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A spectral projection method for transmission eigenvalues

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Abstract We consider a nonlinear integral eigenvalue problem, which is a reformulation of the transmission eigenvalue problem arising in the inverse scattering theory. The boundary element method is employed for discretization, which leads to a generalized matrix eigenvalue problem. We propose a novel method based on the spectral projection. The method probes a given region on the complex plane using contour integrals and decides whether the region contains eigenvalue(s) or not. It is particularly suitable to test whether zero is an eigenvalue of the generalized eigenvalue problem, which in turn implies that the associated wavenumber is a transmission eigenvalue. Effectiveness and efficiency of the new method are demonstrated by numerical examples.

Keywords spectral projection, boundary element method, transmission eigenvalues

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1 Introduction

We consider a non-linear non-selfadjoint transmission eigenvalue problem, which arises in the inverse scattering theory [4, 6]. Since 2010, the problem has attracted quite some attention of numerical mathematicians [1, 5, 7, 15, 17, 20, 22, 28, 29]. The first numerical treatment was studied by Colton et al. [7], where three finite element methods were proposed. A mixed method was developed by Ji et al. [17]. An and Shen [1] proposed an efficient spectral-element based numerical method for transmission eigenvalues of two-dimensional, radially-stratified media. The first method supported by a rigorous convergence analysis was introduced by Sun [28]. Recently, Cakoni et al. [5] reformulated the problem and proved convergence (based on Osborn's compact operator theory [24]) of a mixed finite element method. Li et al. [22] developed a finite element method based on a related quadratic eigenvalue problem. Other methods [10, 16, 18, 31] have been proposed as well.

Despite significant effort to develop various numerical methods for the transmission eigenvalue problem, computation of both real and complex eigenvalues remains difficult due to the fact that the numerical discretization usually ends up with large sparse generalized non-Hermitian eigenvalue problems, which are very challenging in numerical linear algebra. Traditional methods such as shift and invert Arnoldi are handicapped by the lack of a priori spectrum information.

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In this paper, we adopt an integral formulation for the transmission eigenvalue problem. Using boundary element method, the integral equations are discretized and a generalized eigenvalue problem of dense matrices is obtained. The matrices are significantly smaller than those from finite element methods. It is shown that if zero is a generalized eigenvalue, the corresponding wavenumber is a transmission eigenvalue [9]. We propose a probing method based on the spectral projection using contour integrals. The closed contour is chosen to be a small circle centered at the origin and a numerical quadrature is used to compute the spectral projection of a random vector. The norm of the projected vector is used as an indicator of whether zero is an eigenvalue or not.

Integral based methods [3,11,25,26] for eigenvalue computation, having their roots in the classical spectral perturbation theory (see [19]), become popular in many areas, e.g., electronic structure calculation. These methods are based on eigenprojections using contour integrals of the resolvent [2]. Randomly chosen functions are projected to the generalized eigenspace corresponding to the eigenvalues inside a closed contour, which leads to a relative small finite dimension eigenvalue problem. For recently developments along this line, we refer the readers to [21, 30, 32, 33]

For most existing integral based methods, estimation on the locations, number of eigenvalues and dimensions of eigenspace are critical for their successes. In contrast, the proposed method is related to the methods developed in [14, 20].

The rest of the paper is arranged as follows. In Section 2, we introduce the transmission eigenvalue problem and rewrite it using integral operators. In Section 3, we present the probing method based on contour integrals. We present numerical results in Section 4. Conclusions and future work are contained in Section 5.

