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Existence and Gevrey regularity for a two-species chemotaxis system in homogeneous Besov spaces

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Abstract We study the Cauchy problem of a two-species chemotactic model. Using the Fourier frequency localization and the Bony paraproduct decomposition, we establish a unique local solution and blow-up criterion of the solution, when the initial data (u_0, v_0, w_0) belongs to homogeneous Besov spaces $\dot{B}^{-2+3/p}_{p,1}(\mathbb{R}^3) \times \dot{B}^{-2+3/r}_{r,1}(\mathbb{R}^3) \times \dot{B}^{3/q}_{q,1}(\mathbb{R}^3)$ for p, q and r satisfying some technical assumptions. Furthermore, we prove that if the initial data is sufficiently small, then the solution is global. Meanwhile, based on the so-called Gevrey estimates, we particularly prove that the solution is analytic in the spatial variable. In addition, we analyze the long time behavior of the solution and obtain some decay estimates for higher derivatives in Besov and Lebesgue spaces.

Keywords two-species chemotaxis system, Gevrey regularity, Besov spaces, blow-up criterion, Triebel-Lizorkin spaces

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1 Introduction and main results

In this paper, we consider the following three-component generalization of the Keller-Segel system of the form:

$$\begin{cases} u_t = \Delta u - \chi_1 \nabla \cdot (u \nabla w) & \text{in} \quad (0,T) \times \Omega, \\ v_t = \Delta v - \chi_2 \nabla \cdot (v \nabla w) & \text{in} \quad (0,T) \times \Omega, \\ w_t = \Delta w - \gamma w + \alpha_1 u + \alpha_2 v & \text{in} \quad (0,T) \times \Omega, \end{cases} \tag{1.1}$$

where u and v denote the unknown densities of two interacting populations and w describes the unknown concentration of the common chemical attractant. The parameter $\gamma \geqslant 0$ in (1.1) is the rate of consumption. Also, in (1.1) the chemotactic sensitivities χ_1 and χ_2 and the rates of production α_1 and α_2 are all real numbers, $T \in (0, \infty]$ and Ω is a spatial domain.

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In particular, if $v = 0$, then (1.1) becomes the problems related to the following classical parabolic-parabolic Keller-Segel system:

$$\begin{cases} u_t = \Delta u - \chi_1 \nabla \cdot (u \nabla w) & \text{in} \quad (0,T) \times \Omega, \\ w_t = \Delta w - \gamma w + \alpha_1 u & \text{in} \quad (0,T) \times \Omega, \end{cases} \tag{1.2}$$

where u is the density of cells and w is the concentration of the chemo-attractant. System (1.2) is a mathematical model of chemotaxis, which was formulated by Keller and Segel [37] in 1970, while it is also connected with astrophysical models of the gravitational self-interaction of massive particles in a cloud or a nebula (see [10]). Nagai [47] proved that if the initial data (u_0, w_0) satisfies $u_0 \in L^\infty(\mathbb{R}^2)$ and $\|u_0\|_{L^1(\mathbb{R}^2)}, \|\nabla w_0\|_{L^1(\mathbb{R}^2)}$ and $\|\nabla w_0\|_{L^\infty(\mathbb{R}^2)}$ are suitably small, then the Cauchy problem of (1.2) has globally bounded solutions on $\mathbb{R}^2$. The asymptotic behavior of global solutions was investigated in [20,48].

When $\Omega = \mathbb{R}^n$, Corrias and Perthame [20] treated the case $n \geqslant 3$ and constructed a global weak solution for small initial data. The papers [16,20–22] dealt with the problem with the density function in the Lebesgue space $L^{n/2}(\mathbb{R}^n)$ for $n \geqslant 2$. Recently, Kozono and Sugiyama [39] obtained the global and the decay estimates of solutions in homogeneous potential spaces $\dot{H}^{\frac{n}{r}-2,r}(\mathbb{R}^n) \times \dot{H}^{\frac{n}{r},r}(\mathbb{R}^n)$ for $\gamma \geqslant 0$ and $n \geqslant 2$. Bae [3] proved that (1.2) is globally well-posed for small initial data in $\dot{B}^{-2+\frac{3}{p}}_{p,1}(\mathbb{R}^3) \times \dot{B}^{\frac{3}{p}}_{p,1}(\mathbb{R}^3)$ for $1 \leqslant p < 3$. For generalized drift-diffusion equations about analyticity rate estimates (see [62,63]). Zhai [59] established the global existence and uniqueness of solutions with initial data $(u_0, w_0) \in \dot{B}^{-2+3/p}_{p,r}(\mathbb{R}^3) \times \dot{B}^{3/p}_{p,r}(\mathbb{R}^3)$ for $\frac{3}{2} < p < \infty$ and $1 \leqslant r \leqslant \infty$. For more results related to this topic, we refer the reader to [8,11,26,38,40,43,51,55].

Furthermore, if $\partial_t w$ is ignored in (1.2), then the induced system becomes the classical drift-diffusion equation with the parabolic-elliptic form

$$\begin{cases} u_t = \Delta u - \chi_1 \nabla \cdot (u \nabla w) & \text{in} \quad (0,T) \times \Omega, \\ 0 = \Delta w - \gamma w + \alpha_1 u & \text{in} \quad (0,T) \times \Omega. \end{cases} \tag{1.3}$$

In virtue of iterative derivative estimates, Masakazu et al. [45] obtained the analyticity of mild solutions of (1.3) with initial data in weighted-Sobolev spaces and Yamamoto [56] considered the regularizing rates and the analyticity for (1.3) for the initial data in $L^{n/2}(\mathbb{R}^n)$ for $n \geqslant 2$ and extended the results to Besov-Morrey spaces and Triebel-Lizorkin-Morrey spaces (see [24,57,58] for properties of related function spaces). We can also refer the reader to [31,46,50] about the analyticity using iterative derivative estimates for Navier-Stokes equations [1].

Using the Gevrey estimates, Foias and Temam [30] established the analyticity and provided explicit estimates of the analyticity radius of solutions to the Navier-Stokes equations, which avoids some cumbersome recursive estimates of higher order derivatives and intricate combinatorial arguments. Since then, Gevrey class technique has become a standard tool for studying analytic properties of solutions to nonlinear partial differential equations. For involving equations about Gevrey class regularity estimates, we can also refer the reader to [4–7,12,13,15,36]. One goal in this paper is to show the space analyticity of mild solutions of (1.1) and to provide explicit estimates for the analyticity radius as a function of time. There are many applications about the space analyticity (see [6,32,41,49]).

Recently, the two-species chemotaxis model (1.1) has been considered by many authors (see [2,9,19, 25,34]). In the radial symmetric situation, Arenas et al. [2] proved that there is simultaneous blow up for both chemotactic species in the disk of $\mathbb{R}^2$. Conca et al. [19] studied the blow up and the global existence for (1.1) with $\gamma = 0$ in the whole space $\mathbb{R}^2$. Recently, Zhang and Li [60] showed the global existence and the asymptotic properties of the solution to (1.1) when initial data $\|u_0\|_{L^1(\mathbb{R}^2)}$, $\|v_0\|_{L^1(\mathbb{R}^2)}$ and $\|\nabla w_0\|_{L^2(\mathbb{R}^2)}$ are small on $\mathbb{R}^2$. In higher dimensions, the blow up of the parabolic-elliptic counterpart of (1.1) has been studied in [9]. More general forms of the two-species chemotaxis model were considered in [54,61].

Before going further, we first recall the definitions of some function spaces which are needed in this paper, especially, we need to use the Besov spaces (for more details, see [33]). Let $\mathscr{S}(\mathbb{R}^3)$ be the Schwartz class of rapidly decreasing functions, $\mathscr{S}'(\mathbb{R}^3)$ be the space of tempered distributions. Let $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier and the inverse Fourier transforms of any $L^1(\mathbb{R}^3)$ function f, which are, respectively, defined by setting, for any $\xi \in \mathbb{R}^3$,

$$\mathcal{F}f(\xi) := \hat{f}(\xi) := (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} \mathrm{e}^{-\mathrm{i}x\cdot\xi} f(x)dx \quad \text{and} \quad \mathcal{F}^{-1}f(\xi) := \check{f}(\xi) := \hat{f}(-\xi).$$

More generally, the Fourier transform of any $f \in \mathscr{S}'(\mathbb{R}^3)$ is given by $(\mathcal{F}f, g) := (f, \mathcal{F}g)$ for any $g \in \mathscr{S}(\mathbb{R}^3)$. Let $\mathcal{C}$ be the annulus $\{\xi \in \mathbb{R}^3 : \frac{3}{4} \leqslant |\xi| \leqslant \frac{8}{3}\}$ and $\mathcal{D}(\Omega)$ be a space of smooth compactly supported functions on the domain Ω. There exist radial functions χ and φ, valued in the interval $[0,1]$, belonging, respectively, to $\mathcal{D}(B(0,\frac{4}{3}))$ and $\mathcal{D}(\mathcal{C})$, such that

$$\begin{aligned}
&\chi(\xi) + \sum_{j\geqslant 0} \varphi(2^{-j}\xi) = 1, \quad \forall\, \xi \in \mathbb{R}^3, \quad \sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1, \quad \forall\, \xi \in \mathbb{R}^3\backslash\{0\},\\
&|j-j'| \geqslant 2 \Rightarrow \mathrm{supp}\varphi(2^{-j}\cdot) \cap \mathrm{supp}\varphi(2^{-j'}\cdot) = \emptyset,\\
&j' \geqslant 1 \Rightarrow \mathrm{supp}\chi(\cdot) \cap \mathrm{supp}\varphi(2^{-j'}\cdot) = \emptyset.
\end{aligned}$$

Define the set $\widetilde{\mathcal{C}} := B(0,\frac{2}{3}) + \mathcal{C}$. Then we have $|j-j'| \geqslant 5 \Rightarrow 2^{j'}\widetilde{\mathcal{C}} \cap 2^{j}\mathcal{C} = \emptyset$. Furthermore, we have

$$\forall\, \xi \in \mathbb{R}^3, \quad \frac{1}{2} \leqslant \chi^2(\xi) + \sum_{j\geqslant 0} \varphi^2(2^{-j}\xi) \leqslant 1, \quad \forall\, \xi \in \mathbb{R}^3\backslash\{0\}, \quad \frac{1}{2} \leqslant \sum_{j\in\mathbb{Z}} \varphi^2(2^{-j}\xi) \leqslant 1.$$

From now on, we write $h := \mathcal{F}^{-1}\varphi$ and $\widetilde{h} := \mathcal{F}^{-1}\chi$. The homogeneous dyadic blocks $\dot{\Delta}_j$ and $\dot{S}_j$ for all $j \in \mathbb{Z}$ are defined by

$$\begin{aligned}
\dot{\Delta}_j u &:= \varphi(2^{-j}D)u := 2^{3j} \int_{\mathbb{R}^3} h(2^j y)u(x-y)dy,\\
\dot{S}_j u &:= \chi(2^{-j}D)u := 2^{3j} \int_{\mathbb{R}^3} \tilde{h}(2^j y)u(x-y)dy.
\end{aligned}$$

Here, $D := (D_1, D_2, D_3)$ and $D_j := \mathrm{i}^{-1}\partial_{x_j} (\mathrm{i}^2 = -1)$. The set $\{\dot{\Delta}_j, \dot{S}_j\}_{j\in\mathbb{Z}}$ is called the Littlewood-Paley decomposition (see, for example, [53]). Formally, $\dot{\Delta}_j = \dot{S}_j - \dot{S}_{j-1}$ is a frequency projection to the annulus $\{\xi \in \mathbb{R}^3 : 2^{j-1} < |\xi| \leqslant 2^j\}$, and $\dot{S}_j = \sum_{j'\leqslant j-1} \dot{\Delta}_{j'}$ is a frequency projection to the ball $\{\xi \in \mathbb{R}^3 : |\xi| \leqslant 2^j\}$. We denote by $\mathscr{S}'_h(\mathbb{R}^3)$ the space of tempered distributions f such that $\lim_{j\to-\infty} \dot{S}_j f = 0$ in $\mathscr{S}'(\mathbb{R}^3)$. Recall that for any $s \in \mathbb{R}$ and $(p,r) \in [1,\infty] \times [1,\infty]$, the homogeneous Besov spaces $\dot{B}^s_{p,r}(\mathbb{R}^3)$ are defined by

$$\dot{B}^s_{p,r}(\mathbb{R}^3) := \{f \in \mathscr{S}'_h(\mathbb{R}^3) : \|f\|_{\dot{B}^s_{p,r}(\mathbb{R}^3)} < \infty\},$$

where

$$\|f\|_{\dot{B}^s_{p,r}(\mathbb{R}^3)} := \begin{cases} \left[\sum_{k\in\mathbb{Z}} \{2^{ks}\|\dot{\Delta}_k f\|_{L^p(\mathbb{R}^3)}\}^r\right]^{\frac{1}{r}}, & \text{when} \quad 1 \leqslant p \leqslant \infty, \quad 1 \leqslant r < \infty, \quad s \in \mathbb{R},\\ \sup_{k\in\mathbb{Z}} [2^{ks}\|\dot{\Delta}_k f\|_{L^p(\mathbb{R}^3)}], & \text{when} \quad 1 \leqslant p \leqslant \infty, \quad r = \infty, \quad s \in \mathbb{R}. \end{cases}$$

It is well known that if either $s < 3/p$ or $s = 3/p$ and $r = 1$, then $\dot{B}^s_{p,r}(\mathbb{R}^3)$ is a Banach space.

