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Stable homotopy classification of A_n^4-polyhedra with 2-torsion free homology

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Abstract We study the stable homotopy types of $\boldsymbol{F}_{n(2)}^4$-polyhedra, i.e., $(n-1)$-connected, at most $(n+4)$-dimensional polyhedra with 2-torsion free homologies. We are able to classify the indecomposable $\boldsymbol{F}_{n(2)}^4$-polyhedra. The proof relies on the matrix problem technique which was developed in the classification of representations of algebras and applied to homotopy theory by Baues and Drozd (1999, 2001, 2004).

Keywords homotopy, indecomposable, matrix problem

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1 Introduction

Let $\boldsymbol{A}_n^k$ $(n \geqslant k+1)$ be the subcategories of the stable homotopy category consisting of $(n-1)$-connected polyhedra with dimension at most $n+k$. It is a fully additive category if we consider the wedge of two polyhedra as the coproduct of two objects in the category $\boldsymbol{A}_n^k$. The classification problem of $\boldsymbol{A}_n^k$ $(n \geqslant k+1)$ is to find a complete list of indecomposable isomorphic classes, i.e., the indecomposable homotopy types in $\boldsymbol{A}_n^k$ $(n \geqslant k+1)$. For $k \leqslant 3$, all indecomposable stable homotopy types have been described in [3]. For $k \geqslant 4$, Drozd shows the classification problem is wild (in the sense similar to that in representation of finite dimensional algebras) in [8] by finding a wild subcategory of $\boldsymbol{A}_n^4$ $(n \geqslant 5)$ whose objects are polyhedra with 2-torsion homologies.

In another direction, Baues and Drozd [1, 2] also considered full subcategory $\boldsymbol{F}_n^k$ of $\boldsymbol{A}_n^k$ $(n \geqslant k+1)$ consisting of polyhedra with torsion free homology groups. For $k \leqslant 5$, such polyhedra have been classified to have finite indecomposable homotopy types in [1, 2] or [6, 8]. For $k = 6$, Drozd [7] got tame type classification of congruence classes of homotopy types, and proved that, for $k > 6$, this problem is wild in [7].

This is the second of a series of papers devoted to the homotopy theory of $\boldsymbol{A}_n^k$-polyhedra. In [9], Pan and Zhu noticed that, for $(n-1)$-connected and at most $(n+k)$-dimensional $(k < 7)$ spaces with 2- and 3-torsion free homologies, the classification of indecomposable stable homotopy types essentially is

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reduced to that of spaces with torsion free homologies. When homologies of the spaces involved have 3-torsion, the reduction process does not lead to the matrix problem for spaces with torsion free homologies but to a matrix problem which can be solved. By this we are able to classify homotopy types of the full subcategory $\boldsymbol{F}^4_{n(2)}$ of $\boldsymbol{A}^4_n$ ($n \geqslant 5$) consisting of polyhedra with 2-torsion free homology groups. We will discuss the splitting of smash product of $\boldsymbol{A}^k_n$-polyhedra in a latter publication.

The paper is organized as follows. Section 2 contains some basic notation and facts about stable homotopy category and classification problem. Our main theorem is given at the end of this section. In Section 3, Theorem 3.2 and Corollary 3.3 establish a connection between bimodule categories and stable homotopy categories. In Section 4, we use the known results of indecomposable homotopy types of $\boldsymbol{F}^4_n$ in [1] to classify the indecomposable isomorphic classes of another matrix problem $(\mathscr{A}^0, \mathcal{G}^0)$ corresponding to $\boldsymbol{F}^4_n$. In Section 5, the matrix problem $(\mathscr{A}', \mathcal{G}')$ used to classify the indecomposable homotopy types of $\boldsymbol{F}^4_{n(2)}$ is given. In Section 6, we solve the matrix problem $(\mathscr{A}', \mathcal{G}')$ by the results of indecomposable isomorphic classes of matrix problem $(\mathscr{A}^0, \mathcal{G}^0)$. Section 7 presents the concluding remarks.

2 Preliminaries

In this paper, "polyhedron" is used as "finite CW-complex" and "space" means a based space. We denote by $*_X$ (or by $*$ if there is no ambiguity) the based point of the space X. Denote by $\mathrm{Hot}(X,Y)$ the set of homotopy classes of continuous maps $X \to Y$ and by CW the homotopy category of polyhedra. The suspension functor $\Sigma : X \mapsto X[1]$ ($X[n] = \Sigma^n X$) defines a natural map $\mathrm{Hot}(X,Y) \to \mathrm{Hot}(X[n], Y[n])$. Set $\mathrm{Hos}(X,Y) = \lim_{n\to\infty} \mathrm{Hot}(X[n], Y[n])$. If $\alpha \in \mathrm{Hot}(X[n], Y[n])$, $\beta \in \mathrm{Hot}(Y[m], Z[m])$, the class $\beta[n] \cdot \alpha[m] \in \mathrm{Hot}(X[m+n], Z[m+n])$ after stabilization is, by definition, the product $\beta\alpha$ of the classes of α and β in $\mathrm{Hos}(X,Z)$. Thus we obtain the stable homotopy category of polyhedra CWS. Extending CWS by adding formal negative shifts $X[-n] (n \in \mathbb{N})$ of polyhedra and setting $\mathrm{Hos}(X[-n], Y[-m]) := \mathrm{Hos}(X[m], Y[n])$, one gets the category S of [5], which is a fully additive category, and we denote it by CWS too.

We will say a polyhedron X is p-torsion free if all homology groups of X are p-torsion free, where p is a prime. Denote by $\boldsymbol{A}^k_n$ the full subcategory of CW consisting of $(n-1)$-connected and at most $(n+k)$-dimensional polyhedra, and denote by $\boldsymbol{F}^k_n$ (resp. $\boldsymbol{F}^k_{n(2)}$) the full subcategory of $\boldsymbol{A}^k_n$ consisting of torsion free (resp. 2-torsion free) polyhedra. The suspension gives a functor $\Sigma : \boldsymbol{F}^k_n \to \boldsymbol{F}^k_{n+1}$ (resp. $\Sigma : \boldsymbol{F}^k_{n(2)} \to \boldsymbol{F}^k_{n+1(2)}$). By the Freudenthal Theorem (see [11, Theorem 6.26]), one has the following proposition.

Proposition 2.1. *If* $\dim X \leqslant d$ *and* Y *is* $(n-1)$*-connected, where* $d < 2n-1$*, then the map* $\mathrm{Hot}(X,Y) \to \mathrm{Hot}(X[1], Y[1])$ *is bijective. If* $d = 2n-1$*, this map is surjective. In particular, the map* $\mathrm{Hot}(X[m], Y[m]) \to \mathrm{Hos}(X,Y)$ *is bijective if* $m > d-2n+1$ *and surjective if* $m = d-2n+1$*.*

From Proposition 2.1, we get the following proposition.

Proposition 2.2. *The suspension functor induces equivalences* $\boldsymbol{A}^k_n \xrightarrow{\sim} \boldsymbol{A}^k_{n+1}$ *for all* $n > k+1$*. Moreover, if* $n = k+1$*, the suspension functor* $\boldsymbol{A}^k_n \to \boldsymbol{A}^k_{n+1}$ *is a full representation equivalence, i.e., it is full, dense and reflects isomorphisms.*

Remark 2.3. If an additive functor $F : \mathcal{C} \to \mathcal{D}$ is a full representation equivalence, denoted by $\mathcal{C} \xrightarrow{F \simeq_{\mathrm{rep}}} \mathcal{D}$, then it induces a 1-1 correspondence of indecomposable isomorphic classes of objects of these two additive categories.

Corollary 2.4. *Functors* $\Sigma : \boldsymbol{F}^k_n \to \boldsymbol{F}^k_{n+1}$ *and* $\Sigma : \boldsymbol{F}^k_{n(2)} \to \boldsymbol{F}^k_{n+1(2)}$ *are equivalences of categories for* $n \geqslant k+2$ *and full representation equivalences for* $n = k+1$*.*

Therefore $\boldsymbol{F}^k := \boldsymbol{F}^k_n$ and $\boldsymbol{F}^k_{(2)} := \boldsymbol{F}^k_{n(2)}$ with $n \geqslant k+2$ do not depend on n.

Let $\mathcal{C}$ be an additive category with zero object $*$ and biproducts $A \oplus B$ for any objects $A, B \in \mathcal{C}$, where $X \in \mathcal{C}$ means that X is an object of $\mathcal{C}$. $X \in \mathcal{C}$ is decomposable if there is an isomorphism $X \cong A \oplus B$ where A and B are not isomorphic to $*$, otherwise X is indecomposable. For example, $X \in$ CW (resp. CWS) is indecomposable if we always get one of X_1 and X_2 is contractible whenever X is homotopy equivalent (resp. stable homotopy equivalent) to $X_1 \vee X_2$. A decomposition of $X \in \mathcal{C}$ is an isomorphism $X \cong A_1 \oplus \cdots \oplus A_n$, $n < \infty$, where A_i is indecomposable for $i \in \{1, 2, \ldots, n\}$. The classification problem of category $\mathcal{C}$ is to find a complete list of indecomposable isomorphism types in $\mathcal{C}$ and describe the possible decompositions of objects in $\mathcal{C}$.

Theorem 2.5 (Main theorem). *All indecomposable (stable) homotopy types in* $\boldsymbol{F}^4_{n(2)}$ $(n \geqslant 5)$ *are as follows*:

(i) *The polyhedra given in Theorem* 6.3;

(ii) *the Moore spaces* $M(\mathbb{Z}/p^r, n)$, $M(\mathbb{Z}/p^r, n+1)$, $M(\mathbb{Z}/p^r, n+2)$ *and* $M(\mathbb{Z}/p^r, n+3)$, *where the prime* $p \neq 2$, $r \in \mathbb{N}_+$;

(iii) S^n, S^{n+1}, S^{n+2}, S^{n+3}, S^{n+4} *and* $C_\eta = S^n \bigcup_\eta \mathrm{e}^{n+2}$, *where* η *is the Hopf map.*

3 Techniques

Definition 3.1. Let $\mathcal{A}$ and $\mathcal{B}$ be additive categories. $\mathcal{U}$ is an $\mathcal{A}$-$\mathcal{B}$-bimodule, i.e., a biadditive functor $\mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \boldsymbol{Ab}$, the category of abelian groups. We define the bimodule category $\mathrm{El}(\mathcal{U})$ as follows:

• the set of objects is the disjoint union $\bigcup_{A \in \mathcal{A}, B \in \mathcal{B}} \mathcal{U}(A, B)$.

• A morphism $\alpha \to \beta$, where $\alpha \in \mathcal{U}(A, B)$, $\beta \in \mathcal{U}(A', B')$, is a pair of morphisms $f : A \to A'$, $g : B \to B'$ such that $g\alpha = \beta f \in \mathcal{U}(A, B')$ (We write $g\alpha$ instead of $\mathcal{U}(1, g)\alpha$ and βf instead of $\mathcal{U}(f, 1)\beta$).

Obviously $\mathrm{El}(\mathcal{U})$ is a (full) additive category if so are $\mathcal{A}$ and $\mathcal{B}$.

Suppose $\mathcal{A}$ and $\mathcal{B}$ are two full subcategories of CW (or CWS). Then we denote by $\mathcal{A}\dagger\mathcal{B}$ the full subcategory of CW (or CWS) consisting of cofibers of maps $f : A \to B$, where $A \in \mathcal{A}, B \in \mathcal{B}$. We also denote by $\mathcal{A}\dagger_m\mathcal{B}$ the full subcategory of $\mathcal{A}\dagger\mathcal{B}$ consisting of cofibers of $f : A \to B$ such that $H_m(f) = 0$ and denote by $\Gamma(A, B)$ the subgroup of $\mathrm{Hos}(A, B)$ consisting of maps $f : A \to B$ such that $H_m(f) = 0$, where $A \in \mathcal{A}$, $B \in \mathcal{B}$.

Theorem 3.2. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be two full subcategories of* CWS. *Suppose that* $\mathrm{Hos}(B, A[1]) = 0$ *for all* $A \in \mathcal{A}, B \in \mathcal{B}$. *Consider* $H : \mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \boldsymbol{Ab}$, *i.e.,* $(A, B) \mapsto \mathrm{Hos}(A, B)$, *as an* $\mathcal{A}$-$\mathcal{B}$-*bimodule. Denote by* $\mathcal{I}$ *the ideal of category* $\mathcal{A}\dagger\mathcal{B}$ *consisting of morphisms which factor both through* $\mathcal{B}$ *and* $\mathcal{A}[1]$, *and by* $\mathcal{J}$ *the ideal of the category* $\mathrm{El}(H)$ *consisting of morphisms* $(\alpha, \beta) : f \to f'$ *such that* β *factors through* f' *and* α *factors through* f. *Then*

(1) *the functor* $C : \mathrm{El}(H) \to \mathcal{A}\dagger\mathcal{B}$ $(f \mapsto C_f)$ *induces an equivalence* $\mathrm{El}(H)/\mathcal{J} \simeq \mathcal{A}\dagger\mathcal{B}/\mathcal{I}$.

(2) *Moreover,* $\mathcal{I}^2 = 0$, *hence the projection* $\mathcal{A}\dagger\mathcal{B} \to \mathcal{A}\dagger\mathcal{B}/\mathcal{I}$ *is a representation equivalence.*

(3) *In particular, let* $n < m \leqslant n + k$ *and denote by* $\widetilde{\mathcal{B}}$ *the full subcategory of* $\boldsymbol{F}^k_{n(2)}$ $(n \geqslant k+1)$ *consisting of all* $(n-1)$-*connected polyhedra of dimension at most* m *and by* $\widetilde{\mathcal{A}}$ *the full subcategory of* $\boldsymbol{F}^k_{n(2)}$ $(n \geqslant k+1)$ *consisting of all* $(m-1)$-*connected polyhedra of dimension at most* $n+k-1$. *Then*

$$\mathrm{El}(H)/\mathcal{J} \xrightarrow{\overline{C}\ \simeq} \widetilde{\mathcal{A}}\dagger\widetilde{\mathcal{B}}/\mathcal{I} \xleftarrow{P \simeq_{rep}} \widetilde{\mathcal{A}}\dagger\widetilde{\mathcal{B}}$$

gives a natural one-to-one correspondence between isomorphic classes of objects of $\mathrm{El}(H)/\mathcal{J}$ *and* $\widetilde{\mathcal{A}}\dagger\widetilde{\mathcal{B}}$. $\boldsymbol{F}^k_{n(2)}$ *is the full subcategory of* $\widetilde{\mathcal{A}}\dagger\widetilde{\mathcal{B}}$ *consisting of* 2-*torsion free polyhedra.*

Proof. (1) and (2) of Theorem 3.2 follow directly from [8, Theorem 1.1]. It remains to show that $\boldsymbol{F}^k_{n(2)}$ is a full subcategory of $\widetilde{\mathcal{A}}\dagger\widetilde{\mathcal{B}}$. For any $X \in \boldsymbol{F}^k_{n(2)}$, let $B = X^{n+2}$ be the $(n+2)$-skeleton of X. We get a cofiber sequence $B \to X \to X/B$. Since $X/B \simeq A[1]$ for some A by Proposition 2.2, there is a cofiber sequence $A \xrightarrow{f} B \to X \to X/B$, i.e., $X \simeq C_f$. By the homology exact sequence of cofiber sequence, it is easy to know that $A \in \widetilde{\mathcal{A}}, B \in \widetilde{\mathcal{B}}$. □

The following corollary follows from [8, Corollary 1.2].