2 The transmission eigenvalue problem

Let $D \subset \mathbb{R}^2$ be an open bounded domain with C^2 boundary $\Gamma := \partial D$. The transmission eigenvalue problem is to find $k \in \mathbb{C}$ such that there exist non-trivial solutions w and v satisfying

$$\Delta w + k^2 n w = 0, \quad \text{in } D, \tag{2.1a}$$

$$\Delta v + k^2 v = 0, \quad \text{in } D, \tag{2.1b}$$

$$w - v = 0, \quad \text{on } \Gamma, \tag{2.1c}$$

$$\frac{\partial w}{\partial \nu} - \frac{\partial v}{\partial \nu} = 0, \quad \text{on } \Gamma, \tag{2.1d}$$

where ν is the unit outward normal to Γ. The wavenumber k's for which the transmission eigenvalue problem has non-trivial solutions are called transmission eigenvalues. Here n is the index of refraction, which is assumed to be a constant greater than 1 in this paper. Note that, for the integral formulation to be used, the index of refraction needs to be constant (see [12]).

In the following, we describe an integral formulation of the transmission eigenvalue problem following [9] (see also [20]). Let Φ_k be the Green's function given by

$$\Phi_k(x, y) = \frac{\mathrm{i}}{4} H_0^{(1)}(k|x - y|),$$

where $H_0^{(1)}$ is the Hankel function of the first kind of order 0. The single and double layer potentials are defined as

$$(S_k \phi)(x) = \int_\Gamma \Phi_k(x, y) \phi(x)\, ds(y), \tag{2.2}$$

$$(K_k \phi)(x) = \int_\Gamma \frac{\partial \Phi_k}{\partial \nu(y)}(x, y) \phi(x)\, ds(y), \tag{2.3}$$

where ϕ is the density function.

Let $(v, w) \in H^1(D) \times H^1(D)$ be a solution to (2.1). Denote $k_1 = \sqrt{n}k$ and set

$$\alpha := \frac{\partial v}{\partial \nu}\Big|_\Gamma = \frac{\partial w}{\partial \nu}\Big|_\Gamma \in H^{-1/2}(\Gamma),$$
$$\beta := v|_\Gamma = w|_\Gamma \in H^{1/2}(\Gamma).$$

Then v and w has the following integral representation,

$$v = S_k\alpha - K_k\beta, \quad \text{in } D, \tag{2.4a}$$
$$w = S_{k_1}\alpha - K_{k_1}\beta, \quad \text{in } D. \tag{2.4b}$$

Let $u := w - v$. Then $u|_\Gamma = 0$ and $\frac{\partial u}{\partial \nu}|_\Gamma = 0$. The boundary conditions (2.1c) and (2.1d) imply that the transmission eigenvalues are k's such that

$$Z(k)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = 0, \tag{2.5}$$

where

$$Z(k) = \begin{pmatrix} S_{k_1} - S_k & -K_{k_1} + K_k \\ -K'_{k_1} + K'_k & T_{k_1} - T_k \end{pmatrix}$$

and the potentials S_k, K_k, K'_k and T_k are given by

$$(K'_k\phi)(x) = \int_\Gamma \frac{\partial \Phi_k}{\partial \nu(x)}(x, y)\phi(y)ds(y), \tag{2.6a}$$
$$(T_k\psi)(x) = \frac{\partial}{\partial \nu(x)}\int_\Gamma \frac{\partial \Phi_k}{\partial \nu(y)}(x, y)\phi(y)ds(y). \tag{2.6b}$$

It is shown in [9] that

$$Z(k) := H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma) \to H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$$

is of Fredholm type with index zero and analytic on $\mathbb{C} \setminus \mathbb{R}^-$.

From (2.5), k is a transmission eigenvalue if zero is an eigenvalue of $Z(k)$. Unfortunately, $Z(k)$ is compact. The eigenvalues of $Z(k)$ accumulate at zero, which makes it impossible to distinguish zero and other eigenvalues numerically. The workaround proposed in [8] is to consider a generalized eigenvalue problem

$$Z(k)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \lambda B(k)\begin{pmatrix}\alpha\\ \beta\end{pmatrix}, \tag{2.7}$$

where $B(k) = Z(\mathrm{i}k)$. Since there does not exist purely imaginary transmission eigenvalues [7], the accumulation point is shifted to -1. Then 0 becomes isolated.