Let $-\infty < s_2 < s_1 < \infty$, $s_1 - 3/p_1 = s_2 - 3/p_2$ and $1 \leqslant r \leqslant \infty$. Then

$$\dot{B}^{s_1}_{p_1,r}(\mathbb{R}^3) \subset \dot{B}^{s_2}_{p_2,r}(\mathbb{R}^3). \tag{1.4}$$

Let us now recall the definition of the Chemin-Lerner space $\mathfrak{L}^\rho(0,T;\dot{B}^s_{p,r}(\mathbb{R}^3))$ with $0 < T \leqslant \infty$, $s \in \mathbb{R}$ and $1 \leqslant p, r, \rho \leqslant \infty$ (with the usual convention if $r = \infty$ or $\rho = \infty$). The Chemin-Lerner space is defined by

$$\mathfrak{L}^\rho(0,T;\dot{B}^s_{p,r}(\mathbb{R}^3)) := \{f \in \mathscr{S}'((0,T), \mathscr{S}'_h(\mathbb{R}^3)) : \|f\|_{\mathfrak{L}^\rho(0,T;\dot{B}^s_{p,r}(\mathbb{R}^3))} < \infty\},$$

where

$$\|f\|_{\mathfrak{L}^\rho(0,T;\dot{B}^s_{p,r}(\mathbb{R}^3))} := \|f\|_{\mathfrak{L}^\rho_T(\dot{B}^s_{p,r}(\mathbb{R}^3))} := \left\{ \sum_{k\in\mathbb{Z}} [2^{ks}\|\dot{\Delta}_k f\|_{L^\rho_T(L^p(\mathbb{R}^3))}]^r \right\}^{\frac{1}{r}},$$

with the usual modification made when $r=\infty$ or $\rho=\infty$. We equip the space

$$L^\rho(0,T;\dot{B}^s_{p,r}(\mathbb{R}^3)) := \{f \in \mathscr{S}'((0,T),\mathscr{S}'_h(\mathbb{R}^3)) : \|f\|_{L^\rho(0,T;\dot{B}^s_{p,r}(\mathbb{R}^3))} < \infty\},$$

with the following norm:

$$\|f\|_{L^\rho(0,T;\dot{B}^s_{p,r}(\mathbb{R}^3))} := \|f\|_{L^\rho_T(\dot{B}^s_{p,r}(\mathbb{R}^3))} := \left\{ \int_0^T \left[\sum_{k\in\mathbb{Z}} \{2^{ks}\|\Delta_k f\|_{L^p(\mathbb{R}^3)}\}^r \right]^{\frac{\rho}{r}} dt \right\}^{1/\rho},$$

where

$$\|f\|_{L^\rho_T(L^p(\mathbb{R}^3))} := \left[\int_0^T \|f\|^\rho_{L^p(\mathbb{R}^3)} d\tau \right]^{\frac{1}{\rho}}$$

and the usual modifications are needed when $r=\infty$ or $\rho=\infty$. From Minkowski's inequality, it is easy to deduce that

$$\begin{cases} \|f\|_{\mathfrak{L}^\rho(0,T;\dot{B}^s_{p,r}(\mathbb{R}^3))} \leqslant \|f\|_{L^\rho(0,T;\dot{B}^s_{p,r}(\mathbb{R}^3))}, & \text{when} \quad \rho \leqslant r, \\ \|f\|_{L^\rho(0,T;\dot{B}^s_{p,r}(\mathbb{R}^3))} \leqslant \|f\|_{\mathfrak{L}^\rho(0,T;\dot{B}^s_{p,r}(\mathbb{R}^3))}, & \text{when} \quad r \leqslant \rho. \end{cases}$$

If s_1 and s_2 are real numbers such that $s_1 < s_2$ and $\theta \in (0,1)$, then we have, for any $(p,r,\rho,\rho_1,\rho_2) \in [1,\infty]^5$ and any $1/\rho = \theta/\rho_1 + (1-\theta)/\rho_2$,

$$\begin{cases} \|f\|_{\dot{B}^{\theta s_1+(1-\theta)s_2}_{p,r}(\mathbb{R}^3)} \leqslant C\|f\|^\theta_{\dot{B}^{s_1}_{p,r}(\mathbb{R}^3)}\|f\|^{1-\theta}_{\dot{B}^{s_2}_{p,r}(\mathbb{R}^3)}, \\ \|f\|_{L^\rho(0,T;\dot{B}^{\theta s_1+(1-\theta)s_2}_{p,r}(\mathbb{R}^3))} \leqslant C\|f\|^\theta_{L^{\rho_1}(0,T;\dot{B}^{s_1}_{p,r}(\mathbb{R}^3))}\|f\|^{1-\theta}_{L^{\rho_2}(0,T;\dot{B}^{s_2}_{p,r}(\mathbb{R}^3))}, \\ \|f\|_{\mathfrak{L}^\rho(0,T;\dot{B}^{\theta s_1+(1-\theta)s_2}_{p,r}(\mathbb{R}^3))} \leqslant C\|f\|^\theta_{\mathfrak{L}^{\rho_1}(0,T;\dot{B}^{s_1}_{p,r}(\mathbb{R}^3))}\|f\|^{1-\theta}_{\mathfrak{L}^{\rho_2}(0,T;\dot{B}^{s_2}_{p,r}(\mathbb{R}^3))}, \end{cases}$$

where C is a positive constant independent of f.

The homogeneous paraproduct of v and u is defined by

$$T_u v := \sum_{j\in\mathbb{Z}} \dot{S}_{j-1}u\dot{\Delta}_j v.$$

The homogeneous remainder of v and u is defined by

$$R(u,v) := \sum_{|k-j|\leqslant 1} \dot{\Delta}_k u \dot{\Delta}_j v := \sum_{k\in\mathbb{Z}} \Delta_k u \widetilde{\dot{\Delta}}_k v \quad \text{and} \quad \widetilde{\dot{\Delta}}_k := \Delta_{k-1} + \dot{\Delta}_k + \dot{\Delta}_{k+1}.$$

We have the following Bony decomposition:

$$uv := T_u v + R(u,v) + T_v u. \tag{1.5}$$

For any operator $T : \dot{B}^s_{p,r}(\mathbb{R}^3) \to \dot{B}^s_{p,r}(\mathbb{R}^3)$, we let $\|u\|_{T\dot{B}^s_{p,r}(\mathbb{R}^3)} := \|Tu\|_{\dot{B}^s_{p,r}(\mathbb{R}^3)}$. Let Λ be the Fourier multiplier whose symbol is given by $|\xi|_1 := \sum_{i=1}^3 |\xi_i|$, where $\xi := (\xi_1,\xi_2,\xi_3) \in \mathbb{R}^3$. Let $\mathrm{e}^{\theta\sqrt{t}\Lambda}$ be the Fourier multiplier whose symbol is given by $\mathrm{e}^{\theta\sqrt{t}|\xi|_1}$.

Let $\theta \in \{0,1\}$ and

$$E_0 := \dot{B}^{-2+3/p}_{p,1}(\mathbb{R}^3) \times \dot{B}^{-2+3/r}_{r,1}(\mathbb{R}^3) \times \dot{B}^{3/q}_{q,1}(\mathbb{R}^3). \tag{1.6}$$

We introduce a vector space $\Theta_T := \mathbb{X}_T \times \mathbb{Y}_T \times \mathbb{Z}_T$ equipped with the usual product norm

$$\begin{aligned} &\|(u,v,w)\|_{\Theta_T} := \|u\|_{\mathbb{X}_T} + \|v\|_{\mathbb{Y}_T} + \|w\|_{\mathbb{Z}_T}, \\ &\mathbb{X}_T := \{u : u \in \mathfrak{L}^1(0,T;\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{3/p}_{p,1}(\mathbb{R}^3)) \cap \mathfrak{L}^\infty(0,T;\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/p}_{p,1}(\mathbb{R}^3))\}, \\ &\mathbb{Y}_T := \{v : v \in \mathfrak{L}^1(0,T;\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{3/r}_{r,1}(\mathbb{R}^3)) \cap \mathfrak{L}^\infty(0,T;\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/r}_{r,1}(\mathbb{R}^3))\}, \\ &\mathbb{Z}_T := \{w : w \in \mathfrak{L}^1(0,T;\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{2+3/q}_{q,1}(\mathbb{R}^3)) \cap \mathfrak{L}^\infty(0,T;\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{3/q}_{q,1}(\mathbb{R}^3))\} \end{aligned}$$

and

$$\|u\|_{\mathbb{X}_T} := \|u\|_{\mathfrak{L}^\infty(0,T;e^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/p}_{p,1}(\mathbb{R}^3))} + \|u\|_{\mathfrak{L}^1(0,T;e^{\theta\sqrt{t}\Lambda}\dot{B}^{3/p}_{p,1}(\mathbb{R}^3))},$$
$$\|v\|_{\mathbb{Y}_T} := \|v\|_{\mathfrak{L}^\infty(0,T;e^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/r}_{r,1}(\mathbb{R}^3))} + \|v\|_{\mathfrak{L}^1(0,T;e^{\theta\sqrt{t}\Lambda}\dot{B}^{3/r}_{r,1}(\mathbb{R}^3))},$$
$$\|w\|_{\mathbb{Z}_T} := \|w\|_{\mathfrak{L}^\infty(0,T;e^{\theta\sqrt{t}\Lambda}\dot{B}^{3/q}_{q,1}(\mathbb{R}^3))} + \|w\|_{\mathfrak{L}^1(0,T;e^{\theta\sqrt{t}\Lambda}\dot{B}^{2+3/q}_{q,1}(\mathbb{R}^3))}.$$

Meanwhile, for any $T \in (0,\infty]$, let

$$\Theta^{\mathcal{C}}_T := \mathcal{C}([0,T];\dot{B}^{-2+3/p}_{p,1}(\mathbb{R}^3)) \cap \mathcal{C}([0,T];\dot{B}^{-2+3/r}_{r,1}(\mathbb{R}^3)) \cap \mathcal{C}([0,T];\dot{B}^{3/q}_{q,1}(\mathbb{R}^3)),$$

where $\mathcal{C}([0,T];X)$ for any Banach space denotes the set of all continuous functions on $[0,T]$ with value in X. For the notational simplification, when $T=\infty$, we let $\mathbb{X}_\infty := \mathbb{X}$, $\mathbb{Y}_\infty := \mathbb{Y}$, $\mathbb{Z}_\infty := \mathbb{Z}$, $\Theta_\infty := \Theta$ and $\Theta^{\mathcal{C}}_\infty := \Theta^{\mathcal{C}}$.

This article focuses on the Cauchy problem of the following two-species chemotactic model:

$$\begin{cases} u_t = \Delta u - \chi_1 \nabla\cdot(u\nabla w) & \text{in} \quad (0,T)\times\mathbb{R}^3,\\ v_t = \Delta v - \chi_1 \nabla\cdot(v\nabla w) & \text{in} \quad (0,T)\times\mathbb{R}^3,\\ w_t = \Delta w - \gamma w + \alpha_1 u + \alpha_2 v & \text{in} \quad (0,T)\times\mathbb{R}^3,\\ (u,v,w)\,|_{t=0} = (u_0,v_0,w_0) & \text{in} \quad \mathbb{R}^3, \end{cases} \tag{1.7}$$

where $T \in (0,\infty]$. We recall that (1.7) enjoys nice scaling properties for $\gamma = 0$. If (u,v,w) solves (1.7), so does

$$(u^\lambda, v^\lambda, w^\lambda) := (\lambda^2 u(\lambda^2 t, \lambda x), \lambda^2 v(\lambda^2 t, \lambda x), w(\lambda^2 t, \lambda x))$$

with initial data

$$(u^\lambda(0,x), v^\lambda(0,x), w^\lambda(0,x)) := (\lambda^2 u_0(\lambda x), \lambda^2 v_0(\lambda x), w_0(\lambda x)).$$

We say that $(\mathbb{A},\mathbb{B},\mathbb{C})$ is a critical space for (1.7) ($\gamma=0$) if the norm of (u_0,v_0,w_0) in $\mathbb{A}\times\mathbb{B}\times\mathbb{C}$ is invariant for all $\lambda>0$.

We say that the solution (u,v,w) is self-similar for (1.7) ($\gamma = 0$) if $(u^\lambda, v^\lambda, w^\lambda) = (u,v,w)$ for each $\lambda > 0$.

Thus, we observe that $\dot{B}^{-2+3/p}_{p,1}(\mathbb{R}^3)\times\dot{B}^{-2+3/r}_{r,1}(\mathbb{R}^3)\times\dot{B}^{3/q}_{q,1}(\mathbb{R}^3)$ are critical spaces associated with (1.7) when $\gamma = 0$. Notice that when $\gamma \neq 0$, (1.7) has not a scaling property. Despite this, the case $\gamma\neq 0$ is intrinsic scaling that is inherited from the case $\gamma = 0$.

The first novelty of this paper is that we resort the Fourier localization technique and the Bony paraproduct theory to address existence issues of (1.7) (see Theorems 1.1 and 1.2 in the case $\theta = 0$) in homogeneous Besov spaces. Secondly, following the Gevrey class approach pioneered by Foias and Temam [30] and Foias [29] which is also used in [4, 6, 12, 15, 36], we are able to establish the analyticity of (1.7) by obtaining Gevrey estimates in homogeneous Besov spaces (see Theorems 1.1 and 1.2 in the case $\theta = 1$). More precisely, we show that mild solutions of (1.7) are in the Gevrey class (in this case $\theta = 1$) and they satisfy the estimate

$$\sup_{t>0} \|(e^{\sqrt{t}\Lambda}u, e^{\sqrt{t}\Lambda}v, e^{\sqrt{t}\Lambda}w)\|_{\dot{B}^{-2+3/p}_{p,1}(\mathbb{R}^3)\times\dot{B}^{-2+3/r}_{r,1}(\mathbb{R}^3)\times\dot{B}^{3/q}_{q,1}(\mathbb{R}^3)} < \infty.$$

Finally, the global Gevrey regular results in Theorem 1.2 in turn enable us to establish decay results for higher derivatives, which are given in Corollary 1.4 below.

Let us now show the local existence ($\theta = 0$) and the Gevrey regularity (analytic regularity) ($\theta = 1$) of (1.7) with large initial data.

Theorem 1.1. *Let* $\max\{\frac{2pq}{2pq+3p-3q}, \frac{2rq}{2rq+3r-3q}\} \leqslant r_1 < \infty$, $1/r_1 + 1/r_2 = 1$, $1 < p \leqslant q < \infty$, $\frac{1}{q}+\frac{1}{p} > \frac{1}{3}$, $\frac{1}{p}-\frac{1}{q} < \frac{2}{3}$, $1 < r \leqslant q < \infty$, $\frac{1}{q}+\frac{1}{r} > \frac{1}{3}$, $\frac{1}{r}-\frac{1}{q} < \frac{2}{3}$ *and* $\theta\in\{0,1\}$. *Then there exists a* $T\in(0,\infty]$ *such that* (1.7) *has a unique solution* $(u,v,w)\in\Theta^{\mathcal{C}}_T$ *with*

$$\begin{cases} u \in \mathfrak{L}^{r_1}_T(e^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/p+2/r_1}_{p,1}(\mathbb{R}^3)) \cap \mathfrak{L}^{r_2}_T(e^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/p+2/r_2}_{p,1}(\mathbb{R}^3)) \cap \mathfrak{L}^{\infty}_T(e^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/p}_{p,1}(\mathbb{R}^3)),\\ v \in \mathfrak{L}^{r_1}_T(e^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/r+2/r_1}_{r,1}(\mathbb{R}^3)) \cap \mathfrak{L}^{r_2}_T(e^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/r+2/r_2}_{r,1}(\mathbb{R}^3)) \cap \mathfrak{L}^{\infty}_T(e^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/r}_{r,1}(\mathbb{R}^3)),\\ w \in \mathfrak{L}^{r_1}_T(e^{\theta\sqrt{t}\Lambda}\dot{B}^{3/q+2/r_1}_{q,1}(\mathbb{R}^3)) \cap \mathfrak{L}^{r_2}_T(e^{\theta\sqrt{t}\Lambda}\dot{B}^{3/q+2/r_2}_{q,1}(\mathbb{R}^3)) \cap \mathfrak{L}^{\infty}_T(e^{\theta\sqrt{t}\Lambda}\dot{B}^{3/q}_{q,1}(\mathbb{R}^3)). \end{cases} \tag{1.8}$$

Furthermore, let the maximal time T^* be finite. Then

$$\begin{cases}\|u\|_{\mathfrak{L}^{r_1}(0,T^*;e^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/p+2/r_1}_{p,1}(\mathbb{R}^3))\cap\mathfrak{L}^{r_2}(0,T^*;e^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/p+2/r_2}_{p,1}(\mathbb{R}^3))}=\infty,\\ \|v\|_{\mathfrak{L}^{r_1}(0,T^*;e^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/r+2/r_1}_{r,1}(\mathbb{R}^3))\cap\mathfrak{L}^{r_2}(0,T^*;e^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/r+2/r_2}_{r,1}(\mathbb{R}^3))}=\infty,\\ \|w\|_{\mathfrak{L}^{r_1}(0,T^*;e^{\theta\sqrt{t}\Lambda}B^{3/q+2/r_1}_{q,1}(\mathbb{R}^3))\cap\mathfrak{L}^{r_2}(0,T^*;e^{\theta\sqrt{t}\Lambda}B^{3/q+2/r_2}_{q,1}(\mathbb{R}^3))}=\infty.\end{cases}\tag{1.9}$$

In particular, the above results also hold true when $\theta=0$, $p=1$ and $r=1$.

