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Efficient splitting schemes for magneto-hydrodynamic equations

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Abstract We present in this paper several efficient numerical schemes for the magneto-hydrodynamic (MHD) equations. These semi-discretized (in time) schemes are based on the standard and rotational pressure-correction schemes for the Navier-Stokes equations and do not involve a projection step for the magnetic field. We show that these schemes are unconditionally energy stable, present an effective algorithm for their fully discrete versions and carry out demonstrative numerical experiments.

Keywords magneto-hydrodynamic (MHD), incompressible, stability, error analysis, spectral method

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1 Introduction

We consider in this paper numerical approximation of the following magneto-hydrodynamic (MHD) equations

$$\begin{aligned} &u_t + (u\cdot\nabla)u - \nu\Delta u + \nabla p - \alpha(\nabla\times b)\times b = 0, \quad \text{in} \quad \Omega,\\ &b_t - \eta\Delta b + \nabla\times(b\times u) = 0, \quad \text{in} \quad \Omega,\\ &\text{div}\ u = 0, \quad \text{in} \quad \Omega,\\ &\text{div}\ b = 0, \quad \text{in} \quad \Omega, \end{aligned} \tag{1.1}$$

with given initial data

$$u(\cdot,0) = u^0(\cdot), \quad b(\cdot,0) = b^0, \tag{1.2}$$

where Ω is an open bounded domain in $\mathbb{R}^d$ $(d=2,3)$, n is unit outward normal of $\partial\Omega$, the unknowns are the velocity field u, magnetic field b and pressure p, the parameters ν and η are respectively the kinematic viscosity, the magnetic diffusivity, and $\alpha = 1/(4\pi\mu\rho)$ with μ being the magnetic permeability and ρ the fluid density. To fix the idea, we assume the no-slip boundary condition for the velocity, and the perfectly conducting boundary condition for the magnetic field, namely,

$$u|_{\partial\Omega} = 0, \quad n\cdot b|_{\partial\Omega} = 0, \quad n\times(\nabla\times b)|_{\partial\Omega} = 0. \tag{1.3}$$

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The above MHD equations model an incompressible, resistive and electrically conducting fluid in a perfectly conducting container Ω. The wellposedness of this system is similar to the corresponding Navier-Stokes system and has been established by many authors (see [4, 13]). Various numerical approximations for MHD equations (stationary or time dependent) have been proposed, mostly concentrated on developing stable finite-element discretizations which require solving a double saddle point problem (see [1, 5, 10, 12]). On the other hand, a decoupled approach based on the commutator of Laplacian and Leray projection is proposed in [11], and several very interesting splitting schemes based on double projection steps were proposed in [2].

In this paper, we develop several unconditionally stable, semi-discretized linear schemes, based on standard and rotational pressure-correction methods [7,9,15,16] proposed for the Navier-Stokes equations. Unlike in [2], we do not perform a projection step for the magnetic field, avoiding additional splitting errors and computational cost associated with the projection. We show that our schemes will lead to divergence-free magnetic field if the initial condition b^0 is divergence-free.

The rest of the paper is organized as follows. In Section 2, we construct first-order semi-discretized schemes based standard and rotational pressure-correction methods, and show that they are unconditionally stable. Second-order schemes and their stability analysis are presented in Section 3. In Section 4, we perform an error analysis for the first-order standard pressure-correction scheme. In Section 5, we first describe a generic spatial discretization followed by a detailed description for an efficient implementation of Legendre-Galerkin method in space. In Section 6, we present numerical results to demonstrate the convergence rates for our schemes and describe an adaptive time stepping strategy. Finally in Section 7, we present some concluding remarks.

2 First-order schemes

We construct two semi-implicit time discretization schemes based on standard pressure-correction and rotational pressure-correction for the MHD equations and show that they are unconditionally stable.

2.1 A first-order scheme based on standard pressure-correction

Inspired by the pressure-correction scheme for the Navier-Stokes equations [7, 16], we construct the following scheme for (1.1) with (1.3).

Given $(\tilde{u}^n, u^n, b^n, p^n)$, find $(\tilde{u}^{n+1}, u^n, b^{n+1}, p^{n+1})$ such that

$$\begin{cases} \dfrac{\tilde{u}^{n+1}-u^n}{\delta t} + (u^n\cdot\nabla)\tilde{u}^{n+1} - \nu\Delta\tilde{u}^{n+1} + \nabla p^n - \alpha(\nabla\times b^{n+1})\times b^n = 0, \\ \dfrac{b^{n+1}-b^n}{\delta t} - \eta\Delta b^{n+1} + \nabla\times(b^n\times\tilde{u}^{n+1}) = 0, \\ \tilde{u}^{n+1}|_{\partial\Omega} = 0, \\ b^{n+1}\cdot n|_{\partial\Omega} = 0, n\times(\nabla\times b^{n+1})|_{\partial\Omega} = 0, \end{cases} \tag{2.1}$$

and

$$\begin{cases} \dfrac{u^{n+1}-\tilde{u}^{n+1}}{\delta t} + \nabla(p^{n+1}-p^n) = 0, \\ \mathrm{div} u^{n+1} = 0, \\ u^{n+1}\cdot n|_{\partial\Omega} = 0. \end{cases} \tag{2.2}$$

Note that (2.1) is a coupled but linear elliptic type system for $(\tilde{u}^{n+1}, b^{n+1})$, while (2.2) can be rewritten as

$$\begin{aligned} &-\Delta(p^{n+1}-p^n) = -\frac{1}{\delta t}\tilde{u}^{n+1}, \quad \frac{\partial}{\partial n}(p^{n+1}-p^n)|_{\partial\Omega} = 0, \\ &u^{n+1} = \tilde{u}^{n+1} - \delta t\nabla(p^{n+1}-p^n). \end{aligned} \tag{2.3}$$

Hence, the above scheme is very easy to solve numerically compared with a fully implicit coupled nonlinear system.

Let us denote $(u,v)=\int_\Omega u\,v\,dx$ and $\|u\|^2=(u,u)$.

Theorem 2.1. *The scheme* (2.1)–(2.2) *is unconditionally energy stable in the sense that*

$$\begin{aligned}&\|u^{n+1}\|^2+\alpha\|b^{n+1}\|^2+\delta t^2\|\nabla p^{n+1}\|^2+(\|\tilde u^{n+1}-u^n\|^2+2\nu\delta t\|\nabla\tilde u^{n+1}\|^2+2\alpha\eta\delta t\|\nabla b^{n+1}\|^2)\\&\quad\leqslant\|u^n\|^2+\alpha\|b^n\|^2+\delta t^2\|\nabla p^n\|^2,\quad\forall\, n\geqslant 0.\end{aligned}\tag{2.4}$$

Furthermore, we have

$$\|\mathrm{div} b^{m+1}\|^2+2\delta t\eta\sum_{k=0}^{m}\|\nabla\mathrm{div} b^{k+1}\|^2\leqslant\|\mathrm{div} b^0\|^2,\quad\forall\, m>0.\tag{2.5}$$

Proof. Taking the inner product of the first equation in (2.1) with $2\delta t\tilde u^{n+1}$, we obtain

$$\begin{aligned}&\|\tilde u^{n+1}\|^2-\|u^n\|^2+\|\tilde u^{n+1}-u^n\|^2+2\nu\delta t\|\nabla\tilde u^{n+1}\|^2\\&\quad+2\delta t(\nabla p^n,\tilde u^{n+1})-2\delta t\alpha((\nabla\times b^{n+1})\times b^n,\tilde u^{n+1})=0,\end{aligned}\tag{2.6}$$

where we used the fact

$$((u\cdot\nabla)v,v)=0,\quad\forall\, u\in H,\quad v\in H^1(\Omega)^d.\tag{2.7}$$