Now we describe a boundary element discretization of the potentials and refer the readers to [23, 27] for more details. One discretizes the boundary Γ into element segments. Suppose the computational boundary Γ is discretized into N segments $\Gamma_1, \Gamma_2, \ldots, \Gamma_N$ by nodes $x_1, x_2, \ldots, x_N$ and $\tilde{\Gamma} = \bigcup_{i=1}^N \Gamma_i$. Let $\{\psi_j\}, j = 1, 2, \ldots, N$, be piecewise constant basis functions and $\{\varphi_j\}, j = 1, 2, \ldots, N$, be piecewise linear basis functions. We seek an approximate solution α_h and β_h in the form

$$\alpha_h = \sum_{j=1}^N \alpha_j\psi_j, \quad \beta_h = \sum_{j=1}^N \beta_j\varphi_j.$$

We arrive at a linear system

$$(V_{k,h} - V_{k_1,h})\vec{\alpha} + (-K_{k,h} + K_{k_1,h})\vec{\beta} = 0,$$
$$(K'_{k,h} - K'_{k_1,h})\vec{\alpha} + (W_{k,h} - W_{k_1,h})\vec{\beta} = 0,$$

where $\vec{\alpha} = (\alpha_1, \dots, \alpha_N)^{\mathrm{T}}$, $\vec{\beta} = (\beta_1, \dots, \beta_N)^{\mathrm{T}}$, and $V_{k,h}, K_{k,h}, K'_{k,h}$ and $W_{k,h}$ are matrices with entries

$$\begin{aligned}V_{k,h}(i,j) &= \int_{\tilde{\Gamma}} (S_k\psi_j)\psi_i ds, \quad K_{k,h}(i,j) = \int_{\tilde{\Gamma}} (K_k\varphi_j)\psi_i ds,\\ K'_{k,h}(i,j) &= \int_{\tilde{\Gamma}} (K'_k\psi_j)\varphi_i ds, \quad W_{k,h}(i,j) = \int_{\tilde{\Gamma}} (T_k\varphi_j)\varphi_i ds.\end{aligned}$$

In the above matrices, we can use the series expansion of the first kind Hankel function as

$$\begin{aligned}H_0^{(1)}(x) = &\sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2}\left(\frac{x}{2}\right)^{2m} + \frac{2\mathrm{i}}{\pi}\sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2}\left(\frac{x}{2}\right)^{2m}\left(\ln\frac{x}{2} + c_e\right)\\ &- \frac{2\mathrm{i}}{\pi}\sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2}\left(\frac{x}{2}\right)^{2m}\left(1 + \frac{1}{2} + \frac{1}{m}\right),\end{aligned}$$

where c_e is the Euler constant. Thus,

$$H_0^{(1)}(k|x-y|) = \sum_{m=0}^{\infty}\left(C_5(m) + C_6(m)\ln\frac{k}{2}\right)k^{2m}|x-y|^{2m} + C_6(m)\ln|x-y|k^{2m}|x-y|^{2m},$$

where

$$C_5(m) = \frac{(-1)^m}{2^{2m}(m!)^2}\left[1 + \frac{2c_e\mathrm{i}}{\pi} - \frac{2\mathrm{i}}{\pi}\left(1 + \frac{1}{2} + \frac{1}{m}\right)\right], \quad C_6(m) = \frac{(-1)^m\mathrm{i}}{2^{2m-1}(m!)^2\pi}.$$

