$ of colors. We can also regard a k-coloring of G as a partition of the set $V(G)$ into *color classes* $V_1, V_2, \ldots, V_k$ such that each V_i is the set of vertices with color i. In many situations, it is desired that the subgraph induced by each set V_i has a given property. For example, requiring that each set V_i is independent defines a proper k-coloring. If each set V_i induces a graph with a given property we obtain a generalised coloring. One can also require that for any pair of distinct colors i and j, the subgraph induced by the edges whose endpoints have colors i and j satisfies a given property, for example, is acyclic. This yields to the concept of acyclic coloring. In this paper, we concentrate on such generalised acyclic colorings, in which each color class induces an acyclic graph or a graph with bounded degree. Below we state precisely this notion. The terminology and notation concerning generalised colorings, which we follow here, can be found in [6], while the concept of generalised acyclic colorings was introduced in [4].

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Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k$ be nonempty classes of graphs closed with respect to isomorphism. A k-coloring of a graph G is called a $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-*coloring* of G if for each $i \in \{1, 2, \ldots, k\}$ the subgraph induced in G by the color class V_i belongs to $\mathcal{P}_i$. Such a coloring is called an *acyclic* $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-*coloring* if for every two distinct colors i and j the subgraph induced by the edges whose endpoints have colors i and j is acyclic. In other words, every bichromatic cycle in G contains at least one monochromatic edge. Throughout this paper, we use the following notation: $\mathcal{S}_d$ for the class of graphs with maximum degree at most d, and $\mathcal{D}_1$ for the class of acyclic graphs. For convenience, an acyclic $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-coloring, where $\mathcal{P}_i = \mathcal{P}$, for each $i \in \{1, 2, \ldots, k\}$, is referred to as an acyclic $\mathcal{P}^{(k)}$-coloring.

With this notation, an acyclic k-coloring of a graph G corresponds to an acyclic $\mathcal{P}^{(k)}$-coloring of G, where the class $\mathcal{P}$ is the set of all edgeless graphs. We use $\chi_a(G)$ to denote the acyclic chromatic number. An acyclic $\mathcal{P}^{(k)}$-coloring of G such that $\mathcal{P} = \mathcal{S}_d$ is called an *acyclic d-improper k-coloring*.

The notion of acyclic coloring of graphs was introduced in 1973 by Grűnbaum [17] and has been widely considered in the recent past. Even more attention has been paid to this problem since it was proved by Coleman and Cai [12], and Coleman and Moré [13] that acyclic colorings can be used in computing Hessian matrices via the substitution method (see also [16]).

However, determining $\chi_a(G)$ is quite difficult. Kostochka [19] proved that it is an NP-complete problem to decide for a given arbitrary graph G whether $\chi_a(G) \leqslant 3$. The acyclic chromatic number was determined for several families of graphs. In particular, this parameter was studied intensively for the family of graphs with fixed maximum degree. Using the probabilistic method, Alon et al. [3] showed that any graph of maximum degree Δ can be acyclically colored using $O(\Delta^{4/3})$ colors. A t-improper analogues of this result was obtained by Addario-Berry et al. [1]. They proved that any graph of maximum degree Δ has an acyclic t-improper coloring with $O(\Delta \ln \Delta + (\Delta - t)\Delta)$ colors.

Focusing on the family of graphs with small maximum degree, it was shown in [17] that $\chi_a(G) \leqslant 4$ for any graph with maximum degree 3 (see also [21]). Burstein [11] proved that $\chi_a(G) \leqslant 5$ for any graph with maximum degree 4. Recently, Kostochka and Stocker [20] proved that $\chi_a(G) \leqslant 7$ for any graph with maximum degree 5. For graphs with maximum degree 6, Hocquard [18] proved that 11 colors are enough for an acyclic coloring.

In 1999, Boiron et al. [5] began with the study on the problem of acyclic $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-colorings of outerplanar and planar graphs, and bounded degree graphs [4]. In particular, they proved that any graph $G \in \mathcal{S}_3$ has an acyclic $(\mathcal{D}_1, \mathcal{S}_2)$-coloring as well as an acyclic $\mathcal{S}_1^{(3)}$-coloring [4]. Addario-Berry et al. [2] proved that each graph from $\mathcal{S}_3$ has an acyclic $(\mathcal{S}_2, \mathcal{S}_2)$-coloring. This theorem was also proved in [9], where a polynomial-time algorithm was presented. In [10], a polynomial-time algorithm that provides an acyclic $(\mathcal{D}_1, \mathcal{LF})$-coloring of any graph from $\mathcal{S}_3 \setminus \{K_4, K_{3,3}\}$ was given ($\mathcal{LF}$ is the set of acyclic graphs with maximum degree at most 2). Related problems concerning the class of graphs with maximum degree at most 4 are considered in [14], where it was proved that any graph from $\mathcal{S}_4$ has an acyclic $\mathcal{S}_3^{(3)}$-coloring, as well as an acyclic $(\mathcal{S}_3 \cap \mathcal{D}_1)^{(4)}$-coloring. In the present paper we continue the previous work and consider acyclic colorings of graphs with maximum degree at most 5. We prove that each graph $G \in \mathcal{S}_5$ has an acyclic $\mathcal{D}_1^{(5)}$-coloring. The number of colors in this theorem cannot be reduced, since the complete graph K_6 needs at least 5 colors in any such coloring. Next, we slightly improve this result, by proving that any graph from $\mathcal{S}_5$ can be acyclically colored with 5 colors in such a way that each color class induces an acyclic graph with maximum degree at most 4. We also show that the problem of deciding whether a given graph $G \in \mathcal{S}_5$ is acyclically $(\mathcal{S}_3, \mathcal{S}_3)$-colorable is NP-complete. We finish the paper with a general result giving linear-time algorithms for acyclic t-improper colorings of graphs with maximum degree d assuming that the number of colors is large enough with respect to d.

The following definitions and notation, which will be used later in the proofs, deal with a *partial k-coloring* of a graph G, defined as an assignment c of colors from the set $\{1, 2, \ldots, k\}$ to a subset C of $V(G)$. Given a partial k-coloring c of G, the set C is the set of colored vertices. For a vertex v let $n_v = |C \cap N(v)|$ and $p_v = |\bigcup_{u \in N(v) \cap C} c(u)|$. Clearly, $p_v \leqslant n_v$. Let C_v denote the multiset of colors assigned by c to the colored neighbors of v. For $S \subseteq V(G)$ let $C(S) = \bigcup_{v \in S} C_v$. A vertex v is called *rainbow*, if all its colored neighbours have distinct colors.

Let c be a partial k-coloring of G and i, j be distinct colors. A bichromatic cycle (resp. path) having no monochromatic edge is called an *alternating cycle* (resp. *path*). An (i,j)-alternating cycle (path) is an alternating cycle (path) with each vertex colored i or j. Let F be a cycle in G containing v. Then F is called (i,j)-*dangerous for* v, if coloring v with i results in an (i,j)-alternating cycle. F is called i-*mono-dangerous* for v, if coloring v with i results in a monochromatic cycle containing v. When it is convenient, all (i,j)-dangerous cycles and k-mono-dangerous cycles for v will be called simply *dangerous* cycles for v.

2 Acyclic colorings such that each color class induces an acyclic graph

First we show that any graph from $\mathcal{S}_5$ has an acyclic $\mathcal{D}_1^{(5)}$-coloring. We start with the following auxiliary lemma.