In the following, we obtain the global existence ($\theta=0$) and the Gevrey regularity (analytic regularity) ($\theta=1$) of (1.7) with small initial data.

Theorem 1.2. *Let $1<p\leqslant q<\infty$, $\frac{1}{q}+\frac{1}{p}>\frac{1}{3}$, $\frac{1}{p}-\frac{1}{q}\leqslant\frac{2}{3}$, $1<r\leqslant q<\infty$, $\frac{1}{q}+\frac{1}{r}>\frac{1}{3}$, $\frac{1}{r}-\frac{1}{q}\leqslant\frac{2}{3}$ and $\theta=\{0,1\}$. Let $(u_0,v_0,w_0)\in E_0$ with*

$$\|(u_0,v_0,w_0)\|_{E_0}\leqslant\epsilon_0\tag{1.10}$$

for some sufficiently small ϵ_0, where E_0 is as in (1.6)*. Then* (1.7) *admits a global-in-time solution such that $(u,v,w)\in\Theta\cap\Theta^{\mathcal{C}}$. In particular, the above results also hold true when $\theta=0$, $p=1$ and $r=1$.*

Remark 1.3. (i) We can also have a version of Theorems 1.1 and 1.2 in any space dimension after making some slight modifications of their proofs. Just for a clear presentation, we choose to work in three space dimension case here.

(ii) For any $t\in(0,T^*)$ in Theorem 1.1 and $t>0$ in Theorem 1.2, we obtain the solution $(u,v,w)\in C^\infty(\mathbb{R}^3)\times C^\infty(\mathbb{R}^3)\times C^\infty(\mathbb{R}^3)$.

(iii) In particular, when $v=0$, (1.7) becomes the classical semi-linear Keller-Segel system of the double parabolic type (1.2). In [59], Zhai established the global existence and uniqueness of solutions with initial data $(u_0,w_0)\in\dot{B}^{-2+3/p}_{p,r}(\mathbb{R}^3)\times\dot{B}^{3/p}_{p,r}(\mathbb{R}^3)$ for $\frac{3}{2}<p<\infty$ and $1\leqslant r\leqslant\infty$. In [3], Bae established the global existence and uniqueness of solutions with initial data $(u_0,w_0)\in\dot{B}^{-2+3/p}_{p,1}(\mathbb{R}^3)\times\dot{B}^{3/p}_{p,1}(\mathbb{R}^3)$ for $1\leqslant p<3$. However, the indices of the Besov spaces in Theorems 1.1 and 1.2 cannot be obtained by the method used in the above works. Thus, we note that even for the classical semi-linear Keller-Segel system of double parabolic type, our existence ($\theta=0$) results in Theorems 1.1 and 1.2 are also new.

(iv) The main reason that we need to impose the restrictive conditions $\frac{1}{r}-\frac{1}{q}<\frac{2}{3}$ and $\frac{1}{p}-\frac{1}{q}<\frac{2}{3}$ mentioned in Theorem 1.1 in place of $\frac{1}{r}-\frac{1}{q}\leqslant\frac{2}{3}$ and $\frac{1}{r}-\frac{1}{q}\leqslant\frac{2}{3}$ in Theorem 1.2 lies in the facts that (5.1) below needs

$$\max\left\{\frac{2pq}{2pq+3p-3q},\frac{2rq}{2rq+3r-3q}\right\}\leqslant r_1<\infty.$$

Moreover, by Lemmas 2.6 and 3.1, we only can choose $r_1=\infty$ if $\frac{1}{p}-\frac{1}{r}=\frac{1}{p}-\frac{1}{q}=\frac{2}{3}$.

(v) Let all assumptions in Theorem 1.2 hold true. When $\gamma=0$ in (1.7), suppose furthermore that u_0, v_0 and w_0 are, respectively, homogeneous of degree -2, -2 and 0, namely, they satisfy the relations $u_0(x)=\lambda^2u_0(\lambda x)$, $v_0(x)=\lambda^2v_0(\lambda x)$ and $w_0(x)=w_0(\lambda x)$ for all $x\in\mathbb{R}^3$ and $\lambda>0$. The global solution (u,v,w) of (1.7) given by Theorem 1.2 satisfies $u(t,x)=\lambda^2u(\lambda^2t,\lambda x)$, $v(t,x)=\lambda^2v(\lambda^2t,\lambda x)$ and $w(t,x)=w(\lambda^2t,\lambda x)$.

As a consequence of working with Gevrey norms, we obtain the decay of higher-order derivatives of the corresponding solutions in Besov and Lebesgue spaces. The global solution (u,v,w) in Theorem 1.2 in turn enables us to establish the following time-decay estimate on the high-order derivatives of (u,v,w) of Besov and Lebesgue spaces.

Corollary 1.4. *Let $k>0$ and D^k be the Fourier multiplier whose symbol is given by $|\xi|^{2k}$, where $\xi=(\xi_1,\xi_2,\xi_3)\in\mathbb{R}^3$. There exist positive constants C_1, C_2, C_3 and C such that*

(i) *If $m>0$, then*

$$\|(D^mu,D^mv,D^mw)\|_{\dot{B}^{-2+3/p}_{p,1}(\mathbb{R}^3)\times\dot{B}^{-2+3/r}_{r,1}(\mathbb{R}^3)\times\dot{B}^{3/q}_{q,1}(\mathbb{R}^3)}\lesssim C^mm^mt^{-\frac{m}{2}}.$$

(ii) *If* $k_1 > -2 + 3/p$ *and* $1 < p \leqslant \frac{3}{2}$, *then*

$$\|D^{k_1} u(t)\|_{L^p(\mathbb{R}^3)} \lesssim C_1^{k_1+2-3/p}(k_1 + 2 - 3/p)^{k_1+2-3/p} t^{-\frac{k_1+2-3/p}{2}}.$$

(iii) *If* $k_2 > -2 + 3/r$ *and* $1 < r \leqslant \frac{3}{2}$, *then*

$$\|D^{k_2} v(t)\|_{L^r(\mathbb{R}^3)} \lesssim C_2^{k_2+2-3/r}(k_2 + 2 - 3/r)^{k_2+2-3/r} t^{-\frac{k_2+2-3/r}{2}}.$$

(iv) *If* $k_3 > 3/q$ *and* $1 < q \leqslant 2$, *then*

$$\|D^{k_3} w(t)\|_{L^q(\mathbb{R}^3)} \leqslant C_3^{k_3-3/q}(k_3 - 3/q)^{k_3-3/q} t^{-\frac{k_3-3/q}{2}}.$$

Notation. Throughout the paper, c and C stand for harmless positive constants, and we sometimes use the notation $a \lesssim b$ as an equivalent of $a \leqslant Cb$. For a Banach space X and an interval I of $\mathbb{R}$, we denote by $\mathcal{C}(I; X)$ the set of all continuous functions on I with value in X. The symbol $(d_j)_{j\in\mathbb{Z}}$ is a generic element of $\ell^1(\mathbb{Z})$ so that $d_j \geqslant 0$ and $\sum_{j\in\mathbb{Z}} d_j = 1$.

2 Preliminaries

The proofs of Theorem 1.1 in Section 3 and, likewise, Theorem 1.2 in Section 4, require a lot of elementary inequalities which are summarized in the following.

Lemma 2.1 (See [33])**.** *Let* $\mathcal{C}$ *be an annulus and* $\mathcal{B}$ *be a ball in* $\mathbb{R}^3$. *Then there exists a positive constant* C *such that for any non-negative integer* k, *any couple* $(p, q) \in [1, \infty]^2$ *with* $q \geqslant p \geqslant 1$, *and any function* u *of* $L^p(\mathbb{R}^3)$,

(i) *if* $\operatorname{supp} \hat{u} \subseteq \lambda\mathcal{B}$, *then*

$$\|D^k u\|_{L^q(\mathbb{R}^3)} := \sup_{|\alpha|\leqslant k} \|\partial^\alpha u\|_{L^q(\mathbb{R}^3)} \leqslant C^{k+1}\lambda^{k+n(\frac{1}{p}-\frac{1}{q})}\|u\|_{L^p(\mathbb{R}^3)};$$

(ii) *if* $\operatorname{supp} \hat{u} \subseteq \lambda\mathcal{C}$, *then*

$$C^{-k-1}\lambda^k\|u\|_{L^p(\mathbb{R}^3)} \leqslant \|D^k u\|_{L^p(\mathbb{R}^3)} \leqslant C^{k+1}\lambda^k\|u\|_{L^p(\mathbb{R}^3)}.$$

Lemma 2.2 (See [33])**.** *Let* $\mathcal{C}$ *be an annulus in* $\mathbb{R}^3$. *Then there exist two positive constants* c *and* C *such that for any* $p \in [1, \infty]$ *and any couple* (t, λ) *of positive numbers, if* $\operatorname{supp}\hat{f} \subset \lambda\mathcal{C}$, *then*

$$\|\mathrm{e}^{t\Delta} f\|_{L^p(\mathbb{R}^3)} \leqslant C\mathrm{e}^{-c\lambda^2 t}\|f\|_{L^p(\mathbb{R}^3)}.$$

Here, $\mathrm{e}^{t\Delta}$ *is the heat operator with kernel* $G(x, t) = (4t)^{-3/2}\exp(-|x|^2/4t)$ *for all* $x \in \mathbb{R}^3$ *and* $t \in (0, \infty)$.

Lemma 2.3. *Let* Λ *be the Fourier multiplier whose symbol is given by* $|\xi|_1 := \sum_{i=1}^3 |\xi_i|$, *where* $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$. *Consider the operator* $E := \mathrm{e}^{-[\sqrt{t-s}+\sqrt{s}-\sqrt{t}]\Lambda}$ *for* $0 \leqslant s \leqslant t$. *Then* E *is either the identity operator or an operator having* $L^1(\mathbb{R}^3)$ *kernel whose* $L^1(\mathbb{R}^3)$ *norm has a bound independent of* s *and* t.

Proof. For the proof of this lemma, we refer the reader to [6, 42]. □

Lemma 2.4. *Let* Λ *be the Fourier multiplier whose symbol is given by* $|\xi|_1 := \sum_{i=1}^3 |\xi_i|$, *where* $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$. *Then the operator* $E = \mathrm{e}^{\frac{1}{2}a\Delta+\sqrt{a}\Lambda}$ *is a Fourier multiplier which is bounded on* $L^p(\mathbb{R}^3)$ *with* $p \in (1, \infty)$, *and its operator norm is uniformly bounded with respect to* $a \geqslant 0$.

Proof. For the proof of this lemma, we refer the reader to [6, 52]. □

Lemma 2.5. *Let* Λ *be the Fourier multiplier whose symbol is given by* $|\xi|_1 := \sum_{i=1}^3 |\xi_i|$, *where* $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$. *Let* $s \in \mathbb{R}$, $\theta \in \{0, 1\}$, $1 \leqslant \rho, p, r \leqslant \infty$, $T \in (0, \infty]$ *and* $\rho_2 = (1+1/\rho_1-1/\rho)^{-1}$. *Assume that* $u_0 \in \dot{B}^s_{p,1}(\mathbb{R}^3)$, $f \in \mathfrak{L}^\rho(0, T; \mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{s-2+2/\rho}_{p,1}(\mathbb{R}^3))$ *and* u *solves*

$$\begin{cases} u_t - \Delta u = f & \text{in} \quad \mathbb{R}^3 \times (0, T), \\ u\,|_{t=0}= u_0 & \text{in} \quad \mathbb{R}^3. \end{cases} \tag{2.1}$$

Then there exist two positive constants c and C such that for all $\rho_1 \in [\rho, \infty]$,

$$\begin{aligned}
&\|u\|_{\mathfrak{L}^{\rho_1}(0,T;\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{s+2/\rho_1}_{p,1}(\mathbb{R}^3))} \\
&\quad \leqslant C\sum_{j\in\mathbb{Z}} 2^{js}\|\dot{\Delta}_q u_0\|_{L^p(\mathbb{R}^3)}\left(\frac{1-\mathrm{e}^{-cT2^{2j}\rho_1}}{c\rho_1}\right)^{\frac{1}{\rho_1}} \\
&\qquad + C\sum_{j\in\mathbb{Z}} 2^{j(s-2+\frac{2}{\rho})}\|\dot{\Delta}_j \mathrm{e}^{\theta\sqrt{t}\Lambda} f\|_{L^\rho_T(L^p(\mathbb{R}^3))}\left(\frac{1-\mathrm{e}^{-cT2^{2j}\rho_2}}{c\rho_2}\right)^{\frac{1}{\rho_2}};
\end{aligned} \tag{2.2}$$

in particular, there holds true that

$$\|u\|_{\mathfrak{L}^{\rho_1}(0,T;\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{s+2/\rho_1}_{p,1}(\mathbb{R}^3))} \leqslant C[\|u_0\|_{\dot{B}^s_{p,1}(\mathbb{R}^3)} + \|f\|_{\mathfrak{L}^{\rho}(0,T;\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{s-2+2/\rho}_{p,1}(\mathbb{R}^3))}].$$

Moreover, $u \in \mathcal{C}([0,T];\dot{B}^s_{p,1}(\mathbb{R}^3))$. If $f=0$ and $1 \leqslant \rho_1 < \infty$, then

$$\lim_{T\to 0^+} \|u\|_{\mathfrak{L}^{\rho}(0,T;\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{s+2/\rho_1}_{p,1}(\mathbb{R}^3))} = 0. \tag{2.3}$$

Here and hereafter, $T \to 0^+$ means $T > 0$ and $T \to 0$.