We also need the following integrals which can be computed exactly,

$$\begin{aligned}\mathrm{Int}_7(m) &= \int_{-1}^{1}\int_{-1}^{1}(\xi_1-\xi_2)^{2m}d\xi_2 d\xi_1 = \frac{2^{2m+2}}{(2m+1)(m+1)},\\ \mathrm{Int}_8(m) &= \int_{-1}^{1}\int_{-1}^{1}(\xi_1-\xi_2)^{2m}\ln|\xi_1-\xi_2|d\xi_2 d\xi_1\\ &= \frac{2^{2m+2}\ln 2}{(2m+1)(m+1)} - \frac{(4m+3)2^{2m+3}}{(2m+1)^2(2m+2)^2},\\ \mathrm{Int}_9(m) &= \int_{-1}^{1}\int_{-1}^{1}(\xi_1-\xi_2)^{2m}\xi_1\xi_2 d\xi_2 d\xi_1\\ &= \sum_{l=0}^{2m}\frac{(-1)^l C_{2m}^l}{(l+2)(2m+2-l)}[1-(-1)^l]^2,\end{aligned}$$

and

$$\begin{aligned}\mathrm{Int}_{10}(m) &= \int_{-1}^{1}\int_{-1}^{1}(\xi_1-\xi_2)^{2m}\xi_1\xi_2\ln|\xi_1-\xi_2|d\xi_2 d\xi_1\\ &= \frac{-m2^{2m+2}\ln 2}{(2m+1)(m+1)(m+2)} + \frac{1}{(2m+1)(m+1)}\left[\frac{2^{2m+3}}{2m+3} - \frac{2^{2m+2}}{(m+2)^2} - \frac{2^{2m+1}}{m+1}\right]\\ &\quad + \frac{1}{2(m+1)^2(2m+1)^2}\sum_{l=0}^{2m+1}C_{2m+1}^l\left[\frac{(2m+1)^2}{l+2}(1-(-1)^l) - \frac{4m+3}{l+3}(1-(-1)^{l+1})\right].\end{aligned}$$

Now we consider

$$\begin{aligned}V_{k,h}(i,j) &= \int_{\tilde{\Gamma}}(V_k\psi_j)\psi_i ds = \int_{\tilde{\Gamma}}\int_{\tilde{\Gamma}}\Phi_k(x,y)\psi_j(y)\psi_i(x)ds_y ds_x\\ &= \int_{\Gamma_i}\int_{\Gamma_j}\Phi_k(x,y)\psi_j(y)\psi_i(x)ds_y ds_x.\end{aligned}$$

The integral over $\Gamma_i \times \Gamma_j$ can be calculated as

$$\int_{\Gamma_i}\int_{\Gamma_j}\Phi_k(x,y)\psi_j(y)\psi_i(x)ds_y ds_x = \frac{\mathrm{i}}{4}\int_{\Gamma_i}\int_{\Gamma_j}H_0^{(1)}(k|x-y|)\psi_j(y)\psi_i(x)ds_y ds_x$$
$$= \frac{\mathrm{i}L_iL_j}{16}\int_{-1}^{1}\int_{-1}^{1}H_0^{(1)}(k|x(\xi_1)-y(\xi_2)|)d\xi_2 d\xi_1,$$

where

$$x(\xi_1) = x_i + \frac{1+\xi_1}{2}(x_{i+1}-x_i),$$
$$y(\xi_2) = x_j + \frac{1+\xi_2}{2}(x_{j+1}-x_j).$$

When $i \neq j$, it can be calculated by Gaussian quadrature rule. When $i = j$, we have

$$\begin{aligned}
&\frac{\mathrm{i}L_i^2}{16}\int_{-1}^{1}\int_{-1}^{1}H_0^{(1)}(k|x(\xi_1)-y(\xi_2)|)d\xi_2 d\xi_1\\
&=\frac{\mathrm{i}L_i^2}{16}\sum_{m=0}^{\infty}\frac{k^{2m}L_i^{2m}}{2^{2m}}\left(C_5(m)+C_6(m)\ln\frac{kL^i}{4}\right)\int_{-1}^{1}\int_{-1}^{1}(\xi_1-\xi_2)^{2m}d\xi_2 d\xi_1\\
&\quad+\frac{\mathrm{i}L_i^2}{16}\sum_{m=0}^{\infty}\frac{k^{2m}L_i^{2m}}{2^{2m}}C_6(m)\int_{-1}^{1}\int_{-1}^{1}(\xi_1-\xi_2)^{2m}\ln|\xi_1-\xi_2|d\xi_2 d\xi_1\\
&=\sum_{m=0}^{\infty}\frac{\mathrm{i}k^{2m}L_i^{2m+2}}{2^{2m+4}}\left[\left(C_5(m)+C_6(m)\ln\frac{kL^i}{4}\right)\mathrm{Int}_7(m)+C_6(m)\mathrm{Int}_8(m)\right].
\end{aligned}$$