Lemma 2.1. *Let $G \in \mathcal{S}_5$ and c be a partial acyclic $\mathcal{D}_1^{(5)}$-coloring of G. Assume v is an uncolored vertex. If $n_v \leqslant 4$, then there exists a color for v that allows us to extend c.*

Proof. Let c be a partial acyclic $\mathcal{D}_1^{(5)}$-coloring of G and v be an uncolored vertex with $n_v \leqslant 4$. If $n_v \leqslant 1$, then clearly we can color v. Hence, we assume $n_v \geqslant 2$. We show that we can color v. The vertex v cannot be colored with a particular color i $(i = 1, \ldots, 5)$ only if there is an (i,j)-dangerous cycle for v, where $j \in \{1, \ldots, 5\}, j \neq i$ or if there is an i-mono-dangerous cycle for v. It is easy to observe the following:

Proposition 2.2. *If c is any partial acyclic $\mathcal{D}_1^{(5)}$-coloring of $G \in \mathcal{S}_5$ and $u \in V(G)$ is rainbow, then we can color or recolor u with any of 5 colors.*

Thus, we need to consider four cases.

Case 1. Assume that exactly two of the colored neighbours of v have the same color, say $x, y \in N(v)$ and $c(x) = c(y) = 1$, and the others (if exist) have distinct colors or are uncolored. Observe that for each color $i \in \{2, \ldots, 5\}$ there must be an $(i,1)$-dangerous cycle and a 1-mono-dangerous cycle for v, passing through x, since otherwise we can color v. It follows that x is adjacent to at least five distinct colored vertices, but this is impossible, since $d(x) \leqslant 5$ and x is also adjacent to v (which is uncolored).

Case 2. Assume that exactly three of the colored neighbours of v have the same color, say $x, y, z \in N(v)$ and $c(x) = c(y) = c(z) = 1$. If one of x, y, z has four colored neighbours and is rainbow, then Proposition 2.2 yields we can recolor this vertex with a color $i \notin C_v$ and obtain Case 1. It follows $p_u \leqslant 3$, for $u \in \{x, y, z\}$. Moreover, we cannot color v only if for each color $i \in \{2, \ldots, 5\}$ there is an $(i,1)$-dangerous cycle and a 1-mono-dangerous cycle for v. Hence each color $i \in \{1, \ldots, 5\}$ belongs to at least two different multisets among C_x, C_y, C_z. Thus, $p_x + p_y + p_z \geqslant 10$, which is impossible.

Case 3. If there are two pairs (say, x, y and z, w) of neighbours of v with the same color, w.l.o.g. $c(x) = c(y) = 1$ and $c(z) = c(w) = 2$, then, similarly as above, we may assume none of x, y, z, w has neighbours with four different colors. Thus, $p_u \leqslant 3$, for $u \in \{x, y, z, w\}$. We may assume that for each $i \in \{3, 4, 5\}$ there is an $(i,1)$-dangerous or an $(i,2)$-dangerous cycle for v and that there is also a 1-mono-dangerous and a 2-mono-dangerous cycle for v. Hence each color $i \in \{1, \ldots, 5\}$ belongs to at least two different multisets among C_x, C_y or to at least two different multisets among C_z, C_w. Thus, at least one of the following occurs: $p_x = p_y = 3$ or $p_z = p_w = 3$. W.l.o.g., we may assume the former holds. Notice, $c(x) = c(y)$. We focus on x. Assume $x_1, x_2, x_3 \in N(x) \setminus \{v\}$ are all in some dangerous (or mono-dangerous) cycles for v and $c(x_1) = c_1, c(x_2) = c_2, c(x_3) = c_3$, $c_a \neq c_b$ for $a \neq b$. If $n_x = 3$, then we recolor x with 2 and obtain Case 2. Otherwise, we may assume that there is a neighbour x_4 of x such that $c(x_4) = c_3$. Observe that we cannot recolor x with a color $i \in \{2, \ldots, 5\}$ only if for each i there is an alternating (c_3, i)-path from x_3 to x_4. Hence $2, \ldots, 5 \in C_{x_3}$, but this is impossible, since C_{x_3} already contains color 1 twice (because x_3 is in the dangerous cycle for v).

Case 4. Assume that $n_v = 4$ and all colored neighbours of v have the same color, say $x, y, z, w \in N(v)$ and $c(x) = c(y) = c(z) = c(w) = 1$. As above, we may assume none of x, y, z, w has neighbours with four different colors. Moreover, for each $i \in \{2, \ldots, 5\}$ there must be an $(i,1)$-dangerous cycle and also a 1-mono-dangerous cycle for v, since otherwise we can color v. Hence each $i \in \{1, \ldots, 5\}$ belongs to at least two different multisets among C_x, C_y, C_z, C_w. Thus, there are at least two vertices among x, y, z, w, say x, y, such that $p_x = p_y = 3$. We proceed similarly, as in Case 3. □

To prove the next theorem we adapt the method presented in [18]. In particular, we use a notion of good spanning trees. Let G be a d-regular connected graph. A *good spanning tree* of G is defined as its spanning tree that contains a vertex, called *root*, adjacent to $d-1$ leaves.

Theorem 2.3 (See [18]). *Every regular connected graph admits a good spanning tree.*

Theorem 2.4. *Every graph $G \in \mathcal{S}_5$ has an acyclic $\mathcal{D}_1^{(5)}$-coloring.*

Proof. Let $G \in \mathcal{S}_5$. Clearly, we can assume G is connected. If $\delta(G) \leqslant 4$, then obviously G is 4-degenerate. Thus, Lemma 2.1 yields there is an acyclic $\mathcal{D}_1^{(5)}$-coloring of G.

Now we may assume that G is 5-regular. Let T be a good spanning tree of G, rooted at v_n, where n is the order of G (the existence of such a tree follows from Theorem 2.3). Let $N(v_n) = \{v_1, v_2, v_3, v_4, u\}$, where v_1, v_2, v_3 and v_4 are leaves in T. We order the vertices of G, from v_5 to v_n, according to the post order walk of T. We construct an acyclic $\mathcal{D}_1^{(5)}$-coloring as follows. First, we color the vertices $v_1, \ldots, v_4$ with four different colors. Then we successively color vertices v_5 up to $v_{n-1} = u$, using Lemma 2.1, but we will never recolor the vertices $v_1, \ldots, v_4$. Now let us check that this is possible. Assume that we are going to color v_i, where $i \in \{5, \ldots, n-1\}$. If v_i is rainbow or has at most one colored neighbour, then clearly we can color it. Otherwise, one of the Cases 1–4, considered in Lemma 2.1, occurs. In Case 1 there is no recoloring. In Cases 2–4 it may happen, that we need to recolor a rainbow vertex, say w, from the neighbourhood of v_i. However, we do it only if w has four colored neighbours. We claim that $w \notin \{v_1, \ldots, v_4\}$. Indeed, if v_j is adjacent to v_i, for some $j \in \{1, \ldots, 4\}$, then v_j has at most three colored neighbours ($d(v_j) = 5$ and v_j has at least two uncolored neighbours, namely v_i and v_n). In Cases 3 and 4 it is also possible that we recolor another neighbour of v_i, say x. But in this case there is always another neighbour of v_i, say y, with the same color as x such that we can recolor y instead of x. Hence if $x \in \{v_1, \ldots, v_4\}$, then we recolor y. Clearly, if x is one of $v_1, \ldots, v_4$, then y is not, because x and y have the same color.