Taking the inner product of the second equation in (2.1) with $2\delta t\alpha b^{n+1}$, we find

$$\begin{aligned}&\alpha\|b^{n+1}\|^2-\alpha\|b^n\|^2+\alpha\|b^{n+1}-b^n\|^2+2\delta t\alpha\eta\|\nabla b^{n+1}\|^2\\&\quad+2\delta t\alpha(\nabla\times(b^n\times\tilde u^{n+1}),b^{n+1})=0.\end{aligned}\tag{2.8}$$

Next, we rewrite (2.2) as

$$\frac{1}{\delta t}u^{n+1}+\nabla p^{n+1}=\frac{1}{\delta t}\tilde u^{n+1}+\nabla p^n,$$

and take the inner product of the above with itself on both sides. We find

$$\frac{1}{\delta t^2}\|u^{n+1}\|^2+\|\nabla p^{n+1}\|^2=\frac{1}{\delta t^2}\|\tilde u^{n+1}\|^2+\|\nabla p^n\|^2+2\frac{1}{\delta t}(\nabla p^n,\tilde u^{n+1}),$$

which we rearrange as

$$2\delta t(\nabla p^n,\tilde u^{n+1})=\|u^{n+1}\|^2-\|\tilde u^{n+1}\|^2+\delta t^2(\|\nabla p^{n+1}\|^2-\|\nabla p^n\|^2).\tag{2.9}$$

We have from integration by parts that

$$(\nabla\times(b^n\times\tilde u^{n+1}),b^{n+1})=((b^n\times\tilde u^{n+1}),\nabla\times b^{n+1})=((\nabla\times b^{n+1})\times b^n,\tilde u^{n+1}).\tag{2.10}$$

Taking the sum of (2.6) and (2.8), using (2.9) and (2.10), we arrive at

$$\begin{aligned}&\|u^{n+1}\|^2-\|u^n\|^2+\|\tilde u^{n+1}-u^n\|^2+2\delta t\nu\|\nabla\tilde u^{n+1}\|^2+\delta t^2(\|\nabla p^{n+1}\|^2-\|\nabla p^n\|^2)\\&\quad+\alpha\|b^{n+1}\|^2-\alpha\|b^n\|^2+\alpha\|b^{n+1}-b^n\|^2+2\delta t\alpha\eta\|\nabla b^{n+1}\|^2=0,\end{aligned}$$

which implies (2.4).

To prove (2.5), we recall that for any smooth function b, u and ϕ, we have

$$\begin{aligned}&(\nabla\times(b\times u),\nabla\phi)=\langle n\times(b\times u),\nabla\phi\rangle|_{\partial\Omega},\\&n\times(b\times u)=(n\cdot u)b-(n\cdot b)u.\end{aligned}\tag{2.11}$$

We take the inner product of the second equation in (2.1) with $\nabla\mathrm{div} b^{n+1}$. Thanks to the above identities and the boundary conditions for $\tilde u^{n+1}$ and b^n, we find

$$\left(\frac{b^{n+1}-b^n}{\delta t},\nabla\mathrm{div} b^{n+1}\right)-\eta(\Delta b^{n+1},\nabla\mathrm{div} b^{n+1})=0.\tag{2.12}$$

Using the identity $\Delta b = \nabla \times \nabla \times b - \nabla \mathrm{div} b$, and integration by parts, we obtain from the above that

$$\frac{1}{\delta t}(\mathrm{div}(b^{n+1} - b^n), \mathrm{div} b^{n+1}) + \eta(\nabla \mathrm{div} b^{n+1}, \nabla \mathrm{div} b^{n+1}) = 0, \tag{2.13}$$

from which we derive

$$\|\mathrm{div} b^{n+1}\|^2 - \|\mathrm{div} b^n\|^2 + \|\mathrm{div}(b^{n+1} - b^n)\|^2 + 2\delta t \eta \|\nabla \mathrm{div} b^{n+1}\|^2 = 0.$$

We obtain (2.5) by summing up the last relation for $n = 0, 1, \dots, m$. □

Remark 2.2. Note that we do not explicitly enforce $\mathrm{div} b^{n+1} = 0$ in the above scheme. However, (2.5) indicates that b^n will remain divergence-free as long as its initial condition b^0 is divergence-free. Furthermore, even with incompatible b^0, $\mathrm{div} b^n$ will be quickly dissipated towards zero. This remark applies also to other schemes presented below.

2.2 First-order scheme based on rotational pressure-correction

It is well known that the pressure-correction scheme leads to an artificial boundary condition, i.e., $\frac{\partial}{\partial n}(p^{n+1} - p^n)|_{\partial\Omega} = 0$ in (2.3), which induces large errors for the pressure near the boundary. A rotational pressure-correction scheme is introduced in [9,15] for the Navier-Stokes equations to remove the artificial pressure boundary layer. Hence, we adopt a similar approach here and propose a rotational pressure-correction scheme for the MHD equations below:

$$\begin{cases} \dfrac{\tilde{u}^{n+1} - u^n}{\delta t} + (u^n \cdot \nabla)\tilde{u}^{n+1} - \nu \Delta \tilde{u}^{n+1} + \nabla p^n - \alpha(\nabla \times b^{n+1}) \times b^n = 0, \\ \dfrac{b^{n+1} - b^n}{\delta t} - \eta \Delta b^{n+1} + \nabla \times (b^n \times \tilde{u}^{n+1}) = 0, \\ \tilde{u}^{n+1}|_{\partial\Omega} = 0, \\ b^{n+1} \cdot n|_{\partial\Omega} = 0, \quad n \times (\nabla \times b^{n+1})|_{\partial\Omega} = 0, \end{cases} \tag{2.14}$$

$$\begin{cases} \dfrac{u^{n+1} - \tilde{u}^{n+1}}{\delta t} + \nabla \psi^{n+1} = 0, \\ \mathrm{div} u^{n+1} = 0, \\ u^{n+1} \cdot n|_{\partial\Omega} = 0, \end{cases} \tag{2.15}$$

and

$$p^{n+1} = \psi^{n+1} + p^n - \nu \nabla \cdot \tilde{u}^{n+1}. \tag{2.16}$$

Note that (2.15) can also be reformulated as a Poisson equation, so the computational procedure for this scheme is essentially the same as the scheme (2.1)–(2.2).

Theorem 2.3. *The scheme* (2.14)–(2.16) *is unconditionally energy stable in the sense that*

$$\begin{aligned} &\|u^{n+1}\|^2 + \alpha\|b^{n+1}\|^2 + \delta t^2 \|\nabla \psi^{n+1}\|^2 + \frac{\delta t}{\nu}\|q^{n+1}\|^2 \\ &\quad + \delta t(2\nu\|\nabla \times \tilde{u}^{n+1}\|^2 + \nu\|\nabla \cdot \tilde{u}^{n+1}\|^2 + 2\alpha\eta\|\nabla b^{n+1}\|^2) \\ &\qquad \leqslant \|u^n\|^2 + \alpha\|b^n\|^2 + \delta t^2\|\nabla \psi^n\|^2 + \frac{\delta t}{\nu}\|q^n\|^2, \quad \forall\, n \geqslant 0. \end{aligned} \tag{2.17}$$

Furthermore, we have

$$\|\mathrm{div} b^{m+1}\|^2 + 2\delta t \eta \sum_{k=0}^{m} \|\nabla \mathrm{div} b^{k+1}\|^2 \leqslant \|\mathrm{div} b^0\|^2, \quad \forall\, m > 0. \tag{2.18}$$

Proof. The proof of (2.18) is exactly the same as before.