The following regularization formulation is needed to discretize the hyper-singular boundary integral operator

$$W_k\beta(x) = -\frac{d}{ds_x}V_k\left(\frac{d\beta}{ds}\right)(x) - k^2\nu_x\cdot V_k(\beta\nu)(x). \tag{2.8}$$

We refer the readers to [13] for details of the discretization.

The above boundary element method leads to the following generalized eigenvalue problem

$$A\boldsymbol{x} = \lambda B\boldsymbol{x}, \tag{2.9}$$

where $A, B \in \mathbb{C}^{n\times n}$, $\lambda \in \mathbb{C}$ is a scalar, and $\boldsymbol{x} \in \mathbb{C}^n$.

To compute transmission eigenvalues, the following method is proposed in [8]. A searching interval for wavenumbers is discretized. For each k, the boundary integral operators $Z(k)$ and $Z(\mathrm{i}k)$ are discretized to obtain (2.9). Then all eigenvalues $\lambda_i(k)$ of (2.9) are computed and arranged such that

$$0 \leqslant |\lambda_1(k)| \leqslant |\lambda_2(k)| \leqslant \cdots.$$

If k is a transmission eigenvalue, $|\lambda_1|$ is very close to 0 numerically. If one plots the inverse of $|\lambda_1(k)|$ against k, the transmission eigenvalues are located at spikes.

3 The probing method

The method in [8] only uses the smallest eigenvalue. Hence it is not necessary to compute all eigenvalues of (2.9). In fact, there is no need to know the exact value of λ_1. The only thing we need to verify is that, for a given wavenumber k, whether the generalized eigenvalue problem (2.9) has an isolated eigenvalue 0. This motivates us to propose a probing method to test whether 0 is a generalized eigenvalue of (2.9). The method does not compute the actual eigenvalue and only solves a couple of linear systems. The workload is reduced significantly in two-dimensional case and even more in three-dimensional case.

We start to recall some basic results from spectral theory of compact operators [19]. Let $T : \mathcal{X} \to \mathcal{X}$ be a compact operator on a complex Hilbert space $\mathcal{X}$. The resolvent set of T is defined as

$$\rho(T) = \{z \in \mathbb{C} : (z - T)^{-1} \text{ exists as a bounded operator on } \mathcal{X}\}. \tag{3.1}$$

For any $z \in \rho(T)$, the resolvent operator of T is defined as

$$R_z(T) = (z - T)^{-1}. \tag{3.2}$$

The spectrum of T is $\sigma(T) = \mathbb{C} \setminus \rho(T)$. We denote the null space of an operator A by $N(A)$. Let α be such that

$$N((\lambda - T)^{\alpha}) = N((\lambda - T)^{\alpha+1}).$$

Then $m = \dim N((\lambda - T)^{\alpha})$ is called the algebraic multiplicity of λ. The vectors in $N((\lambda - T)^{\alpha})$ are called generalized eigenvectors of T corresponding to λ. Geometric multiplicity of λ is defined as $\dim N(\lambda - T)$.