Now let c be the obtained partial coloring of G, with v_n being the only one uncolored vertex. If v_n is rainbow, then Proposition 2.2 yields we can color v_n. Otherwise, assume that there are two neighbours of v_n with the same color, say $c(v_1) = c(u) = 1$, and all other neighbours have distinct colors. We cannot color v_n only if for each color $\alpha \in \{2, \ldots, 5\}$ there is an $(\alpha, 1)$-dangerous cycle and there is a 1-mono-dangerous cycle for v_n. Each such cycle passes through both v_1 and u. Hence $p_{v_1} \geqslant 5$ and $p_u \geqslant 5$, but this is impossible, because $d(v_1) = d(u) = 5$. □

3 Acyclic colorings such that each color class induces an acyclic graph with bounded degree

Since K_6 needs 5 colors in any acyclic coloring in which each color class induces an acyclic graph, we cannot reduce the number of colors in Theorem 2.4. Nevertheless, we can improve Theorem 2.4 in the following way.

Theorem 3.1. *Every graph $G \in \mathcal{S}_5$ has an acyclic $(\mathcal{S}_4 \cap \mathcal{D}_1)^{(5)}$-coloring.*

Proof. Let $G \in \mathcal{S}_5$. Theorem 2.4 implies that there is an acyclic $\mathcal{D}_1^{(5)}$-coloring of G. We choose such a coloring c with the smallest possible number of vertices that have 5 neighbours with its color. Let v be a vertex that has 5 neighbours colored with $c(v)$. We show that we can recolor v or a neighbour of v (sometimes we must recolor some other vertices first) in such a way that the obtained coloring is an acyclic $\mathcal{D}_1^{(5)}$-coloring of G with smaller number of vertices having 5 neighbours colored with its color. It is easily seen that it is enough to prove the theorem for 5-regular graphs. Thus, let G be 5-regular.

Assume that $c(v) = 1$ and $C_v = \{1, 1, 1, 1, 1\}$. Let $N(v) = \{x, y, z, w, t\}$. A colored vertex u is called 4-*saturated*, if it has exactly 4 neighbours colored with $c(u)$. First observe the following:

Claim 1. Suppose that there is a neighbour of v, say x, such that each vertex of $N(x) \setminus \{v\}$ is colored with distinct colors and x has a neighbour, say x_1, such that $c(x_1) \neq 1$ and x_1 is not 4-saturated. Then we can recolor x with $c(x_1)$.

Observe that if we cannot recolor v, then for any color $i \in \{2, \ldots, 5\}$ there is an $(i, 1)$-dangerous cycle for v. Thus, each color $i \in \{2, \ldots, 5\}$ must be contained in at least two different multisets among C_x, C_y, C_z, C_w, C_t. It follows that there is a neighbour of v, say x, which belongs to at least two dangerous cycles for v. We will focus on this vertex and consider all possible assignments of colors to its neighbours. Let $N(x) = \{v, x_1, x_2, x_3, x_4\}$. We may assume w.l.o.g. that $c(x_1) = 2, c(x_2) = 3$ and both x_1 and x_2 belong to some dangerous cycles for v. Thus we have the following:

Observation 3.2. *In both C_{x_1} and C_{x_2} color 1 occurs at least twice*; *neither x_1 nor x_2 is 4-saturated.*

Claim 2. Suppose that in $C(N(x) \setminus \{v\})$ exactly one color occurs twice. Then we can recolor x.

Proof. Let $c(x_3) = a, c(x_4) = b$. Since exactly one color in $C(N(x) \setminus \{v\})$ occurs twice, there are (up to symmetry) three cases to consider.

Case 1. Assume that $a = b = 1$.

If we cannot recolor x with any color i, for $i \in \{2, 3, 4, 5\}$, then for each $i \in \{2, 3, 4, 5\}$ there is an $(i, 1)$-dangerous cycle for x, passing through both x_3 and x_4. Thus, $C_{x_3} = \{1, \ldots, 5\}$. Furthermore, none of the neighbours of x_3 is 4-saturated. Hence, we can recolor x_3 with 4. Thus, by Claim 1 we can recolor x.

Case 2. Suppose that $a \in \{1, 4, 5\}$ and $b \in \{2, 3\}$.

For convenience, we may assume that $b = 2$ and $a \neq 5$. If we cannot recolor x neither with 3 nor with 5, then there is an $(i, 2)$-dangerous cycle for x, passing through both x_1 and x_4, for each $i \in \{3, 5\}$. Thus, x_4 is not 4-saturated. Next, if we cannot recolor x with 2, then there must be a 2-mono-dangerous cycle for x, passing through both x_1 and x_4. It follows that $C_{x_1} = \{1, 1, 2, 3, 5\}$. We may recolor x_1 with 5 (since the neighbour of x_1 colored with 5 is not 4-saturated). Claim 1 implies we can recolor x.

Case 3. Let $b = a$ and $a \in \{4, 5\}$.

Assume w.l.o.g. that $a = 4$. If we cannot recolor x with any color i, for $i \in \{2, 3, 5\}$, then for each $i \in \{2, 3, 5\}$ there is an $(i, 4)$-dangerous cycle for x, passing through both x_3 and x_4. Thus, neither x_3 nor x_4 is 4-saturated. It follows that if we cannot recolor x with 4, then there must be a 4-mono-dangerous cycle for x, passing through both x_3 and x_4. Hence, $C_{x_3} = \{1, \ldots, 5\}$. We may recolor x_3 with 5 (since the neighbour of x_3 colored with 5 is not 4-saturated). From Claim 1 it follows that we can recolor x. □

Claims 1 and 2 imply that we have to consider only two cases: Both colors 2, 3 occur twice in C_x and one of colors from $\{2, 3\}$ occurs three times.

Case 1. Assume that $c(x_1) = 2, c(x_2) = 3, c(x_3) = 2, c(x_4) = 3$.

Observe the following:

Claim 3. If there is a neighbour x_i of x such that in $C(N(x_i) \setminus \{x\})$ each color occurs exactly once and at most one vertex of $N(x_i) \setminus \{x\}$ is both 4-saturated and colored with either 4 or 5, then we can recolor x.

Proof. First, we recolor x_i either with 4 or 5. Since at most one vertex of $N(x_i) \setminus \{x\}$ colored with 4 or 5 is 4-saturated and $4, 5 \notin C_x$, such a recoloring is possible. Thus, we obtain the coloring that satisfies the assertions of Claim 2 and hence we can recolor x. □

Claim 4. If x has a neighbour x_i such that x_i does not have a 4-saturated neighbour colored with 1 and there is neither a 1-mono-dangerous cycle nor a $(1, j)$-dangerous cycle for x_i (for each $j \in \{2, 3, 4, 5\}$) passing through two vertices of $N(x_i) \setminus \{x\}$, then we can recolor x.

Proof. Suppose to the contrary that there is a neighbour x_i of x such that none of the colored 1 neighbours of x_i is 4-saturated and there is neither a 1-mono-dangerous cycle nor a $(1, j)$-dangerous cycle for x_i passing through two vertices of $N(x_i) \setminus \{x\}$ and we cannot recolor x. If we can recolor x_i with 1, then the coloring satisfies the assertions of Claim 2 and hence we can recolor x, a contradiction. Otherwise, there is a 1-mono-dangerous cycle for x_i passing through x. For convenience and w.l.o.g. we may assume that $c(x_i) = 2$. (Thus, $i \in \{1, 3\}$.) If there is no $(a, 3)$-dangerous cycle for x ($a \in \{4, 5\}$), then we may recolor x_i with 1 and x with a. Thus, we assume that both such dangerous cycles are present, and are passing through x_2 and x_4. Hence, x_4 is not 4-saturated. (Recall that Observation 3.2 yields that also x_2 is not 4-saturated.) It follows that we cannot recolor x with 3 only if there is a 3-mono-dangerous cycle for x, passing through x_2 and x_4. From the above and Observation 3.2 we clearly obtain $C_{x_2} = \{1, 1, 3, 4, 5\}$. Therefore x_2 satisfies the assertions of Claim 3. This implies that we can recolor x. □

Observe that if we can recolor x with 4 or 5, then we are done. This is impossible only if there are at least two dangerous cycles for x, namely a $(4, i)$-dangerous cycle and a $(5, j)$-dangerous cycle, where $i, j \in \{2, 3\}$. Up to symmetry, there are two possibilities.