To prove (2.17), we introduce a set of new variables

$$q^{n+1} = q^n - \nu\nabla\cdot\tilde{u}^{n+1}, \quad \psi^{n+1} = p^{n+1} - q^{n+1}. \tag{2.19}$$

Then, we can rewrite (2.15) as

$$\begin{cases} \dfrac{u^{n+1}-\tilde{u}^{n+1}}{\delta t} + \nabla(\psi^{n+1}-\psi^n) = 0, \\ \mathrm{div} u^{n+1} = 0, \\ u^{n+1}\cdot n|_{\partial\Omega} = 0. \end{cases} \tag{2.20}$$

Taking the inner product of the first equation in (2.14) with $2\delta t\tilde{u}^{n+1}$, and that of the section equation in (2.14) with $2\alpha\delta t b^{n+1}$, and adding them together, we obtain

$$\begin{aligned} &\|\tilde{u}^{n+1}\|^2 - \|u^n\|^2 + \|\tilde{u}^{n+1}-u^n\|^2 + 2\delta t\nu\|\nabla\tilde{u}^{n+1}\|^2 + 2\delta t(\nabla(\psi^n+q^n),\tilde{u}^{n+1}) \\ &\quad + \alpha\|b^{n+1}\|^2 - \alpha\|b^n\|^2 + \alpha\|b^{n+1}-b^n\|^2 + 2\delta t\alpha\eta\|\nabla b^{n+1}\|^2 = 0, \end{aligned} \tag{2.21}$$

where we used the fact that the curl terms are canceled with each other as in the proof of Theorem 2.1. For the inner product term in the above, we have

$$\begin{aligned} &2\delta t(\nabla\psi^n + \nabla q^n, \tilde{u}^{n+1}) \\ &\quad = 2\delta t(\nabla\psi^n, \tilde{u}^{n+1}) + 2\delta t(\nabla q^n, \tilde{u}^{n+1}) \\ &\quad = 2\delta t(\nabla\psi^n, u^{n+1} + \delta t(\nabla\psi^{n+1} - \nabla\psi^n)) - 2\delta t(q^n, \nabla\cdot\tilde{u}^{n+1}) \\ &\quad = \delta t^2(\|\nabla\psi^{n+1}\|^2 - \|\nabla\psi^n\|^2 - \|\nabla\psi^{n+1}-\nabla\psi^n\|^2) + \frac{2\delta t}{\nu}(q^n, q^{n+1}-q^n) \\ &\quad = \delta t^2\left(\|\nabla\psi^{n+1}\|^2 - \|\nabla\psi^n\|^2 - \frac{1}{\delta t^2}\|u^{n+1}-\tilde{u}^{n+1}\|^2\right) + \frac{\delta t}{\nu}(\|q^{n+1}\|^2 - \|q^n\|^2 - \|q^{n+1}-q^n\|^2) \\ &\quad = \delta t^2(\|\nabla\psi^{n+1}\|^2 - \|\nabla\psi^n\|^2) - \|u^{n+1}-\tilde{u}^{n+1}\|^2 + \frac{\delta t}{\nu}(\|q^{n+1}\|^2 - \|q^n\|^2 - \nu^2\|\nabla\cdot\tilde{u}^{n+1}\|^2). \end{aligned}$$

On the other hand, taking the inner product of (2.15) with $2\delta t u^{n+1}$, we obtain

$$\|u^{n+1}\|^2 - \|\tilde{u}^{n+1}\|^2 + \|u^{n+1}-\tilde{u}^{n+1}\|^2 = 0.$$

Combining the above equalities together and using the identity $\|\nabla u\|^2 = \|\nabla\times u\|^2 + \|\nabla\cdot u\|^2$ for $u\in H_0^1(\Omega)^d$, we obtain

$$\begin{aligned} &\|u^{n+1}\|^2 - \|u^n\|^2 + \|\tilde{u}^{n+1}-u^n\|^2 + 2\delta t\nu\|\nabla\times\tilde{u}^{n+1}\|^2 + \nu\delta t\|\nabla\cdot\tilde{u}^{n+1}\|^2 \\ &\quad + \delta t^2(\|\nabla\psi^{n+1}\|^2 - \|\nabla\psi^n\|^2) + \frac{\delta t}{\nu}(\|q^{n+1}\|^2 - \|q^n\|^2) \\ &\quad + \alpha\|b^{n+1}\|^2 - \alpha\|b^n\|^2 + \alpha\|b^{n+1}-b^n\|^2 + 2\delta t\alpha\eta\|\nabla b^{n+1}\|^2 \\ &\qquad = 0, \end{aligned}$$

which implies (2.17). □

3 Second-order schemes

The schemes presented in last section is only first-order accurate. Similarly to that for the Navier-Stokes equations, we can construct second-order schemes based on standard pressure-correction and rotational pressure-correction.

3.1 Second-order scheme based standard pressure-correction

A semi-implicit second-order scheme for (1.1)–(1.3) based on pressure-correction is as follows:

$$\begin{cases} \dfrac{3\tilde{u}^{n+1}-4u^n+u^{n-1}}{2\delta t}+((2u^n-u^{n-1})\cdot\nabla)\tilde{u}^{n+1}-\nu\Delta\tilde{u}^{n+1} \\ \quad +\nabla p^n-\alpha(\nabla\times b^{n+1})\times(2b^n-b^{n-1})=0, \\ \dfrac{3b^{n+1}-4b^n+b^{n-1}}{2\delta t}-\eta\Delta b^{n+1}+\nabla\times((2b^n-b^{n-1})\times\tilde{u}^{n+1})=0, \\ \tilde{u}^{n+1}|_{\partial\Omega}=0, \\ b^{n+1}\cdot n|_{\partial\Omega}=0, \quad n\times(\nabla\times b^{n+1})|_{\partial\Omega}=0, \end{cases} \tag{3.1}$$

and

$$\begin{cases} \dfrac{3u^{n+1}-3\tilde{u}^{n+1}}{2k}+\nabla(p^{n+1}-p^n)=0, \\ \mathrm{div}u^{n+1}=0, \\ u^{n+1}\cdot n|_{\partial\Omega}=0. \end{cases} \tag{3.2}$$

To start up the scheme, we can use the first-order scheme (2.1)–(2.2) to obtain the approximation at the first time step. Hence, the computational procedure of this scheme is still essentially the same as the previous first-order schemes.

Theorem 3.1. *The scheme* (3.1)–(3.2) *is unconditionally energy stable in the sense that*

$$\begin{aligned} &\|u^{n+1}\|^2+\alpha\|b^{n+1}\|^2+\|2u^{n+1}-u^n\|^2+\alpha\|b^n\|^2+\alpha\|2b^{n+1}-b^n\|^2 \\ &\quad+\frac{4}{3}\delta t^2\|\nabla p^{n+1}\|^2+\delta t(4\nu\|\nabla\tilde{u}^{n+1}\|^2+4\alpha\eta\|\nabla b^{n+1}\|^2) \\ &\qquad\leqslant\|u^n\|^2+\alpha\|b^n\|^2+\|2u^n-u^{n-1}\|^2+\alpha\|2b^n-b^{n-1}\|^2+\frac{4}{3}\delta t^2\|\nabla p^n\|^2, \quad \forall\, n\geqslant 0. \end{aligned} \tag{3.3}$$

Furthermore, we have

$$\begin{aligned} &\|\mathrm{div}b^{m+1}\|^2+\|\mathrm{div}(2b^{m+1}-b^m)\|^2+4\delta t\eta\sum_{k=0}^{m}\|\nabla\mathrm{div}b^{k+1}\|^2 \\ &\quad\leqslant\|\mathrm{div}b^1\|^2+\|\mathrm{div}(2b^1-b^0)\|^2, \quad \forall\, m>0. \end{aligned} \tag{3.4}$$

Proof. Taking the inner product of the first equation in (3.1) with $4\delta t\tilde{u}^{n+1}$, using (2.7), we derive

$$\begin{aligned} &(3\tilde{u}^{n+1}-4u^n+u^{n-1},2\tilde{u}^{n+1})+4\delta t\nu\|\nabla\tilde{u}^{n+1}\|^2+4\delta t(\nabla p^n,\tilde{u}^{n+1}) \\ &\quad-4\alpha\delta t((\nabla\times b^{n+1})\times(2b^n-b^{n-1}),\tilde{u}^{n+1})=0. \end{aligned} \tag{3.5}$$