Let γ be a simple closed curve on the complex plane $\mathbb{C}$ lying in $\rho(T)$, which contains m eigenvalues, counting multiplicity, of T: $\lambda_i, i = 1, \ldots, m$. We set

$$P = \frac{1}{2\pi \mathrm{i}} \int_\gamma R_z(T) dz.$$

It is well known that P is a projection from $\mathcal{X}$ onto the space of generalized eigenfunctions $\boldsymbol{u}_i, i = 1, \ldots, m$ associated with $\lambda_i, i = 1, \ldots, m$ (see [19]).

Let $\boldsymbol{f} \in \mathcal{X}$ be randomly chosen. If there are no eigenvalues inside γ, we have that $P\boldsymbol{f} = \boldsymbol{0}$. Therefore, $P\boldsymbol{f}$ can be used to decide whether a region contains eigenvalues of T or not.

For the generalized matrix eigenvalue problem (2.9), the resolvent is

$$R_z(A, B) = (zB - A)^{-1} \tag{3.3}$$

for z in the resolvent set of the matrix pencil (A, B). The projection onto the generalized eigenspace corresponding to eigenvalues enclosed by γ is given by

$$P_k(A, B) = \frac{1}{2\pi \mathrm{i}} \int_\gamma (zB - A)^{-1} dz. \tag{3.4}$$

We write P_k to emphasize that the projection depends on the wavenumber k.

The approximation of $P_k\boldsymbol{f}$ is computed by quadrature rules

$$P_k\boldsymbol{f} = \frac{1}{2\pi \mathrm{i}} \int_\gamma R_z(A, B)\boldsymbol{f} dz \approx \frac{1}{2\pi \mathrm{i}} \sum_{j=1}^{W} \omega_j R_{z_j}(A, B)\boldsymbol{f} = \frac{1}{2\pi \mathrm{i}} \sum_{j=1}^{W} \omega_j \boldsymbol{x}_j, \tag{3.5}$$

where w_j are weights and z_j are quadrature points. Here $\boldsymbol{x}_j$'s are the solutions of the following linear systems,

$$(z_j B - A)\boldsymbol{x}_j = \boldsymbol{f}, \quad j = 1, \ldots, W. \tag{3.6}$$

Similar to the continuous case, if there are no eigenvalues inside γ, then $P_k = 0$ and thus $P_k\boldsymbol{f} = \boldsymbol{0}$ for all $\boldsymbol{f} \in \mathbb{C}^n$. For robustness [14], we project the random vector twice, i.e., we compute $P_k^2\boldsymbol{f}$.

For a fixed wavenumber k, the algorithm of the probing method is as follows:

Input: a small circle γ center at the origin with radius $r \ll 1$ and a random $\boldsymbol{f}$

Output: $0 - k$ is not a transmission eigenvalue; $1 - k$ is a transmission eigenvalue

1. Compute $P_k^2\boldsymbol{f}$ by (3.5);
2. decide whether γ contains an eigenvalue:
   - No, output 0.
   - Yes, output 1.

4 Numerical examples

We start with an interval (a,b) of wavenumbers and uniformly divide it into K subintervals. At each wavenumber

$$k_j = a + jh, \quad j = 0, 1, \ldots, K, \quad h = \frac{b-a}{K},$$

we employ the boundary element method to discretize the potentials. We choose $N = 32$ and end up with a generalized eigenvalue problem (2.9) with 64×64 matrices A and B. To test whether 0 is a generalized eigenvalue of (2.9), we choose γ to be a circle of radius 1/100. Then we use 16 uniformly distributed quadrature points on γ and evaluate the eigenprojection (3.5). If at a wavenumber k_j, the projection is approximately 1, then k_j is a transmission eigenvalue. For the actual computation, we use a threshold value $\sigma = 1/2$ to decide whether k_j is a transmission eigenvalue or not, i.e., k_j is a transmission eigenvalue if $\|P_{k_j}^2 \boldsymbol{f}\|/\|P_{k_j}\boldsymbol{f}\| \geqslant \sigma$ and not otherwise.