Proof. When $\theta = 0$, this lemma can be deduced from [23] immediately, the details being omitted here. We only prove Lemma 2.5 in the case $\theta = 1$ by making some minor modifications. Thanks to Duhamel's principle, we can reformulate (2.1) into the following integral equation:

$$u(t,x) = \mathrm{e}^{t\Delta}u_0 + \int_0^t \mathrm{e}^{(t-\tau)\Delta} f(\tau,x)d\tau. \tag{2.4}$$

Applying $\dot{\Delta}_j \mathrm{e}^{\sqrt{t}\Lambda}$ to (2.4), we conclude that

$$\dot{\Delta}_j \mathrm{e}^{\sqrt{t}\Lambda}u(t,x) = \dot{\Delta}_j \mathrm{e}^{\sqrt{t}\Lambda}\mathrm{e}^{t\Delta}u_0 + \int_0^t \mathrm{e}^{(t-\tau)\Delta}\dot{\Delta}_j \mathrm{e}^{\sqrt{t}\Lambda} f(\tau,x)d\tau.$$

Therefore, by Lemmas 2.1–2.4, we have the following inequality:

$$\begin{aligned}
&\|\dot{\Delta}_j \mathrm{e}^{\sqrt{t}\Lambda}u(t,x)\|_{L^p(\mathbb{R}^3)} \\
&\quad \leqslant \|\dot{\Delta}_j \mathrm{e}^{\sqrt{t}\Lambda}\mathrm{e}^{t\Delta}u_0\|_{L^p(\mathbb{R}^3)} + \int_0^t \|\mathrm{e}^{(t-\tau)\Delta}\dot{\Delta}_j \mathrm{e}^{\sqrt{t}\Lambda} f(\tau,x)\|_{L^p(\mathbb{R}^3)}d\tau \\
&\quad \leqslant \|\mathrm{e}^{\sqrt{t}\Lambda+\frac{1}{2}t\Delta}\dot{\Delta}_j \mathrm{e}^{\frac{t}{2}\Delta}u_0\|_{L^p(\mathbb{R}^3)} \\
&\qquad + \int_0^t \|\mathrm{e}^{(\sqrt{t}-\sqrt{\tau}-\sqrt{t-\tau})\Lambda}\mathrm{e}^{\sqrt{t-\tau}\Lambda+\frac{t-\tau}{2}\Delta}\dot{\Delta}_j \mathrm{e}^{\frac{t-\tau}{2}\Delta}\mathrm{e}^{\sqrt{\tau}\Lambda} f(\tau,x)\|_{L^p(\mathbb{R}^3)}d\tau \\
&\quad \leqslant \|\dot{\Delta}_j \mathrm{e}^{\frac{t}{2}\Delta}u_0\|_{L^p(\mathbb{R}^3)} + \int_0^t \|\dot{\Delta}_j \mathrm{e}^{\frac{t-\tau}{2}\Delta}\mathrm{e}^{\sqrt{\tau}\Lambda} f(\tau,x)\|_{L^p(\mathbb{R}^3)}d\tau \\
&\quad \leqslant \mathrm{e}^{-ct2^{2j}}\|\dot{\Delta}_j u_0\|_{L^p(\mathbb{R}^3)} + \int_0^t \mathrm{e}^{-c(t-\tau)2^{2j}}\|\dot{\Delta}_j \mathrm{e}^{\sqrt{\tau}\Lambda} f(\tau,x)\|_{L^p(\mathbb{R}^3)}d\tau.
\end{aligned} \tag{2.5}$$

Taking $L^{\rho_1}_T$ norm on the both sides of (2.5), and using Young's inequality, for $\rho_1 \leqslant \rho \leqslant \infty$, we conclude that

$$\begin{aligned}
\|\dot{\Delta}_j \mathrm{e}^{\sqrt{t}\Lambda}u(t,x)\|_{L^{\rho_1}_T(L^p(\mathbb{R}^3))} &\leqslant \left(\frac{1-\mathrm{e}^{-cT\rho_1 2^{2j}}}{c\rho_1 2^{2j}}\right)^{1/\rho_1}\|\dot{\Delta}_j u_0\|_{L^p(\mathbb{R}^3)} \\
&\quad + \left(\frac{1-\mathrm{e}^{-cT\rho_2 2^{2j}}}{c\rho_2 2^{2j}}\right)^{1/\rho_2}\|\dot{\Delta}_j \mathrm{e}^{\sqrt{t}\Lambda} f(t,x)\|_{L^\rho_T(L^p(\mathbb{R}^3))}
\end{aligned}$$

with $1/\rho_2 = 1 + 1/\rho_1 - 1/\rho$.

Multiplying by $2^{j(s+2/\rho_1)}$ and summing up over j, we then obtain (2.2).

Finally, applying the Lebesgue dominated convergence theorem and (2.2), we immediately obtain (2.3), which completes the proof of Lemma 2.5. □

Lemma 2.6. *Let Λ be the Fourier multiplier whose symbol is given by $|\xi|_1 := \sum_{i=1}^3 |\xi_i|$, where $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$. Let $1/r_1 + 1/r_2 = 1$, $1 < p \leqslant q < \infty$, $\frac{1}{q} + \frac{1}{p} > \frac{1}{3}$, $\frac{1}{p} - \frac{1}{q} \leqslant \frac{2}{3}$, $r_1 \geqslant \max\{2, \frac{2pq}{2pq+3p-3q}\}$ and $\theta \in \{0, 1\}$. If $T \in (0, \infty]$, $\phi \in \mathfrak{L}^1(0, T; \mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{2+3/q}_{q,1}(\mathbb{R}^3)) \cap \mathfrak{L}^\infty(0, T; \mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{3/q}_{q,1}(\mathbb{R}^3))$ and*

$$f \in \mathfrak{L}^1(0, T; \mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{3/p}_{p,1}(\mathbb{R}^3)) \cap \mathfrak{L}^\infty(0, T; \mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/p}_{p,1}(\mathbb{R}^3)),$$

then it holds true that

$$\begin{aligned}
&\|f\nabla\phi\|_{\mathfrak{L}^1_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{-1+3/p}_{p,1}(\mathbb{R}^3))} \\
&\quad\leqslant C\|f\|^{1-\frac{1}{r_2}}_{\mathfrak{L}^\infty_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/p}_{p,1}(\mathbb{R}^3))}\|f\|^{\frac{1}{r_2}}_{\mathfrak{L}^1_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{3/p}_{p,1}(\mathbb{R}^3))}\|\phi\|^{1-\frac{1}{r_1}}_{\mathfrak{L}^\infty_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{3/q}_{q,1}(\mathbb{R}^3))}\|\phi\|^{\frac{1}{r_1}}_{\mathfrak{L}^1_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{2+3/q}_{q,1}(\mathbb{R}^3))} \\
&\qquad + C\|f\|^{1-\frac{1}{r_1}}_{\mathfrak{L}^\infty_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/p}_{p,1}(\mathbb{R}^3))}\|f\|^{\frac{1}{r_1}}_{\mathfrak{L}^1_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{3/p}_{p,1}(\mathbb{R}^3))}\|\phi\|^{1-\frac{1}{r_2}}_{\mathfrak{L}^\infty_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{3/q}_{q,1})}\|\phi\|^{\frac{1}{r_2}}_{\mathfrak{L}^1_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{2+3/q}_{q,1}(\mathbb{R}^3))},
\end{aligned}$$

where C is a positive constant independent of f and ϕ. For $r_1 = \infty$, it holds true that

$$\begin{aligned}
&\|f\nabla\phi\|_{\mathfrak{L}^1_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{-1+3/p}_{p,1}(\mathbb{R}^3))} \\
&\quad\leqslant C\|f\|_{\mathfrak{L}^1_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{3/p}_{p,1})}\|\phi\|_{\mathfrak{L}^\infty_T(\mathrm{e}^{\sqrt{t}\Lambda}\dot{B}^{3/q}_{q,1}(\mathbb{R}^3))} + C\|f\|_{\mathfrak{L}^\infty_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/p}_{p,1}(\mathbb{R}^3))}\|\phi\|_{\mathfrak{L}^1_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{2+3/q}_{q,1}(\mathbb{R}^3))},
\end{aligned}$$

where C is a positive constant independent of f and ϕ. In particular, the above results when $\theta = 0$ and $p = 1$ also hold true.

Proof. Let $(F, \Phi) := (\mathrm{e}^{\theta\sqrt{t}\Lambda}f, \mathrm{e}^{\theta\sqrt{t}\Lambda}\phi)$. It follows from the Bony paraproduct decomposition (1.5) that

$$\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{\Delta}_j(f\nabla\phi) = \mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{\Delta}_j(T_{\mathrm{e}^{-\theta\sqrt{t}\Lambda}F}\nabla\mathrm{e}^{-\theta\sqrt{t}\Lambda}\Phi + R(\mathrm{e}^{-\theta\sqrt{t}\Lambda}F, \nabla\mathrm{e}^{-\theta\sqrt{t}\Lambda}\Phi) + T_{\nabla\mathrm{e}^{-\theta\sqrt{t}\Lambda}\Phi}\mathrm{e}^{-\theta\sqrt{t}\Lambda}F),$$

which ensures that

$$\begin{aligned}
\|\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{\Delta}_j(f\nabla\phi)\|_{L^1_TL^p(\mathbb{R}^3)} \leqslant &\sum_{|j-j'|\leqslant 4}\|\dot{\Delta}_j\mathrm{e}^{\theta\sqrt{t}\Lambda}(S_{j'-1}\mathrm{e}^{-\sqrt{t}\Lambda}F\Delta_{j'}\nabla\mathrm{e}^{-\theta\sqrt{t}\Lambda}\Phi)\|_{L^1_T(L^p(\mathbb{R}^3))} \\
&+ \sum_{|j-j'|\leqslant 4}\|\dot{\Delta}_j\mathrm{e}^{\theta\sqrt{t}\Lambda}(S_{j'-1}\mathrm{e}^{-\theta\sqrt{t}\Lambda}\nabla\Phi\dot{\Delta}_{j'}\mathrm{e}^{-\sqrt{t}\Lambda}F)\|_{L^1_T(L^p(\mathbb{R}^3))} \\
&+ \sum_{j'\geqslant j-N_0}\|\dot{\Delta}_j\mathrm{e}^{\theta\sqrt{t}\Lambda}(\Delta_{j'}\mathrm{e}^{-\theta\sqrt{t}\Lambda}F\widetilde{\dot{\Delta}}_{j'}\nabla\mathrm{e}^{-\theta\sqrt{t}\Lambda}\Phi)\|_{L^1_T(L^p(\mathbb{R}^3))},
\end{aligned} \tag{2.6}$$

where N_0 is some fixed positive integer.

Now we introduce the bilinear operators $B^\theta_t(\mathfrak{f}, \mathfrak{g})$ of the form

$$B^\theta_t(\mathfrak{f}, \mathfrak{g}) := \mathrm{e}^{\theta\sqrt{t}\Lambda}(\mathrm{e}^{-\theta\sqrt{t}\Lambda}\mathfrak{f}\mathrm{e}^{-\theta\sqrt{t}\Lambda}\mathfrak{g}) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\mathrm{e}^{\mathrm{i}x\xi}\mathrm{e}^{\theta\sqrt{t}(|\xi|_1-|\xi-\eta|_1-|\eta|_1)}\widehat{\mathfrak{f}}(\xi-\eta)\widehat{\mathfrak{g}}(\eta)d\eta d\xi. \tag{2.7}$$

Claim that

$$\|B^\theta_t(\mathfrak{f}, \mathfrak{g})\|_{L^{\mathfrak{p}}(\mathbb{R}^3)} \lesssim \|\mathfrak{f}\|_{L^{p_1}(\mathbb{R}^3)}\|\mathfrak{g}\|_{L^{p_2}} \quad \text{when} \quad 1/p_1 + 1/p_2 = 1/\mathfrak{p}, \quad 1 < p_1, p_2 \leqslant \infty, \quad 1 < \mathfrak{p} < \infty. \tag{2.8}$$

When $\theta = 0$, it is obvious to obtain

$$\|B^0_t(\mathfrak{f}, \mathfrak{g})\|_{L^{\mathfrak{p}}(\mathbb{R}^3)} \lesssim \|\mathfrak{f}\|_{L^{p_1}(\mathbb{R}^3)}\|\mathfrak{g}\|_{L^{p_2}(\mathbb{R}^3)} \quad \text{when} \quad 1/p_1 + 1/p_2 = 1/\mathfrak{p}, \quad 1 \leqslant \mathfrak{p} < \infty.$$

Next, we only prove the case $\theta = 1$, here we borrow some ideas from [36]. For $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, $\mu = (\mu_1, \mu_2, \mu_3)$ and $\nu = (\nu_1, \nu_2, \nu_3)$ with λ_i, μ_i, $\nu_i \in \{1, -1\}$ for $i \in \{1, 2, 3\}$, let

$$\begin{aligned}
&D_\lambda := \{\eta : \lambda_i\eta_i \geqslant 0,\ i = 1, 2, 3\}, \\
&D_\mu := \{\xi - \eta : \mu_i(\xi_i - \eta_i) \geqslant 0,\ i = 1, 2, 3\}, \\
&D_\nu := \{\xi : \nu_i\xi_i \geqslant 0,\ i = 1, 2, 3\}.
\end{aligned}$$

We denote by χ_D the characteristic function on domain D. Then we can rewrite (2.7) ($\theta=1$) as

$$\sum_{\lambda_i,\mu_j,\nu_k\in\{-1,1\}}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}x\xi}\chi_{D_\nu}(\xi)\mathrm{e}^{\sqrt{t}(|\xi|_1-|\xi-\eta|_1-|\eta|_1)}\chi_{D_\mu}(\xi-\eta)\widehat{\mathfrak{f}}(\xi-\eta)\chi_{D_\lambda}(\eta)\widehat{\mathfrak{g}}(\eta)d\eta d\xi.$$

Obverse that for $\eta\in D_\lambda$, $\xi-\eta\in D_\mu$ and $\xi\in D_\mu$, $\mathrm{e}^{\sqrt{t}(|\xi_i|-|\xi_i-\eta_i|-|\eta_j|)}$ must belong to the following set:

$$\mathfrak{M}:=\{1,\mathrm{e}^{-2\sqrt{t}|\xi_i|},\mathrm{e}^{-2\sqrt{t}|\xi_i-\eta_i|},\mathrm{e}^{-2\sqrt{t}|\eta_i|}\}\quad\text{when}\quad i\in\{1,2,3\}.$$

Let $1\leqslant p\leqslant\infty$ and $\varrho\in\mathscr{S}'(\mathbb{R}^3)$. If there exists a positive constant C such that, for all $f\in\mathscr{S}(\mathbb{R}^3)$,

$$\|\mathcal{F}^{-1}\varrho\mathcal{F}f\|_{L^p(\mathbb{R}^3)}\leqslant C\|f\|_{L^p(\mathbb{R}^3)},$$

then ϱ is called a multiplier on $L^p(\mathbb{R}^3)$. The set of all multipliers on $L^p(\mathbb{R}^3)$ is denoted by $M_p(\mathbb{R}^3)$ (for more details, see [35]).