Taking the inner product of the second equation in (3.1) with $4\delta t\alpha b^{n+1}$, we derive

$$\begin{aligned} &\alpha(3b^{n+1}-4b^n+b^{n-1},2b^{n+1})+4\eta\alpha\delta t\|\nabla b^{n+1}\|^2 \\ &\quad+4\alpha\delta t(\nabla\times((2b^n-b^{n-1})\times\tilde{u}^{n+1}),b^{n+1})=0. \end{aligned} \tag{3.6}$$

We obtain from integration by parts that

$$\begin{aligned} (\nabla\times((2b^n-b^{n-1})\times\tilde{u}^{n+1}),b^{n+1})&=((2b^n-b^{n-1})\times\tilde{u}^{n+1},\nabla\times b^{n+1}) \\ &=((\nabla\times b^{n+1})\times(2b^n-b^{n-1}),\tilde{u}^{n+1}). \end{aligned} \tag{3.7}$$

Summing up (3.5) and (3.6), and using (3.7), we obtain

$$\begin{aligned} &(3\tilde{u}^{n+1}-4u^n+u^{n-1},2\tilde{u}^{n+1})+\alpha(3b^{n+1}-4b^n+b^{n-1},2b^{n+1}) \\ &\quad+4\delta t\nu\|\nabla\tilde{u}^{n+1}\|^2+4\eta\alpha\delta t\|\nabla b^{n+1}\|^2+4\delta t(\nabla p^n,\tilde{u}^{n+1})=0. \end{aligned} \tag{3.8}$$

We denote

$$
\begin{aligned}
&I_1 := (3\tilde{u}^{n+1} - 4u^n + u^{n-1}, 2\tilde{u}^{n+1}),\\
&I_2 := \alpha(3b^{n+1} - 4b^n + b^{n-1}, 2b^{n+1}),\\
&I_3 := 4\delta t(\nabla p^n, \tilde{u}^{n+1}),
\end{aligned}
\tag{3.9}
$$

and bound them separately below,

$$
\begin{aligned}
I_1 &= (3\tilde{u}^{n+1} - 4u^n + u^{n-1}, 2\tilde{u}^{n+1})\\
&= (3u^{n+1} - 4u^n + u^{n-1}, 2\tilde{u}^{n+1}) - (3u^{n+1} - 3\tilde{u}^{n+1}, 2\tilde{u}^{n+1})\\
&= (3u^{n+1} - 4u^n + u^{n-1}, 2u^{n+1}) - (3u^{n+1} - 4u^n + u^{n-1}, 2(u^{n+1} - \tilde{u}^{n+1}))\\
&\quad - (3u^{n+1} - 3\tilde{u}^{n+1}, 2\tilde{u}^{n+1})\\
&=: I_{11} - I_{12} - I_{13}.
\end{aligned}
\tag{3.10}
$$

For any sequence $\{v^n\}$, we denote $\delta v^n = v^n - v^{n-1}$ and $\delta^2 v^n = \delta(v^n - v^{n-1})$. Using the identities

$$
(a-b, 2a) = |a|^2 - |b|^2 + |a-b|^2, \quad (2a - 2b, 2a) = |2a - b|^2 - |b|^2,
$$

we find

$$
\begin{aligned}
I_{11} &= (3u^{n+1} - 4u^n + u^{n-1}, 2u^{n+1})\\
&= (u^{n+1} - 2u^n + u^{n-1}, 2u^{n+1}) + (2u^{n+1} - 2u^n, 2u^{n+1})\\
&= \|u^{n+1}\|^2 - \|2u^n - u^{n-1}\|^2 + \|\delta^2 u^{n+1}\|^2 + \|2u^{n+1} - u^n\|^2 - \|u^n\|^2,
\end{aligned}
\tag{3.11}
$$

$$
\begin{aligned}
I_{12} &= (3u^{n+1} - 4u^n + u^{n-1}, 2(u^{n+1} - \tilde{u}^{n+1}))\\
&= -\left(3u^{n+1} - 4u^n + u^{n-1}, \frac{4\delta t}{3}\nabla(p^{n+1} - p^n)\right) = 0,
\end{aligned}
\tag{3.12}
$$

and

$$
I_{13} = (3u^{n+1} - 3\tilde{u}^{n+1}, 2\tilde{u}^{n+1}) = 3(\|u^{n+1}\|^2 - \|\tilde{u}^{n+1}\|^2 - \|u^{n+1} - \tilde{u}^{n+1}\|^2).
\tag{3.13}
$$

Similarly, we have

$$
\begin{aligned}
I_2 &= \alpha(3b^{n+1} - 4b^n + b^{n-1}, 2b^{n+1})\\
&= \alpha((b^{n+1} - 2b^n + b^{n-1}, 2b^{n+1}) + (2b^{n+1} - 2b^n, 2b^{n+1}))\\
&= \alpha(\|b^{n+1}\|^2 - \|2b^n - b^{n-1}\|^2 + \|\delta^2 b^{n+1}\|^2 + \|2b^{n+1} - b^n\|^2 - \|b^n\|^2).
\end{aligned}
\tag{3.14}
$$

From the equation (3.2), we have

$$
3u^{n+1} + 2\delta t\nabla p^{n+1} = 3\tilde{u}^{n+1} + 2\delta t\nabla p^n.
$$

Taking the inner product of the above with itself, we derive

$$
9\|u^{n+1}\|^2 + 4\delta t^2\|\nabla p^{n+1}\|^2 = 9\|\tilde{u}^{n+1}\|^2 + 12\delta t(\tilde{u}^{n+1}, \nabla p^n) + 4\delta t^2\|\nabla p^n\|^2,
$$

which implies that

$$
I_3 = 4\delta t(\tilde{u}^{n+1}, \nabla p^n) = 3\|u^{n+1}\| - 3\|\tilde{u}^{n+1}\| + \frac{4}{3}\delta t^2(\|\nabla p^{n+1}\|^2 - \|\nabla p^n\|^2).
\tag{3.15}
$$

Combining the above identities into (3.8), we arrive at

$$
\begin{aligned}
&\|u^{n+1}\|^2 - \|u^n\|^2 + \|2u^{n+1} - u^n\|^2 - \|2u^n - u^{n-1}\|^2 + 4\delta t\nu\|\nabla\tilde{u}^{n+1}\|^2\\
&\quad + 4\alpha\eta\delta t\|\nabla b^{n+1}\|^2 + \|\delta^2 u^{n+1}\|^2 + 3\|u^{n+1}\tilde{u}^{n+1}\|^2\\
&\quad + \alpha(\|b^{n+1}\|^2 - \|b^n\|^2 + \|2b^{n+1} - b^n\|^2 - \|2b^n - b^{n-1}\|^2)\\
&\quad + \frac{4}{3}\delta t^2(\|\nabla p^{n+1}\|^2 - \|\nabla p^n\|^2)\\
&\qquad = 0,
\end{aligned}
$$

which implies (3.3).