Corollary 3.3. *Under conditions of Theorem* 3.2, *let* H_0 *be an* $\mathcal{A}$-$\mathcal{B}$-*subbimodule of* H *such that* $f_1 a f_2 = 0$ *whenever* $a \in \mathrm{El}(H_0)$, $f_i \in \mathrm{Hos}(B_i, A_i)$ $(i = 1, 2)$. *Denote by* $\mathcal{A}\dagger_{H_0}\mathcal{B}$ *the full subcategory of*

$\mathcal{A}\dagger\mathcal{B}$ consisting of cofibers of $a \in \mathrm{El}(H_0)$. $\mathcal{I}_{H_0} = \mathrm{Mor}(\mathcal{A}\dagger_{H_0}\mathcal{B}) \cap \mathcal{I}$ and $\mathcal{J}_{H_0} = \mathrm{Mor}(\mathrm{El}(H_0)) \cap \mathcal{J}$. Then we have

(1) $\mathcal{J}_{H_0}^2 = \mathcal{I}_{H_0}^2 = 0$;

(2) $C : \mathrm{El}(H_0) \xrightarrow{P\simeq_{\mathrm{rep}}} \mathrm{El}(H_0)/\mathcal{J}_{H_0} \xrightarrow{\overline{C}\ \simeq} \mathcal{A}\dagger_{H_0}\mathcal{B}/\mathcal{I}_{H_0} \xleftarrow{P\simeq_{\mathrm{rep}}} \mathcal{A}\dagger_{H_0}\mathcal{B}$.

If $H_0 = \Gamma : \mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \boldsymbol{Ab}$, then $\mathcal{A}\dagger_{H_0}\mathcal{B} = \mathcal{A}\dagger_m\mathcal{B}$.

Matrix problem. Let $\mathscr{A}$ be a set of matrices which is closed under finite direct sums of matrices and let $\mathcal{G}$ denote the set of admissible transformations on $\mathscr{A}$. We say $A \cong B$ in $\mathscr{A}$ if A can be transformed to B by admissible transformations, and we say A is decomposable if $A \cong A_1 \oplus A_2$ for nontrivial $A_1, A_2 \in \mathscr{A}$. The block matrices $\left(\begin{smallmatrix} A_1 \\ 0 \end{smallmatrix}\right)$ and $(A_1\ \ 0)$ are also thought to be decomposable. The matrix problem $(\mathscr{A}, \mathcal{G})$, or simply $\mathscr{A}$, means to classify the indecomposable isomorphic classes of $\mathscr{A}$ (denoted by $\mathrm{ind}\mathscr{A}$) under admissible transformations $\mathcal{G}$. Matrix problem $(\mathscr{A}, \mathcal{G})$ is said to be equivalent to matrix problem $(\mathscr{A}', \mathcal{G}')$ if there is a bijective map $\varphi : \mathscr{A} \to \mathscr{A}'$ such that $A \cong A'$ in $\mathscr{A}$ if and only if $\varphi(A) \cong \varphi(A')$ in $\mathscr{A}'$ and $\varphi(A_1 \oplus A_2) = \varphi(A_1) \oplus \varphi(A_2)$. It is clear that if two matrix problems are equivalent, then there is a one-to-one correspondence between their indecomposable isomorphic classes.

Definition 3.4. Let $\mathscr{A}$ be a set of some matrices and "$\cdot$" be a "product" of two matrices defined in $\mathscr{A}$ ("$\cdot$" may not be the usual matrix product). We say that $M \in \mathscr{A}$ is invertible in $\mathscr{A}$ if there is a matrix $N \in \mathscr{A}$ such that $M \cdot N = N \cdot M = I \in \mathscr{A}$, where I is the identity matrix.

In the following context, for a matrix problem $(\mathscr{A}, \mathcal{G})$, saying a matrix $M \in \mathscr{A}$ is invertible always means that M is invertible in $\mathscr{A}$.

4 The solution of a new matrix problem of the category $\boldsymbol{F}_n^4$ $(n \geqslant 5)$

In the following context, the tabulations

*	*	*
*	*	*

represent the matrices or block matrices. For any category $\mathcal{C}$, denote by $\mathrm{ind}\mathcal{C}$ the set of indecomposable isomorphic classes of $\mathcal{C}$.

$\mathrm{ind}\boldsymbol{F}_n^4$ is known in [1] and Drozd [8] got a matrix problem corresponding to $\boldsymbol{F}_n^4$. Here we need a new matrix problem $(\mathscr{A}^0, \mathcal{G}^0)$ for the classification problem of $\boldsymbol{F}_n^4$.

When $n \geqslant 5$, denote by $\mathcal{B}_0$ the full subcategory of $\boldsymbol{F}_n^4$ $(n \geqslant 5)$ consisting of all $(n-1)$-connected polyhedra of dimension at most $n+2$ and by $\mathcal{A}_0$ the full subcategory of $\boldsymbol{F}_n^4$ $(n \geqslant 5)$ consisting of all $(n+1)$-connected polyhedra of dimension at most $n+2$. Then $\boldsymbol{F}_n^4 = \mathcal{A}_0\dagger_{n+2}\mathcal{B}_0$. From [4] we know

$$\mathrm{ind}\mathcal{A}_0 = \{S^{n+2}, S^{n+3}\}, \quad \mathrm{ind}\mathcal{B}_0 = \left\{S^n, S^{n+1}, S^{n+2}, C_\eta = S^n \bigcup_\eta \mathrm{e}^{n+2}\right\}.$$

Now taking $m = n+2$, we obtain the $\mathcal{A}_0$-$\mathcal{B}_0$ subbimodule Γ of H:

$$\Gamma : \mathcal{A}_0^{op} \times \mathcal{B}_0 \to \boldsymbol{Ab}(A, B) \mapsto \Gamma(A, B),$$

where $\Gamma(A, B)$ is the subgroup of $\mathrm{Hos}(A, B)$ defined in Section 3. Take $H_0 = \Gamma$ in Corollary 3.3. Then $f_1 a f_2 = 0$ whenever $f_i \in \mathrm{Hos}(B_i, A_i)$ $(i = 1, 2)$, $a \in \Gamma(A_2, B_1)$, $A_i \in \mathcal{A}_0$ and $B_i \in \mathcal{B}_0$. Hence by Corollary 3.3, we have

$$C : \mathrm{El}(\Gamma) \xrightarrow{P\simeq_{rep}} \mathrm{El}(\Gamma)/\mathcal{J}_\Gamma \xrightarrow{\overline{C}\ \simeq} \mathcal{A}_0\dagger_{n+2}\mathcal{B}_0/\mathcal{I}_\Gamma \xleftarrow{P\simeq_{rep}} \mathcal{A}_0\dagger_{n+2}\mathcal{B}_0.$$

Objects of $\mathrm{El}(\Gamma)$ can be represented by 5×2 block matrices (γ_{ij}), where block γ_{ij} has entries from the (ij)-th cell of Table 1. Morphisms $\gamma \to \gamma'$ are given by block matrices $\alpha = (\alpha_{ij})_{2\times 2}$, $\beta = (\beta_{ij})_{5\times 5}$, α_{ij} has entries from the (ij)-th cell of Table 2 and β_{ij} has entries from the (ij)-th cell of Table 3. Their

Table 1 $\Gamma(\mathcal{A}_0, \mathcal{B}_0)$

$\mathcal{B}_0$ \ $\mathcal{A}_0$	S^{n+2}	S^{n+3}
S^n	$\mathbb{Z}/2$	$\mathbb{Z}/24$
S^{n+1}	$\mathbb{Z}/2$	$\mathbb{Z}/2$
S^{n+2}	0	$\mathbb{Z}/2$
$C_\eta : n$	0	$\mathbb{Z}/12$
$n+2$	0	0

Table 2 $\mathrm{Hos}(\mathcal{A}_0, \mathcal{A}_0)$

$\mathcal{A}_0$ \ $\mathcal{A}_0$	S^{n+2}	S^{n+3}
S^{n+2}	$\mathbb{Z}$	$\mathbb{Z}/2$
S^{n+3}	0	$\mathbb{Z}$

Table 3 $\mathrm{Hos}(\mathcal{B}_0, \mathcal{B}_0)$

$\mathcal{B}_0$ \ $\mathcal{B}_0$	S^n	S^{n+1}	S^{n+2}	$C_\eta : n$	$n+2$
S^n	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$2\mathbb{Z}$	0
S^{n+1}	0	$\mathbb{Z}$	$\mathbb{Z}/2$	0	0
S^{n+2}	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$
$C_\eta : n$	$\mathbb{Z}$	0	$\mathbb{Z}/2^=$	$\mathbb{Z}^=$	0
$n+2$	0	0	$2\mathbb{Z}^=$	0	$\mathbb{Z}^=$

sizes are compatible with those of γ_{ij} and γ'_{ij} and $\beta\gamma = \gamma'\alpha$. Such a morphism is invertible if and only if α and β are invertible in $\mathrm{Hos}(\mathcal{A}_0, \mathcal{A}_0)$ and $\mathrm{Hos}(\mathcal{B}_0, \mathcal{B}_0)$, respectively. It is equivalent to say that all diagonal blocks of α and β are square, and both $\det(\alpha)$ and $\det(\beta)$ equal ± 1. Since only entries from $\mathbb{Z}$ and $2\mathbb{Z}$ give nonzero input to the determinants, they belong indeed to $\mathbb{Z}$. We get the corresponding matrix problem of $\mathrm{El}(\Gamma)$ which is denoted by $(\mathscr{A}^0, \mathcal{G}^0)$.

In Table 3, $\mathrm{Hos}(C_\eta, C_\eta)$ is identified with the ring

$$\begin{pmatrix} \mathbb{Z}^= & 0 \\ 0 & \mathbb{Z}^= \end{pmatrix} = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \middle| \, a \equiv b \pmod 2 \right\};$$

$\mathrm{Hos}(S^{n+2}, C_\eta)$ is identified with the subgroup $\left(\begin{smallmatrix} \mathbb{Z}/2^= \\ 2\mathbb{Z}^= \end{smallmatrix}\right)$ of

$$\begin{pmatrix} \mathbb{Z}/2 \\ 2\mathbb{Z} \end{pmatrix} = \left\{ \begin{pmatrix} \varepsilon \\ 2a \end{pmatrix} \middle| \, \varepsilon \in \mathbb{Z}/2, a \in \mathbb{Z} \right\},$$

which is the image of the following injective map

$$\mathrm{Hos}(S^{n+2}, C_\eta) \xrightarrow{\mathcal{F}} \begin{pmatrix} \mathbb{Z}/2 \\ 2\mathbb{Z} \end{pmatrix}, \quad f \mapsto \begin{pmatrix} \varepsilon \\ 2a \end{pmatrix}.$$

For any $f \in \mathrm{Hos}(S^{n+2}, C_\eta)$, let $S^{n+1} \xrightarrow{\eta} S^n \xrightarrow{i} C_\eta \xrightarrow{q} S^{n+2}$ be the cofiber sequence and $S^{n+3} \xrightarrow{\eta_{n+2}} S^{n+2}$ be the suspension of η. Then $qf = 2a\iota_{n+2} \in \mathrm{Hos}(S^{n+2}, S^{n+2}) \cong \mathbb{Z}$ for some $a \in \mathbb{Z}$, where $\iota_{n+2} : S^{n+2} \to$

S^{n+2} is the identity map. Let

$$\varepsilon = \begin{cases} 1, & \text{if } f\eta_{n+2} \neq 0, \\ 0, & \text{if } f\eta_{n+2} = 0, \end{cases}$$

and $\mathcal{F}$ be defined by mapping f to $\binom{\varepsilon}{2a}$.

In order to make the product of matrices in Table 3 compatible with the composition of the corresponding maps, special rules for the matrix product in Table 3 is needed:

(1) For $(2a \ \ 0)$ and $\binom{1}{2b}$ respectively in

$$\begin{array}{r|cc|} & \multicolumn{2}{c}{C_\eta : n, n+2} \\ \cline{2-3} S^n & 2\mathbb{Z} & 0 \\ \cline{2-3} \end{array}$$

and

$$\begin{array}{r|c|} & S^{n+2} \\ \cline{2-2} C_\eta : n & \mathbb{Z}/2^{=} \\ n+2 & 2\mathbb{Z}^{=} \\ \cline{2-2} \end{array},$$

$(2a \ \ 0)\binom{1}{2b} = \overline{a}$ in $\begin{array}{r|c|} & S^{n+2} \\ \cline{2-2} S^n & \mathbb{Z}/2 \\ \cline{2-2} \end{array}$, where $\overline{a}$ is the image of a under the quotient map $\mathbb{Z} \to \mathbb{Z}/2$.

(2) For $\binom{a}{0}$ and (ε), respectively in

$$\begin{array}{r|c|} & S^{n} \\ \cline{2-2} C_\eta : n & \mathbb{Z} \\ n+2 & 0 \\ \cline{2-2} \end{array}$$

and $\begin{array}{r|c|} & S^{n+2} \\ \cline{2-2} S^n & \mathbb{Z}/2 \\ \cline{2-2} \end{array}$, $\binom{a}{0}(\varepsilon) = \binom{0}{0}$ in

$$\begin{array}{r|c|} & S^{n+2} \\ \cline{2-2} C_\eta : n & \mathbb{Z}/2^{=} \\ n+2 & 2\mathbb{Z}^{=} \\ \cline{2-2} \end{array}.$$

(3) Keep elements in zero blocks being zero. For example, for any (a) and $(0 \ \ b)$, respectively in

$$\begin{array}{r|c|} & S^{n+2} \\ \cline{2-2} S^n & \mathbb{Z}/2 \\ \cline{2-2} \end{array} \quad \text{and} \quad \begin{array}{r|cc|} & \multicolumn{2}{c}{C_\eta : n, n+2} \\ \cline{2-3} S^{n+2} & 0 & \mathbb{Z} \\ \cline{2-3} \end{array},$$

$(a)(0 \ \ b) = (0 \ \ 0)$ in

$$\begin{array}{r|cc|} & \multicolumn{2}{c}{C_\eta : n, n+2} \\ \cline{2-3} S^{n+2} & \mathbb{Z} & 0 \\ \cline{2-3} \end{array},$$

where the second element is not ab but 0.