Subcase 1.1. Assume that both a $(4, 2)$-dangerous cycle and a $(5, 3)$-dangerous cycle for x are present. It follows that $4 \in C_{x_1}$, $4 \in C_{x_3}$ and $5 \in C_{x_2}$, $5 \in C_{x_4}$. Next, we cannot recolor x with 2 only if there is a $(2, 3)$-dangerous cycle for x, passing through x_2 and x_4, or there is a 2-mono-dangerous cycle for x, passing through x_1 and x_3. The same argument can be applied if we consider recoloring x with 3. Assume first that we have for x both a 2-mono-dangerous cycle and a $(3, 2)$-dangerous cycle, both passing through x_1. Recall that Observation 3.2 yields that color 1 occurs twice in C_{x_1}. Thus $C_{x_1} = \{1, 1, 2, 3, 4\}$ and by Claim 3 we can recolor x, so we are done. We can proceed in this way also in the case when we have for x both a 3-mono-dangerous cycle and a $(2, 3)$-dangerous cycle (both passing through x_2). Hence we have two possibilities left. There are either both a 2-mono and a 3-mono-dangerous cycle for x, or both a $(2, 3)$-dangerous and a $(3, 2)$-dangerous cycle for x. We consider only the first situation, since the second one can be solved analogously. It follows that $C_{x_1} = \{1, 1, 2, 4, \alpha\}$ and $C_{x_2} = \{1, 1, 3, 5, \beta\}$. We focus on the vertex x_1 (see Figure 1). Let $N(x_1) \setminus \{x\} = \{x_1', x_2', x_3', x_4'\}$ and $c(x_1') = 1, c(x_2') = 2, c(x_3') = 4, c(x_4') = \alpha$. By Claim 3 we may assume that $\alpha \in \{1, 2, 4\}$. Claim 4 implies that there is a $(1, \alpha)$-dangerous or 1-mono-dangerous cycle for x_1, passing through two vertices of $\{x_1', x_2', x_3', x_4'\}$ (these two vertices are colored with α) or $\alpha = 1$ and x_4' is 4-saturated. In the former case, if recoloring x_1 with 4, 5 is impossible, then there are two dangerous cycles for x_1 passing through two vertices of $\{x_1', x_2', x_3', x_4'\}$ colored with α. Let $x' \in \{x_1', x_2', x_3'\}$ be the vertex with color α. Thus, $C_{x'} = \{2, 1, 2, 4, 5\}$. We can recolor x' with 5 (since the neighbour of x' colored with 5 is not 4-saturated) and hence by Claim 3 we can recolor x. In the latter case, when $\alpha = 1$ and x_4' is 4-saturated, it is easy to observe that we can recolor x_1 with 4 and then Claim 2 implies we can recolor x.

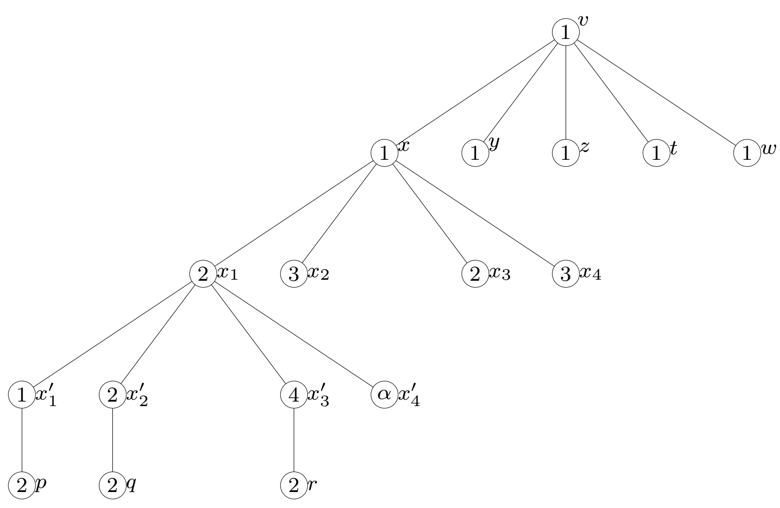

Figure 1 Subcase 1.1. Since there is a $(2, 1)$-dangerous cycle for v, $c(p) = 2$. Since there is a 2-mono-dangerous cycle for x, $c(q) = 2$. Since there is a $(4, 2)$-dangerous cycle for x, $c(r) = 2$

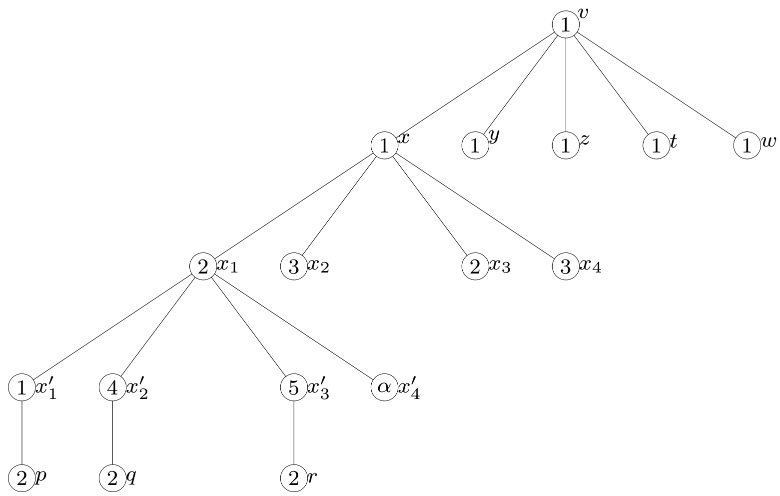

Figure 2 Subcase 1.2. Since there is a $(2,1)$-dangerous cycle for v, $c(p)=2$. Since there is a $(4,2)$-dangerous cycle for x, $c(q)=2$. Since there is a $(5,2)$-dangerous cycle for x, $c(r)=2$

Subcase 1.2. Suppose that there are both a $(4,2)$-dangerous cycle and a $(5,2)$-dangerous cycle for x. It follows that $C_{x_1}=\{1,1,4,5,\alpha\}$ (see Figure 2). Let

$$N(x_1)\setminus\{x\}=\{x_1',x_2',x_3',x_4'\}$$

and $c(x_1')=1, c(x_2')=2, c(x_3')=4, c(x_4')=\alpha$. By Claim 3 we may assume that $\alpha\in\{1,2,4\}$. If recoloring x_1 with $1,4,5$ is impossible, then there are three dangerous cycles for x_1 passing through two vertices of $\{x_1',x_2',x_3',x_4'\}$ colored with α (Claim 4 implies that the $(1,\alpha)$-dangerous or 1-mono-dangerous cycle exists) or $\alpha=1$ and x_4' is 4-saturated. We start with considering the first situation. Let $x'\in\{x_1',x_2',x_3'\}$ be the vertex with color α. Thus, $C_{x'}=\{2,1,2,4,5\}$. We can recolor x' with 3 and hence by Claim 3 we can recolor x. In the second case, the fact that x_4' is 4-saturated implies we can recolor x_1 with 4 and, by Claim 2, we can recolor x.

Case 2. Assume that $c(x_1)=2, c(x_2)=3, c(x_3)=3, c(x_4)=3$.