The proof of (3.4) is similar to that of (2.5) except that we use an identity similar to (3.10). □

3.2 Second-order scheme based on rotational pressure-correction

We can also construct a second-order scheme based on rotational pressure-correction as follows:

$$\begin{cases} \dfrac{3\tilde{u}^{n+1}-4u^n+u^{n-1}}{2\delta t}+((2u^n-u^{n-1})\cdot\nabla)\tilde{u}^{n+1}-\nu\Delta\tilde{u}^{n+1} \\ \quad +\nabla p^n-\alpha(\nabla\times b^{n+1})\times(2b^n-b^{n-1})=0, \\ \dfrac{3b^{n+1}-4b^n+b^{n-1}}{2\delta t}-\eta\Delta b^{n+1}+\nabla\times((2b^n-b^{n-1})\times\tilde{u}^{n+1})=0, \\ \tilde{u}^{n+1}|_{\partial\Omega}=0, \\ b^{n+1}\cdot n|_{\partial\Omega}=0, \quad n\times(\nabla\times b^{n+1})|_{\partial\Omega}=0, \end{cases} \tag{3.16}$$

and

$$\begin{cases} \dfrac{3u^{n+1}-3\tilde{u}^{n+1}}{2\delta t}+\nabla(p^{n+1}-p^n+\nu\nabla\cdot\tilde{u}^{n+1})=0, \\ \mathrm{div} u^{n+1}=0, \\ u^{n+1}\cdot n|_{\partial\Omega}=0. \end{cases} \tag{3.17}$$

Based on the results that we established in Theorem 2.3, it can be expected, and confirmed by our numerical experiments, that the above scheme is also unconditionally stable. However, its proof appears to be much more involved that we will not pursue here for the sake of brevity.

4 Error estimates

To illustrate the procedure for error analysis, we establish below some error estimates for the first-order pressure-correction scheme.

We start with some notation. Given a time step δt and $T>0$, let ϕ^k $(k=0,1,\dots,m=[T/\delta t])$ be a sequence of functions in a normed space E, and denote

$$\|\phi_{\delta t}\|^2_{l^2(E)}=\delta t\sum_{k=0}^{m}\|\phi^k\|^2_E, \quad \|\phi_{\delta t}\|_{l^\infty(E)}=\max_{0\leqslant k\leqslant m}\|\phi^k\|_E. \tag{4.1}$$

Let $t_n=n\delta t$, and denote

$$e_u^n=u(t_n)-u^n, \quad \tilde{e}_u^n=u(t_n)-\tilde{u}^n, \quad e_b^n=b(t_n)-b^n, \quad e_p^n=p(t_n)-p^n. \tag{4.2}$$

Theorem 4.1. *Assume that the solution to* (1.1)–(1.3) *is sufficiently smooth. Then the solution to the scheme* (2.1)–(2.2) *satisfies the following error estimates*:

$$\|e_{u,\delta t}\|_{l^\infty(L^2(\Omega)^d)}+\|e_{b,\delta t}\|_{l^\infty(L^2(\Omega)^d)}+\|\tilde{e}_{u,\delta t}\|_{l^2(H^1(\Omega)^d)}+\|e_{b,\delta t}\|_{l^2(H^1(\Omega)^d)}\leqslant C\delta t.$$

Proof. We define the following truncation errors:

$$\begin{aligned} R_u^n=&\frac{u(t^{n+1})-u(t^n)}{\delta t}+(u(t^n)\cdot\nabla)u(t^{n+1})-\nu\Delta u(t^{n+1}) \\ &+\nabla p(t^n)-\alpha(\nabla\times b(t^{n+1}))\times b(t^n), \end{aligned} \tag{4.3}$$

$$R_b^n=\frac{b(t^{n+1})-b(t^n)}{\delta t}-\eta\Delta b(t^{n+1})+\nabla\times(b(t^n)\times u(t^{n+1})). \tag{4.4}$$

We also define

$$R_p^n=\frac{u(t^{n+1})-u(t^n)}{\delta t}+\nabla(p(t^{n+1})-p(t^n)). \tag{4.5}$$

It is clear that we have

$$\|R_{u,\delta t}\|_{l^\infty(H^1(\Omega))}+\|R_{b,\delta t}\|_{l^\infty(H^1(\Omega))}+\|R_{p,\delta t}\|_{l^\infty(L^2(\Omega))}\leqslant c_R\delta t, \tag{4.6}$$

where $c_R > 0$ is independent of δt.

Subtracting the first and second equations of (2.1) and (2.2) from (4.3)–(4.5), respectively, we get the following error equations for $n \geqslant 0$,

$$\begin{aligned}
&\frac{\tilde{e}_u^{n+1} - e_u^n}{\delta t} + ((u(t^n) \cdot \nabla)u(t^{n+1}) - (u^n \cdot \nabla)\tilde{u}^{n+1}) - \nu\Delta\tilde{e}_u^{n+1} \\
&\quad + \nabla q^n - \alpha(\nabla \times b(t^{n+1})) \times e_b^n - \alpha(\nabla \times e_b^{n+1}) \times b^n = R_u^{n+1}, \qquad (4.7)\\
&\frac{e_b^{n+1} - e_b^n}{\delta t} - \eta\Delta e_b^{n+1} + \nabla \times (e_b^n \times u(t^{n+1})) + \nabla \times (b^n \times \tilde{e}_u^{n+1}) = R_b^{n+1}, \qquad (4.8)\\
&\frac{e_u^{n+1} - \tilde{e}_u^{n+1}}{\delta t} + \nabla q^{n+1} - \nabla q^n = R_p^{n+1}. \qquad (4.9)
\end{aligned}$$

Taking the inner product (4.7) with $2\delta t\tilde{e}_u^{n+1}$, we obtain

$$\begin{aligned}
&\|\tilde{e}_u^{n+1}\|^2 - \|e_u^n\|^2 + \|\tilde{e}_u^{n+1} - e_u^n\|^2 + 2\delta t((u(t^n) \cdot \nabla)u(t^{n+1}) - (u^n \cdot \nabla)\tilde{u}^{n+1}, \tilde{e}_u^{n+1}) \\
&\quad + 2\delta t\nu\|\nabla\tilde{e}_u^{n+1}\|^2 + 2\delta t(\nabla q^n, \tilde{e}_u^{n+1}) - 2\delta t\alpha((\nabla \times b(t^{n+1})) \times e_b^n, \tilde{e}_u^{n+1}) \\
&\quad - 2\delta t\alpha((\nabla \times e_b^{n+1}) \times b^n, \tilde{e}_u^{n+1}) \\
&\quad\quad = 2\delta t(R_u^n, \tilde{e}_u^{n+1}). \qquad (4.10)
\end{aligned}$$

Thanks to (2.7), we have

$$\begin{aligned}
&((u(t^n) \cdot \nabla)u(t^{n+1}) - (u^n \cdot \nabla)\tilde{u}^{n+1}, \tilde{e}_u^{n+1}) \\
&\quad = ((u(t^n) \cdot \nabla)\tilde{e}_u^{n+1} + (e_u^n \cdot \nabla)\tilde{u}^{n+1}, \tilde{e}_u^{n+1}) \\
&\quad = ((u(t^n) \cdot \nabla)\tilde{e}_u^{n+1} - (e_u^n \cdot \nabla)\tilde{e}_u^{n+1} + (e_u^n \cdot \nabla)u(t^{n+1}), \tilde{e}_u^{n+1}) \\
&\quad = ((e_u^n \cdot \nabla)u(t^{n+1}), \tilde{e}_u^{n+1}).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
2\delta t(\nabla q^n, \tilde{e}_u^{n+1}) &= 2\delta t(\nabla q^n, e_u^{n+1} + \delta t(\nabla q^{n+1} - \nabla q^n) - \delta t R_p^n) \\
&= \delta t^2(\|\nabla q^{n+1}\|^2 - \|\nabla q^n\|^2 - \|\nabla q^{n+1} - \nabla q^n\|^2) - 2\delta t^2(\nabla q^n, R_p^n). \qquad (4.11)
\end{aligned}$$