Denote by W_x (respectively W^y) the x- horizontal (respectively y-vertical) stripe, where $x \in \{S^n, S^{n+1}, S^{n+2}, C_\eta : n, C_\eta : n+2\}$, $y \in \{S^{n+2}, S^{n+3}\}$, and denote by W_x^y the block corresponding to x-horizontal stripe and y-vertical stripe. Let $\dim W_x =$ the number of rows in W_x, $\dim W^y =$ the number of columns in W^y. Table 1 represents the matrix set $\mathscr{A}^0$. By right multiplication with invertible matrices in Table 2 and left multiplication with invertible matrices in Table 3, Tables 2 and 3 provide admissible transformations $\mathcal{G}^0$ (see [6]) for matrices in Table 1, i.e.,

(a) "elementary-row transformations" of W_x consisting of following three types:

- $(j + ai)$-type: The replacement of the j-th row α_j of W_x by $\alpha_j + a\alpha_i$, where α_i is the i-th row of W_x, $a \in \mathbb{Z}$.
- (ai)-type: The multiplication of the i-th row α_i of W_x by $a \in \{\pm 1\}$.
- (i, j)-type: The transposition of the i-th and j-th row.

(b) "Elementary-column transformations" of W^y which also have three types as for elementary-row transformations.

• (Restriction on (a) and (b)) If one performs a $(j+ai)$-type (respectively (ai)-type and (i,j)-type) elementary-row transformation of $W_{C_\eta:n}$, then one has to perform $(j+a'i)$-type (respectively $(a'i)$-type and (i,j)-type) elementary-row transformation of $W_{C_\eta:n+2}$ simultaneously where $a\equiv a'$ (mod 2) and vice versa.

(c) Adding k times of a column of $W^{S^{n+2}}$ to a column of $W^{S^{n+3}}$.

(d) Adding k times of a row of $W_{S^{n+1}}$ or $W_{S^{n+2}}$ to a row of W_{S^n}.

(e) Adding k times of a row of $W_{S^{n+2}}$ to a row of $W_{S^{n+1}}$.

(f) (i) Adding k times of a row of W_{S^n} to a row of $W_{C_\eta:n}$.

(ii) Adding $2k$ times of a row of $W_{C_\eta:n}$ to a row of W_{S^n}.

(g) Adding $6k$ times of a row of $W_{S^{n+2}}$ to a row of $W_{C_\eta:n}$,

where k is an integer.

Remark 4.1. When admissible transformations above are performed on block matrix $\gamma=(\gamma_{ij})$, where block γ_{ij} has entries from (ij)-cell of Table 1, we should note that

(1) If (ij)-cell of Table 1 is zero, then γ_{ij} keeps being zero after admissible transformations.

(2) Adding $1\in\mathbb{Z}/2$ to an element $a\in\mathbb{Z}/24$ gives $a+12$ in $\mathbb{Z}/24$, since η^3 is 12 in $\mathbb{Z}/24=\mathrm{Hos}(S^{n+3},S^n)$.

(3) The reason for (g) is as follow: In the definition of the injective map $\mathcal{F}$ above, for any $f\in\mathrm{Hos}(S^{n+2},C_\eta)$, $f\eta=ix$ for some $x\in\mathrm{Hos}(S^{n+3},S^n)=\mathbb{Z}/24$. If $qf=2\iota_{n+2}\in\mathrm{Hos}(S^{n+2},S^{n+2})$ then $x=6$ (see [12, Proposition 6(iii)]).

From the known fact that

$$\mathrm{ind}(\mathscr{A}^0)\cong\mathrm{indEl}(\Gamma)\cong\mathrm{ind}(\mathcal{A}_0\dagger_{n+2}\mathcal{B}_0)=\mathrm{ind}\boldsymbol{F}_n^4,$$

we have the following results.

List (∗) (I) $X(\eta v\eta)=S^n\vee S^{n+2}\bigcup_{i_1\eta}\mathrm{e}^{n+2}\bigcup_{i_1v+i_2\eta}\mathrm{e}^{n+4}$ corresponds to

	S^{n+3}
S^{n+2}	1
$C_\eta:n$	v
$n+2$	0

,

where $v\in\{1,2,3\}\subset\mathbb{Z}/12$.

(II) (1) $X(\eta\eta v\eta\eta)=S^n\vee S^{n+1}\bigcup_{i_1\eta\eta}\mathrm{e}^{n+3}\bigcup_{i_1v+i_2\eta\eta}\mathrm{e}^{n+4}$ corresponds to

	S^{n+2}	S^{n+3}
S^n	1	v
S^{n+1}	0	1

.

(2) $X(\eta\eta v\eta)=S^n\vee S^{n+2}\bigcup_{i_1\eta\eta}\mathrm{e}^{n+3}\bigcup_{i_1v+i_2\eta}\mathrm{e}^{n+4}$ corresponds to

	S^{n+2}	S^{n+3}
S^n	1	v
S^{n+2}	0	1

.

(3) $X(\eta v\eta\eta)=S^n\vee S^{n+1}\bigcup_{i_1\eta}\mathrm{e}^{n+2}\bigcup_{i_1v+i_2\eta}\mathrm{e}^{n+4}$ corresponds to

	S^{n+3}
S^{n+1}	1
$C_\eta:n$	v
$n+2$	0

.

(4) $X(\eta\eta v)=S^n\bigcup_{\eta\eta}\mathrm{e}^{n+3}\bigcup_v\mathrm{e}^{n+4}$ corresponds to

	S^{n+2}	S^{n+3}
S^n	1	v

.

(5) $X(v\eta\eta) = S^n \vee S^{n+1} \bigcup_{i_1 v + i_2 \eta\eta} \mathrm{e}^{n+4}$ corresponds to

$$\begin{array}{c|c|}
 & S^{n+3} \\ \cline{2-2}
S^n & v \\
S^{n+1} & 1 \\ \cline{2-2}
\end{array}.$$

(6) $X(\eta v) = S^n \bigcup_\eta \mathrm{e}^{n+2} \bigcup_v \mathrm{e}^{n+4}$ corresponds to

$$\begin{array}{c|c|}
 & S^{n+3} \\ \cline{2-2}
C_\eta : n & v \\
n+2 & 0 \\ \cline{2-2}
\end{array}.$$

(7) $X(v\eta) = S^n \vee S^{n+2} \bigcup_{i_1 v + i_2 \eta} \mathrm{e}^{n+4}$ corresponds to

$$\begin{array}{c|c|}
 & S^{n+3} \\ \cline{2-2}
S^n & v \\
S^{n+2} & 1 \\ \cline{2-2}
\end{array},$$

where $v \in \{1, 2, 3, 4, 5, 6\} \subset \mathbb{Z}/24$ in the cases (1), (2), (4), (5), (7) of (II), and $v \in \{1, 2, 3, 4, 5, 6\} \subset \mathbb{Z}/12$ in the cases (3), (6) of (II).

(III) $X(v) = S^n \bigcup_v \mathrm{e}^{n+4}$ corresponds to

$$\begin{array}{c|c|}
 & S^{n+3} \\ \cline{2-2}
S^n & v \\ \cline{2-2}
\end{array},$$

where $v \in \{1, 2, \ldots, 12\} \subset \mathbb{Z}/24$.

(IV) (1) $X(\eta_1) = S^{n+1} \bigcup_\eta \mathrm{e}^{n+3}$ corresponds to

$$\begin{array}{c|c|}
 & S^{n+2} \\ \cline{2-2}
S^{n+1} & 1 \\ \cline{2-2}
\end{array}.$$

(2) $X(\eta_2) = S^{n+2} \bigcup_\eta \mathrm{e}^{n+4}$ corresponds to

$$\begin{array}{c|c|}
 & S^{n+3} \\ \cline{2-2}
S^{n+2} & 1 \\ \cline{2-2}
\end{array}.$$

(3) $X(\eta\eta)_0 = S^n \bigcup_{\eta\eta} \mathrm{e}^{n+3}$ corresponds to

$$\begin{array}{c|c|}
 & S^{n+2} \\ \cline{2-2}
S^n & 1 \\ \cline{2-2}
\end{array}.$$

(4) $X(\eta\eta)_1 = S^{n+1} \bigcup_{\eta\eta} \mathrm{e}^{n+4}$ corresponds to

$$\begin{array}{c|c|}
 & S^{n+3} \\ \cline{2-2}
S^{n+1} & 1 \\ \cline{2-2}
\end{array}.$$

For a wedge of spaces $X \vee Y$, $i_1 : X \hookrightarrow X \vee Y$ and $i_2 : X \hookrightarrow X \vee Y$ above are the canonical inclusions.

Remark 4.2. Indecomposable homotopy types in $\{A[1] \mid A \in \mathrm{ind}\mathcal{A}_0\}$ and $\mathrm{ind}\mathcal{B}_0$ of $\boldsymbol{F}_n^4$ are not contained in List (*). An element $A[1]$ of $\{A[1] \mid A \in \mathrm{ind}\mathcal{A}_0\}$ (resp. B of $\mathrm{ind}\mathcal{B}_0$) can be considered as a mapping cone of map $A \to *$ (resp. $* \to B$) in $\mathcal{A}_0 \dagger_{n+2} \mathcal{B}_0$ which corresponds to 0×1 matrix (resp. 1×0 matrix) in $\mathscr{A}^0$. For a general matrix problem $(\mathscr{A}, \mathcal{G})$, these 0×1 and 1×0 matrices are regarded as elements in $\mathrm{ind}\mathscr{A}$, but will not be listed to simplify notation.

5 The reduction of the classification problem of $\boldsymbol{F}^4_{n(2)}$ ($n \geqslant 5$)

Let M_t^k be the Moore space $M(\mathbb{Z}/t, k)$, $t, k \in \mathbb{N}_+ = \{1, 2, \ldots\}$.

Take $m = n + 2$ and two full subcategories $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{B}}$ of $\boldsymbol{F}^4_{n(2)}$ as in Theorem 3.2(3). By the results of the indecomposable homotopy types of $\boldsymbol{A}_n^2$ ($n \geqslant 3$) in [4], we have

$$\mathrm{ind}\widetilde{\mathcal{A}} = \{S^{n+2}, S^{n+3}, M_{p^r}^{n+2} \mid \text{ prime } p \neq 2, r \in \mathbb{N}_+\},$$

$$\mathrm{ind}\widetilde{\mathcal{B}} = \left\{ S^n, S^{n+1}, S^{n+2}, C_\eta = S^n \bigcup_\eta \mathrm{e}^{n+2}, M^n_{p^s}, M^{n+1}_{p^s} \,\middle|\, \text{prime } p \neq 2, s \in \mathbb{N}_+ \right\}.$$

Lemma 5.1. • $\mathrm{Hos}(M^{n+2}_{p^r}, B) = 0$ *for any* $B \in \mathrm{ind}\widetilde{\mathcal{B}}$*, where prime* $p \neq 2, 3; r \in \mathbb{N}_+$.

- $\mathrm{Hos}(A, M^n_{p^s}) = 0$ *for any* $A \in \mathrm{ind}\widetilde{\mathcal{A}}$*, where prime* $p \neq 2, 3; s \in \mathbb{N}_+$.
- $\mathrm{Hos}(A, M^{n+1}_{p^s}) = 0$ *for any* $A \in \mathrm{ind}\widetilde{\mathcal{A}}$*, where prime* $p \neq 2; s \in \mathbb{N}_+$.

Proof. It follows from the triviality of p-primary component of relevant homotopy groups of spheres and the universal coefficients theorem for homotopy groups with coefficients. □

For $C_f \in \widetilde{\mathcal{A}}\dagger\widetilde{\mathcal{B}}$, where $f : A \to B$, $A = \vee A_i$, $A_i \in \mathrm{ind}\widetilde{\mathcal{A}}$, $B = \vee B_j$, $B_j \in \mathrm{ind}\widetilde{\mathcal{B}}$. If $A_i = M^{n+2}_{p^r}$ $(p \neq 2, 3)$ for some i, then $A_i[1]$ splits out of C_f. Similarly if $B_j = M^n_{p^s}$ $(p \neq 2, 3)$ or $M^{n+1}_{p^s}$ $(p \neq 2)$ for some j, then this B_j also splits out of C_f. So we get the following lemma.

Lemma 5.2. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be the full subcategories of* $\widetilde{\mathcal{A}}$ *and* $\widetilde{\mathcal{B}}$*, respectively, such that*

$$\mathrm{ind}\mathcal{A} = \{S^{n+2}, S^{n+3}, M^{n+2}_{3^r} \mid r \in \mathbb{N}_+\},$$
$$\mathrm{ind}\mathcal{B} = \left\{ S^n, S^{n+1}, S^{n+2}, C_\eta = S^n \bigcup_\eta \mathrm{e}^{n+2}, M^n_{3^s} \,\middle|\, s \in \mathbb{N}_+ \right\}.$$

Then

$$\mathrm{ind}(\widetilde{\mathcal{A}}\dagger\widetilde{\mathcal{B}}) = \mathrm{ind}(\mathcal{A}\dagger\mathcal{B}) \cup \{M^{n+3}_{p^r}, M^n_{p^r}, M^{n+1}_{q^r} \mid \textit{primes } p \neq 2, 3, q \neq 2; r \in \mathbb{N}_+\}.$$

By Theorem 3.2(3), we have the following corollary.

Corollary 5.3. $\mathrm{ind}\boldsymbol{F}^4_{n(2)} = \{X \in \mathrm{ind}(\mathcal{A}\dagger\mathcal{B}) \mid X$ *is* 2*-torsion free*$\} \cup \{M^{n+3}_{p^r}, M^n_{p^r}, M^{n+1}_{q^r} \mid$ *prime* $p \neq 2, 3$*, prime* $q \neq 2$ *and* $r \in \mathbb{N}_+\}$.

In order to get $\mathrm{ind}\boldsymbol{F}^4_{n(2)}$, it suffices to compute $\{X \in \mathrm{ind}(\mathcal{A}\dagger\mathcal{B}) \mid X \text{ is 2-torsion free}\}$.

Let $\Gamma : \mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \boldsymbol{Ab}$, $\Gamma(A, B) = \{g \in \mathrm{Hos}(A, B) \mid H_{n+2}(g) = 0\}$, defined in Section 3, be a sub-bimodule of $\mathcal{A}$-$\mathcal{B}$-bimodule $H : \mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \boldsymbol{Ab}$, $H(A, B) = \mathrm{Hos}(A, B)$.