Similarly as Claims 3 and 4 we can prove the following:

Claim 5. If there is a neighbour x_i ($i\in\{2,3,4\}$) of x such that in $C(N(x_i)\setminus\{x\})$ each color occurs exactly once and at most one vertex of $N(x_i)\setminus\{x\}$ is both 4-saturated and colored with either 4 or 5, then we can recolor x.

Claim 6. If x has a neighbour x_i ($i\in\{2,3,4\}$) such that x_i does not have a 4-saturated neighbour colored with 1 and there is neither a 1-mono-dangerous cycle nor a $(1,j)$-dangerous cycle for x_i passing through two vertices of $N(x_i)\setminus\{x\}$, then we can recolor x.

If we can recolor x with 2, 4 or 5, then we are done. This is impossible only if there is an $(i,3)$-dangerous cycle for x, for each $i\in\{2,4,5\}$. Thus, recoloring x with 3 is impossible if we have a 3-mono-dangerous cycle for x or one of x_2,x_3,x_4 is 4-saturated. Assume at the beginning that we have a 4-saturated neighbour of x, say x_4. Hence, $C_{x_2}=\{1,1,2,4,5\}$ and we can recolor x_2 with 4, so by Claim 2 we can recolor x. Therefore, we may assume there is a 3-mono-dangerous cycle for x and none of x_2,x_3,x_4 is 4-saturated. Thus, each color $i\in\{2,3,4,5\}$ occurs twice among C_{x_2},C_{x_3},C_{x_4} and hence one multiset contains at least three of these colors. W.l.o.g. we may assume that $C_{x_4}=\{1,2,3,4,\alpha\}$ (see Figure 3). By Claim 5 we may assume that $\alpha=2,3$ or 4. If recoloring x_4 with $1,4,5$ is impossible, then there are three dangerous cycles for x_4 passing through two vertices of $\{x_1',x_2',x_3',x_4'\}$ colored with α (Claim 6 implies that the $(1,\alpha)$-dangerous cycle exists). Let $x'\in\{x_1',x_2',x_3'\}$ be the vertex with color α. Thus, $C_{x'}=\{3,1,3,4,5\}$. We can recolor x' with 5 and hence by Claim 5 we can recolor x. □

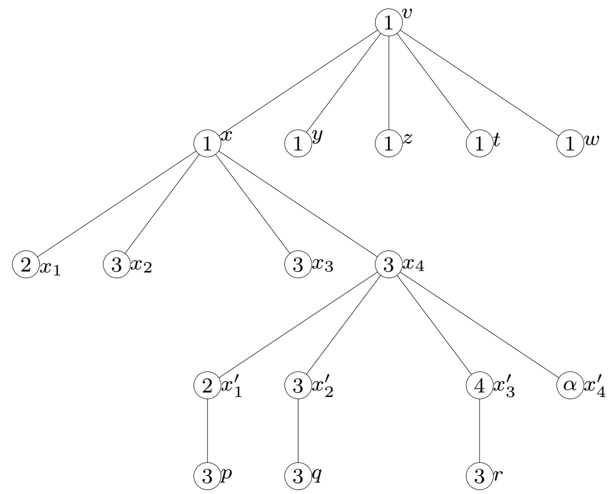

Figure 3 Case 2. Since there is a $(2,3)$-dangerous cycle for x, $c(p) = 3$. Since there is a 3-mono-dangerous cycle for x, $c(q) = 3$. Since there is a $(4,3)$-dangerous cycle for x, $c(r) = 3$

4 Complexity result

Now we show that the problem of deciding whether a graph $G \in \mathcal{S}_5$ has an acyclic $(\mathcal{S}_3, \mathcal{S}_3)$-coloring is NP-complete. First we present some special graphs and their properties. Let $G(C_j)$ and F be the graphs depicted in Figure 4. It is easy to observe that both $G(C_j)$ and F belong to $\mathcal{S}_5$. Their acyclic $(\mathcal{S}_3, \mathcal{S}_3)$-colorings are presented in Figure 4. The following two observations concerning graph $G(C_j)$ are straightforward.

Observation 4.1. *In any acyclic $(\mathcal{S}_3, \mathcal{S}_3)$-coloring of $G(C_j)$ the vertices $u_{j,1}$, $u_{j,2}$, $u_{j,3}$ are not all colored with the same color.*

Observation 4.2. *Any partition of the set $\{u_{j,1}, u_{j,2}, u_{j,3}\}$ into two nonempty parts can be extended to an acyclic $(\mathcal{S}_3, \mathcal{S}_3)$-coloring of $G(C_j)$.*

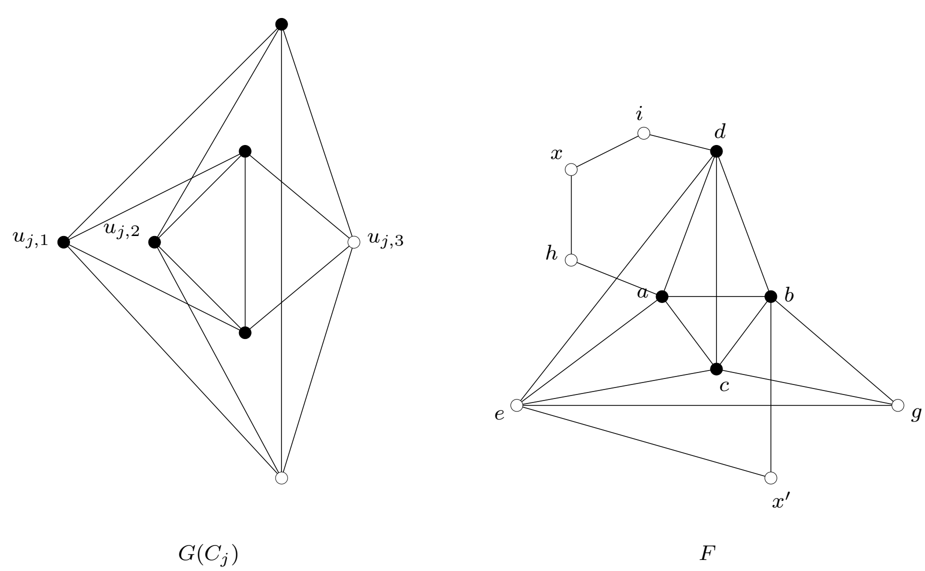

Figure 4 Graphs $G(C_j)$ and F

A colored vertex v is called 3-*saturated*, if it has exactly 3 neighbours colored with $c(v)$. Considering the graph F, its important properties are the following.

Observation 4.3. *In any acyclic $(\mathcal{S}_3, \mathcal{S}_3)$-coloring f of F, both x and x' have the same color, x has exactly two neighbours colored with $f(x)$, and x' has exactly one neighbour colored with $f(x')$.*

Proof. Let f be an acyclic $(\mathcal{S}_3, \mathcal{S}_3)$-coloring of F. We prove that f has the desired properties, by considering all possible colorings of the vertices a, b, c.

Case 1. Assume that a, b, c all have the same color, say 1.

If d also has color 1, then i, h, e, g and x' must be colored with 2, because each of a, b, c, d is 3-saturated. There is an alternating path between h and i, hence x must be colored with 2. Clearly, such a coloring has the desired properties.

Suppose now that d has color 2. It follows that e and g must be colored with 1, since otherwise we have an alternating cycle. Thus, c has four neighbours in its own color, a contradiction.

Case 2. Suppose two vertices among a, b, c are colored with 1 and the remaining one is colored with 2. There are three possibilities.