We derive from (4.9) that

$$\begin{aligned}
&\left\|\frac{1}{\delta t}e_u^{n+1} + \nabla q^{n+1} - \nabla q^n\right\|^2 = \left\|R_p^n + \frac{1}{\delta t}\tilde{e}_u^{n+1}\right\|^2, \\
&\|\nabla q^{n+1} - \nabla q^n\|^2 = \frac{1}{\delta t^2}(\|\tilde{e}_u^{n+1}\|^2 - \|e_u^{n+1}\|^2) + \|R_p^n\|^2 + \frac{2}{\delta t}(R_p^n, \tilde{e}_u^{n+1}),
\end{aligned}$$

where we used the fact that $(e_u^{n+1}, \nabla q^{n+1} - \nabla q^n) = 0$. Hence

$$\begin{aligned}
2\delta t(\nabla q^n, \tilde{e}_u^{n+1}) &= \|e_u^{n+1}\|^2 - \|\tilde{e}_u^{n+1}\|^2 + \delta t^2(\|\nabla q^{n+1}\|^2 - \|\nabla q^n\|^2) \\
&\quad - \delta t^2\|R_p^n\|^2 - 2\delta t(R_p^n, \tilde{e}_u^{n+1}) - 2\delta t^2(\nabla q^n, R_p^n). \qquad (4.12)
\end{aligned}$$

Taking the inner product (4.8) with $2\alpha\delta t e_b^{n+1}$, we obtain

$$\begin{aligned}
&\alpha(\|e_b^{n+1}\|^2 - \|e_b^n\|^2 + \|e_b^{n+1} - e_b^n\|^2) + 2\alpha\delta t\eta\|\nabla e_b^{n+1}\|^2 \\
&\quad + 2\alpha\delta t(\nabla \times (e_b^n \times u(t^{n+1})), e_b^{n+1}) + 2\alpha\delta t(\nabla \times (b^n \times \tilde{e}_u^{n+1}), e_b^{n+1}) \\
&\quad = 2\alpha\delta t(R_b^n, e_b^{n+1}). \qquad (4.13)
\end{aligned}$$

Using integration by parts, we get

$$\begin{aligned}
2\alpha\delta t(\nabla \times (b^n \times \tilde{e}_u^{n+1}), e_b^{n+1}) &= 2\alpha\delta t((b^n \times \tilde{e}_u^{n+1}), \nabla \times e_b^{n+1}) \\
&= 2\delta t\alpha((\nabla \times e_b^{n+1}) \times b^n, \tilde{e}_u^{n+1}). \qquad (4.14)
\end{aligned}$$

Summing up (4.10) and (4.13), using (4.12) and (4.14), we obtain

$$\begin{aligned}
&\|e_u^{n+1}\|^2 - \|e_u^n\|^2 + \|\tilde{e}_u^{n+1} - e_u^n\|^2 + 2\delta t\nu\|\nabla\tilde{e}_u^{n+1}\|^2 \\
&\quad + \alpha(\|e_b^{n+1}\|^2 - \|e_b^n\|^2 + \|e_b^{n+1} - e_b^n\|^2) + 2\alpha\delta t\eta\|\nabla e_b^{n+1}\|^2 \\
&\quad + \delta t^2(\|\nabla q^{n+1}\|^2 - \|\nabla q^n\|^2) - \delta t^2\|R_p^n\|^2 - 2\delta t(R_p^n, \tilde{e}_u^{n+1}) - 2\delta t^2(\nabla q^n, R_p^n) \\
&\quad + 2\delta t((e_u^n\cdot\nabla)u(t^{n+1}), \tilde{e}_u^{n+1}) - 2\delta t\alpha((\nabla\times b(t^{n+1}))\times e_b^n, \tilde{e}_u^{n+1}) \\
&\quad + 2\alpha\delta t(\nabla\times(e_b^n\times u(t^{n+1})), e_b^{n+1}) \\
&\qquad = 2\delta t(R_u^n, \tilde{e}_u^{n+1}) + 2\alpha\delta t(R_b^n, e_b^{n+1}).
\end{aligned} \tag{4.15}$$

We bound each inner product term in the above as follows:

$$\begin{aligned}
|2k\alpha((\nabla\times b(t^{n+1}))\times e_b^n, \tilde{e}_u^{n+1})| &\leqslant ck\|(\nabla\times b(t^{n+1}))\times e_b^n\|\|\tilde{e}_u^{n+1}\| \\
&\leqslant c\delta t\|e_b^n\|\|\tilde{e}_u^{n+1}\| \\
&\leqslant c\delta t\|e_b^n\|\|\nabla\tilde{e}_u^{n+1}\| \\
&\leqslant c\delta t\|e_b^n\|^2 + \frac{\nu}{8}\delta t\|\nabla\tilde{e}_u^{n+1}\|^2, \\
|2\alpha\delta t(\nabla\times(e_b^n\times u(t^{n+1})), e_b^{n+1})| &= |2\alpha\delta t(e_b^n\times u(t^{n+1}), \nabla\times e_b^{n+1})| \\
&\leqslant c\delta t\|e_b^n\times u(t^{n+1})\|\|\nabla\times e_b^{n+1}\| \\
&\leqslant c\delta t\|e_b^n\|\|\nabla e_b^{n+1}\| \\
&\leqslant c\delta t\|e_b^n\|^2 + \delta t\frac{\alpha\eta}{8}\|\nabla e_b^{n+1}\|^2, \\
|2\delta t(R_p^n, \tilde{e}_u^{n+1})| \leqslant 2\delta t\|R_p^n\|\|\tilde{e}_u^{n+1}\| &\leqslant c\delta t^3 + \delta t\frac{\nu}{8}\|\nabla\tilde{e}_u^{n+1}\|^2, \\
|2k((e_u^n\cdot\nabla)u(t^{n+1}), \tilde{e}_u^{n+1})| &\leqslant 2\delta t\|(e_u^n\cdot\nabla)u(t^{n+1})\|\|\tilde{e}_u^{n+1}\| \\
&\leqslant c\delta t\|e_u^n\|\|\tilde{e}_u^{n+1}\| \\
&\leqslant c\delta t\|e_u^n\|^2 + \delta t\frac{\nu}{8}\|\nabla\tilde{e}_u^{n+1}\|^2, \\
2\delta t(R_b^n, e_b^{n+1}) \leqslant 2\delta t\|R_b^n\|\|e_b^{n+1}\| &\leqslant c\delta t^3 + \delta t\eta\frac{\alpha}{8}\|\nabla e_b^{n+1}\|^2, \\
|2\delta t^2(\nabla q^n, R_p^n)| = 2\delta t^2\|\nabla q^n\|\|R_p^n\| &\leqslant c\delta t^3\|\nabla q^n\|^2 + c\delta t^3.
\end{aligned}$$

Combining the above estimates into (4.15), we find

$$\begin{aligned}
&\|e_u^{n+1}\|^2 - \|e_u^n\|^2 + \|\tilde{e}_u^{n+1} - e_u^n\|^2 + \delta t\nu\|\nabla\tilde{e}_u^{n+1}\|^2 \\
&\quad + \alpha(\|e_b^{n+1}\|^2 - \|e_b^n\|^2 + \|e_b^{n+1} - e_b^n\|^2) + \alpha\delta t\eta\|\nabla e_b^{n+1}\|^2 + \delta t^2(\|\nabla q^{n+1}\|^2 - \|\nabla q^n\|^2) \\
&\qquad \leqslant c\delta t^3 + c\delta t(\|e_u^n\|^2 + \|e_b^n\|^2 + \delta t^2\|\nabla q^n\|^2).
\end{aligned}$$

Summing up the above for $n = 0, \ldots, m$, we obtain

$$\begin{aligned}
&\|e_u^{m+1}\|^2 + \alpha\|e_b^{m+1}\|^2 + \delta t^2\|\nabla q^{m+1}\|^2 + \sum_{n=0}^{m}(\|\tilde{e}_u^{n+1} - e_u^n\|^2 \\
&\quad + \delta t\nu\|\nabla\tilde{e}_u^{n+1}\|^2 + \alpha\delta t\eta\|\nabla e_b^{n+1}\|^2 + \alpha\|e_b^{n+1} - e_b^n\|^2) \\
&\qquad \leqslant c\delta t^2 + c\delta t\sum_{n=0}^{m}(\|e_u^n\|^2 + \alpha\|e_b^n\|^2 + \delta t^2\|\nabla q^n\|^2).
\end{aligned}$$

Finally, we can obtain the desired result by applying a discrete Gronwall inequality. □

Remark 4.2. It is also possible to derive rigorous error estimates for the rotational pressure-correction and second-order schemes presented above by combining the procedures in the above proof and in [9]. However, the process is very tedious so we leave the details to the interested readers.