Lemma 5.4.

$$\begin{aligned}&\{X \in \mathrm{ind}(\mathcal{A}\dagger\mathcal{B}) \mid X \textit{ is 2-torsion free}\}\\ &\quad = \{M^{n+2}_{p^r} \mid \textit{prime } p \neq 2, r \in \mathbb{N}_+\} \cup \{C(f) \textit{ is indecomposable} \mid f \in \mathrm{El}(\Gamma)\}.\end{aligned}$$

Proof. For integers $k, l, t, u, v, w \geqslant 0$, let

$$A(k, l) := \bigvee_k S^{n+2} \vee \bigvee_l S^{n+3} \in \mathcal{A}_0,$$
$$B(t, u, v, w) := \bigvee_t S^{n+2} \vee \bigvee_u C_\eta \vee \bigvee_v S^n \vee \bigvee_w S^{n+1} \in \mathcal{B}_0.$$

For any 2-torsion free polyhedra $X = C_f \in \mathcal{A}\dagger\mathcal{B}$, $f \in \mathrm{Hos}(A, B)$, where $A \in \mathcal{A}$, $B \in \mathcal{B}$. Suppose that

$$A = A(k, l) \vee M_A, \quad B = B(t, u, v, w) \vee M_B,$$

where M_A (resp. M_B) is a wedge of Moore spaces $\{M^{n+2}_{3^r} \mid r \in \mathbb{N}_+\}$ (resp. $\{M^n_{3^s} \mid s \in \mathbb{N}_+\}$). Let

$$A(k, l) \xhookrightarrow{j_A} A, \quad B \xrightarrow{p_B} B(t, u, v, w)$$

be the canonical inclusion and projection of the summands, respectively. For

$$h := p_B f j_A \in \mathrm{Hos}(A(k, l), B(t, u, v, w)),$$

by the proof of [9, Theorem 5.5], we have the commutable top square in the following diagram,

$$
\begin{array}{ccccccc}
(\bigvee_{k_1} S^{n+2})\vee A(k',l) & & & \xrightarrow{\quad h_1\vee h' \quad} & & & (\bigvee_{k_1} S^{n+2})\vee B(t',u,v,w) \\
\| & & & & & & \| \\
A(k,l) & \xrightarrow{\alpha\ \simeq} & A(k,l) & \xrightarrow{h} & B(t,u,v,w) & \xrightarrow{\beta\ \simeq} & B(t,u,v,w) \\
\big\downarrow{\scriptstyle j_A} & & \big\downarrow{\scriptstyle j_A} & & \big\uparrow{\scriptstyle p_B} & & \big\uparrow{\scriptstyle p_B} \\
A & \xrightarrow{\alpha\vee 1_{M_A}\ \simeq} & A & \xrightarrow{f} & B & \xrightarrow{\beta\vee 1_{M_B}\ \simeq} & B \\
\| & & & & & & \| \\
(\bigvee_{k_1} S^{n+2})\vee(A(k',l)\vee M_A) & & & \overset{h_1\vee f'}{\cdots\cdots\cdots\rightarrow} & & & (\bigvee_{k_1} S^{n+2})\vee(B(t',u,v,w)\vee M_B),
\end{array}
$$

where $k_1+k'=k, k_1+t'=t$, α and β are self-homotopy equivalences of $A(k,l)$ and $B(t,u,v,w)$, respectively and the maps

$$\bigvee_{k_1} S^{n+2} \xrightarrow{h_1} \bigvee_{k_1} S^{n+2}, \quad A(k',l) \xrightarrow{h'} B(t',u,v,w)$$

satisfy that

(i) the mapping cone $C_{h_1} = \bigvee_i M^{n+2}_{\alpha_i}$, where $\alpha_i \in \mathbb{N}_+$ is odd for each i;

(ii) the composition of maps

$$\bigvee_{k'} S^{n+2} \xrightarrow{j} A(k',l) \xrightarrow{h'} B(t',u,v,w) \xrightarrow{p} \bigvee_{t'} S^{n+2} \vee \bigvee_{u} C_\eta$$

is zero, where j and p are canonical inclusion and projection of the summands, respectively. This is equivalent to the statement that $H_{n+2}(h')=0$.

Note that $\mathrm{Hos}(S^{n+2}, M_B)=0$ and $\mathrm{Hos}(M_A, S^{n+2})=0$. Hence

$$(\beta\vee 1_{M_B})f(\alpha\vee 1_{M_A}) = h_1 \vee f'$$

such that $A(k',l)\vee M_A \xrightarrow{f'} B(t',u,v,w)\vee M_B$ satisfies $H_{n+2}(f')=0$. It implies that

$$X = C_f \simeq C_{h_1} \vee C_{f'} = \bigvee_i M^{n+2}_{\alpha_i} \vee C_{f'}, \quad f' \in \mathrm{El}(\Gamma).$$

Since for any $f \in \mathrm{El}(\Gamma)$, $C(f)=C_f$ is 2-torsion free. By the above analysis, we complete the proof of Lemma 5.4. □

Take $H_0=\Gamma$ in Corollary 3.3, then $f_1 a f_2 = 0$ whenever $f_i \in \mathrm{Hos}(B_i, A_i)$ $(i=1,2)$, $a \in \Gamma(A_2, B_1)$, $A_i \in \mathcal{A}$ and $B_i \in \mathcal{B}$. Hence, by Corollary 3.3, we have

$$C: \mathrm{El}(\Gamma) \xrightarrow{P \simeq_{rep}} \mathrm{El}(\Gamma)/\mathcal{J}_\Gamma \xrightarrow{\overline{C}\ \simeq} \mathcal{A}\dagger_{n+2}\mathcal{B}/\mathcal{I}_\Gamma \xleftarrow{P\simeq_{rep}} \mathcal{A}\dagger_{n+2}\mathcal{B},$$

which implies the following result.

Corollary 5.5. *In Lemma* 5.4,

$$\{C(f) \textit{ is indecomposable} \mid f \in \mathrm{El}(\Gamma)\} = \mathrm{ind}(\mathcal{A}\dagger_{n+2}\mathcal{B}) \cong \mathrm{indEl}(\Gamma).$$

In the remainder of this section, we will find the matrix problem corresponding to $\mathrm{El}(\Gamma)$.

Compute $\Gamma(A,B)$ for $A \in \mathrm{ind}\mathcal{A}$, $B \in \mathrm{ind}\mathcal{B}$; $\mathrm{Hos}(A,A')$ for $A, A' \in \mathrm{ind}\mathcal{A}$ and $\mathrm{Hos}(B,B')$ for $B, B' \in \mathrm{ind}\mathcal{B}$ as in [7]. For example,

- $\Gamma(S^{n+2}, S^{n+2}) = \Gamma(S^{n+2}, C_\eta) = 0$.
- $\mathrm{Hos}(M^{n+2}_{3^r}, M^n_{3^s}) =$

	$M^{n+2}_{3^r}: n+2$	$n+3$
$M^n_{3^s}: n$	0	$\mathbb{Z}/3$
$n+1$	0	0

.

- $\mathrm{Hos}(M^{n+2}_{3^r}, C_\eta) = \begin{array}{r|cc|} & \multicolumn{2}{c}{M^{n+2}_{3^r}: n+2 \quad n+3} \\ \cline{2-3} C_\eta : n & 0 & \mathbb{Z}/3 \\ n+1 & 0 & 0 \\ \cline{2-3} \end{array}.$

- $\mathrm{Hos}(M^{n+2}_{3^r}, M^{n+2}_{3^s}) = \begin{array}{r|cc|} & \multicolumn{2}{c}{M^{n+2}_{3^r}: n+2 \quad n+3} \\ \cline{2-3} M^{n+2}_{3^s} : n+2 & \mathbb{Z}/3^{s=} & 0 \\ n+3 & 0 & \mathbb{Z}/3^{r=} \\ \cline{2-3} \end{array},$

where

$$\begin{array}{|cc|} \hline \mathbb{Z}/3^{s=} & 0 \\ 0 & \mathbb{Z}/3^{r=} \\ \hline \end{array} = \left\{ \begin{pmatrix} \bar a & 0 \\ 0 & \bar b \end{pmatrix} \;\middle|\; \bar a \in \mathbb{Z}/3^s, \bar b \in \mathbb{Z}/3^r \text{ and } 3^r a = 3^s b \text{ in } \mathbb{Z} \right\}$$

$$= \begin{cases} \left\{ \begin{pmatrix} \bar a & 0 \\ 0 & \overline{3^{r-s}a} \end{pmatrix} \;\middle|\; \bar a \in \mathbb{Z}/3^s, \overline{3^{r-s}a} \in \mathbb{Z}/3^r \right\}, & r > s, \\ \left\{ \begin{pmatrix} \overline{3^{s-r}b} & 0 \\ 0 & \bar b \end{pmatrix} \;\middle|\; \overline{3^{s-r}b} \in \mathbb{Z}/3^s, \bar b \in \mathbb{Z}/3^r \right\}, & r \leqslant s. \end{cases}$$

Now we get the matrix problem $(\widetilde{\mathscr{A}}, \widetilde{\mathcal{G}})$ corresponding to $\mathrm{El}(\Gamma)$ as follows. The objects of $\mathrm{El}(\Gamma)$ can be represented by block matrices $\gamma = (\gamma_{ij})$ with finite order in Table 4 which provides the matrix set $\widetilde{\mathscr{A}}$, where block γ_{ij} has entries from the (ij)-th cell of Table 4. Morphisms $\gamma \to \gamma'$ are given by block matrices $\alpha = (\alpha_{ij})$ and $\beta = (\beta_{ij})$ from Tables 5 and 6, respectively with proper order, which provide the admissible transformations $\widetilde{\mathcal{G}}$.

It is well known that $\mathrm{ind}\widetilde{\mathscr{A}} \cong \mathrm{indEl}(\Gamma)$.

We eliminate the zero stripes $M^{n+2}_{3^r} : n+2$ and $M^n_{3^s} : n+1$ of matrices in $\widetilde{\mathscr{A}}$ to simplify the matrix problem $(\widetilde{\mathscr{A}}, \widetilde{\mathcal{G}})$ to the following equivalent matrix problem $(\mathscr{A}', \mathcal{G}')$, i.e., Tables 7–9.

In Tables 7–9, $M^{n+2}_{3^r}$ represents the $M^{n+2}_{3^r} : n+3$-vertical stripe and $M^n_{3^s}$ represents the $M^n_{3^s} : n$-horizontal stripe. Table 7 provides the matrix set $\mathscr{A}'$. Tables 8 and 9 provide the (non-trivial) admissible transformations $\mathcal{G}'$:

$\mathcal{G}'_{\mathrm{el}}$: "elementary-row (column)" transformations of each horizontal (vertical) stripe.

$\mathcal{G}'_{\mathrm{co}}$: (i) $W^{S^{n+2}} < W^{S^{n+3}}$.

(ii) $W^{S^{n+3}} < W^{M^{n+2}_{3^r}}$; $W^{M^{n+2}_{3^{r+1}}} < W^{M^{n+2}_{3^r}}$ for any $r \in \mathbb{N}_+$.

$\mathcal{G}'_{\mathrm{ro}}$: (i) $W_{S^{n+2}} < W_{S^{n+1}} < W_{S^n}$.

(ii) $W_{S^n} < W_{C_\eta : n} < W_{M^n_{3^s}}$; $W_{M^n_{3^{s+1}}} < W_{M^n_{3^s}}$; $2W_{C_\eta : n} < W_{S^n}$ for any $s \in \mathbb{N}_+$.

(iii) $6W_{S^{n+2}} < W_{C_\eta : n}$.

Table 4 $\Gamma(\mathcal{A}, \mathcal{B})$

$\mathcal{B}$ \ $\mathcal{A}$	S^{n+2}	S^{n+3}	$M^{n+2}_3 : n+2$	$n+3$	$M^{n+2}_{3^2} : n+2$	$n+3$	$M^{n+2}_{3^3} : n+2$	$n+3$	$\cdots$
S^n	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	$\mathbb{Z}/3$	0	$\mathbb{Z}/3$	0	$\mathbb{Z}/3$	$\cdots$
S^{n+1}	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	0	0	0	0	0	$\cdots$
S^{n+2}	0	$\mathbb{Z}/2$	0	0	0	0	0	0	$\cdots$
$C_\eta : n$	0	$\mathbb{Z}/12$	0	$\mathbb{Z}/3$	0	$\mathbb{Z}/3$	0	$\mathbb{Z}/3$	$\cdots$
$n+2$	0	0	0	0	0	0	0	0	$\cdots$
$M^n_3 : n$	0	$\mathbb{Z}/3$	0	$\mathbb{Z}/3$	0	$\mathbb{Z}/3$	0	$\mathbb{Z}/3$	$\cdots$
$n+1$	0	0	0	0	0	0	0	0	$\cdots$
$M^n_{3^2} : n$	0	$\mathbb{Z}/3$	0	$\mathbb{Z}/3$	0	$\mathbb{Z}/3$	0	$\mathbb{Z}/3$	$\cdots$
$n+1$	0	0	0	0	0	0	0	0	$\cdots$
$M^n_{3^3} : n$	0	$\mathbb{Z}/3$	0	$\mathbb{Z}/3$	0	$\mathbb{Z}/3$	0	$\mathbb{Z}/3$	$\cdots$
$n+1$	0	0	0	0	0	0	0	0	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

Table 5 Hos($\mathcal{A},\mathcal{A}$)

$\mathcal{A}$ \ $\mathcal{A}$	S^{n+2}	S^{n+3}	$M_3^{n+2}: n+2$	$n+3$	$M_{3^2}^{n+2}: n+2$	$n+3$	$\cdots$	$M_{3^r}^{n+2}: n+2$	$n+3$	$\cdots$
S^{n+2}	$\mathbb{Z}$	$\mathbb{Z}/2$	0	0	0	0	$\cdots$	0	0	$\cdots$
S^{n+3}	0	$\mathbb{Z}$	0	$\mathbb{Z}/3$	0	$\mathbb{Z}/3^2$	$\cdots$	0	$\mathbb{Z}/3^r$	$\cdots$
$M_3^{n+2}: n+2$	$\mathbb{Z}/3$	0	$\mathbb{Z}/3^=$	0	$\mathbb{Z}/3^=$	0	$\cdots$	$\mathbb{Z}/3^=$	0	$\cdots$
$n+3$	0	0	0	$\mathbb{Z}/3^=$	0	$\mathbb{Z}/3^{2=}$	$\cdots$	0	$\mathbb{Z}/3^{r=}$	$\cdots$
$M_{3^2}^{n+2}: n+2$	$\mathbb{Z}/3^2$	0	$\mathbb{Z}/3^{2=}$	0	$\mathbb{Z}/3^{2=}$	0	$\cdots$	$\mathbb{Z}/3^{2=}$	0	$\cdots$
$n+3$	0	0	0	$\mathbb{Z}/3^=$	0	$\mathbb{Z}/3^{2=}$	$\cdots$	0	$\mathbb{Z}/3^{r=}$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$M_{3^r}^{n+2}: n+2$	$\mathbb{Z}/3^r$	0	$\mathbb{Z}/3^{r=}$	0	$\mathbb{Z}/3^{r=}$	0	$\cdots$	$\mathbb{Z}/3^{r=}$	0	$\cdots$
$n+3$	0	0	0	$\mathbb{Z}/3^=$	0	$\mathbb{Z}/3^{2=}$	$\cdots$	0	$\mathbb{Z}/3^{r=}$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

Table 6 Hos($\mathcal{B},\mathcal{B}$)