Assume a, b have color 1, c has color 2. Observe that d, e, g must have color 1, since otherwise an alternating cycle occurs. Hence b is 3-saturated, thus x' must be colored with 2. It follows that there is an alternating cycle induced by x', b, c, e, a contradiction.

Suppose now that a, c have color 1, b has color 2. Clearly, d, e must have color 1, otherwise an alternating cycle occurs. Thus, d is 3-saturated. Hence i must be colored with 2. Furthermore, each of c, e, a is 3-saturated, thus g, x', h must have color 2. There is an alternating path between h and i, hence x must have color 2. Again, such a coloring has the desired properties.

Finally, assume b, c have color 1 and a has color 2. It follows d, g must be colored with 1, otherwise we would have an alternating cycle. Thus, c is 3-saturated, therefore e must have color 2, but then vertices d, a, c, e induce an alternating cycle. A contradiction follows. □

Observation 4.4. *In any acyclic $(\mathcal{S}_3, \mathcal{S}_3)$-coloring of F there is no alternating path between the vertices x and x'.*

Let m be a positive integer. Based on the graph F, we construct a graph $G^m(v_i)$. We take m copies of F, say $F_1, F_2, \ldots, F_m$. In each copy F_s, vertices x and x' are denoted by $x_{i,s}$ and $x'_{i,s}$, respectively. Next, for $s \in \{1, \ldots, m-1\}$, we identify $x'_{i,s}$ with $x_{i,s+1}$, and denote the obtained vertex by $v_{i,s+1}$. Finally, we identify $x_{i,1}$ with $x'_{i,m}$ and denote the new vertex by $v_{i,1}$. Observe that $G^m(v_i) \in \mathcal{S}_5$ and the vertices $v_{i,s}$ are all of degree 4. Such a graph for $m = 3$ is presented in Figure 5.

Observation 4.5. *For any positive integer m, the graph $G^m(v_i)$ has an acyclic $(\mathcal{S}_3, \mathcal{S}_3)$-coloring. In any such a coloring the vertices $v_{i,s}$ are 3-saturated and are all in the same color, for $s \in \{1, \ldots, m\}$.*

Proof. This follows from the construction of $G^m(v_i)$ and Observations 4.3 and 4.4. □

Theorem 4.6. *The problem of deciding whether a graph admits an acyclic $(\mathcal{S}_3, \mathcal{S}_3)$-coloring is NP-complete, even for graphs with maximum degree at most 5.*

Proof. Clearly the problem of acyclic $(\mathcal{S}_3, \mathcal{S}_3)$-coloring is in NP. We will show that it is NP-complete. We use the reduction from the problem Not-All-Equal-3-SAT without negative literals (3NN-SAT problem, for short) whose NP-completeness was proved in [15]. Let $\mathcal{C} = \{C_1, \ldots, C_m\}$ be a set of clauses and let $\mathcal{V} = \{v_1, \ldots, v_n\}$ be a set of Boolean variables. Furthermore, $C_j \subseteq \mathcal{V}$ and for $j \in \{1, \ldots, m\}$, $C_j = (c_{j1}, c_{j2}, c_{j3})$. In 3NN-SAT we ask if there is a truth assignment such that in each clause there exist at least one true variable and at least one false variable. Note that there are no negative variables.

Given an example $\mathcal{C}$ of 3NN-SAT we create an instance of our problem, i.e., the graph G, in the following way. For each variable $v_i \in \mathcal{V}$ we take the graph $G^m(v_i)$ and for each clause $C_j \in \mathcal{C}$ we take the graph $G(C_j)$. To obtain the graph G we connect graphs $G^m(v_i)$ and $G(C_j)$ in such a way that for $i = 1, \ldots, n$, $j = 1, \ldots, m$, $k = 1, 2, 3$, we add an edge $v_{i,j}u_{j,k}$ if and only if c_{jk} is the variable v_i. Observe that for $j \in \{1, \ldots, m\}, k \in \{1, 2, 3\}$ each vertex $u_{j,k}$ is adjacent to exactly one vertex from

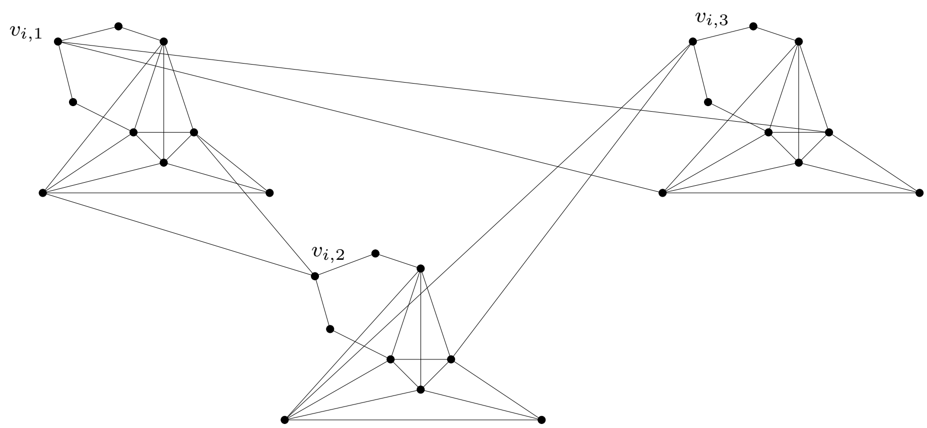

Figure 5 Graph $G^3(v_i)$

graphs $G^m(v_1),\dots,G^m(v_n)$ and for $i\in\{1,\dots,n\}, s\in\{1,\dots,m\}$ each vertex $v_{i,s}$ is adjacent to at most one vertex from graphs $G(C_1),\dots,G(C_m)$. Thus, $G\in\mathcal{S}_5$. We will prove that G has an acyclic $(\mathcal{S}_3,\mathcal{S}_3)$-coloring if and only if $\mathcal{C}\in$ 3NN-SAT.

Assume first that $\mathcal{C}\in$ 3NN-SAT. We construct the coloring c as follows. If the variable v_i was assigned *true*, then all vertices $v_{i,s}$ are colored with 1, otherwise we color them with 2. We will prove that this coloring can be extended to an acyclic $(\mathcal{S}_3,\mathcal{S}_3)$-coloring of G. Observation 4.5 yields that this coloring can be extended to an acyclic $(\mathcal{S}_3,\mathcal{S}_3)$-coloring of each graph $G^m(v_i)$. Furthermore, all vertices $v_{i,s}$ are 3-saturated, hence a vertex $u_{j,k}$ is colored with 1, if its corresponding variable in the clause C_j was assigned *false*, and with 2 otherwise. Since $C\in$ 3NN-SAT, we have the partition of each set $\{u_{j,1},u_{j,2},u_{j,3}\}$ into two nonempty parts. Observation 4.2 yields it can be extended to an acyclic $(\mathcal{S}_3,\mathcal{S}_3)$-coloring of each graph $G(C_j)$. We claim that such a coloring is an acyclic $(\mathcal{S}_3,\mathcal{S}_3)$-coloring of G. Indeed, if there is an alternating cycle, then such a cycle passes through $v_{i,s}$ and $v_{i,s+1}$, but Observation 4.4 implies there is no alternating path joining them. Thus, the obtained coloring is an acyclic $(\mathcal{S}_3,\mathcal{S}_3)$-coloring of G.