Let D be a disk with radius 1/2. The index of refraction is $n = 16$. In this case, the exact transmission eigenvalues are known [7]. They are k's such that

$$J_1(k/2)J_0(2k) - 4J_0(k/2)J_1(2k) = 0 \tag{4.1}$$

and

$$J_{m-1}(k/2)J_m(2k) - 4J_m(k/2)J_{m-1}(2k) = 0 \tag{4.2}$$

for $m = 1, 2, \ldots$ The actual values are given in Table 1.

We choose the interval to be $(1.5, 3.5)$ and uniformly divide it into 2000 subintervals. At each k_j we compute the projection (3.5) twice. The probing method finds three eigenvalues in $(1.5, 3.5)$,

$$k_1 = 1.988, \quad k_2 = 2.614, \quad k_3 = 3.228,$$

which approximate the exact eigenvalues (the first column of Table 1) accurately. Note that the continuous finite element method in [7] computes

$$k_1 = 2.0301, \quad k_2 = 2.6937, \quad k_3 = 3.3744,$$

on a triangular mesh with mesh size ≈ 0.1. The method proposed in this paper is more accurate. However, we would like to remark that the methodology of the finite element method in [7] is different.

We also plot the log of $|P^2\boldsymbol{f}|$ against the wavenumber k in Figure 1. The method is robust since the eigenvalues can be easily identified.

We repeat the experiment by choosing $n = 9$ and $(a,b) = (3,5)$. The rest parameters keep the same. The following eigenvalues are obtained $k_1 = 3.554$, $k_2 = 4.360$. The log of $|P^2\boldsymbol{f}|$ against the wavenumber k is shown in Figure 2.

Finally, we compare the proposed method with the method in [8]. We take $n = 16$ and compute for 2,000 wavenumbers. The CPU time in second is shown in Table 2. Note that all the computation is done using Matlab R2014a on a MacBook Pro with a 3 GHz Intel Core i7 and 16 GB memory. We can see that the proposed method saves more time if the size of the generalized eigenvalue problem is larger. We expect that it has a greater advantage for three-dimensional problems since the size of the matrices are much larger than two-dimensional cases.

We also show the log plot of $1/|\lambda_{\min}|$ by the method of [8] in Figure 3. Comparing Figures 1 and 2 with Figure 3, it is clear that the probing method has much narrower span.

Table 1 Transmission eigenalues of a disk with radius $r = 1/2$ and index of refraction $n = 16$

$m = 0$	1.9880	3.7594	6.5810
$m = 1$	2.6129	4.2954	5.9875
$m = 2$	3.2240	4.9462	6.6083

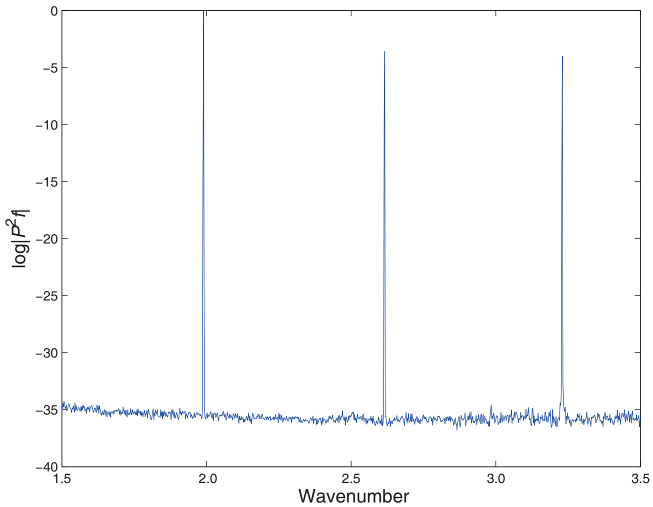

Figure 1 The plot of $\log|P^2\boldsymbol{f}|$ against the wavenumber k for $n = 16$