$\mathcal{B}$ \ $\mathcal{B}$	S^n	S^{n+1}	S^{n+2}	$C_\eta: n$	$n+2$	$M_3^n: n$	$n+1$	$M_{3^2}^n: n$	$n+1$	$\cdots$	$M_{3^r}^n: n$	$n+1$	$\cdots$
S^n	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$2\mathbb{Z}$	0	0	0	0	0	$\cdots$	0	0	$\cdots$
S^{n+1}	0	$\mathbb{Z}$	$\mathbb{Z}/2$	0	0	0	0	0	0	$\cdots$	0	0	$\cdots$
S^{n+2}	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	0	0	0	$\cdots$	0	0	$\cdots$
$C_\eta: n$	$\mathbb{Z}$	0	$\mathbb{Z}/2^=$	$\mathbb{Z}^=$	0	0	0	0	0	$\cdots$	0	0	$\cdots$
$n+2$	0	0	$2\mathbb{Z}^=$	0	$\mathbb{Z}^=$	0	0	0	0	$\cdots$	0	0	$\cdots$
$M_3^n: n$	$\mathbb{Z}/3$	0	0	$\mathbb{Z}/3$	0	$\mathbb{Z}/3^=$	0	$\mathbb{Z}/3^=$	0	$\cdots$	$\mathbb{Z}/3^=$	0	$\cdots$
$n+1$	0	0	0	0	0	0	$\mathbb{Z}/3^=$	0	$\mathbb{Z}/3^{2=}$	$\cdots$	0	$\mathbb{Z}/3^{r=}$	$\cdots$
$M_{3^2}^n: n$	$\mathbb{Z}/3^2$	0	0	$\mathbb{Z}/3^2$	0	$\mathbb{Z}/3^{2=}$	0	$\mathbb{Z}/3^{2=}$	0	$\cdots$	$\mathbb{Z}/3^{2=}$	0	$\cdots$
$n+1$	0	0	0	0	0	0	$\mathbb{Z}/3^=$	0	$\mathbb{Z}/3^{2=}$	$\cdots$	0	$\mathbb{Z}/3^{r=}$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$M_{3^r}^n: n$	$\mathbb{Z}/3^r$	0	0	$\mathbb{Z}/3^r$	0	$\mathbb{Z}/3^{r=}$	0	$\mathbb{Z}/3^{r=}$	0	$\cdots$	$\mathbb{Z}/3^{r=}$	0	$\cdots$
$n+1$	0	0	0	0	0	0	$\mathbb{Z}/3^=$	0	$\mathbb{Z}/3^{2=}$	$\cdots$	0	$\mathbb{Z}/3^{r=}$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

Remark 5.6. (1) $W_x < W_y$ means that adding k times of a row of W_x to a row of W_y is admissible and $aW_x < W_y$ ($a \in \mathbb{N}_+$) means adding ak times of a row of W_x to a row of W_y is admissible where k is an any nonzero integer. $W^x < W^y$ has the similar meaning for corresponding vertical stripes.

(2) Similarly, zero blocks in Table 7 should keep being zero after admissible transformations. Adding $1 \in \mathbb{Z}/2$ to an element $a \in \mathbb{Z}/24$ gives $a + 12$ in $\mathbb{Z}/24$.

(3) Special rules of matrix product in Table 3 are also needed for matrix product in Table 9.

6 Computation of ind$\mathscr{A}'$ for matrix problem $(\mathscr{A}', \mathcal{G}')$

In this section we solve the matrix problem $(\mathscr{A}', \mathcal{G}')$ to get ind$\mathscr{A}'$. Then we get the ind$\boldsymbol{F}^4_{n(2)}$ by ind$\mathscr{A}'$.

6.1 p-primary component of matrix problem $(\mathscr{A}', \mathcal{G}')$, $p = 2, 3$

Let $\Gamma'(\mathcal{A},\mathcal{B})_{(2)}$ be the 2-primary component of $\Gamma'(\mathcal{A},\mathcal{B})$, which means we replace $\mathbb{Z}/24$ by $\mathbb{Z}/8$, $\mathbb{Z}/12$ by $\mathbb{Z}/4$ and $\mathbb{Z}/3$ by 0 in Table 7 to get Table 10. Similarly, $\Gamma'(\mathcal{A},\mathcal{B})_{(3)}$ is the 3-primary component of $\Gamma'(\mathcal{A},\mathcal{B})$, which means we replace $\mathbb{Z}/24$ by $\mathbb{Z}/3$, $\mathbb{Z}/12$ by $\mathbb{Z}/3$ and $\mathbb{Z}/2$ by 0 in Table 7 to get Table 11.

Table 7 $\Gamma'(\mathcal{A},\mathcal{B})$

$\mathcal{B}$ \ $\mathcal{A}$	S^{n+2}	S^{n+3}	M_3^{n+2}	$M_{3^2}^{n+2}$	$M_{3^3}^{n+2}$	$\cdots$
S^n	$\mathbb{Z}/2$	$\mathbb{Z}/24$	$\mathbb{Z}/3$	$\mathbb{Z}/3$	$\mathbb{Z}/3$	$\cdots$
S^{n+1}	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	0	0	$\cdots$
S^{n+2}	0	$\mathbb{Z}/2$	0	0	0	$\cdots$
$C_\eta : n$	0	$\mathbb{Z}/12$	$\mathbb{Z}/3$	$\mathbb{Z}/3$	$\mathbb{Z}/3$	$\cdots$
$n+2$	0	0	0	0	0	$\cdots$
M_3^n	0	$\mathbb{Z}/3$	$\mathbb{Z}/3$	$\mathbb{Z}/3$	$\mathbb{Z}/3$	$\cdots$
$M_{3^2}^n$	0	$\mathbb{Z}/3$	$\mathbb{Z}/3$	$\mathbb{Z}/3$	$\mathbb{Z}/3$	$\cdots$
$M_{3^3}^n$	0	$\mathbb{Z}/3$	$\mathbb{Z}/3$	$\mathbb{Z}/3$	$\mathbb{Z}/3$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

Table 8 $\mathrm{Hos}'(\mathcal{A},\mathcal{A})$

$\mathcal{A}$ \ $\mathcal{A}$	S^{n+2}	S^{n+3}	M_3^{n+2}	$M_{3^2}^{n+2}$	$\cdots$	$M_{3^r}^{n+2}$	$\cdots$
S^{n+2}	$\mathbb{Z}$	$\mathbb{Z}/2$	0	0	$\cdots$	0	$\cdots$
S^{n+3}	0	$\mathbb{Z}$	$\mathbb{Z}/3$	$\mathbb{Z}/3^2$	$\cdots$	$\mathbb{Z}/3^r$	$\cdots$
M_3^{n+2}	0	0	$\mathbb{Z}/3$	0	$\cdots$	0	$\cdots$
$M_{3^2}^{n+2}$	0	0	$\mathbb{Z}/3$	$\mathbb{Z}/3^2$	$\cdots$	0	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$M_{3^r}^{n+2}$	0	0	$\mathbb{Z}/3$	$\mathbb{Z}/3^2$	$\cdots$	$\mathbb{Z}/3^r$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

Table 9 $\mathrm{Hos}'(\mathcal{B},\mathcal{B})$

$\mathcal{B}$ \ $\mathcal{B}$	S^n	S^{n+1}	S^{n+2}	$C_\eta : n$	$n+2$	M_3^n	$M_{3^2}^n$	$\cdots$	$M_{3^r}^n$	$\cdots$
S^n	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$2\mathbb{Z}$	0	0	0	$\cdots$	0	$\cdots$
S^{n+1}	0	$\mathbb{Z}$	$\mathbb{Z}/2$	0	0	0	0	$\cdots$	0	$\cdots$
S^{n+2}	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	0	$\cdots$	0	$\cdots$
$C_\eta : n$	$\mathbb{Z}$	0	$\mathbb{Z}/2^=$	$\mathbb{Z}^=$	0	0	0	$\cdots$	0	$\cdots$
$n+2$	0	0	$2\mathbb{Z}^=$	0	$\mathbb{Z}^=$	0	0	$\cdots$	0	$\cdots$
M_3^n	$\mathbb{Z}/3$	0	0	$\mathbb{Z}/3$	0	$\mathbb{Z}/3$	$\mathbb{Z}/3$	$\cdots$	$\mathbb{Z}/3$	$\cdots$
$M_{3^2}^n$	$\mathbb{Z}/3^2$	0	0	$\mathbb{Z}/3^2$	0	0	$\mathbb{Z}/3^2$	$\cdots$	$\mathbb{Z}/3^2$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$M_{3^r}^n$	$\mathbb{Z}/3^r$	0	0	$\mathbb{Z}/3^r$	0	0	0	$\cdots$	$\mathbb{Z}/3^r$	$\cdots$

Then we get the following two matrix problems $(\mathscr{A}'_{(2)},\mathcal{G}')$ and $(\mathscr{A}'_{(3)},\mathcal{G}')$ with admissible transformations also provided by Tables 8 and 9.

The list of non-trivial admissible transformations on $\Gamma'(\mathcal{A},\mathcal{B})_{(2)}$ is:

el: "elementary-row (column)" transformations of each horizontal (vertical) stripe.

co: $W^{S^{n+2}} < W^{S^{n+3}}$.

ro: $W_{S^n} < W_{C_\eta:n}$; $2W_{C_\eta:n} < W_{S^n}$; $2W_{S^{n+2}} < W_{C_\eta:n}$.

The list of non-trivial admissible transformations on $\Gamma'(\mathcal{A},\mathcal{B})_{(3)}$ is:

el: "elementary-row (column)" transformations of each horizontal (vertical) stripe.

co: $W^{S^{n+3}} < W^{M_{3^r}^{n+2}}$; $W^{M_{3^{r+1}}^{n+2}} < W^{M_{3^r}^{n+2}}$ for any $r \in \mathbb{N}_+$.

Table 10 $\Gamma'(\mathcal{A},\mathcal{B})_{(2)}$

$\mathcal{B}$ \ $\mathcal{A}$	S^{n+2}	S^{n+3}	M_3^{n+2}	$M_{3^2}^{n+2}$	$\cdots$
S^n	$\mathbb{Z}/2$	$\mathbb{Z}/8$	0	0	$\cdots$
S^{n+1}	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	0	$\cdots$
S^{n+2}	0	$\mathbb{Z}/2$	0	0	$\cdots$
$C_\eta : n$	0	$\mathbb{Z}/4$	0	0	$\cdots$
$n+2$	0	0	0	0	$\cdots$
M_3^n	0	0	0	0	$\cdots$
$M_{3^2}^n$	0	0	0	0	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

Table 11 $\Gamma'(\mathcal{A},\mathcal{B})_{(3)}$

$\mathcal{B}$ \ $\mathcal{A}$	S^{n+2}	S^{n+3}	M_3^{n+2}	$M_{3^2}^{n+2}$	$M_{3^3}^{n+2}$	$\cdots$
S^n	0	$\mathbb{Z}/3$	$\mathbb{Z}/3$	$\mathbb{Z}/3$	$\mathbb{Z}/3$	$\cdots$
S^{n+1}	0	0	0	0	0	$\cdots$
S^{n+2}	0	0	0	0	0	$\cdots$
$C_\eta : n$	0	$\mathbb{Z}/3$	$\mathbb{Z}/3$	$\mathbb{Z}/3$	$\mathbb{Z}/3$	$\cdots$
$n+2$	0	0	0	0	0	$\cdots$
M_3^n	0	$\mathbb{Z}/3$	$\mathbb{Z}/3$	$\mathbb{Z}/3$	$\mathbb{Z}/3$	$\cdots$
$M_{3^2}^n$	0	$\mathbb{Z}/3$	$\mathbb{Z}/3$	$\mathbb{Z}/3$	$\mathbb{Z}/3$	$\cdots$
$M_{3^3}^n$	0	$\mathbb{Z}/3$	$\mathbb{Z}/3$	$\mathbb{Z}/3$	$\mathbb{Z}/3$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

ro: $W_{S^n} < W_{C_\eta:n} < W_{M^n_{3^s}}$; $W_{M^n_{3^{s+1}}} < W_{M^n_{3^s}}$; $2W_{C_\eta:n} < W_{S^n}$ for any $s \in \mathbb{N}_+$.

Let $\Delta_{(2)(3)}$ be the subset of $\Gamma'(\mathcal{A},\mathcal{B})_{(2)} \times \Gamma'(\mathcal{A},\mathcal{B})_{(3)}$ which consists of (M_2, M_3) such that for every $x \in \mathrm{ind}\mathcal{A}$, $y \in \mathrm{ind}\mathcal{B}$, the two blocks W_x^y, respectively in matrices M_2 and M_3 have the same order. Define the map $\Gamma'(\mathcal{A},\mathcal{B}) \xrightarrow{L=(L_2,L_3)} \Delta_{(2)(3)} \subset \Gamma'(\mathcal{A},\mathcal{B})_{(2)} \times \Gamma'(\mathcal{A},\mathcal{B})_{(3)}$.

L is given by the following ring isomorphisms

$$\mathbb{Z}/24 \xrightarrow{L_{24}} \mathbb{Z}/8 \times \mathbb{Z}/3, \quad 1 \mapsto (1,1),$$
$$\mathbb{Z}/12 \xrightarrow{L_{12}} \mathbb{Z}/4 \times \mathbb{Z}/3, \quad 1 \mapsto (1,1).$$

The inverse map T of L $\Delta_{(2)(3)} \xrightarrow{T} \Gamma'(\mathcal{A},\mathcal{B})$ is given by the following two ring isomorphisms

$$\mathbb{Z}/8 \times \mathbb{Z}/3 \xrightarrow{T_8} \mathbb{Z}/24, \quad (a,b) \mapsto 9a + 16b,$$
$$\mathbb{Z}/4 \times \mathbb{Z}/3 \xrightarrow{T_4} \mathbb{Z}/12, \quad (a,b) \mapsto 9a + 4b,$$

which are the inverse of L_{24} and L_{12}, respectively.

It is easy to know that if $M \cong N$ in matrix problem $(\mathscr{A}', \mathcal{G}')$ then $L_2(M) \cong L_2(N)$ and $L_3(M) \cong L_3(N)$ in matrix problem $(\mathscr{A}'_{(2)}, \mathcal{G}')$ and $(\mathscr{A}'_{(3)}, \mathcal{G}')$, respectively. We do not know whether the inverse is true. However, in the following we will show that the inverse will be true if we take some restrictions to the admissible transformations on $\mathscr{A}'_{(2)}$ and $\mathscr{A}'_{(3)}$.

We give some notation as follows:

(1) Let $\mathrm{Hos}'(\mathcal{A},\mathcal{A})(y_1, y_2, \ldots, y_n)$ (resp. $\mathrm{Hos}'(\mathcal{B},\mathcal{B})(x_1, x_2, \ldots, x_m)$) be the set of all square matrices in the $\mathrm{Hos}'(\mathcal{A},\mathcal{A})$ (resp. $\mathrm{Hos}'(\mathcal{B},\mathcal{B})$) with only $y_1, y_2, \ldots, y_n$-stripes (resp. $x_1, x_2, \ldots, x_m$-stripes).

Especially, we denote $\mathcal{V} = \mathrm{Hos}'(\mathcal{B},\mathcal{B})(S^n, S^{n+1}, S^{n+2}, C_\eta : n)$ (note that there is no $C_\eta : n+2$-stripe). We call the sub-matrix which contains entries in S^n, S^{n+1}, S^{n+2}, $C_\eta : n$-stripes of $M \in \mathrm{Hos}'(\mathcal{B},\mathcal{B})$ "$\mathcal{V}$-part of M".