When $1<\mathfrak{p}<\infty$, $\chi_{D_\lambda}\in M_p(\mathbb{R}^3)$, $m\in M_{\mathfrak{p}}(\mathbb{R}^3)$ for any $m\in\mathfrak{M}$, it follows from the algebra property of $M_{\mathfrak{p}}(\mathbb{R}^3)$ that

$$\|B_t^1(\mathfrak{f},\mathfrak{g})\|_{L^{\mathfrak{p}}(\mathbb{R}^3)}\lesssim\|\mathfrak{f}\|_{L^{p_1}(\mathbb{R}^3)}\|\mathfrak{g}\|_{L^{p_2}(\mathbb{R}^3)}\quad\text{when}\quad 1/p_1+1/p_2=1/\mathfrak{p}.$$

Choosing $1\leqslant r_1,\, r_2\leqslant\infty$ such that $\frac{1}{r_2}+\frac{1}{r_1}=1$, noticing that $r_1\geqslant 2$, thanks to (2.8), applying Lemmas 2.1 and 2.2, we obtain

$$\begin{aligned}
&\|\dot{\Delta}_j\mathrm{e}^{\theta\sqrt{t}\Lambda}(T_{\nabla\mathrm{e}^{-\theta\sqrt{t}\Lambda}\Phi}\mathrm{e}^{-\sqrt{t}\theta\Lambda}F)\|_{L^1_T(L^p(\mathbb{R}^3))}\\
&\quad\lesssim\sum_{|j-j'|\leqslant4}\|\Delta_j\mathrm{e}^{\theta\sqrt{t}\Lambda}(\dot{S}_{j'-1}\mathrm{e}^{-\theta\sqrt{t}\Lambda}\nabla\Phi\dot{\Delta}_{j'}\mathrm{e}^{-\theta\sqrt{t}\Lambda}F)\|_{L^1_T(L^p(\mathbb{R}^3))}\\
&\quad\lesssim\sum_{|j-j'|\leqslant4}\|\dot{\Delta}_{j'}F\|_{L^{r_2}_T(L^p(\mathbb{R}^3))}\|\dot{S}_{j'-1}\nabla\Phi\|_{L^{r_1}_T(L^\infty(\mathbb{R}^3))}\\
&\quad\lesssim\sum_{|j-j'|\leqslant4}\sum_{k\leqslant j'-2}2^{k(1+\frac{3}{q})}\|\dot{\Delta}_k\Phi\|_{L^{r_1}_T(L^q(\mathbb{R}^3))}\|\dot{\Delta}_{j'}F\|_{L^{r_2}_T(L^p(\mathbb{R}^3))}\\
&\quad\lesssim\sum_{|j-j'|\leqslant4}\|\dot{\Delta}_{j'}F\|_{L^{r_2}_T(L^p(\mathbb{R}^3))}\sum_{k\leqslant j'-2}2^{k(1-\frac{2}{r_1})}2^{k(3/q+2/r_1)}\|\dot{\Delta}_k\Phi\|_{L^{r_1}_T(L^q(\mathbb{R}^3))}\\
&\quad\lesssim\sum_{|j-j'|\leqslant4}2^{j'(2-3/p-2/r_2)}d_{j'}\sum_{k\leqslant j'-2}2^{(1-2/r_1)k}\|\Phi\|_{\mathfrak{L}^{r_1}_T(\dot{B}^{3/q+2/r_1}_{q,1}(\mathbb{R}^3))}\|F\|_{\mathfrak{L}^{r_2}_T(\dot{B}^{-2+2/r_2+3/p}_{p,1}(\mathbb{R}^3))}\\
&\quad\lesssim 2^{j(1-3/p)}d_j\|\Phi\|_{\mathfrak{L}^{r_1}_T(\dot{B}^{3/q+2/r_1}_{q,1}(\mathbb{R}^3))}\|F\|_{\mathfrak{L}^{r_2}_T(\dot{B}^{-2+3/p+2/r_2}_{p,1}(\mathbb{R}^3))}\\
&\quad\lesssim 2^{j(1-3/p)}d_j\|F\|^{1-\frac{1}{r_2}}_{\mathfrak{L}^\infty_T(B^{-2+3/p}_{p,1}(\mathbb{R}^3))}\|F\|^{\frac{1}{r_2}}_{\mathfrak{L}^1_T(\dot{B}^{3/p}_{p,1}(\mathbb{R}^3))}\|\Phi\|^{1-\frac{1}{r_1}}_{\mathfrak{L}^\infty_T(\dot{B}^{3/q}_{q,1}(\mathbb{R}^3))}\|\Phi\|^{\frac{1}{r_1}}_{\mathfrak{L}^1_T(\dot{B}^{2+3/q}_{q,1}(\mathbb{R}^3))},
\end{aligned}$$

where

$$d_{j'}=\frac{2^{j'(-2+3/p+2/r_2)}\|\dot{\Delta}_{j'}F\|_{L^{r_2}_T(L^p(\mathbb{R}^3))}}{\|F\|_{\mathfrak{L}^{r_2}_T(\dot{B}^{-2+2/r_2+3/p}_{p,1}(\mathbb{R}^3))}}.$$

If $1<p\leqslant q<\infty$, then there exists $1<\lambda\leqslant\infty$ such that $\frac{1}{p}=\frac{1}{q}+\frac{1}{\lambda}$ and $\frac{2}{r_1}\leqslant 2-\frac{3}{\lambda}\geqslant 0$ (as

$r_1 \geqslant \frac{2pq}{2pq-3q+3p}$). It follows, from Lemmas 2.1, 2.2 and (2.8), that

$$\begin{aligned}
&\|\dot{\Delta}_j \mathrm{e}^{-\theta\sqrt{t}\Lambda}(T_{\mathrm{e}^{-\theta\sqrt{t}\Lambda}F}\nabla \mathrm{e}^{-\theta\sqrt{t}\Lambda}\Phi)\|_{L^1_T(L^p(\mathbb{R}^3))}\\
&\quad\lesssim \sum_{|j-j'|\leqslant 4}\|\dot{\Delta}_j \mathrm{e}^{\theta\sqrt{t}\Lambda}(\dot{S}_{j'-1}\mathrm{e}^{-\sqrt{t}\Lambda}F\dot{\Delta}_{j'}\nabla \mathrm{e}^{-\theta\sqrt{t}\Lambda}\Phi)\|_{L^1_T(L^p(\mathbb{R}^3))}\\
&\quad\lesssim \sum_{|j-j'|\leqslant 4}\|\dot{\Delta}_{j'}\nabla\Phi\|_{L^{r_2}_T(L^q(\mathbb{R}^3))}\|\dot{S}_{j'-1}F\|_{L^{r_1}_T(L^\lambda(\mathbb{R}^3))}\\
&\quad\lesssim \sum_{|j-j'|\leqslant 4}2^{j'}\|\dot{\Delta}_{j'}\Phi\|_{L^{r_2}_T(L^q(\mathbb{R}^3))}\sum_{k\leqslant j'-2}2^{3k(\frac{1}{p}-\frac{1}{\lambda})}\|\dot{\Delta}_k F\|_{L^{r_1}_T(L^p(\mathbb{R}^3))}\\
&\quad\lesssim \sum_{|j-j'|\leqslant 4}2^{j'(1-3/q-2/r_2)}d_{j'}\|\Phi\|_{\mathfrak{L}^{r_2}_T(\dot{B}^{3/q+2/r_2}_{q,1}(\mathbb{R}^3))}\\
&\qquad\times\sum_{k\leqslant j'-2}2^{k(2-3/\lambda-2/r_1)}2^{k(-2+3/p+2/r_1)}\|\dot{\Delta}_k F\|_{L^{r_1}_T(L^p(\mathbb{R}^3))}\\
&\quad\lesssim 2^{j(1-3/p)}d_j\|\Phi\|_{\mathfrak{L}^{r_2}_T(\dot{B}^{3/q+2/r_2}_{q,1}(\mathbb{R}^3))}\|F\|_{\mathfrak{L}^{r_1}_T(\dot{B}^{-2+3/p+2/r_1}_{p,1}(\mathbb{R}^3))}\\
&\quad\lesssim 2^{j(1-3/p)}d_j\|F\|^{1-\frac{1}{r_1}}_{\mathfrak{L}^\infty_T(\dot{B}^{-2+3/p}_{p,1}(\mathbb{R}^3))}\|F\|^{\frac{1}{r_1}}_{\mathfrak{L}^1_T(\dot{B}^{3/p}_{p,1}(\mathbb{R}^3))}\|\Phi\|^{1-\frac{1}{r_2}}_{\mathfrak{L}^\infty_T(\dot{B}^{3/q}_{q,1}(\mathbb{R}^3))}\|\Phi\|^{\frac{1}{r_2}}_{\mathfrak{L}^1_T(\dot{B}^{2+3/q}_{q,1}(\mathbb{R}^3))}.
\end{aligned}$$

To estimate the remaining term $R(\mathrm{e}^{-\theta\sqrt{t}\Lambda}F,\nabla \mathrm{e}^{-\theta\sqrt{t}\Lambda}\Phi)$, we consider two cases: $\frac{1}{p}+\frac{1}{q}>1$ and $\frac{1}{3}<\frac{1}{p}+\frac{1}{q}\leqslant 1$.

Case 1. $\frac{1}{p}+\frac{1}{q}>1$. We find $1<p'\leqslant\infty$ such that $\frac{1}{p}+\frac{1}{p'}=1$. Applying (2.8) and Lemmas 2.1 and 2.2 again, for some fixed integer N_0, we have

$$\begin{aligned}
&\|\dot{\Delta}_j \mathrm{e}^{\theta\sqrt{t}\Lambda}R(\mathrm{e}^{-\theta\sqrt{t}\Lambda}F,\nabla \mathrm{e}^{-\theta\sqrt{t}\Lambda}\Phi)\|_{L^1_T(L^p(\mathbb{R}^3))}\\
&\quad\lesssim \sum_{j'\geqslant j-N_0}\|\dot{\Delta}_j \mathrm{e}^{\theta\sqrt{t}\Lambda}(\Delta_{j'}\mathrm{e}^{-\theta\sqrt{t}\Lambda}F\tilde{\dot{\Delta}}_{j'}\nabla \mathrm{e}^{-\theta\sqrt{t}\Lambda}\Phi)\|_{L^1_T(L^p(\mathbb{R}^3))}\\
&\quad\lesssim 2^{3j(1-1/p)}\sum_{j'\geqslant j-N_0}\|\dot{\Delta}_j \mathrm{e}^{\theta\sqrt{t}\Lambda}(\dot{\Delta}_{j'}\mathrm{e}^{-\theta\sqrt{t}\Lambda}F\tilde{\dot{\Delta}}_{j'}\nabla \mathrm{e}^{-\theta\sqrt{t}\Lambda}\Phi)\|_{L^1_T(L^1(\mathbb{R}^3))}\\
&\quad\lesssim 2^{3j(1-1/p)}\sum_{j'\geqslant j-N_0}2^{j'}\|\tilde{\dot{\Delta}}_{j'}\Phi\|_{L^{r_1}_T(L^{p'}(\mathbb{R}^3))}\|\dot{\Delta}_{j'}F\|_{L^{r_2}_T(L^p(\mathbb{R}^3))}\\
&\quad\lesssim 2^{3j(1-1/p)}\sum_{j'\geqslant j-N_0}2^{-j'(-2+3/p+2/r_2)}d_{j'}\|F\|_{\mathfrak{L}^{r_2}_T(B^{-2+3/p+2/r_2}_{p,1}(\mathbb{R}^3))}\\
&\qquad\times 2^{j'(1-3/p'-2/r_1)}2^{j'(3/q+2/r_1)}\|\tilde{\dot{\Delta}}_{j'}\Phi\|_{L^{r_1}_T(L^q(\mathbb{R}^3))}\\
&\quad\lesssim 2^{3j(1-1/p)}\sum_{j'\geqslant j-N_0}d_{j'}2^{-2j'}\|F\|_{\mathfrak{L}^{r_2}_T(B^{-2+3/p+2/r_2}_{p,1}(\mathbb{R}^3))}2^{j'(3/q+2/r_1)}\|\tilde{\dot{\Delta}}_{j'}\Phi\|_{L^{r_1}_T(L^q(\mathbb{R}^3))}\\
&\quad\lesssim 2^{j(1-3/p)}d_j\|F\|_{\mathfrak{L}^{r_2}_T(\dot{B}^{-2+3/p+2/r_2}_{p,1}(\mathbb{R}^3))}\|\Phi\|_{\mathfrak{L}^{r_1}_T(\dot{B}^{3/q+2/r_1}_{q,1}(\mathbb{R}^3))}\\
&\quad\lesssim 2^{j(1-3/p)}d_j\|F\|^{1-\frac{1}{r_2}}_{\mathfrak{L}^\infty_T(\dot{B}^{-2+3/p}_{p,1}(\mathbb{R}^3))}\|F\|^{\frac{1}{r_2}}_{\mathfrak{L}^1_t(\dot{B}^{3/p}_{p,1}(\mathbb{R}^3))}\|\Phi\|^{1-\frac{1}{r_1}}_{\mathfrak{L}^\infty_T(\dot{B}^{3/q}_{q,1}(\mathbb{R}^3))}\|\Phi\|^{\frac{1}{r_1}}_{\mathfrak{L}^1_T(\dot{B}^{2+3/q}_{q,1}(\mathbb{R}^3))}.
\end{aligned}$$

Case 2. $\frac{1}{3}<\frac{1}{p}+\frac{1}{q}\leqslant 1$. In this case, by (2.8), Lemmas 2.1 and 2.2, we find that

$$\begin{aligned}
&\|\dot{\Delta}_j \mathrm{e}^{\theta\sqrt{t}\Lambda}R(\mathrm{e}^{-\theta\sqrt{t}\Lambda}F,\nabla \mathrm{e}^{-\theta\sqrt{t}\Lambda}\Phi)\|_{L^1_T(L^p(\mathbb{R}^3))}\\
&\quad\lesssim 2^{3j/q}\sum_{j'\geqslant j-N_0}\|\dot{\Delta}_j \mathrm{e}^{\theta\sqrt{t}\Lambda}(\dot{\Delta}_{j'}\mathrm{e}^{-\theta\sqrt{t}\Lambda}F\tilde{\dot{\Delta}}_{j'}\nabla \mathrm{e}^{-\theta\sqrt{t}\Lambda}\Phi)\|_{L^1_T(L^{(pq)/(p+q)}(\mathbb{R}^3))}\\
&\quad\lesssim 2^{3j/q}\sum_{j'\geqslant j-N_0}2^{j'}\|\Delta_{j'}F\|_{L^{r_2}_T(L^p(\mathbb{R}^3))}\|\tilde{\dot{\Delta}}_{j'}\Phi\|_{L^{r_1}_T(L^q(\mathbb{R}^3))}\\
&\quad\lesssim 2^{3j/q}\sum_{j'\geqslant j-N_0}2^{j'(3-3/p-2/r_2-2/r_1-3/q)}d_{j'}\|F\|_{\mathfrak{L}^{r_2}_T(B^{-2+3/p+2/r_2}_{p,1}(\mathbb{R}^3))}2^{j'(2/r_1+3/q)}\|\tilde{\dot{\Delta}}_{j'}\Phi\|_{L^{r_1}_T(L^q(\mathbb{R}^3))}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim 2^{j(1-3/p)} d_j \|F\|_{\mathfrak{L}^{r_2}_T(\dot{B}^{-2+3/p+2/r_2}_{p,1}(\mathbb{R}^3))} \|\Phi\|_{\mathfrak{L}^{r_1}_T(\dot{B}^{3/q+2/r_1}_{q,1}(\mathbb{R}^3))} \\
&\lesssim 2^{j(1-3/p)} d_j \|F\|^{1-\frac{1}{r_2}}_{\mathfrak{L}^{\infty}_T(\dot{B}^{-2+3/q}_{q,1}(\mathbb{R}^3))} \|F\|^{\frac{1}{r_2}}_{\mathfrak{L}^{1}_T(\dot{B}^{3/q}_{q,1}(\mathbb{R}^3))} \|\Phi\|^{1-\frac{1}{r_1}}_{\mathfrak{L}^{\infty}_T(\dot{B}^{3/p}_{p,1}(\mathbb{R}^3))} \|\Phi\|^{\frac{1}{r_1}}_{\mathfrak{L}^{1}_T(\dot{B}^{2+3/p}_{p,1}(\mathbb{R}^3))} \\
&\lesssim 2^{j(1-3/p)} d_j \|F\|^{1-\frac{1}{r_2}}_{\mathfrak{L}^{\infty}_T(\dot{B}^{-2+3/p}_{p,1}(\mathbb{R}^3))} \|F\|^{\frac{1}{r_2}}_{\mathfrak{L}^{1}_T(\dot{B}^{3/p}_{p,1}(\mathbb{R}^3))} \|\Phi\|^{1-\frac{1}{r_1}}_{\mathfrak{L}^{\infty}_T(\dot{B}^{3/q}_{q,1}(\mathbb{R}^3))} \|\Phi\|^{\frac{1}{r_1}}_{\mathfrak{L}^{1}_T(\dot{B}^{2+3/q}_{q,1}(\mathbb{R}^3))}.
\end{aligned}
$$

Inserting the above estimates into (2.6), we obtain the desired results, which completes the proof of Lemma 2.6. □

3 Proof of Theorem 1.1: Local existence and analyticity with large initial data

The goal of this section is to show Theorem 1.1. To prove the existence and the analyticity part of Theorem 1.1, thanks to Lemma 2.6, we obtain the following proposition, the details being omitted.