Now let c be an acyclic $(\mathcal{S}_3,\mathcal{S}_3)$-coloring of G. Observation 4.5 yields for $s=1,\dots,m$, the vertices $v_{i,s}$ in each graph $G(v_i)$ all have the same color and are 3-saturated. Thus, $u_{j,k}$ from the graph $G(C_j)$ must have a color distinct from the color of its neighbour $v_{i,j}$. Hence for $i=1,\dots,n$ vertices $u_{j,k}$, corresponding in the graphs $G(C_j)$ to the variable $c_{j,k}=v_i$ all have the same color. Furthermore, in $G(C_j)$, the vertices $u_{j,1}$, $u_{j,2}$, $u_{j,3}$ do not all have the same color, which follows from Observation 4.1. Thus, the following assignment $c_{j,k}:=$ true if $u_{j,k}$ has color 1, $c_{j,k}:=$ false for otherwise, shows $\mathcal{C}\in$ 3NN-SAT. □

5 Results for graphs with fixed maximum degree

Kostochka and Stocker [20] proved that there exists a linear-time algorithm computing an acyclic coloring of any graph with maximum degree d with at most $\lfloor\frac{(d+1)^2}{4}\rfloor+1$ colors. Using a similar method, we prove analogous results for acyclic improper colorings of graphs with maximum degree d. We need a definition first. Recall that for a given partial k-coloring of a graph G, a vertex is rainbow, if all its colored neighbours have distinct colors. A partial k-coloring of G is called *rainbow* if every uncolored vertex is rainbow.

Theorem 5.1. *For every fixed d, $d\geqslant 4$, there exists a linear (in n) algorithm finding an acyclic $(d-1)$-improper coloring for any n-vertex graph G with maximum degree at most d using $\lfloor\frac{d^2}{4}\rfloor+1$ colors.*

Proof. Let G be a graph with maximum degree d and c be its partial coloring. By C we denote the set of colored vertices of G. The algorithm proceeds as follows: We choose an uncolored vertex v with the most number of colored neighbours, then we greedily color v with color c_i such that:

(1) if v has exactly one colored neighbour, then $c_i \notin c(N(v)) \cup c(\bigcup_{x\in N(v)\setminus C} N(x))$;

(2) if v has more than one colored neighbour, then $c_i \notin c(\bigcup_{x\in N(v)\setminus C} N(x))$.

First, we claim that we can always find such a color c_i in $\{1,\ldots,\lfloor\frac{d^2}{4}\rfloor+1\}$. Suppose that v has exactly one colored neighbour. Since v is an uncolored vertex with the most number of colored neighbours, each uncolored neighbour of v has at most one colored neighbour. Thus,

$$\left|c(N(v)) \cup c\left(\bigcup_{x\in N(v)\setminus C} N(x)\right)\right| \leqslant d \leqslant \left\lfloor\frac{d^2}{4}\right\rfloor.$$

Suppose that v has exactly k colored neighbours. It clearly follows that

$$\left|c\left(\bigcup_{x\in N(v)\setminus C} N(x)\right)\right| \leqslant (d-k)k \leqslant \left\lfloor\frac{d}{2}\right\rfloor\left\lceil\frac{d}{2}\right\rceil = \left\lfloor\frac{d^2}{4}\right\rfloor.$$

Now we show that we eventually obtain an acyclic $(d-1)$-improper coloring. After each step the partial coloring is rainbow. Thus, if we color the vertex v, then we do not create any alternating cycle. Finally, it suffices to prove that after each step any colored vertex has at least one neighbour colored with a color other than its own (if G has more than one colored vertex). Indeed, if an uncolored vertex v has exactly one colored neighbour, then according to our algorithm we color v with a color that is not in C_v. Suppose now that the uncolored vertex v has more than one colored neighbour and till now after each step any colored vertex has at least one neighbour with a color distinct from its own. The vertex v is rainbow, hence if we color v, then v will still have this property. Each colored neighbour of v has this property, hence coloring v will not destroy this property.

For the runtime analysis, we propose the following detailed algorithm for an acyclic $(d-1)$-improper k-coloring of a graph $G \in \mathcal{S}_d$, with

$$k = \left\lfloor\frac{d^2}{4}\right\rfloor + 1.$$

The graph G is represented by the lists of incidences. For each vertex v we add to its list of incidences two additional values: $u(v)$ which stores the number of colored neighbours of v, $c(v)$ which stores the color of v. If v is uncolored, then we have $c(v) = 0$. We also maintain $d+1$ lists $A_0, A_1, \ldots, A_d$ to store vertices and we put a vertex v to A_j if $u(v) = j$. Initially, we put $c(v) = u(v) = 0$ for each vertex v, A_0 contains all vertices, and $i = 0$, $n = |V(G)|$.

(1) **while** $i < n$ **do**
(2) choose the largest index j such that A_j is nonempty;
(3) choose v to be the first vertex on A_j;
(4) **if** $u(v) = j$ **then**
(5) {
(6) **if** $u(v) = 1$ **then** choose the first color $a \notin c(N(v)) \cup c(\bigcup_{y\in N(v):c(y)=0} N(y))$
(7) **else** choose the first color $a \notin c(\bigcup_{y\in N(v):c(y)=0} N(y))$;
(8) $c(v) := a$; /* color v with a */
(9) delete v from A_j;
(10) **for** each neighbour y of v **do**
(11) **if** $c(y) = 0$ **then** $\{u(y)++$; add y to $A_{u(y)}$;$\}$
(12) $i++$;
(13) }
(14) **else** delete v from A_j. /* v is already colored */

Let us compute the running time of the algorithm. Observe first that the **while** loop iterates at most $(d+1)n$ times. Steps (2) and (3) take $\mathcal{O}(d)$ time. Choosing an admissible color for a vertex v (in Steps (6) or (7)) and coloring v takes $\mathcal{O}(d^2)$ time, since we have to check the colors of the vertices which are at distance at most two from v. Updating the list A_j takes a constant amount of time. The **for** loop in step (10) iterates at most d times and each iteration takes a constant amount of time. The last step also takes a constant amount of time. Hence, for each iteration of the **while** loop we need $\mathcal{O}(d^2)$ time. Thus for fixed d, the running time of the algorithm is $\mathcal{O}(n)$. □

Theorem 5.2. *Let d and t be fixed such that $2 \leqslant t \leqslant \lfloor \frac{d}{2} \rfloor - 1$. There exists a linear (in n) algorithm finding an acyclic $(d-t)$-improper coloring for any n-vertex graph G with maximum degree at most d using $\lfloor \frac{d^2}{4} \rfloor + t + 1$ colors.*

Proof. Similarly, as in the previous proof, the algorithm will color vertices of G such that after each step we obtain a rainbow partial coloring c of G with some additional restrictions. In each step of the algorithm we choose an uncolored vertex v with the most number of colored neighbours. Let v be the vertex with exactly k colored neighbours. Observe that there is an ordering $v_1, v_2, \ldots, v_k$ of colored neighbours of v such that each vertex v_i $(i = 2, \ldots, k)$ had at least $i-1$ colored neighbours at the moment when it was colored. Indeed, if we order the neighbours of v such that v_i is (in this ordering) before v_j, if v_i was colored before v_j, then we obtain such an ordering. Clearly, each vertex v_i had at least $i-1$ colored neighbours at the moment when it was colored, otherwise v should be colored before v_i. Hence each vertex v_i $(i = 2, \ldots, k)$ has at least $i-1$ neighbours colored with distinct colors. We color v with color c_i such that

(1) if $k \leqslant t$, then $c_i \notin c(\{v_1, v_2, \ldots, v_k\}) \cup c(\bigcup_{x \in N(v) \setminus C} N(x))$;

(2) if $k > t$, then $c_i \notin c(\{v_1, v_2, \ldots, v_t\}) \cup c(\bigcup_{x \in N(v) \setminus C} N(x))$.