(2) Let I be the identity matrix and E_{ij} be the matrix whose unique nonzero entry has index (i,j) and equals 1. Then $(i+aj)$-type of elementary row (column) transformations corresponds to left (right) multiplication by an elementary matrix $I + aE_{ij}$ $(I + aE_{ji})$, and $(-1i)$-type of elementary row (column) transformations corresponds to left (right) multiplication by an elementary matrix $I - 2E_{ii}$. Note that (i,j)-type of transformations can be obtained by composition of $(i+aj)$-type and $(-1i)$-type of elementary transformations.

Let $\boldsymbol{A}^+$ be the subset consisting of invertible matrices $\alpha = \begin{array}{|cc|} \hline U_1 & U_2 \\ 0 & U_4 \\ \hline \end{array}$ in $\mathrm{Hos}'(\mathcal{A},\mathcal{A})$, where $U_1 \in \mathrm{Hos}'(\mathcal{A},\mathcal{A})(S^{n+2}, S^{n+3})$ is a product of elementary matrices $I + aE_{ij}$, $i \neq j$.

Let $\boldsymbol{B}^+$ be the subset consisting of invertible matrices $\beta = \begin{array}{|cc|} \hline V_1 & 0 \\ V_3 & V_4 \\ \hline \end{array}$ in $\mathrm{Hos}'(\mathcal{B},\mathcal{B})$, where

$$V_1 = \begin{array}{|ccccc|} \hline V_{11} & V_{12} & V_{13} & 2V_{14} & 0 \\ 0 & V_{22} & V_{23} & 0 & 0 \\ 0 & 0 & V_{33} & 0 & 0 \\ V_{41} & 0 & V_{43} & V_{44} & 0 \\ 0 & 0 & 0 & 0 & V_{55} \\ \hline \end{array},$$

which is an element in

$$\mathrm{Hos}'(\mathcal{B},\mathcal{B})(S^n, S^{n+1}, S^{n+2}, C_\eta : n, C_\eta : n+2) = \begin{array}{|ccccc|} \hline \mathbb{Z} & \mathbb{Z}/2 & \mathbb{Z}/2 & 2\mathbb{Z} & 0 \\ 0 & \mathbb{Z} & \mathbb{Z}/2 & 0 & 0 \\ 0 & 0 & \mathbb{Z} & 0 & \mathbb{Z} \\ \mathbb{Z} & 0 & \mathbb{Z}/2^{=} & \mathbb{Z}^{=} & 0 \\ 0 & 0 & 2\mathbb{Z}^{=} & 0 & \mathbb{Z}^{=} \\ \hline \end{array}$$

such that the $\mathcal{V}$-part

$$W_1 = \begin{array}{|cccc|} \hline V_{11} & V_{12} & V_{13} & 2V_{14} \\ 0 & V_{22} & V_{23} & 0 \\ 0 & 0 & V_{33} & 0 \\ V_{41} & 0 & V_{43} & V_{44} \\ \hline \end{array}$$

of V_1 is a product of elementary matrices $I + aE_{ij}$, $i \neq j$.

Denoting by $\mathcal{G}'^+$ the admissible transformations provided by $\boldsymbol{A}^+$ and $\boldsymbol{B}^+$ on $\Gamma'(\mathcal{A},\mathcal{B})_{(2)}$ and $\Gamma'(\mathcal{A},\mathcal{B})_{(3)}$, we get two new matrix problems $(\mathscr{A}'_{(2)}, \mathcal{G}'^+)$ and $(\mathscr{A}'_{(3)}, \mathcal{G}'^+)$.

The differences between $(\mathscr{A}'_{(2)}, \mathcal{G}'^+)$ and $(\mathscr{A}'_{(2)}, \mathcal{G}')$. The list of non-trivial admissible transformations of matrix problem $(\mathscr{A}'_{(2)}, \mathcal{G}'^+)$ is the same as that of matrix problem $(\mathscr{A}'_{(2)}, \mathcal{G}')$ except that $(-1i)$-type of elementary transformations is not allowed, and (i,j)-type should be replaced by $(i,-j)$-type or $(-i,j)$-type, which means when we transport two rows (columns) of a stripe, one row (column) α of them is replaced by $-\alpha$.

The differences between $(\mathscr{A}'_{(3)}, \mathcal{G}'^+)$ and $(\mathscr{A}'_{(3)}, \mathcal{G}')$. The list of non-trivial admissible transformations of matrix problem $(\mathscr{A}'_{(3)}, \mathcal{G}'^+)$ is the same as that of matrix problem $(\mathscr{A}'_{(3)}, \mathcal{G}')$ except that $(-1i)$-type of elementary transformations on the $W^{S^{n+3}}$, W_{S^n} and $W_{C_\eta:n}$ is not allowed; (i,j)-type elementary transformations on $W^{S^{n+3}}$, W_{S^n} and $W_{C_\eta:n}$ should be replaced by $(i,-j)$-type or $(-i,j)$-type.

Theorem 6.1. *If $M_{(2)} \cong N_{(2)}$ in matrix problem $(\mathscr{A}'_{(2)}, \mathcal{G}'^+)$ and $M_{(3)} \cong N_{(3)}$ in matrix problem $(\mathscr{A}'_{(3)}, \mathcal{G}'^+)$, then $T(M_{(2)}, M_{(3)}) \cong T(N_{(2)}, N_{(3)})$ in matrix problem $(\mathscr{A}', \mathcal{G}')$.*

Proof. By the condition of this Theorem, we get that $\beta_2 M_{(2)} \alpha_2 = N_{(2)}$, $\beta_3 M_{(3)} \alpha_3 = N_{(3)}$, where $\alpha_2, \alpha_3 \in \boldsymbol{A}^+$ and $\beta_2, \beta_3 \in \boldsymbol{B}^+$. Let

$$\alpha_2 = \begin{array}{|cc|} \hline U_1 & U_2 \\ 0 & U_4 \\ \hline \end{array}, \quad \alpha_3 = \begin{array}{|cc|} \hline U'_1 & U'_2 \\ 0 & U'_4 \\ \hline \end{array},$$

where $U_1, U_1' \in \mathrm{Hos}'(\mathcal{A},\mathcal{A})(S^{n+2}, S^{n+3})$.

$$\beta_2 = \boxed{\begin{matrix} V_1 & 0 \\ V_3 & V_4 \end{matrix}}, \quad \beta_3 = \boxed{\begin{matrix} V_1' & 0 \\ V_3' & V_4' \end{matrix}},$$

where $V_1, V_1' \in \mathrm{Hos}'(\mathcal{B},\mathcal{B})(S^n, S^{n+1}, S^{n+2}, C_\eta : n, C_\eta : n+2)$. $\mathcal{V}$-parts of V_1 and V_1' are denoted by W_1 and W_1', respectively.

Lemma 6.2. *For any*

$$U_1, U_1' \in \mathrm{Hos}'(\mathcal{A},\mathcal{A})(S^{n+2}, S^{n+3}) = \boxed{\begin{matrix} \mathbb{Z} & \mathbb{Z}/2 \\ 0 & \mathbb{Z} \end{matrix}}$$

and

$$W_1, W_1' \in \mathcal{V} = \boxed{\begin{matrix} \mathbb{Z} & \mathbb{Z}/2 & \mathbb{Z}/2 & 2\mathbb{Z} \\ 0 & \mathbb{Z} & \mathbb{Z}/2 & 0 \\ 0 & 0 & \mathbb{Z} & 0 \\ \mathbb{Z} & 0 & \mathbb{Z}/2 & \mathbb{Z} \end{matrix}},$$

where U_1, U_1', W_1 and W_1' are products of elementary matrices $I + aE_{ij}$ $(i \neq j,\ a \in \mathbb{Z})$ and orders of U_1, U_1' (respectively orders of W_1, W_1') are the same, there exist invertible matrices $U \in \mathrm{Hos}'(\mathcal{A},\mathcal{A})(S^{n+2}, S^{n+3})$, $W \in \mathcal{V}$ such that

$$\begin{cases} U \equiv U_1 \pmod 8, \\ U \equiv U_1' \pmod 3, \end{cases} \quad \textit{and} \quad \begin{cases} W \equiv W_1 \pmod 8, \\ W \equiv W_1' \pmod 3. \end{cases}$$

Note that for any abelian group A, $a, b \in A$, and positive integer k, $a \equiv b \pmod k$ means that the images of a and b are equal under the quotient homomorphism $A \to A/kA$.

We give some remarks before the proof of this lemma.

By W and U given in Lemma 6.2, let

$$V = \begin{array}{c|cc|} & & C_\eta : n+2 \\ \cline{2-3} & W & 0 \\ C_\eta : n+2 & 0 & V_{55} \\ \cline{2-3} \end{array},$$

where V_{55} is an invertible matrix that makes V be an element in $\mathrm{Hos}'(\mathcal{B},\mathcal{B})(S^n, S^{n+1}, S^{n+2}, C_\eta : n, C_\eta : n+2)$. Let

$$\alpha = \boxed{\begin{matrix} U & U_2' \\ 0 & U_4' \end{matrix}} \quad \text{and} \quad \beta = \boxed{\begin{matrix} V & 0 \\ V_3' & V_4' \end{matrix}}.$$

Then α and β are invertible and $\beta M_{(2)} \alpha = \beta_2 M_{(2)} \alpha_2 = N_{(2)}$, $\beta M_{(3)} \alpha = \beta_3 M_{(3)} \alpha_3 = N_{(3)}$. Since $\beta T(M_{(2)}, M_{(3)}) \alpha = T(\beta M_{(2)} \alpha, \beta M_{(3)} \alpha) = T(N_{(2)}, N_{(3)})$, $T(M_{(2)}, M_{(3)}) \cong T(N_{(2)}, N_{(3)})$ in matrix problem $(\mathscr{A}', \mathcal{G}')$. □

Proof of Lemma 6.2. Statement (1): For any $A, B \in SL_n(\mathbb{Z})$, there is a $C \in SL_n(\mathbb{Z})$, such that $C \equiv A \pmod 8$ and $C \equiv B \pmod 3$.

Statement (1) follows from the following two conclusions in [10]:

$$\mathrm{SL}_n(\mathbb{Z}) \xrightarrow{q} \mathrm{SL}_n(\mathbb{Z}/24) \text{ is surjective,} \quad \mathrm{SL}_n(\mathbb{Z}/24) \xrightarrow{(q_1, q_2)} \mathrm{SL}_n(\mathbb{Z}/8) \times \mathrm{SL}_n(\mathbb{Z}/3) \quad \text{is isomorphic,}$$

where q, q_1, q_2 are quotient maps.

Statement (2): Suppose that

$$A = I + aE_{ij} = \begin{pmatrix} 1 & & & \\ & & a_{ij} & \\ & \ddots & & \\ & & & 1 \end{pmatrix}, \quad B = I + bE_{st} = \begin{pmatrix} 1 & & & \\ & & b_{st} & \\ & \ddots & & \\ & & & 1 \end{pmatrix} \quad \begin{pmatrix} i \neq j, a \in \mathbb{Z} \\ s \neq t, b \in \mathbb{Z} \end{pmatrix}$$

are any two elementary matrices in $\mathcal{V}$ (resp. $\mathrm{Hos}'(\mathcal{A},\mathcal{A})(S^{n+2},S^{n+3})$) of the same order, then there is an invertible block matrix C in $\mathcal{V}$ (resp. $\mathrm{Hos}'(\mathcal{A},\mathcal{A})(S^{n+2},S^{n+3})$) such that $C\equiv A \pmod 8, C\equiv B \pmod 3$.

Proof of Statement (2). We only prove the case for $A,B\in\mathcal{V}$ since the remaining case is much easier.

Note if a_{ij} (resp. b_{st}) is from $\mathbb{Z}$ or $2\mathbb{Z}$ block, then $a_{ij}=a$ (resp. $b_{st}=b$); if a_{ij} (resp. b_{st}) is from $\mathbb{Z}/2$ block, then a_{ij} (resp. b_{st}) is the image of a (resp. b) under the quotient map $\mathbb{Z}\to\mathbb{Z}/2$.

• If b_{st} is from $\mathbb{Z}/2$ block, then $b_{st}\equiv 0 \pmod 3$. For a_{ij} from $\mathbb{Z}/2$ block, take $C=A$. For a_{ij} from $\mathbb{Z}$ or $2\mathbb{Z}$ block, there is a $c\in\mathbb{Z}$, such that $c\equiv a \pmod 8$ and $c\equiv 0 \pmod 3$. Take $C=I+cE_{ij}$.

• If b_{st} is from $\mathbb{Z}$ or $2\mathbb{Z}$ block,

(i) $i\neq t$ or $j\neq s$. There is a $d\in\mathbb{Z}$ such that $d\equiv 0 \pmod 8$ and $d\equiv b \pmod 3$.

If a_{ij} is from $\mathbb{Z}/2$ block, take integer c such that $c\equiv a \pmod 2$.

If a_{ij} is from $\mathbb{Z}$ or $2\mathbb{Z}$ block, take integer c such that $c\equiv a \pmod 8$ and $c\equiv 0 \pmod 3$. Then take $C=I+cE_{ij}+dE_{st}=(I+cE_{ij})(I+dE_{st})$ which is invertible in $\mathcal{V}$.

(ii) $i=t$ and $j=s$. In this case a_{ij} must come from $\mathbb{Z}$ or $2\mathbb{Z}$ block. Suppose $i>j$. By Statement (1), there is a matrix

$$X=\begin{pmatrix} x_{11} & x_{12}\\ x_{21} & x_{22}\end{pmatrix}\in \mathrm{SL}_2(\mathbb{Z})$$

such that

$$X\equiv\begin{pmatrix} 1 & 0\\ a & 1\end{pmatrix} \pmod 8 \quad\text{and}\quad X\equiv\begin{pmatrix} 1 & b\\ 0 & 1\end{pmatrix} \pmod 3.$$

Take $C=I-(1-x_{11})E_{jj}-(1-x_{22})E_{ii}+x_{12}E_{ji}+x_{21}E_{ij}$. Note that $x_{12}\in 2\mathbb{Z}$ and if $a\in 2\mathbb{Z}$ then $x_{21}\in 2\mathbb{Z}$, so C is an element in $\mathcal{V}$. It is easy to check that C is invertible in $\mathcal{V}$ and $C\equiv A \pmod 8$, $C\equiv B \pmod 3$. The proof of the case $i<j$ is similar. □

Now the proof of Lemma 6.2 is easily obtained by Statement (2). □

6.2 The indecomposable isomorphic classes of $(\mathscr{A}'_{(2)},\mathcal{G}'^{+})$ and $(\mathscr{A}'_{(3)},\mathcal{G}'^{+})$

Note that the matrix problem $(\mathscr{A}'_{(2)},\mathcal{G}')$ is essentially the same as the 2-primary component of the matrix problem $(\mathscr{A}^0,\mathcal{G}^0)$. Thus we can get the list (denoted by List($**$)) of the indecomposable isomorphic classes of $(\mathscr{A}'_{(2)},\mathcal{G}')$ from List($*$) by taking v to its image of the quotient map $\mathbb{Z}/24\to\mathbb{Z}/8$ or $\mathbb{Z}/12\to\mathbb{Z}/4$. It means List($**$) is just the same as List($*$) except that the ranges of v are different, i.e.,

- $v\in\{1\}\subset\mathbb{Z}/4$ for case (I);
- $v\in\{1,2\}\subset\mathbb{Z}/8$ for cases (1), (2), (4), (5), (7) of (II);
- $v\in\{1,2\}\subset\mathbb{Z}/4$ for cases (3), (6) of (II);
- $v\in\{1,2,3,4\}\subset\mathbb{Z}/8$ for case (III).