Proposition 3.1. *Let* $\max\{\frac{2pq}{2pq+3p-3q}, \frac{2rq}{2rq+3r-3q}\} \leqslant r_1 < \infty$, $1/r_1 + 1/r_2 = 1$, $1 < p \leqslant q < \infty$, $\frac{1}{q} + \frac{1}{p} > \frac{1}{3}$, $\frac{1}{p} - \frac{1}{q} < \frac{2}{3}$, $1 < r \leqslant q < \infty$, $\frac{1}{q} + \frac{1}{r} > \frac{1}{3}$, $\frac{1}{r} - \frac{1}{q} < \frac{2}{3}$ *and* $\theta \in \{0, 1\}$*. If* $T \in (0, \infty]$ *and* $f \in \mathbb{A}_T$, $g \in \mathbb{B}_T$ *and* $\phi \in \mathbb{C}_T$*, then it holds true that*

$$
\begin{aligned}
\|f\nabla\phi\|_{\mathfrak{L}^1_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{-1+3/p}_{p,1}(\mathbb{R}^3))} \leqslant{} & C\|\phi\|_{\mathfrak{L}^{r_1}_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{3/q+2/r_1}_{q,1}(\mathbb{R}^3))} \|f\|_{\mathfrak{L}^{r_2}_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/p+2/r_2}_{p,1}(\mathbb{R}^3))} \\
& + C\|\phi\|_{\mathfrak{L}^{r_2}_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{3/q+2/r_2}_{q,1}(\mathbb{R}^3))} \|f\|_{\mathfrak{L}^{r_1}_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/p+2/r_1}_{p,1}(\mathbb{R}^3))}
\end{aligned}
$$

and

$$
\begin{aligned}
\|g\nabla\phi\|_{\mathfrak{L}^1_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{-1+3/r}_{r,1}(\mathbb{R}^3))} \leqslant{} & C\|\phi\|_{\mathfrak{L}^{r_1}_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{3/q+2/r_1}_{q,1})} \|f\|_{\mathfrak{L}^{r_2}_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/r+2/r_2}_{r,1}(\mathbb{R}^3))} \\
& + C\|\phi\|_{\mathfrak{L}^{r_2}_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{3/q+2/r_2}_{q,1}(\mathbb{R}^3))} \|f\|_{\mathfrak{L}^{r_1}_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/r+2/r_1}_{r,1}(\mathbb{R}^3))},
\end{aligned}
$$

where C *is a positive constant independent of* f *and* ϕ*. In particular, the above results also hold true with* $p = 1$ *and* $r = 1$ *when* $\theta = 0$*.*

Let

$$
\begin{aligned}
\mathbb{A}_T &:= \mathfrak{L}^{r_1}_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/p+2/r_1}_{p,1}(\mathbb{R}^3)) \cap \mathfrak{L}^{r_2}_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/p+2/r_2}_{p,1}(\mathbb{R}^3)), \\
\mathbb{B}_T &:= \mathfrak{L}^{r_2}_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/r+2/r_2}_{r,1}(\mathbb{R}^3)) \cap \mathfrak{L}^{r_1}_T(B^{-2+3/r+2/r_1}_{r,1}(\mathbb{R}^3))
\end{aligned}
$$

and

$$
\mathbb{C}_T := \mathfrak{L}^{r_1}_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{3/q+2/r_1}_{q,1}(\mathbb{R}^3)) \cap \mathfrak{L}^{r_2}_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{3/q+2/r_2}_{q,1}(\mathbb{R}^3)).
$$

In order to prove the local existence, when $r_1 < \infty$, applying Lemma 2.5, we have

$$
\lim_{T\to 0} \|\mathrm{e}^{t\Delta}u_0\|_{\mathbb{A}_T} = 0, \quad \lim_{T\to 0} \|\mathrm{e}^{t\Delta}v_0\|_{\mathbb{B}_T} = 0 \quad \text{and} \quad \lim_{T\to 0} \|\mathrm{e}^{-\gamma t}\mathrm{e}^{t\Delta}w_0\|_{\mathbb{C}_T} = 0. \tag{3.1}
$$

Thus, for $\delta > 0$, we can define

$$
\begin{aligned}
\mathfrak{T}_1 &:= \sup\left\{T_1 > 0 : \|\mathrm{e}^{t\Delta}u_0\|_{\mathbb{A}_T} \leqslant \frac{\delta}{4}\right\}, \\
\mathfrak{T}_2 &:= \sup\left\{T_1 > 0 : \|\mathrm{e}^{t\Delta}v_0\|_{\mathbb{B}_T} \leqslant \frac{\delta}{4}\right\}, \\
\mathfrak{T}_3 &:= \sup\left\{T_1 > 0 : \|\mathrm{e}^{-\gamma t}\mathrm{e}^{t\Delta}w_0\|_{\mathbb{C}_T} \leqslant \frac{\delta}{4}\right\}.
\end{aligned} \tag{3.2}
$$

Choose $\mathfrak{T} := \min\{\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3\}$ and take $T \leqslant \mathfrak{T}$ and $\delta \in (0,\infty)$ small enough. Now let us consider the map $\Gamma(u,v) := (\Gamma_1(u,v), \Gamma_2(u,v))$ with

$$\begin{cases} \Gamma_1(u,v) := \mathrm{e}^{t\Delta}u_0 - \chi_1 \displaystyle\int_0^t \mathrm{e}^{(t-s)\Delta}\nabla(u\nabla w)(\cdot,s)ds, \\ \Gamma_2(u,v) := \mathrm{e}^{t\Delta}v_0 - \chi_2 \displaystyle\int_0^t \mathrm{e}^{(t-s)\Delta}\nabla(v\nabla w)(\cdot,s)ds, \\ w := \mathrm{e}^{-\gamma t}\mathrm{e}^{t\Delta}w_0 + \displaystyle\int_0^t \mathrm{e}^{-\gamma(t-s)}\mathrm{e}^{(t-s)\Delta}(\alpha_1 u(\cdot,s) + \alpha_2 v(\cdot,s))ds \end{cases}$$

in the metric space

$$\mathfrak{D} := \{(u,v) : \|u\|_{\mathbb{A}_T} + \|v\|_{\mathbb{B}_T} \leqslant \delta\} \quad \text{and} \quad \mathfrak{d}[(u_1,v_1),(u_2,v_2)] := \|u_1-u_2\|_{\mathbb{A}_T} + \|v_1-v_2\|_{\mathbb{B}_T}.$$

By (1.4), we have

$$\dot{B}^{-2+3/p}_{p,1}(\mathbb{R}^3),\ \dot{B}^{-2+3/r}_{r,1}(\mathbb{R}^3) \hookrightarrow \dot{B}^{-2+3/q}_{q,1}(\mathbb{R}^3)$$

when $p \leqslant q$ and $r \leqslant q$; accordingly, from Lemma 2.5, it follows that

$$\begin{aligned} \|w\|_{\mathbb{C}_T} &\lesssim \|\mathrm{e}^{-\gamma t}\mathrm{e}^{t\Delta}w_0\|_{\mathfrak{L}^{r_2}_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{3/q+2/r_2}_{q,1}(\mathbb{R}^3))\cap\mathfrak{L}^{r_1}_T(\dot{B}^{3/q+2/r_1}_{q,1}(\mathbb{R}^3))} + \|(u,v)\|_{\mathfrak{L}^{r_2}_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/q+2/r_2}_{q,1}(\mathbb{R}^3))} \\ &\quad + \|(u,v)\|_{\mathfrak{L}^{r_1}_T(\dot{B}^{-2+3/q+2/r_1}_{q,1}(\mathbb{R}^3))} \\ &\lesssim \|\mathrm{e}^{-\gamma t}\mathrm{e}^{t\Delta}w_0\|_{\mathfrak{L}^{r_2}_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}B^{-2+3/q+2/r_2}_{q,1}(\mathbb{R}^3))\cap\mathfrak{L}^{r_1}_T(\dot{B}^{3/q+\frac{2}{r_1}}_{q,1}(\mathbb{R}^3))} \\ &\quad + \|(u,v)\|_{\mathfrak{L}^{r_2}_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/p+2/r_2}_{p,1}(\mathbb{R}^3))\cap\mathfrak{L}^{r_2}_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/r+2/r_2}_{r,1}(\mathbb{R}^3))} \\ &\quad + \|(u,v)\|_{\mathfrak{L}^{r_1}_T(\dot{B}^{-2+3/p+2/r_1}_{p,1}(\mathbb{R}^3))\cap\mathfrak{L}^{r_1}_T(B^{-2+3/r+2/r_1}_{r,1}(\mathbb{R}^3))} \\ &\lesssim \|\mathrm{e}^{-\gamma t}\mathrm{e}^{t\Delta}w_0\|_{\mathbb{C}_T} + \|(u,v)\|_{\mathbb{A}_T\times\mathbb{B}_T}, \end{aligned} \tag{3.3}$$

which, together with Lemma 2.5 and Proposition 3.1, implies that

$$\begin{aligned} &\|\Gamma(u,v)\|_{\mathbb{A}_T\times\mathbb{B}_T} \\ &\quad := \|\Gamma_1(u,v)\|_{\mathbb{A}_T} + \|\Gamma_2(u,v)\|_{\mathbb{B}_T} \\ &\quad \lesssim \|(\mathrm{e}^{t\Delta}u_0, \mathrm{e}^{t\Delta}v_0)\|_{\mathbb{A}_T\times\mathbb{B}_T} + \|\nabla(u\nabla w)\|_{\mathfrak{L}^1_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/p}_{p,1}(\mathbb{R}^3))} + \|\nabla(v\nabla w)\|_{\mathfrak{L}^1_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/r}_{r,1}(\mathbb{R}^3))} \\ &\quad \lesssim \|(\mathrm{e}^{t\Delta}u_0, \mathrm{e}^{t\Delta}v_0)\|_{\mathbb{A}_T\times\mathbb{B}_T} + \|(u,v)\|_{\mathbb{A}_T\times\mathbb{B}_T}[\|w\|_{\mathfrak{L}^{r_1}_T(\dot{B}^{3/q+\frac{2}{r_1}}_{q,1}(\mathbb{R}^3))} + \|w\|_{\mathfrak{L}^{r_2}_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{3/q+2/r_2}_{q,1}(\mathbb{R}^3))}] \\ &\quad \lesssim \|(\mathrm{e}^{t\Delta}u_0, \mathrm{e}^{t\Delta}v_0)\|_{\mathbb{A}_T\times\mathbb{B}_T} + \|(u,v)\|_{\mathbb{A}_T\times\mathbb{B}_T}[\|\mathrm{e}^{-\gamma t}\mathrm{e}^{t\Delta}w_0\|_{\mathbb{C}_T} + \|(u,v)\|_{\mathbb{A}_T\times\mathbb{B}_T}] \\ &\quad \lesssim \frac{\delta}{2} + \|(u,v)\|_{\mathbb{A}_T\times\mathbb{B}_T}\left[\frac{\delta}{4} + \|(u,v)\|_{\mathbb{A}_T\times\mathbb{B}_T}\right] \\ &\quad \lesssim \frac{\delta}{2} + \frac{5\delta^2}{4} \\ &\quad \lesssim \delta. \end{aligned} \tag{3.4}$$

For any $(u_1,v_1) \in \mathfrak{D}$ and $(u_2,v_2) \in \mathfrak{D}$, for simplicity, we write $(u^*,v^*,w^*) := (u_2-u_1, v_2-v_1, w_2-w_1)$. Repeating the argument used in (3.4), taking $\delta \in (0,\infty)$ sufficiently small, we then conclude that

$$\begin{aligned} &\mathfrak{d}(\Gamma(u_1,v_1), \Gamma(u_2,v_2)) \\ &\quad := \|\Gamma_1(u_2,v_2) - \Gamma_1(u_1,v_1)\|_{\mathbb{A}_T} + \|\Gamma_2(u_2,v_2) - \Gamma_2(u_1,v_1)\|_{\mathbb{B}_T} \\ &\quad \lesssim \|u^*\nabla w_2 + u_1\nabla w^*\|_{\mathfrak{L}^1_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{-1+3/p}_{p,1}(\mathbb{R}^3))} + \|v^*\nabla w_2 + v_1\nabla w^*\|_{\mathfrak{L}^1_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{-1+3/r}_{r,1}(\mathbb{R}^3))} \\ &\quad \lesssim [\|u^*\|_{\mathbb{A}_T} + \|w^*\|_{\mathfrak{L}^{r_1}_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{\frac{3}{q}+\frac{2}{r_1}}_{q,1}(\mathbb{R}^3))} + \|w^*\|_{\mathfrak{L}^{r_2}_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{\frac{3}{q}+\frac{2}{r_2}}_{q,1}(\mathbb{R}^3))}] \\ &\qquad \times [\|u_1\|_{\mathbb{A}_T} + \|w_2\|_{\mathfrak{L}^{r_1}_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{\frac{3}{q}+\frac{2}{r_1}}_{q,1}(\mathbb{R}^3))} + \|w_2\|_{\mathfrak{L}^{r_2}_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{\frac{3}{q}+\frac{2}{r_2}}_{q,1}(\mathbb{R}^3))}] \end{aligned}$$

$$
\begin{aligned}
&+[\|v^*\|_{\mathbb{B}_T}+\|w^*\|_{\mathfrak{L}^{r_1}_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{\frac{3}{q}+\frac{2}{r_1}}_{q,1}(\mathbb{R}^3))}+\|w^*\|_{\mathfrak{L}^{r_2}_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{\frac{3}{q}+\frac{2}{r_2}}_{q,1}(\mathbb{R}^3))}]\\
&\times[\|v_1\|_{\mathbb{B}_T}+\|w_2\|_{\mathfrak{L}^{r_1}_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{\frac{3}{q}+\frac{2}{r_1}}_{q,1}(\mathbb{R}^3))}+\|w_2\|_{\mathfrak{L}^{r_2}_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{\frac{3}{q}+\frac{2}{r_2}}_{q,1}(\mathbb{R}^3))}]\\
\lesssim{}&[\|u_1\|_{\mathbb{A}_T}+\|\mathrm{e}^{-\gamma t}\mathrm{e}^{t\Delta}w_0\|_{\mathfrak{L}^{r_2}_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/q+2/r_2}_{q,1}(\mathbb{R}^3))\cap\mathfrak{L}^{r_1}_T(\dot{B}^{3/q+2/r_1}_{q,1}(\mathbb{R}^3))}+\|(u_2,v_2)\|_{\mathbb{A}_T\times\mathbb{B}_T}]\\
&\times[\|u^*\|_{\mathbb{A}_T}+\|(u^*,v^*)\|_{\mathbb{A}_T\times\mathbb{B}_T}]+[\|v^*\|_{\mathbb{B}_T}+\|(u^*,v^*)\|_{\mathbb{A}_T\times\mathbb{B}_T}]\\
&\times[\|v_1\|_{\mathbb{B}_T}+\|\mathrm{e}^{-\gamma t}\mathrm{e}^{t\Delta}w_0\|_{\mathfrak{L}^{r_2}_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/q+2/r_2}_{q,1}(\mathbb{R}^3))\cap\mathfrak{L}^{r_1}_T(\dot{B}^{3/q+2/r_1}_{q,1}(\mathbb{R}^3))}+\|(u_2,v_2)\|_{\mathbb{A}_T\times\mathbb{B}_T}]\\
\leqslant{}&\frac{9C\delta}{4}\|(u^*,v^*)\|_{\mathbb{A}_T\times\mathbb{B}_T}\\
\leqslant{}&\frac{1}{2}\|(u^*,v^*)\|_{\mathbb{A}_T\times\mathbb{B}_T}.
\end{aligned}\tag{3.5}
$$