5 Spatial discretizations and fully discrete schemes

We first describe briefly how to construct fully discrete schemes with a generic approximation in space, and then describe in some detail a Legendre-Galerkin method. For the sake of simplicity, we shall consider only the first-order scheme (2.1)–(2.2), the other schemes can be treated by the same manner.

5.1 A generic spatial approximation

Let $\boldsymbol{X}_h \subset (H_0^1(\Omega))^d$ denote the approximation space for the intermediate velocity field,

$$\boldsymbol{Y}_h \subset \boldsymbol{H}_n^1(\Omega) := \{b \in (H^1(\Omega))^d : b\cdot n|_{\partial\Omega} = 0\}$$

denote the approximation space for the magnetic field,

$$M_h \subset H_{\int}^1(\Omega) := \left\{q \in H^1(\Omega) : \int_\Omega q = 0\right\}$$

denote the approximation space for the pressure, and $\boldsymbol{Z}_h := \{v = u + \nabla q : u \in X_h,\ q \in M_h\}$.

Then, a fully discrete scheme based on (2.1)–(2.2) and the above discrete spaces is as follows:

(i) Find $\tilde{u}_h^{n+1} \in \boldsymbol{X}_h$ and $b_h^{n+1} \in \boldsymbol{Y}_h$ such that

$$\begin{aligned}&\left(\frac{1}{\delta t}\tilde{u}_h^{n+1}, v_h\right) + b(u_h^n, \tilde{u}_h^{n+1}, v_h) + \nu(\nabla\tilde{u}_h^{n+1}, \nabla v_h) - \alpha(\nabla\times b_h^{n+1}, b_h^n \times v_h)\\ &\quad = \left(\frac{1}{\delta t}u_h^n - \nabla p_h^n, v_h\right), \quad \forall\, v_h \in \boldsymbol{X}_h,\end{aligned} \tag{5.1}$$

$$\left(\frac{\alpha}{\delta t}b_h^{n+1}, w_h\right) + \alpha\eta(\nabla b_h^{n+1}, \nabla w_h) + \alpha(b_h^n \times \tilde{u}_h^{n+1}, \nabla\times w_h) = \left(\frac{\alpha}{\delta t}b_h^n, w_h\right), \quad \forall\, w_h \in \boldsymbol{Y}_h, \tag{5.2}$$

where $b(u_h, w_h, v_h) = ((u_h\cdot\nabla)w_h, v_h) + \frac{1}{2}(\mathrm{div}u_h, w_h, v_h)$.

(ii) Find $p_h^{n+1} \in M_h$ such that

$$(\nabla p_h^{n+1}, \nabla q_h) = \left(\nabla p_h^n + \frac{1}{\delta t}\tilde{u}_h^{n+1}, \nabla q_h\right), \quad \forall\, q_h \in M_h. \tag{5.3}$$

(iii) Set

$$u_h^{n+1} = \tilde{u}_h^{n+1} - \delta t\nabla(p_h^{n+1} - p_h^n) \in \boldsymbol{Z}_h. \tag{5.4}$$

By using a similar procedure to that in the proof of Theorem 2.1, we can establish the following:

Theorem 5.1. *The scheme* (5.1)–(5.4) *is unconditionally energy stable in the sense that*

$$\begin{aligned}&\|u_h^{n+1}\|^2 + \alpha\|b_h^{n+1}\|^2 + \delta t^2\|\nabla p_h^{n+1}\|^2\\ &\quad + (\|\tilde{u}_h^{n+1} - u_h^n\|^2 + 2\nu\delta t\|\nabla\tilde{u}_h^{n+1}\|^2 + 2\alpha\eta\delta t\|\nabla b_h^{n+1}\|^2)\\ &\quad \leqslant \|u_h^n\|^2 + \alpha\|b_h^n\|^2 + \delta t^2\|\nabla p_h^n\|^2, \quad \forall\, n \geqslant 0.\end{aligned}$$

Proof. Taking $v_h = 2\delta t\tilde{u}_h^{n+1}$ in (5.1), we obtain

$$\begin{aligned}&\|\tilde{u}_h^{n+1}\|^2 - \|u_h^n\|^2 + \|\tilde{u}_h^{n+1} - u_h^n\|^2 + 2\nu\delta t\|\nabla\tilde{u}_h^{n+1}\|^2\\ &\quad + 2\delta t(\nabla p_h^n, \tilde{u}_h^{n+1}) - 2\delta t\alpha(\nabla\times b_h^{n+1}, b_h^n\times\tilde{u}_h^{n+1})\\ &\quad = 0,\end{aligned} \tag{5.5}$$

since, by integration by parts, we have

$$b((u_h\cdot\nabla)v_h, v_h) = 0, \quad \forall\, u_h \in Z_h, \quad v_h \in X_h. \tag{5.6}$$

Taking $w_h = 2\delta t\alpha b_h^{n+1}$ in (5.2), we find

$$\alpha\|b_h^{n+1}\|^2 - \alpha\|b_h^n\|^2 + \alpha\|b_h^{n+1} - b_h^n\|^2 + 2\delta t\alpha\eta\|\nabla b_h^{n+1}\|^2$$

$$+ 2\delta t\alpha(b_h^n \times \tilde{u}_h^{n+1}, \nabla \times b_h^{n+1}) = 0. \tag{5.7}$$

We rewrite (5.4) as

$$u_h^{n+1} + \delta t \nabla p_h^{n+1} = \tilde{u}_h^{n+1} + \delta t \nabla p_h^n.$$

Since both sides are in $\boldsymbol{Z}_h$, we can take the inner product of the above with itself on both sides to get

$$\|u_h^{n+1}\|^2 + \delta t^2 \|\nabla p_h^{n+1}\|^2 = \|\tilde{u}_h^{n+1}\|^2 + \delta t^2 \|\nabla p_h^n\|^2 + 2\delta t(\nabla p_h^n, \tilde{u}_h^{n+1}). \tag{5.8}$$

We obtain the desired result by taking the sum of (5.5), (5.7) and (5.8). □

Remark 5.2. By making standard assumptions on the approximation properties of $\boldsymbol{X}_h$, $\boldsymbol{Y}_h$, M_h and the usual inf-sup condition on $(\boldsymbol{X}_h, M_h)$, one can establish an error estimate, for the fully discretized scheme (5.1)–(5.4) using a standard procedure (see [8]). We leave the details to the interested readers.

Note that $\tilde{u}_h^{n+1}$ and b_h^{n+1} are coupled together, and are solutions to a linear system of the form

$$\mathcal{A}_h^{n+1} \begin{pmatrix} \tilde{u}_h^{n+1} \\ b_h^{n+1} \end{pmatrix} = \begin{pmatrix} \frac{1}{\delta t} u_h^n - \nabla p_h^n \\ \frac{1}{\delta t} b_h^n \end{pmatrix}. \tag{5.9}$$

It is easy to check that for $u_h \in \boldsymbol{X}_h$ and $b_h \in \boldsymbol{Y}_h$, we have

$$\left(\begin{pmatrix} u_h \\ b_h \end{pmatrix}, \mathcal{A}_h^{n+1} \begin{pmatrix} u_h \\ b_h \end{pmatrix} \right) = \frac{1}{\delta t}\|u_h\|^2 + \nu\|\nabla u_h\|^2 + \frac{\alpha}{\delta t}\|b_h\|^2 + \alpha\eta\|\nabla b_h\|^2. \tag{5.10}$$

Hence the matrix representation $\mathcal{A}_h^{n+1}$ is positive definite (but not symmetric). Therefore, (5.9) can be solved efficiently by one's favorite method. For example, one can use the preconditioned BiCGSTAB [6], which does not require $(\mathcal{A}_h^{n+1})^{\mathrm{T}}$, with the following block diagonal operator as the preconditioner:

$$\mathcal{P}_h = \begin{pmatrix} \frac{1}{\delta t} I - \nu\Delta_h & 0 \\ 0 & \frac{\alpha}{\delta t} I - \alpha\eta\tilde{\Delta}_h, \end{pmatrix}, \tag{5.11}$$

where Δ_h and $\tilde{\Delta}_h$ are discrete Laplacians defined by

$$\begin{aligned} -\langle \Delta_h u_h, v_h \rangle &:= (\nabla u_h, \nabla v_h), \quad \forall\, u_h, v_h \in \boldsymbol{X}_h, \\ -\langle \tilde{\Delta}_h b_h, w_h \rangle &:= (\nabla b_h, \nabla w_h), \quad \forall\, u_h, v_h \in \boldsymbol{Y}_h. \end{aligned} \tag{5.12}$$

Hence, each BiCGSTAB iteration requires solving only two decoupled Poisson-type equations.