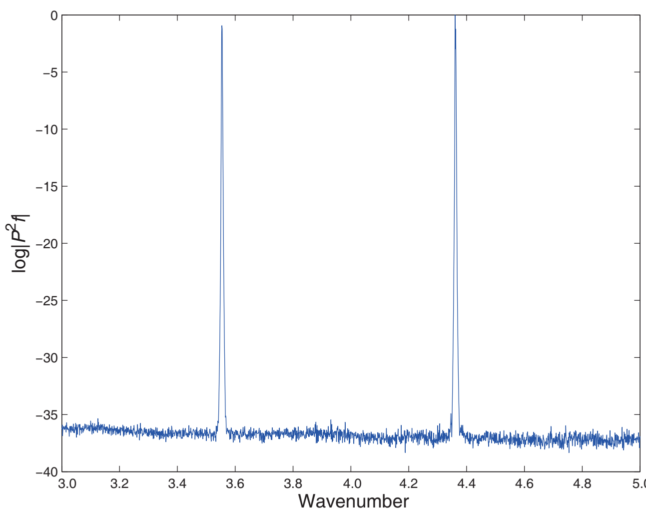

Figure 2 The plot of $\log|P^2\boldsymbol{f}|$ against the wavenumber for $n = 9$

Table 2 Comparison. The first column is the size of the matrix problem. The second column is the time used by the proposed method in second. The second column is the time used by the method given in [8]. The fourth column is the ratio

Size	Probing method	Method in [8]	Ratio
64×64	1.741340	5.742839	3.30
128×128	5.653961	31.152448	5.51
256×256	25.524530	224.435704	8.79
512×512	130.099433	1822.545973	14.01

5 Conclusions and future work

In this paper, we proposed a probing method based on contour integrals for the transmission eigenvalue problem. The method only tests whether a given region contains an eigenvalue or not. Comparing with the

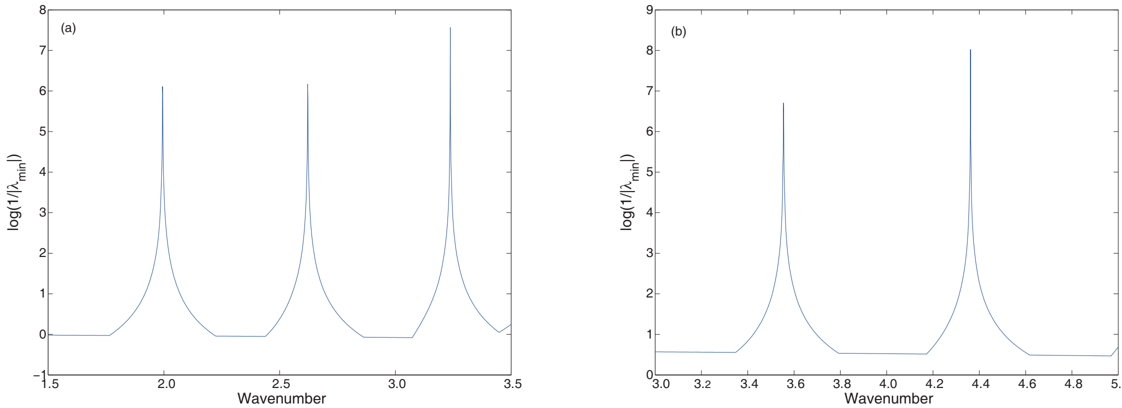

Figure 3 Log plot of $1/|\lambda_{\min}|$. (a) $n = 16$. (b) $n = 9$

existing methods, it needs little a prior spectrum information and is more efficient. The method can be viewed as an eigensolver without computing eigenvalues. One advantage of the contour integral method is that it is suitable for parallel computing. Therefore, even the desired eigenvalues are dispersed, one can use a parallel scheme to capture them simultaneously.

Note that one needs to construct two matrices for each wavenumber. It is time consuming if one wants to divide the searching interval into more subintervals to improve accuracy. The work load is much more in three dimension. Currently, we are developing a parallel version of the method using graphics processing units (GPUs).

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