Combining (3.4) and (3.5), we find that Γ is a contraction mapping on $\mathfrak{D}$. So, there is a $(u,v)\in\mathfrak{D}$ satisfying $\Gamma(u,v)=(u,v)$. Since $(u,v)\in\mathfrak{D}$ and $w_0\in\dot{B}^{3/q}_{q,1}(\mathbb{R}^3)$, from (5.4), it follows that

$$
w\in\mathfrak{L}^{r_2}_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{\frac{3}{q}+\frac{2}{r_2}}_{q,1}(\mathbb{R}^3))\cap\mathfrak{L}^{r_1}_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{\frac{3}{q}+\frac{2}{r_1}}_{q,1}(\mathbb{R}^3)).
$$

Applying the continuous embedding (1.4), we know that $\dot{B}^{-2+3/p}_{p,1}(\mathbb{R}^3)$, $\dot{B}^{-2+3/r}_{r,1}(\mathbb{R}^3)\hookrightarrow\dot{B}^{-2+3/q}_{q,1}(\mathbb{R}^3)$ when $p\leqslant q$ and $r\leqslant q$. Thus, using Lemma 2.5 and Proposition 3.1, we conclude that

$$
\begin{aligned}
&\|(u,v)\|_{\mathfrak{L}^\infty_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+\frac{3}{p}}_{p,1}(\mathbb{R}^3))\times\mathfrak{L}^\infty_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+\frac{3}{r}}_{r,1}(\mathbb{R}^3))}\\
&\lesssim\|(u_0,v_0)\|_{\dot{B}^{-2+3/p}_{p,1}(\mathbb{R}^3)\times\dot{B}^{-2+3/r}_{r,1}(\mathbb{R}^3)}+\|\nabla(u\nabla w)\|_{\mathfrak{L}^1_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/p}_{p,1}(\mathbb{R}^3))}+\|\nabla(u\nabla v)\|_{\mathfrak{L}^1_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/r}_{r,1}(\mathbb{R}^3))}\\
&\lesssim\|(u_0,v_0)\|_{\dot{B}^{-2+3/p}_{p,1}(\mathbb{R}^3)\times\dot{B}^{-2+3/r}_{r,1}(\mathbb{R}^3)}+\|(u,v)\|_{\mathbb{A}_T\times\mathbb{B}_T}[\|w\|_{\mathfrak{L}^{r_1}_T(\dot{B}^{3/q+\frac{2}{r_1}}_{q,1}(\mathbb{R}^3))}+\|w\|_{\mathfrak{L}^{r_2}_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{3/q+2/r_2}_{q,1}(\mathbb{R}^3))}]\\
&\lesssim\|(u_0,v_0)\|_{\dot{B}^{-2+3/p}_{p,1}(\mathbb{R}^3)\times\dot{B}^{-2+3/r}_{r,1}(\mathbb{R}^3)}+\|(u,v)\|_{\mathbb{A}_T\times\mathbb{B}_T}[\|\mathrm{e}^{-\gamma t}\mathrm{e}^{t\Delta}w_0\|_{\mathbb{C}_T}+\|(u,v)\|_{\mathbb{A}_T\times\mathbb{B}_T}]\\
&\lesssim\|(u_0,v_0)\|_{\dot{B}^{-2+3/p}_{p,1}(\mathbb{R}^3)\times B^{-2+3/r}_{r,1}(\mathbb{R}^3)}+\|(u,v)\|_{\mathbb{A}_T\times\mathbb{B}_T}[\|w_0\|_{\dot{B}^{3/q}_{q,1}(\mathbb{R}^3)}+\|(u,v)\|_{\mathbb{A}_T\times\mathbb{B}_T}]
\end{aligned}
$$

and

$$
\begin{aligned}
\|w\|_{\mathfrak{L}^\infty_T(\dot{B}^{\frac{3}{q}}_{q,1}(\mathbb{R}^3))}&\lesssim\|\mathrm{e}^{-\gamma t}\mathrm{e}^{t\Delta}w_0\|_{\dot{B}^{3/q}_{q,1}(\mathbb{R}^3)}+\|(u,v)\|_{\mathfrak{L}^{r_1}_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{3/q-2++2/r_1}_{q,1}(\mathbb{R}^3))}\\
&\lesssim\|w_0\|_{\dot{B}^{3/q}_{q,1}(\mathbb{R}^3)}+\|u\|_{\mathfrak{L}^{r_1}_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/q+2/r_1}_{q,1}(\mathbb{R}^3))}+\|v\|_{\mathfrak{L}^{r_1}_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/q+2/r_1}_{q,1}(\mathbb{R}^3))}\\
&\lesssim\|w_0\|_{\dot{B}^{3/q}_{q,1}(\mathbb{R}^3)}+\|u\|_{\mathfrak{L}^{r_1}_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/p+2/r_1}_{p,1}(\mathbb{R}^3))}+\|v\|_{\mathfrak{L}^{r_1}_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/r+2/r_1}_{r,1}(\mathbb{R}^3))}.
\end{aligned}
$$

This finishes the proof of (1.8). Finally, using (1.8), as in the proof of Lemma 2.6, we find that

$$
\begin{aligned}
&\chi_1\nabla\cdot(u\nabla w)\in\mathfrak{L}^1(0,T;\dot{B}^{-2+3/p}_{p,1}(\mathbb{R}^3)),\quad\chi_2\nabla\cdot(v\nabla w)\in\mathfrak{L}^1(0,T;\dot{B}^{-2+3/r}_{r,1}(\mathbb{R}^3)),\\
&\alpha_1u\in\mathbb{A}_T\quad\text{and}\quad\alpha_2v\in\mathbb{B}_T,
\end{aligned}
$$

which, as the third index $r=1<\infty$, using Lemma 2.5, yields

$$
(u,v,w)\in\mathcal{C}(I;\dot{B}^{-2+3/p}_{p,1}(\mathbb{R}^3))\times\mathcal{C}(I;\dot{B}^{-2+3/r}_{r,1}(\mathbb{R}^3))\times\mathcal{C}(I;\dot{B}^{3/q}_{q,1}(\mathbb{R}^3)),
$$

where $I=[0,T]$. From above, we deduce that $(u,v,w)\in\Theta^{\mathcal{C}}_T$.

If $T^*<\infty$ and

$$
\begin{cases}
\|u\|_{\mathfrak{L}^{r_1}(0,T^*;\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/p+2/r_1}_{p,1}(\mathbb{R}^3))\cap\mathfrak{L}^{r_2}(0,T^*;\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/p+2/r_2}_{p,1}(\mathbb{R}^3))}<\infty,\\
\|v\|_{\mathfrak{L}^{r_1}(0,T^*;\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/r+2/r_1}_{r,1}(\mathbb{R}^3))\cap\mathfrak{L}^{r_2}(0,T^*;\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/r+\frac{2}{r_2}}_{r,1}(\mathbb{R}^3))}<\infty,\\
\|w\|_{\mathfrak{L}^{r_1}(0,T^*;\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{3/q+2/r_1}_{q,1}(\mathbb{R}^3))\cap\mathfrak{L}^{r_2}(0,T^*;\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{3/q+2/r_2}_{q,1}(\mathbb{R}^3))}<\infty,
\end{cases}\tag{3.6}
$$

we claim that the solution can be extended beyond T^*. Indeed, let us consider the integral equation, for any $t \in (0,T)$,

$$\begin{cases} u(t) = \mathrm{e}^{(t-T)\Delta}u_0 - \chi_1 \int_T^t \mathrm{e}^{(t-s)\Delta}\nabla(u\nabla w)(\cdot,s)ds, \\ v(t) = \mathrm{e}^{(t-T)\Delta}v_0 - \chi_2 \int_T^t \mathrm{e}^{(t-s)\Delta}\nabla(v\nabla w)(\cdot,s)ds, \\ w(t) = \mathrm{e}^{-\gamma(t-T)}\mathrm{e}^{(t-T)\Delta}w_0 + \int_T^t \mathrm{e}^{-\gamma(t-s)}\mathrm{e}^{(t-s)\Delta}(\alpha_1 u(\cdot,s) + \alpha_2 v(\cdot,s))ds, \end{cases} \tag{3.7}$$

which, for $T < T^*$ and T being sufficiently close to T^*, together with (3.6), (3.7), Lemma 2.5 and Proposition 3.1, implies that

$$\begin{aligned} &\|\mathrm{e}^{(t-T)\Delta}u_0\|_{\mathfrak{L}^{r_1}(T,T^*;\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/p+2/r_1}_{p,1}(\mathbb{R}^3))\cap\mathfrak{L}^{r_2}(T,T^*;\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/p+2/r_2}_{p,1}(\mathbb{R}^3))} \\ &\quad\lesssim \|u\|_{\mathfrak{L}^{r_1}(T,T^*;\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/p+2/r_1}_{p,1}(\mathbb{R}^3))\cap\mathfrak{L}^{r_2}(T,T^*;\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/p+2/r_2}_{p,1}(\mathbb{R}^3))} \\ &\qquad + \|u\|_{\mathfrak{L}^{r_1}(T,T^*;\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/p+2/r_1}_{p,1}(\mathbb{R}^3))\cap\mathfrak{L}^{r_2}(T,T^*;\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/p+2/r_2}_{p,1}(\mathbb{R}^3))} \\ &\qquad \times (\|w\|_{\mathfrak{L}^{r_1}_T(T,T^*;\dot{B}^{3/q+\frac{2}{r_1}}_{q,1}(\mathbb{R}^3))} + \|w\|_{\mathfrak{L}^{r_2}_T(T,T^*;\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{3/q+2/r_2}_{q,1}(\mathbb{R}^3))}) < \frac{\delta}{4}, \end{aligned} \tag{3.8}$$

where

$$\mathfrak{L}^{\rho}(T,T^*;\dot{B}^s_{p,r}(\mathbb{R}^3)) := \{f \in \mathscr{S}'((T,T^*),\mathscr{S}'_h(\mathbb{R}^3)) : \|f\|_{\mathfrak{L}^{\rho}(T,T^*;\dot{B}^s_{p,r}(\mathbb{R}^3))} < \infty\}$$

and

$$\|f\|_{\mathfrak{L}^{\rho}(T,T^*;\dot{B}^s_{p,r}(\mathbb{R}^3))} := \left\{ \int_T^{T^*} \left[\sum_{k\in\mathbb{Z}} \{2^{ks}\|\Delta_k f\|_{L^p(\mathbb{R}^3)}\}^r \right]^{\frac{\rho}{r}} dt \right\}^{1/\rho}.$$

Similarly, when $T < T^*$ and T is sufficiently close to T^*, we also have

$$\begin{aligned} &\|\mathrm{e}^{(t-T)\Delta}v_0\|_{\mathfrak{L}^{r_1}(T,T^*;\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/r+2/r_1}_{r,1}(\mathbb{R}^3))\cap\mathfrak{L}^{r_2}(T,T^*;\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/r+2/r_2}_{r,1}(\mathbb{R}^3))} < \frac{\delta}{4}, \\ &\|\mathrm{e}^{-\gamma(t-T)}\mathrm{e}^{(t-T)\Delta}w_0\|_{\mathfrak{L}^{r_1}(T,T^*;\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{3/q+2/r_1}_{q,1}(\mathbb{R}^3))\cap\mathfrak{L}^{r_2}(T,T^*;\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{3/q+2/r_2}_{q,1}(\mathbb{R}^3))} < \frac{\delta}{4}. \end{aligned} \tag{3.9}$$

Obverse that (3.8) and (3.9) are analogous to (3.1), which further implies that the solution of (1.7) exists on $[T,T^*]$. This contradicts to the fact that T^* is maximal. Therefore, the desired estimate (1.9) now follows, which completes the proof of Theorem 1.1.

4 Proof of Theorem 1.2: Global existence and analyticity with small initial data

This section is devoted to the proof of the global existence and analyticity. First, we prove the global existence and analyticity of solutions of (1.7) by applying the contraction mapping in Gevrey spaces. System (1.7) can be written into the following integral equations. Let $T \in (0,\infty]$, $t \in (0,T)$ and $(u,v) := \Pi(\tilde{u},\tilde{v}) := (\Pi_1(\tilde{u},\tilde{v}),\Pi_2(\tilde{u},\tilde{v}))$ with

$$\begin{cases} \Pi_1(\tilde{u},\tilde{v}) := \mathrm{e}^{t\Delta}u_0 - \chi_1 \int_0^t \mathrm{e}^{(t-s)\Delta}\nabla(\tilde{u}\nabla\tilde{w})(\cdot,s)ds, \\ \Pi_2(\tilde{u},\tilde{v}) := \mathrm{e}^{t\Delta}v_0 - \chi_2 \int_0^t \mathrm{e}^{(t-s)\Delta}\nabla(\tilde{v}\nabla\tilde{w})(\cdot,s)ds, \\ \tilde{w} := \mathrm{e}^{-\gamma t}\mathrm{e}^{t\Delta}w_0 + \int_0^t \mathrm{e}^{-\gamma(t-s)}\mathrm{e}^{(t-s)\Delta}(\alpha_1\tilde{u}(\cdot,s) + \alpha_2\tilde{v}(\cdot,s))ds. \end{cases} \tag{4.1}$$

Let $(\tilde{u},\tilde{v})\in\mathbb{X}_T\times\mathbb{Y}_T$ and $(u_0,v_0,w_0)\in E_0$. Applying (1.4), together with $\dot{B}^{3/r}_{r,1}(\mathbb{R}^3)\hookrightarrow\dot{B}^{3/q}_{q,1}(\mathbb{R}^3)$ and $\dot{B}^{3/p}_{p,1}(\mathbb{R}^3)\hookrightarrow\dot{B}^{3/q}_{q,1}(\mathbb{R}^3)$ for $p\leqslant q$ and $r\leqslant q$ and Lemmas 2.5 and 2.6, we conclude that, for (4.1),

$$\begin{aligned}\|\tilde{w}\|_{\mathbb{Z}_T}&\lesssim\|\mathrm{e}^{-\gamma t}\mathrm{e}^{t\Delta}w_0\|_{\mathbb{Z}_T}+\|\tilde{u}\|_{\mathfrak{L}^1_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{3/q}_{q,1}(\mathbb{R}^3))}+\|\tilde{v}\|_{\mathfrak{L}^1_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{3/q}_{q,1}(\mathbb{R}^3))}\\&\lesssim\|w_0\|_{\dot{B}^{3/q}_{q,1}(\mathbb{R}^3)}+\|\tilde{u}\|_{\mathfrak{L}^1_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{3/p}_{p,1}(\mathbb{R}^3))}+\|\tilde{v}\|_{\mathfrak{L}^1_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{3/r}_{r,1}(\mathbb{R}^3))}\\&\lesssim\|w_0\|_{\dot{B}^{3/q}_{q,1}(\mathbb{R}^3)}+\|\tilde{u}\|_{\mathbb{X}_T}+\|\tilde{v}\|_{\mathbb{Y}_T},\end{aligned}\tag{4.2}$$

which implies that

$$\begin{aligned}\|u\|_{\mathbb{X}_T}&\leqslant\|\mathrm{e}^{t\Delta}u_0\|_{\mathbb{X}_T}+\|\chi_1\nabla\cdot(\tilde{u}\nabla\tilde{w})\|_{\mathfrak{L}^1_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/p}_{p,1}(\mathbb{R}^3))}\\&\leqslant\|\mathrm{e}^{t\Delta}u_0\|_{\mathbb{X}_T}+\|\chi_1\cdot(\tilde{u}\nabla\tilde{w})\|_{\mathfrak{L}^1_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{-1+3/p}_{p,1}(\mathbb{R}^3))}\\&\lesssim\|\mathrm{e}^{t\Delta}u_0\|_{\mathbb{X}_T}+\|\tilde{u}\|_{\mathfrak{L}^1_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{3/p}_{p,1}(\mathbb{R}^3))}\|\tilde{w}\|_{\mathfrak{L}^\infty_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{3/q}_{q,1}(\mathbb{R}^3))}\\&\quad+\|\tilde{u}\|_{\mathfrak{L}^\infty_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{-2+3/p}_{p,1}(\mathbb{R}^3))}\|\tilde{w}\|_{\mathfrak{L}^1_T(\mathrm{e}^{\theta\sqrt{t}\Lambda}\dot{B}^{2+3/q}_{q,1}(\mathbb{R}^3))}\\&\lesssim\|\mathrm{e}^{t\Delta}u_0\|_{\mathbb{X}_T}+\|\tilde{u}\|_{\mathbb{X}_T}\|\tilde{w}\|_{\mathbb{Z}_T}\\&\lesssim\|u_0\|_{\dot{B}^{-2+3/p}_{p,1}(\mathbb{R}^3)}+\|\tilde{u}\|_{\mathbb{X}_T}(\|w_0\|_{\dot{B}^{3/q}_{q,1}(\mathbb{R}^3)}+\|\tilde{u}\|_{\mathbb{X}_T}+\|\tilde{v}\|_{\mathbb{Y}_T}).\end{aligned}\tag{4.3}$$

Similarly, we have

$$\|v\|_{\mathbb{Y}_T}\lesssim\|v_0\|_{\dot{B}^{-2+3/r}_{r,1}(\mathbb{R}^3)}+\|\tilde{v}\|_{\mathbb{Y}_T}(\|w_0\|_{\dot{B}^{3/q}_{q,1}(\mathbb{R}^3)}+\|\tilde{u}\|_{\mathbb{X}_T}+\|\tilde{v}\|_{\mathbb{Y}}).\tag{4.4}$$

As a consequence, we conclude that $(u,v)\in\mathbb{X}_T\times\mathbb{Y}_T$.