From the differences between $(\mathscr{A}'_{(2)},\mathcal{G}'^{+})$ and $(\mathscr{A}'_{(2)},\mathcal{G}')$, we know that $M\in\Gamma'(\mathcal{A},\mathcal{B})_{(2)}$ is indecomposable in $(\mathscr{A}'_{(2)},\mathcal{G}'^{+})$ if and only if it is indecomposable in $(\mathscr{A}'_{(2)},\mathcal{G}')$. But non-isomorphic matrices of $(\mathscr{A}'_{(2)},\mathcal{G}'^{+})$ may be isomorphic in $(\mathscr{A}'_{(2)},\mathcal{G}')$.

For example, $\begin{array}{c|c|} & S^{n+3}\\ \cline{2-2} S^n & 1 \\ \cline{2-2}\end{array}$ and $\begin{array}{c|c|} & S^{n+3}\\ \cline{2-2} S^n & -1 \\ \cline{2-2}\end{array}$ (where $1,-1\in\mathbb{Z}/8$), which are isomorphic under $\mathcal{G}'$, are not isomorphic under $\mathcal{G}'^{+}$.

Here is the list of the indecomposable isomorphic classes of $(\mathscr{A}'_{(2)},\mathcal{G}'^{+})$:

List (1) (I) $\begin{array}{r|c|} & S^{n+3}\\ \cline{2-2} S^{n+2} & 1\\ C_\eta : n & 1\\ n+2 & 0\\ \cline{2-2}\end{array}$.

(II) (1) $\begin{array}{r|cc|} & S^{n+2} & S^{n+3}\\ \cline{2-3} S^{n} & 1 & v\\ S^{n+1} & 0 & 1\\ \cline{2-3}\end{array}$; (2) $\begin{array}{r|cc|} & S^{n+2} & S^{n+3}\\ \cline{2-3} S^{n} & 1 & v\\ S^{n+2} & 0 & 1\\ \cline{2-3}\end{array}$; (4) $\begin{array}{r|cc|} & S^{n+2} & S^{n+3}\\ \cline{2-3} S^{n} & 1 & v\\ \cline{2-3}\end{array}$;

(5) $\begin{array}{c|c|} & S^{n+3} \\ \cline{2-2} S^n & v \\ S^{n+1} & 1 \\ \cline{2-2} \end{array}$; (7) $\begin{array}{c|c|} & S^{n+3} \\ \cline{2-2} S^n & v \\ S^{n+2} & 1 \\ \cline{2-2} \end{array}$; (3) $\begin{array}{c|c|} & S^{n+3} \\ \cline{2-2} S^{n+2} & 1 \\ C_\eta : n & v \\ n+2 & 0 \\ \cline{2-2} \end{array}$; (6) $\begin{array}{c|c|} & S^{n+3} \\ \cline{2-2} C_\eta : n & v \\ n+2 & 0 \\ \cline{2-2} \end{array}$;

where $v \in \{1,2,3\} \subset \mathbb{Z}/8$ for the cases (1), (2), (4), (5), (7) and where $v \in \{1,2,3\} \subset \mathbb{Z}/4$ for the cases (3) and (6).

(III) $\begin{array}{c|c|} & S^{n+3} \\ \cline{2-2} S^n & v \\ \cline{2-2} \end{array}$, where $v \in \{1,2,\ldots,7\} \subset \mathbb{Z}/8$.

(IV) (1) $\begin{array}{c|c|} & S^{n+2} \\ \cline{2-2} S^{n+1} & 1 \\ \cline{2-2} \end{array}$; (2) $\begin{array}{c|c|} & S^{n+3} \\ \cline{2-2} S^{n+2} & 1 \\ \cline{2-2} \end{array}$; (3) $\begin{array}{c|c|} & S^{n+2} \\ \cline{2-2} S^{n} & 1 \\ \cline{2-2} \end{array}$; (4) $\begin{array}{c|c|} & S^{n+3} \\ \cline{2-2} S^{n+1} & 1 \\ \cline{2-2} \end{array}$.

For the matrix problem $(\mathscr{A}'_{(3)}, \mathscr{G}'^{+})$, the indecomposable isomorphic classes are given by

List (2) $\begin{array}{c|c|} & S^{n+3} \\ \cline{2-2} S^{n} & 1 \\ \cline{2-2} \end{array}$; $\begin{array}{c|c|} & S^{n+3} \\ \cline{2-2} S^{n} & -1 \\ \cline{2-2} \end{array}$; $\begin{array}{c|c|} & S^{n+3} \\ \cline{2-2} C_\eta : n & 1 \\ n+2 & 0 \\ \cline{2-2} \end{array}$; $\begin{array}{c|c|} & S^{n+3} \\ \cline{2-2} C_\eta : n & -1 \\ n+2 & 0 \\ \cline{2-2} \end{array}$;

$\begin{array}{c|c|} & S^{n+3} \\ \cline{2-2} M^n_{3^s} & 1 \\ \cline{2-2} \end{array}$; $\begin{array}{c|c|} & M^{n+2}_{3^r} \\ \cline{2-2} M^n_{3^s} & 1 \\ \cline{2-2} \end{array}$; $\begin{array}{c|c|} & M^{n+2}_{3^r} \\ \cline{2-2} S^{n} & 1 \\ \cline{2-2} \end{array}$; $\begin{array}{c|c|} & M^{n+2}_{3^r} \\ \cline{2-2} C_\eta : n & 1 \\ n+2 & 0 \\ \cline{2-2} \end{array}$;

where $1, -1 \in \mathbb{Z}/3$ and $s, n \in \mathbb{N}_+$.

6.3 The indecomposable isomorphic classes of $(\mathscr{A}', \mathscr{G}')$ and $F^4_{n(2)} (n \geqslant 5)$

By Theorem 6.1, for any $M \in \Gamma'(\mathcal{A}, \mathcal{B})$, we have $M \cong T(N_2, N_3)$ in matrix problem $(\mathscr{A}', \mathscr{G}')$ for some $N_2 \in \Gamma'(\mathcal{A}, \mathcal{B})_{(2)}$ and $N_3 \in \Gamma'(\mathcal{A}, \mathcal{B})_{(3)}$, where

$$N_2 = \bigoplus_i N_2^i \bigoplus \boldsymbol{O}_2, \ \ N_2^i \ \text{ is an indecomposable matrix listed in List (1) for every } i,$$

$$N_3 = \bigoplus_j N_3^j \bigoplus \boldsymbol{O}_3, \ \ N_3^j \ \text{ is an indecomposable matrix listed in List (2) for every } j.$$

$\boldsymbol{O}_2$ and $\boldsymbol{O}_3$ are direct products of some zero matrices.

Since N_3 is a matrix of which every row and every column have at most one nonzero entry, it enables us to select from $T(N_2, N_3)$ a set of indecomposable matrices as follows which covers all the indecomposable isomorphic classes of matrix problem $(\mathscr{A}', \mathscr{G}')$.

(I) $\begin{array}{c|c|} & S^{n+3} \\ \cline{2-2} S^{n+2} & 1 \\ C_\eta : n & T_4(a,b) \\ n+2 & 0 \\ \cline{2-2} \end{array}$, $\begin{array}{c|cc|} & S^{n+3} & M^{n+2}_{3^r} \\ \cline{2-3} S^{n+2} & 1 & 0 \\ C_\eta : n & T_4(1,0) & 1 \\ n+2 & 0 & 0 \\ \cline{2-3} \end{array}$,

$\begin{array}{c|c|} & S^{n+3} \\ \cline{2-2} S^{n+2} & 1 \\ C_\eta : n & T_4(1,0) \\ n+2 & 0 \\ M^n_{3^s} & 1 \\ \cline{2-2} \end{array}$, $\begin{array}{c|cc|} & S^{n+3} & M^{n+2}_{3^r} \\ \cline{2-3} S^{n+2} & 1 & 0 \\ C_\eta : n & T_4(1,0) & 1 \\ n+2 & 0 & 0 \\ M^n_{3^s} & 1 & 0 \\ \cline{2-3} \end{array}$;

where $(a,b) \in \mathbb{Z}/4 \times \mathbb{Z}/3$ such that $a \in \{0,1\}$ and $(a,b) \neq (0,0)$ and $s, r \in \mathbb{N}_+$.

(II) (1) $\begin{array}{c|cc|} & S^{n+2} & S^{n+3} \\ \cline{2-3} S^{n} & 1 & T_8(a,b) \\ S^{n+1} & 0 & 1 \\ \cline{2-3} \end{array}$, $\begin{array}{c|ccc|} & S^{n+2} & S^{n+3} & M^{n+2}_{3^r} \\ \cline{2-4} S^{n} & 1 & T_8(a,0) & 1 \\ S^{n+1} & 0 & 1 & 0 \\ \cline{2-4} \end{array}$,

$\begin{array}{c|cc|} & S^{n+2} & S^{n+3} \\ \cline{2-3} S^{n} & 1 & T_8(a,0) \\ S^{n+1} & 0 & 1 \\ M^n_{3^s} & 0 & 1 \\ \cline{2-3} \end{array}$, $\begin{array}{c|ccc|} & S^{n+2} & S^{n+3} & M^{n+2}_{3^r} \\ \cline{2-4} S^{n} & 1 & T_8(a,0) & 1 \\ S^{n+1} & 0 & 1 & 0 \\ M^n_{3^s} & 0 & 1 & 0 \\ \cline{2-4} \end{array}$;

(2)
$$\begin{array}{c|cc|}
 & S^{n+2} & S^{n+3} \\ \hline
S^n & 1 & T_8(a,b) \\
S^{n+2} & 0 & 1 \\ \hline
\end{array},\quad
\begin{array}{c|ccc|}
 & S^{n+2} & S^{n+3} & M^{n+2}_{3^r} \\ \hline
S^n & 1 & T_8(a,0) & 1 \\
S^{n+2} & 0 & 1 & 0 \\ \hline
\end{array},$$
$$\begin{array}{c|cc|}
 & S^{n+2} & S^{n+3} \\ \hline
S^n & 1 & T_8(a,0) \\
S^{n+2} & 0 & 1 \\
M^n_{3^s} & 0 & 1 \\ \hline
\end{array},\quad
\begin{array}{c|ccc|}
 & S^{n+2} & S^{n+3} & M^{n+2}_{3^r} \\ \hline
S^n & 1 & T_8(a,0) & 1 \\
S^{n+2} & 0 & 1 & 0 \\
M^n_{3^s} & 0 & 1 & 0 \\ \hline
\end{array};$$

(3)
$$\begin{array}{c|c|}
 & S^{n+3} \\ \hline
S^{n+1} & 1 \\
C_\eta : n & T_4(a,b) \\
n+2 & 0 \\ \hline
\end{array},\quad
\begin{array}{c|cc|}
 & S^{n+3} & M^{n+2}_{3^r} \\ \hline
S^{n+1} & 1 & 0 \\
C_\eta : n & T_4(a,0) & 1 \\
n+2 & 0 & 0 \\ \hline
\end{array},\quad
\begin{array}{c|c|}
 & S^{n+3} \\ \hline
S^{n+2} & 1 \\
C_\eta : n & T_4(a,0) \\
n+2 & 0 \\
M^n_{3^s} & 1 \\ \hline
\end{array},\quad
\begin{array}{c|cc|}
 & S^{n+3} & M^{n+2}_{3^r} \\ \hline
S^{n+2} & 1 & 0 \\
C_\eta : n & T_4(a,0) & 1 \\
n+2 & 0 & 0 \\
M^n_{3^s} & 1 & 0 \\ \hline
\end{array};$$

(4)
$$\begin{array}{c|cc|}
 & S^{n+2} & S^{n+3} \\ \hline
S^n & 1 & T_8(a,b) \\ \hline
\end{array},\quad
\begin{array}{c|ccc|}
 & S^{n+2} & S^{n+3} & M^{n+2}_{3^r} \\ \hline
S^n & 1 & T_8(a,0) & 1 \\ \hline
\end{array},$$
$$\begin{array}{c|cc|}
 & S^{n+2} & S^{n+3} \\ \hline
S^n & 1 & T_8(a,0) \\
M^n_{3^s} & 0 & 1 \\ \hline
\end{array},\quad
\begin{array}{c|ccc|}
 & S^{n+2} & S^{n+3} & M^{n+2}_{3^r} \\ \hline
S^n & 1 & T_8(a,0) & 1 \\
M^n_{3^s} & 0 & 1 & 0 \\ \hline
\end{array};$$

(5)
$$\begin{array}{c|c|}
 & S^{n+3} \\ \hline
S^n & T_8(a,b) \\
S^{n+1} & 1 \\ \hline
\end{array},\quad
\begin{array}{c|cc|}
 & S^{n+3} & M^{n+2}_{3^r} \\ \hline
S^n & T_8(a,0) & 1 \\
S^{n+1} & 1 & 0 \\ \hline
\end{array},\quad
\begin{array}{c|c|}
 & S^{n+3} \\ \hline
S^n & T_8(a,b) \\
S^{n+1} & 1 \\
M^n_{3^s} & 1 \\ \hline
\end{array},\quad
\begin{array}{c|cc|}
 & S^{n+3} & M^{n+2}_{3^r} \\ \hline
S^n & T_8(a,0) & 1 \\
S^{n+1} & 1 & 0 \\
M^n_{3^s} & 1 & 0 \\ \hline
\end{array};$$

(6)
$$\begin{array}{c|c|}
 & S^{n+3} \\ \hline
C_\eta : n & T_4(a,b) \\
n+2 & 0 \\ \hline
\end{array},\quad
\begin{array}{c|cc|}
 & S^{n+3} & M^{n+2}_{3^r} \\ \hline
C_\eta : n & T_4(a,0) & 1 \\
n+2 & 0 & 0 \\ \hline
\end{array},\quad
\begin{array}{c|c|}
 & S^{n+3} \\ \hline
C_\eta : n & T_4(a,0) \\
n+2 & 0 \\
M^n_{3^s} & 1 \\ \hline
\end{array},\quad
\begin{array}{c|cc|}
 & S^{n+3} & M^{n+2}_{3^r} \\ \hline
C_\eta : n & T_4(a,0) & 1 \\
n+2 & 0 & 0 \\
M^n_{3^s} & 1 & 0 \\ \hline
\end{array};$$

(7)
$$\begin{array}{c|c|}
 & S^{n+3} \\ \hline
S^n & T_8(a,b) \\
S^{n+2} & 1 \\ \hline
\end{array},\quad
\begin{array}{c|cc|}
 & S^{n+3} & M^{n+2}_{3^r} \\ \hline
S^n & T_8(a,0) & 1 \\
S^{n+2} & 1 & 0 \\ \hline
\end{array},\quad
\begin{array}{c|c|}
 & S^{n+3} \\ \hline
S^n & T_8(a,0) \\
S^{n+2} & 1 \\
M^n_{3^s} & 1 \\ \hline
\end{array},\quad
\begin{array}{c|cc|}
 & S^{n+3} & M^{n+2}_{3^r} \\ \hline
S^n & T_8(a,0) & 1 \\
S^{n+2} & 1 & 0 \\
M^n_{3^s} & 1 & 0 \\ \hline
\end{array};$$

where $(a,b) \in \mathbb{Z}/8 \times \mathbb{Z}/3$ such that $a \in \{0,1,2,3\} \subset \mathbb{Z}/8$ and $(a,b) \neq (0,0)$ for T_8, $(a,b) \in \mathbb{Z}/4 \times \mathbb{Z}/3$ such that $a \in \{0,1,2,3\} \subset \mathbb{Z}/4$ and $(a,b) \neq (0,0)$ for T_4, and $s, r \in \mathbb{N}_+$.