First, we claim that we can always find such a color c_i in

$$\left\{1, \ldots, \left\lfloor \frac{d^2}{4} \right\rfloor + t + 1\right\}.$$

Since the vertex v has k colored neighbours and it is the vertex with the most number of colored neighbours, we have

$$\left| c\left(\bigcup_{x \in N(v) \setminus C} N(x) \right) \right| \leqslant (d-k)k \leqslant \left\lfloor \frac{d}{2} \right\rfloor \left\lceil \frac{d}{2} \right\rceil = \left\lfloor \frac{d^2}{4} \right\rfloor.$$

Now, we show that after each step the obtained partial coloring has no alternating cycle and for each colored vertex v the following holds: Either v has at least t colored neighbours which have colors distinct from the color of v, or all colored neighbours of v have distinct colors (if v has less than $t+1$ colored neighbours). If we color the first vertex, then it obviously holds. Suppose that till now after each step this property holds and now we color the vertex v which has k colored neighbours. Since v is rainbow, coloring v does not create any alternating cycle. If $k \leqslant t$, then we color v with a color distinct from the colors of its neighbours. Thus, all neighbours of v are colored with distinct colors. Furthermore, for each neighbour u of v the following holds: Either u has at least t neighbours colored with colors distinct from the color of u, or all colored neighbours of u have distinct colors (if u has less than $t+1$ colored neighbours). Suppose now that $k > t$. We color v with the color distinct from the colors of $v_1, v_2, \ldots, v_t$. Thus, v has at least t neighbours colored with colors distinct from its own color. Before we color v, any vertex among $v_{t+1}, \ldots, v_k$ has at least t colored neighbours, so each of $v_{t+1}, \ldots, v_k$ has at least t neighbours colored with a color distinct from its own. So after coloring v they still have this property. Since we color v with a color distinct from the colors of $v_1, v_2, \ldots, v_t$, after coloring v the neighbours of $v_1, v_2, \ldots, v_t$ have this property. Hence the algorithm creates a $(d-t)$-improper coloring. For the running time, it is enough to observe that an algorithm, similar to that given in the proof of Theorem 5.1, will work. □

It is an easy observation that if a graph G admits an acyclic t-improper coloring, then G also has an acyclic p-improper coloring, for any $p \geqslant t$. From this fact and since an acyclic coloring can be always treated as an acyclic s-improper coloring, with $s = 0$, the next result follows directly from the aforementioned theorem of Kostochka and Stocker [20].

Corollary 5.3. *Let d and t be fixed such that $t \leqslant \lceil \frac{d}{2} \rceil$. There exists a linear (in n) algorithm finding an acyclic t-improper coloring for any n-vertex graph G with maximum degree at most d using $\lfloor \frac{(d+1)^2}{4} \rfloor + 1$ colors.*

6 Concluding remarks

In the paper, we consider acyclic improper colorings of graphs with bounded degree. In particular, we give a linear-time algorithm for an acyclic t-improper coloring of any graph with maximum degree at most d, provided that the number of colors used is large enough with respect to d. Fixing the maximum degree to five, we obtain results which are more exact. Namely, we show that every such graph has an acyclic coloring with five colors in which each color class induces an acyclic graph (see Theorem 2.4) and we further improve this result (see Theorem 3.1) as follows: Every graph with maximum degree at most five admits an acyclic coloring with five colors such that each color class induces an acyclic graph with maximum degree at most four.

It might be interesting, how the above mentioned result can be further improved. One possible way is to put stronger condition on the color classes. We post the following problem:

Open Problem 6.1. *Let G be a graph with maximum degree at most five. For which properties $\mathcal{P}$ graph G admits an acyclic $(\mathcal{P})^{(5)}$-coloring?*

We prove that the property $\mathcal{D}_1 \cap \mathcal{S}_4$ (of acyclic graphs with maximum degree at most four) can be taken as $\mathcal{P}$. Finding the smallest such property is a challenging question. If we additionally assume that the property $\mathcal{P}$ has to be hereditary, then the above problem coincides with the problem of finding the acyclic reducible bounds for $\mathcal{S}_5$ (see, e.g., [7, 8] for the definition and some results concerning acyclic reducible bounds).

On the other hand, one can think about reducing the number of colors used. Since the complete graph on five vertices is, as far as we know, the only one graph that actually requires 5 colors, maybe it can be possible to use less colors, while excluding K_5.

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References

1 Addario-Berry L, Espert L, Kang R J, et al. Acyclic improper colorings of graphs with bounded degree. Discrete Math, 2010, 310: 223–229
2 Addario-Berry L, Kang R J, Műller T. Acyclic dominating partitions. J Graph Theory, 2010, 64: 292–311
3 Alon N, McDiarmid C, Reed B. Acyclic coloring of graphs. Random Structures Algorithms, 1991, 2: 277–288
4 Boiron P, Sopena E, Vignal L. Acyclic improper colorings of graphs with bounded degree. DIMACS Ser Discrete Math Theoret Comput Sci, 1997, 49: 1–10
5 Boiron P, Sopena E, Vignal L. Acyclic improper colorings of graphs. J Graph Theory, 1999, 32: 97–107
6 Borowiecki M, Broere I, Frick M, et al. A survey of hereditary properties of graphs. Discuss Math Graph Theory, 1997, 17: 5–50
7 Borowiecki M, Fiedorowicz A. On partitions of hereditary properties of graphs. Discuss Math Graph Theory, 2006, 26: 377–387
8 Borowiecki M, Fiedorowicz A, Hałuszczak M. Acyclic reducible bounds for outerplanar graphs. Discuss Math Graph Theory, 2009, 29: 219–239
9 Borowiecki M, Fiedorowicz A, Jesse-Józefczyk K, et al. Acyclic colorings of graphs with bounded degree. DIMACS Ser Discrete Math Theoret Comput Sci, 2010, 12: 59–74
10 Borowiecki M, Jesse-Józefczyk K, Sidorowicz E. The complexity of some acyclic improper coloring. Discrete Math, 2011, 311: 732–737
11 Burstein M I. Every 4-valent graph has an acyclic 5-coloring (in Russian). Soobšč Akad Gruzin SSR, 1979, 93: 21–24
12 Coleman T F, Cai J. The cyclic coloring problem and estimation of sparse Hessian matrices. SIAM J Algebra Discrete Methods, 1986, 7: 221–235
13 Coleman T F, Moré J J. Estimation of sparse Hessian matrices and graph coloring problems. Math Program, 1984, 28: 243–270
14 Fiedorowicz A, Sidorowicz E. Acyclic improper coloring of graphs with maximum degree 4. Sci China Math, 2014, 57: 2485–2494
15 Garey M R, Johnson D S. Computers and Intractability: A Guide to the Theory of NP-Completeness. San Francisco: W H Freeman and Company, 1979

16 Gebremedhin A H, Manne F, Pothen A. What color is your Jacobian? Graph coloring for computing derivatives. SIAM Rev, 2005, 47: 629–705

17 Grűnbaum B. Acyclic coloring of planar graphs. Israel J Math, 1973, 14: 390–412

18 Hocquard H. Graphs with maximum degree 6 are acyclically 11-colorable. Inform Process Lett, 2011, 111: 748–753

19 Kostochka A V. Upper bounds of chromatic functions of graphs (in Russian). PhD Thesis. Novosibirsk: Novosibirsk State University, 1978

20 Kostochka A V, Stocker C. Graphs with maximum degree 5 are acyclically 7-colorable. Ars Math Contemp, 2011, 4: 153–164

21 Skulrattanakulchai S. Acyclic colorings of subcubic graphs. Inform Process Lett, 2004, 92: 161–167

22 West D. Introduction to Graph Theory, 2nd ed. Upper Saddle River: Prentice Hall, 2001