Next, we prove the global existence for small initial data. For this purpose we choose $T=\infty$. Our proof is divided into two steps. Firstly, we show that for $\epsilon_0>0$ small enough, Π is a map from $\mathbb{X}^{\epsilon_0}\times\mathbb{Y}^{\epsilon_0}$ to itself. We say $u\in\mathbb{X}^{\epsilon_0}$ if $u\in\mathbb{X}$ and $\|u\|_{\mathbb{X}^{\epsilon_0}}:=\|u\|_{\mathbb{X}}\leqslant C\epsilon_0$, where C is a positive constant independent of u, and similar notation for $v\in\mathbb{Y}^{\epsilon_0}$ and $w\in\mathbb{Z}^{\epsilon_0}$.

Proposition 4.1. *For a given constant $\epsilon_0>0$ small enough, the initial data class (u_0,v_0,w_0) satisfing (1.10) and $(\tilde{u},\tilde{v})\in\mathbb{X}^{\epsilon_0}\times\mathbb{Y}^{\epsilon_0}$, System (4.1) satisfies $(u,v)\in\mathbb{X}^{\epsilon_0}\times\mathbb{Y}^{\epsilon_0}$.*

Proof. By Lemma 2.5, we know that there exists a positive constant C such that

$$\|(\mathrm{e}^{t\Delta}u_0,\mathrm{e}^{t\Delta}v_0,\mathrm{e}^{-\gamma t}\mathrm{e}^{t\Delta}w_0)\|_{\Theta}\leqslant C\|(u_0,v_0,w_0)\|_{E_0}\leqslant C\epsilon_0,$$

which, along with (4.2)–(4.4), implies that

$$\begin{aligned}\|(u,v)\|_{\mathbb{X}\times\mathbb{Y}}&\leqslant C\epsilon_0+C\|(\tilde{u},\tilde{v})\|_{\mathbb{X}^{\epsilon_0}\times\mathbb{Y}^{\epsilon_0}}[C\|w_0\|_{\dot{B}^{3/p}_{p,r}(\mathbb{R}^3)}+C_0\|\tilde{u}\|_{\mathbb{X}^{\epsilon_0}}+C_0\|\tilde{v}\|_{\mathbb{Y}^{\epsilon_0}}]\\&\leqslant 2C\epsilon_0(1+C\epsilon_0)\leqslant C\epsilon_0.\end{aligned}$$

This further implies that $(u,v)\in\mathbb{X}^{\epsilon_0}\times\mathbb{Y}^{\epsilon_0}$, which completes the proof of Proposition 4.1. □

Secondly, we show that for $\epsilon_0>0$ small enough, the map Π is a contractive map.

Proposition 4.2. *For $\epsilon_0>0$ small enough, letting $(\tilde{u},\tilde{v})\in\mathbb{X}^{\epsilon_0}\times\mathbb{Y}^{\epsilon_0}$ and $(\bar{u},\bar{v})\in\mathbb{X}^{\epsilon_0}\times\mathbb{Y}^{\epsilon_0}$ with $(\tilde{u},\tilde{v},\tilde{w})\,|_{t=0}=(\bar{u},\bar{v},\bar{w})\,|_{t=0}:=(u_0,v_0,w_0)$, then the map Π defined in (4.1) is a contractive map.*

Proof. For simplicity, we write $(u^*,v^*,w^*):=(\tilde{u}-\bar{u},\tilde{v}-\bar{v},\tilde{w}-\bar{w})$ and we define, for $t\in(0,\infty]$,

$$\begin{cases}\tilde{w}:=\mathrm{e}^{-\gamma t}\mathrm{e}^{t\Delta}w_0+\displaystyle\int_0^t\mathrm{e}^{-\gamma(t-s)}\mathrm{e}^{(t-s)\Delta}(\alpha_1\tilde{u}(\cdot,s)+\alpha_2\tilde{v}(\cdot,s))ds,\\ \bar{w}:=\mathrm{e}^{-\gamma t}\mathrm{e}^{t\Delta}w_0+\displaystyle\int_0^t\mathrm{e}^{-\gamma(t-s)}\mathrm{e}^{(t-s)\Delta}(\alpha_1\bar{u}(\cdot,s)+\alpha_2\bar{v}(\cdot,s))ds.\end{cases}$$

Then we have

$$|\Pi_1(\tilde{u},\tilde{v})-\Pi_1(\bar{u},\bar{v})|=\left|\chi_1\int_0^t\nabla\mathrm{e}^{(t-\tau)\Delta}(\tilde{u}\nabla w^*+u^*\nabla\bar{w})d\tau\right|.$$

Similarly,

$$|\Pi_2(\tilde{u},\tilde{v})-\Pi_1(\bar{u},\bar{v})|=\left|\chi_2\int_0^t \nabla \mathrm{e}^{(t-\tau)\Delta}(\tilde{v}\nabla w^*+v^*\nabla\bar{w})d\tau\right|.$$

Repeating the proof of Proposition 4.1, we have

$$\begin{aligned}\|\Pi_1(\tilde{u},\tilde{v})-\Pi_1(\bar{u},\bar{v})\|_{\mathbb{X}} &\lesssim \|\tilde{u}\|_{\mathbb{X}^{\epsilon_0}}[\|u^*\|_{\mathbb{X}^{\epsilon_0}}+\|v^*\|_{\mathbb{Y}^{\epsilon_0}}]+\|u^*\|_{\mathbb{X}^{\epsilon_0}}[\|\bar{u}\|_{\mathbb{X}^{\epsilon_0}}+\|\bar{v}\|_{\mathbb{Y}^{\epsilon_0}}]\\ &\lesssim [\|\bar{u}\|_{\mathbb{X}^{\epsilon_0}}+\|\bar{v}\|_{\mathbb{Y}^{\epsilon_0}}+\|\tilde{u}\|_{\mathbb{X}^{\epsilon_0}}][\|u^*\|_{\mathbb{X}^{\epsilon_0}}+\|v^*\|_{\mathbb{Y}^{\epsilon_0}}]\\ &\leqslant C_0\epsilon_0[\|u^*\|_{\mathbb{X}^{\epsilon_0}}+\|v^*\|_{\mathbb{Y}^{\epsilon_0}}],\end{aligned}$$

where C_0 is a positive constant independent of u^* and v^*.

The same process also ensures

$$\|\Pi_2(\tilde{u},\tilde{v})-\Pi_2(\bar{u},\bar{v})\|_{\mathbb{Y}}\leqslant C_0\epsilon_0\left[\|u^*\|_{\mathbb{X}^{\epsilon_0}}+\|v^*\|_{\mathbb{Y}^{\epsilon_0}}\right].$$

Taking ϵ_0 small enough such that $(C_0+C_0)\epsilon_0\leqslant\frac{1}{2}$, we then complete the proof of Proposition 4.2. □

From Propositions 4.1, we deduce the global existence $(u,v)\in\mathbb{X}^{\epsilon_0}\times\mathbb{Y}^{\epsilon_0}$. It follows, from (4.2), $w_0\in\dot{B}^{3/q}_{q,1}(\mathbb{R}^3)$ and $(u,v)\in\mathbb{X}^{\epsilon_0}\times\mathbb{Y}^{\epsilon_0}$, that

$$\|w\|_{\mathbb{Z}}\leqslant C\,\|w_0\|_{\dot{B}^{3/q}_{q,1}(\mathbb{R}^3)}+C_0\,\|u\|_{\mathbb{X}^{\epsilon_0}}+C_0\,\|v\|_{\mathbb{Y}^{\epsilon_0}}\,, \tag{4.5}$$

where C_0 and C are positive constants independent of u,w_0 and v.

Combining $(u,v)\in\mathbb{X}\times\mathbb{Y}$ and (4.5), we then complete the proof of Theorem 1.2.

5 Proof of Corollary 1.4: Decay of Besov and Lebesgue norms

Theorem 1.2 tells us that if the initial data are sufficiently small, the solution of (1.7) is globally in the Gevrey class, i.e., the energy bound $\|(u,v,w)\|_{\Theta}<\infty$ for $\theta=1$. Specifically, we can show that a solution (u,v,w) satisfies

$$\sup_{t>0}\|(\mathrm{e}^{\sqrt{t}\Lambda}u,\mathrm{e}^{\sqrt{t}\Lambda}v,\mathrm{e}^{\sqrt{t}\Lambda}w)\|_{\dot{B}^{-2+3/p}_{p,1}(\mathbb{R}^3)\times\dot{B}^{-2+3/r}_{r,1}(\mathbb{R}^3)\times\dot{B}^{3/q}_{q,1}(\mathbb{R}^3)}<\infty. \tag{5.1}$$

From (5.1), together with an argument used in [14] for $m>0$, it follows that there exists a positive constant C such that

$$\begin{aligned}&\|(D^m u,D^m v,D^m w)\|_{\dot{B}^{-2+3/p}_{p,1}(\mathbb{R}^3)\times\dot{B}^{-2+3/r}_{r,1}(\mathbb{R}^3)\times\dot{B}^{3/q}_{q,1}(\mathbb{R}^3)}\\ &\quad=\|(D^m\mathrm{e}^{-\sqrt{t}\Lambda}\mathrm{e}^{\sqrt{t}\Lambda}u,D^m\mathrm{e}^{-\sqrt{t}\Lambda}\mathrm{e}^{\sqrt{t}\Lambda}v,D^m\mathrm{e}^{-\sqrt{t}\Lambda}\mathrm{e}^{\sqrt{t}\Lambda}w)\|_{\dot{B}^{-2+3/p}_{p,1}(\mathbb{R}^3)\times\dot{B}^{-2+3/r}_{r,1}(\mathbb{R}^3)\times\dot{B}^{3/q}_{q,1}(\mathbb{R}^3)}\\ &\quad\leqslant C^m m^m t^{-\frac{m}{2}}\sup_{t>0}\|(\mathrm{e}^{\sqrt{t}\Lambda}u,\mathrm{e}^{\sqrt{t}\Lambda}v,\mathrm{e}^{\sqrt{t}\Lambda}w)\|_{\dot{B}^{-2+3/p}_{p,1}(\mathbb{R}^3)\times\dot{B}^{-2+3/r}_{r,1}(\mathbb{R}^3)\times\dot{B}^{3/q}_{q,1}(\mathbb{R}^3)}\\ &\quad\lesssim C^m m^m t^{-\frac{m}{2}}.\end{aligned} \tag{5.2}$$

Using the relation between homogeneous Besov spaces and homogeneous Triebel-Lizorkin spaces $\dot{F}^s_{p,p}(\mathbb{R}^3)$ (see [17, 18, 27, 28, 44, 57] for properties of related function spaces), noting that $\ell^p\hookrightarrow\ell^2$ for $p\leqslant 2$ and $\dot{F}^s_{p,2}(\mathbb{R}^3):=\dot{W}^{s,p}(\mathbb{R}^3):=(-\Delta)^{-s/2}L^p(\mathbb{R}^3)$, we conclude that

$$\dot{B}^s_{p,1}(\mathbb{R}^3)\hookrightarrow\dot{B}^s_{p,p}(\mathbb{R}^3)\hookrightarrow\dot{F}^s_{p,p}(\mathbb{R}^3)\hookrightarrow\dot{F}^s_{p,2}(\mathbb{R}^3)=\dot{W}^{s,p}(\mathbb{R}^3). \tag{5.3}$$

Applying (5.3) and (5.1), together with an argument used in (5.2), we find that for $k_1>-2+3/p$ and $1<p\leqslant\frac{3}{2}$, there exists a positive constant C_1 such that

$$\begin{aligned}\|D^{k_1}u\|_{L^p(\mathbb{R}^3)}&=\|D^{k_1+2-3/p}\mathrm{e}^{-\sqrt{t}\Lambda}D^{-2+3/p}\mathrm{e}^{\sqrt{t}\Lambda}u\|_{L^p(\mathbb{R}^3)}\\ &\leqslant C_1^{k_1+2-3/p}(k_1+2-3/p)^{k_1+2-3/p}t^{-\frac{k_1+2-3/p}{2}}\|D^{-2+3/p}\mathrm{e}^{\sqrt{t}\Lambda}u\|_{L^p(\mathbb{R}^3)}\end{aligned}$$

$$\leqslant C_1^{k_1+2-3/p}(k_1+2-3/p)^{k_1+2-3/p}t^{-\frac{k_1+2-3/p}{2}}\|e^{\sqrt{t}\Lambda}u\|_{\dot{F}^{-2+3/p}_{p,2}(\mathbb{R}^3)}$$
$$\lesssim C_1^{k_1+2-3/p}(k_1+2-3/p)^{k_1+2-3/p}t^{-\frac{k_1+2-3/p}{2}}. \tag{5.4}$$

Similar to (5.4), we find that there exist positive constants C_2 and C_3 such that for $k_2 > -2+3/r$, $1 < r \leqslant \frac{3}{2}$, $k_3 > 3/q$ and $1 < q \leqslant 2$,

$$\|D^{k_2}v\|_{L^r(\mathbb{R}^3)} \lesssim C_2^{k_2+2-3/r}\,(k_2+2-3/r)^{k_2+2-3/r}\,t^{-\frac{k_2+2-3/r}{2}},$$
$$\|D^{k_3}w\|_{L^q(\mathbb{R}^3)} \lesssim C_3^{k_3-3/q}\,(k_3-3/q)^{k_3-3/q}\,t^{-\frac{k_3-3/q}{2}}.$$

This finishes the proof of Corollary 1.4.

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