(III)
$$\begin{array}{c|c|}
 & S^{n+3} \\ \hline
S^n & T_8(a,b) \\ \hline
\end{array},\quad
\begin{array}{c|cc|}
 & S^{n+3} & M^{n+2}_{3^r} \\ \hline
S^n & T_8(a,0) & 1 \\ \hline
\end{array},\quad
\begin{array}{c|c|}
 & S^{n+3} \\ \hline
S^n & T_8(a,0) \\
M^n_{3^s} & 1 \\ \hline
\end{array},\quad
\begin{array}{c|cc|}
 & S^{n+3} & M^{n+2}_{3^r} \\ \hline
S^n & T_8(a,0) & 1 \\
M^n_{3^s} & 1 & 0 \\ \hline
\end{array};$$

where $(a,b) \in \mathbb{Z}/8 \times \mathbb{Z}/3$ such that $(a,b) \neq (0,0)$ and $s, r \in \mathbb{N}_+$.

(IV) (1)
$$\begin{array}{c|c|}
 & S^{n+2} \\ \hline
S^{n+1} & 1 \\ \hline
\end{array};$$

(2)
$$\begin{array}{c|c|}
 & S^{n+3} \\ \hline
S^{n+2} & 1 \\ \hline
\end{array},\quad
\begin{array}{c|c|}
 & S^{n+3} \\ \hline
S^{n+2} & 1 \\
M^n_{3^s} & 1 \\ \hline
\end{array};$$

(3)
$$\begin{array}{c|c|}
 & S^{n+2} \\ \hline
S^n & 1 \\ \hline
\end{array},\quad
\begin{array}{c|cc|}
 & S^{n+2} & M^{n+2}_{3^r} \\ \hline
S^n & 1 & 1 \\ \hline
\end{array};$$

(4)
$$\begin{array}{c|c|}
 & S^{n+3} \\ \hline
S^{n+1} & 1 \\ \hline
\end{array},\quad
\begin{array}{c|c|}
 & S^{n+3} \\ \hline
S^{n+1} & 1 \\
M^n_{3^s} & 1 \\ \hline
\end{array};$$

where $r, s \in \mathbb{N}_+$.

Through a detailed check by admissible transformations of matrix problem $(\mathscr{A}', \mathscr{G}')$, we get the following results.

Theorem 6.3. *All indecomposable isomorphic classes of $(\mathscr{A}', \mathcal{G}')$ are given by the following list:*

(I)

	S^{n+3}
S^{n+2}	1
$C_\eta : n$	v
$n+2$	0

$X(\eta v \eta)$,

	S^{n+3}	$M^{n+2}_{3^r}$
S^{n+2}	1	0
$C_\eta : n$	3	1
$n+2$	0	0

$X(\eta 3 \eta)^r$,

	S^{n+3}
S^{n+2}	1
$C_\eta : n$	3
$n+2$	0
$M^n_{3^s}$	1

$X(\eta 3 \eta)_s$,

	S^{n+3}	$M^{n+2}_{3^r}$
S^{n+2}	1	0
$C_\eta : n$	3	1
$n+2$	0	0
$M^n_{3^s}$	1	0

$X(\eta 3 \eta)^r_s$;

where $v \in \{1, 2, 3\} \subset \mathbb{Z}/12$.

(II) (1)

	S^{n+2}	S^{n+3}
S^n	1	v
S^{n+1}	0	1

$X(\eta\eta v \eta\eta)$,

	S^{n+2}	S^{n+3}	$M^{n+2}_{3^r}$
S^n	1	v_1	1
S^{n+1}	0	1	0

$X(\eta\eta v_1 \eta\eta)^r$,

	S^{n+2}	S^{n+3}
S^n	1	v_1
S^{n+1}	0	1
$M^n_{3^s}$	0	1

$X(\eta\eta v_1 \eta\eta)_s$,

	S^{n+2}	S^{n+3}	$M^{n+2}_{3^r}$
S^n	1	v_1	1
S^{n+1}	0	1	0
$M^n_{3^s}$	0	1	0

$X(\eta\eta v_1 \eta\eta)^r_s$;

(2)

	S^{n+2}	S^{n+3}
S^n	1	v
S^{n+2}	0	1

$X(\eta\eta v \eta)$,

	S^{n+2}	S^{n+3}	$M^{n+2}_{3^r}$
S^n	1	v_1	1
S^{n+2}	0	1	0

$X(\eta\eta v_1 \eta)^r$,

	S^{n+2}	S^{n+3}
S^n	1	v_1
S^{n+2}	0	1
$M^n_{3^s}$	0	1

$X(\eta\eta v_1 \eta)_s$,

	S^{n+2}	S^{n+3}	$M^{n+2}_{3^r}$
S^n	1	v_1	1
S^{n+2}	0	1	0
$M^n_{3^s}$	0	1	0

$X(\eta\eta v_1 \eta)^r_s$;

(3)

	S^{n+3}
S^{n+1}	1
$C_\eta : n$	v
$n+2$	0

$X(\eta v \eta\eta)$,

	S^{n+3}	$M^{n+2}_{3^r}$
S^{n+1}	1	0
$C_\eta : n$	v_1	1
$n+2$	0	0

$X(\eta v_1 \eta\eta)^r$,

	S^{n+3}
S^{n+2}	1
$C_\eta : n$	v_1
$n+2$	0
$M^n_{3^s}$	1

$X(\eta v_1 \eta\eta)_s$,

	S^{n+3}	$M^{n+2}_{3^r}$
S^{n+2}	1	0
$C_\eta : n$	v_1	1
$n+2$	0	0
$M^n_{3^s}$	1	0

$X(\eta v_1 \eta\eta)^r_s$;

(4)

	S^{n+2}	S^{n+3}
S^n	1	v

$X(\eta\eta v)$,

	S^{n+2}	S^{n+3}	$M^{n+2}_{3^r}$
S^n	1	v_1	1

$X(\eta\eta v_1)^r$,

	S^{n+2}	S^{n+3}
S^n	1	v_1
$M^n_{3^s}$	0	1

$X(\eta\eta v_1)_s$,

	S^{n+2}	S^{n+3}	$M^{n+2}_{3^r}$
S^n	1	v_1	1
$M^n_{3^s}$	0	1	0

$X(\eta\eta v_1)^r_s$;

(5)
$$\underset{X(v\eta\eta)}{\begin{array}{c|c|} & S^{n+3} \\ \hline S^n & v \\ S^{n+1} & 1 \\ \hline \end{array}},\quad \underset{X(v_1\eta\eta)^r}{\begin{array}{c|cc|} & S^{n+3} & M^{n+2}_{3^r} \\ \hline S^n & v_1 & 1 \\ S^{n+1} & 1 & 0 \\ \hline \end{array}},\quad \underset{X(v_1\eta\eta)_s}{\begin{array}{c|c|} & S^{n+3} \\ \hline S^n & v \\ S^{n+1} & 1 \\ M^n_{3^s} & 1 \\ \hline \end{array}},\quad \underset{X(v_1\eta\eta)^r_s}{\begin{array}{c|cc|} & S^{n+3} & M^{n+2}_{3^r} \\ \hline S^n & v_1 & 1 \\ S^{n+1} & 1 & 0 \\ M^n_{3^s} & 1 & 0 \\ \hline \end{array}};$$

(6)
$$\underset{X(\eta v)}{\begin{array}{c|c|} & S^{n+3} \\ \hline C_\eta : n & v \\ n+2 & 0 \\ \hline \end{array}},\quad \underset{X(\eta v_1)^r}{\begin{array}{c|cc|} & S^{n+3} & M^{n+2}_{3^r} \\ \hline C_\eta : n & v_1 & 1 \\ n+2 & 0 & 0 \\ \hline \end{array}},\quad \underset{X(\eta v_1)_s}{\begin{array}{c|c|} & S^{n+3} \\ \hline C_\eta : n & v_1 \\ n+2 & 0 \\ M^n_{3^s} & 1 \\ \hline \end{array}},\quad \underset{X(\eta v_1)^r_s}{\begin{array}{c|cc|} & S^{n+3} & M^{n+2}_{3^r} \\ \hline C_\eta : n & v_1 & 1 \\ n+2 & 0 & 0 \\ M^n_{3^s} & 1 & 0 \\ \hline \end{array}};$$

(7)
$$\underset{X(v\eta)}{\begin{array}{c|c|} & S^{n+3} \\ \hline S^n & v \\ S^{n+2} & 1 \\ \hline \end{array}},\quad \underset{X(v_1\eta)^r}{\begin{array}{c|cc|} & S^{n+3} & M^{n+2}_{3^r} \\ \hline S^n & v_1 & 1 \\ S^{n+2} & 1 & 0 \\ \hline \end{array}},\quad \underset{X(v_1\eta)_s}{\begin{array}{c|c|} & S^{n+3} \\ \hline S^n & v_1 \\ S^{n+2} & 1 \\ M^n_{3^s} & 1 \\ \hline \end{array}},\quad \underset{X(v_1\eta)^r_s}{\begin{array}{c|cc|} & S^{n+3} & M^{n+2}_{3^r} \\ \hline S^n & v_1 & 1 \\ S^{n+2} & 1 & 0 \\ M^n_{3^s} & 1 & 0 \\ \hline \end{array}};$$

where $v \in \{1,2,3,4,5,6\} \subset \mathbb{Z}/24$ *or* $\mathbb{Z}/12$, $v_1 \in \{3,6\} \subset \mathbb{Z}/24$ *or* $\mathbb{Z}/12$ *and* $r, s \in \mathbb{N}_+$.

(III)
$$\underset{X(v)}{\begin{array}{c|c|} & S^{n+3} \\ \hline S^n & v \\ \hline \end{array}},\quad \underset{X(v_1)^r}{\begin{array}{c|cc|} & S^{n+3} & M^{n+2}_{3^r} \\ \hline S^n & v_1 & 1 \\ \hline \end{array}},\quad \underset{X(v_1)_s}{\begin{array}{c|c|} & S^{n+3} \\ \hline S^n & v_1 \\ M^n_{3^s} & 1 \\ \hline \end{array}},\quad \underset{X(v_1)^r_s}{\begin{array}{c|cc|} & S^{n+3} & M^{n+2}_{3^r} \\ \hline S^n & v_1 & 1 \\ M^n_{3^s} & 1 & 0 \\ \hline \end{array}};$$

where $v \in \{1,2,\ldots,12\} \subset \mathbb{Z}/24$, $v_1 \in \{3,6,9\} \subset \mathbb{Z}/24$ *and* $r, s \in \mathbb{N}_+$.

(IV) (1)
$$\underset{X(\eta_1)}{\begin{array}{c|c|} & S^{n+2} \\ \hline S^{n+1} & 1 \\ \hline \end{array}};$$

(2)
$$\underset{X(\eta_2)}{\begin{array}{c|c|} & S^{n+3} \\ \hline S^{n+2} & 1 \\ \hline \end{array}},\quad \underset{X(\eta_2)_s}{\begin{array}{c|c|} & S^{n+3} \\ \hline S^{n+2} & 1 \\ M^n_{3^s} & 1 \\ \hline \end{array}};$$

(3)
$$\underset{X(\eta\eta)_0}{\begin{array}{c|c|} & S^{n+2} \\ \hline S^n & 1 \\ \hline \end{array}},\quad \underset{X(\eta\eta)^r_0}{\begin{array}{c|cc|} & S^{n+2} & M^{n+2}_{3^r} \\ \hline S^n & 1 & 1 \\ \hline \end{array}};$$

(4)
$$\underset{X(\eta\eta)_1}{\begin{array}{c|c|} & S^{n+3} \\ \hline S^{n+1} & 1 \\ \hline \end{array}},\quad \underset{X(\eta\eta)_{1s}}{\begin{array}{c|c|} & S^{n+3} \\ \hline S^{n+1} & 1 \\ M^n_{3^s} & 1 \\ \hline \end{array}};$$

where $r, s \in \mathbb{N}_+$.

It is easy to recover the polyhedra from the matrices listed in Theorem 6.3. For example,

	S^{n+3}	$M^{n+2}_{3^r}$
S^{n+2}	1	0
$C_\eta : n$	3	1
$n+2$	0	0
$M^n_{3^s}$	1	0

$X(\eta 3\eta)^r_s$

represents the cone of the map $S^{n+3} \vee M^{n+2}_{3^r} \to S^{n+2} \vee C_\eta \vee M^n_{3^s}$ corresponding to the matrix. Since $S^{n+2} \vee C_\eta \vee M^n_{3^s} = (S^{n+2} \vee S^n \vee S^n) \bigcup_{i_2\eta} \mathrm{e}^{n+2} \bigcup_{i_3 3^s} \mathrm{e}^{n+1}$, we get that the polyhedron corresponding to this matrix is $(S^{n+2} \vee S^n \vee S^n \vee S^{n+3}) \bigcup_{i_2\eta} \mathrm{e}^{n+2} \bigcup_{i_3 3^r} \mathrm{e}^{n+1} \bigcup_{i_1\eta\eta + i_2 3 + i_3 1} \mathrm{e}^{n+4} \bigcup_{i_3 1 + i_4 3^r} \mathrm{e}^{n+4}$, where $i_t : X_t \hookrightarrow \bigvee_j X_j$ is the canonical inclusion of the summand.

Finally, from Lemma 5.4, Corollaries 5.3 and 5.5, we obtain all the 2-torsion free indecomposable homotopy types of $\mathcal{A}\dagger\mathcal{B}$, which completes the proof of Theorem 2.5 (Main theorem).

7 Concluding remarks

In this paper, using the well-known results about homotopy classes of maps between Moore spaces and suspended complex projective space and their compositions, we succeed in classifying indecomposable $\boldsymbol{F}^4_{n(2)}$ polyhedra. However, the corresponding classification problems for the cases $\boldsymbol{F}^5_{n(2)}$ and $\boldsymbol{F}^6_{n(2)}$ are still open. We hope to return to this issue in the future publication. On the other hand, from the previous remark, it is crucial to understand globally a collection of spaces as a subcategory of homotopy category of spaces. We will focus on this point in the future works.

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