5.2 A Legendre-Galerkin method

As an example, we present and implement below a Legendre-Galerkin method in space. To this end, we set $\Omega = (-1,1)^d$. Using the convention of spectral methods, we shall use N instead of h to denote the spatial discretization parameter.

Let P_N be the space of polynomials of degree less than or equal to N in each direction. We define the approximation spaces for the velocity u, magnetic field b and pressure p by

$$\begin{aligned} &\boldsymbol{X}_N = \{u \in (P_N)^d : u|_{\partial\Omega} = 0\}, \\ &\boldsymbol{Y}_N = \{b \in (P_N)^d : n \cdot b|_{\partial\Omega} = 0 \text{ and } n \times (\nabla \times b)|_{\partial\Omega} = 0\}, \\ &M_N = \left\{p \in P_{N-2} : \int_\Omega p dx = 0\right\}, \end{aligned} \tag{5.13}$$

respectively, and set $\boldsymbol{Z}_N := \{v_N + \nabla q_N : v_N \in \boldsymbol{X}_N, q_N \in M_N\}$. Note that here we choose to enforce the boundary condition $n \times (\nabla \times b)|_{\partial\Omega} = 0$ explicitly in $\boldsymbol{Y}_N$. As we shall see below, this allows us to construct a more efficient algorithm.

The Legendre-Galerkin method for (2.1) is to find $\tilde{u}_N^{n+1} \in \boldsymbol{X}_N$ and $b_N^{n+1} \in \boldsymbol{Y}_N$ such that

$$\begin{aligned}
&\left(\frac{1}{\delta t}\tilde{u}_N^{n+1}, v_N\right) + ((u_N^n \cdot \nabla)\tilde{u}_N^{n+1}, v_N) - \nu(\Delta\tilde{u}_N^{n+1}, v_N) \\
&\quad - \alpha((\nabla \times b_N^{n+1}) \times b_N^n, v_N) = \left(\frac{1}{\delta t}u_N^n - \nabla p_N^n, v_N\right), \quad \forall\, v_N \in \boldsymbol{X}_N, \\
&\left(\frac{\alpha}{\delta t}b_N^{n+1}, w_N\right) - \alpha\eta(\Delta b_N^{n+1}, w_N) + \alpha(\nabla \times (b_N^n \times \tilde{u}_N^{n+1}), w_N) = \left(\frac{\alpha}{\delta t}b_N^n, w_N\right), \quad \forall\, w_N \in \boldsymbol{Y}_N.
\end{aligned} \tag{5.14}$$

The Legendre-Galerkin method for (2.2) is to find $\nabla p_N^{n+1} \in M_N$ such that

$$(\nabla p_N^{n+1}, \nabla q_N) = \left(\nabla p_N^n + \frac{1}{\delta t}\tilde{u}_N^{n+1}, \nabla q_N\right), \quad \forall\, q_N \in M_N, \tag{5.15}$$

and

$$u_N^{n+1} = \tilde{u}_N^{n+1} - \delta t \nabla(p_N^{n+1} - p_N^n) \in \boldsymbol{Z}_N. \tag{5.16}$$

As discussed above, we shall solve the coupled linear system (5.14) using the preconditioned BiCGSTAB method. With the block diagonal preconditioner, each BiCGSTAB iteration requires solving consecutively two equations of the following form: Find $u_N \in \boldsymbol{X}_N$ such that

$$\alpha_1(u_N, v_N) + (\nabla u_N, \nabla v_N) = (f_1, v_N), \quad \forall\, v_N \in \boldsymbol{X}_N, \tag{5.17}$$

and find $b_N \in \boldsymbol{Y}_N$ such that

$$\alpha_2(b_N, w_N) + (\nabla b_N, \nabla w_N) = (f_2, w_N), \quad \forall\, w_N \in \boldsymbol{Y}_N, \tag{5.18}$$

where α_1 and α_2 are two constants, and f_1 and f_2 are two given functions.

We shall use the efficient method presented in [14] for solving (5.17), and describe below an efficient method for solving (5.18). We consider first the two-dimensional case.

Let $b = (b_1, b_2)^{\mathrm{T}}$. Then, $n \cdot b|_{\partial\Omega} = 0$ can be written as

$$b_1(\pm 1, y) = 0 \quad \text{and} \quad b_2(x, \pm 1) = 0, \quad x, y \in (-1, 1). \tag{5.19}$$

The second boundary condition

$$n \times (\nabla \times b) = 0$$

can also be separated. Indeed, since

$$n \times (\nabla \times b) = n \times (0, \partial_x b_2 - \partial_y b_1), \tag{5.20}$$

at $x = \pm 1$, we have

$$\partial_x b_2(\pm 1, y) - \partial_y b_1(\pm 1, y) = 0.$$

Since $b_1(\pm 1, y) = 0$ from the first boundary condition, we have $\partial_x b_2(\pm 1, y) = 0$. Similarly, we have $b_y(x, \pm 1) = 0$. Hence, the boundary conditions for b_1 and b_2 are completely decoupled and are given by

$$b_1(\pm 1, y) = 0, \quad \partial_y b_1(x, \pm 1) = 0, \quad y \in (-1, 1) \tag{5.21}$$

and

$$b_2(x, \pm 1) = 0, \quad \partial_x b_1(\pm 1, y) = 0. \tag{5.22}$$

Hence we can solve for b_1 and b_2 separately by using the efficient Legendre-Galerkin method presented in [14].

In the three-dimensional case, the boundary conditions no longer decouple from each other. However, one can still develop an efficient Legendre-Galerkin solver for the coupled system as in [3].

6 Numerical results

It is clear that the first-order schemes lead to first-order convergence rate for all quantities. But the convergence rates of second-order schemes are more complicated [7]. Hence, we shall examine the rate of convergence for the second-order schemes (3.1)–(3.2) and (3.16)–(3.17) with the following fabricated exact solution (with corresponding forcing functions added in (1.1)):

$$u = (\sin(t)\sin(2\pi y)\sin^2(\pi x), -\sin(t)\sin(2\pi x)\sin^2(\pi y))^{\mathrm{T}},$$
$$b = (\sin(t)\sin(\pi x)\cos(\pi y), -\sin(t)\sin(\pi y)\cos(\pi x))^{\mathrm{T}},$$
$$p = \sin(t)\exp(x+y).$$

We take $\nu = \alpha = \eta = 1$, and unless specified otherwise, we use $N = 40$ in our spectral approximation which is more than enough to resolve the given solution to machine accuracy so the computed errors are due only to the time discretizations.

In Figure 1, we plot the errors vs. δt of the scheme (3.1)–(3.2). We observe that the convergence rate for the magnetic field b is of second-order in both L_2 and H_1. However, the convergence rate for the velocity u is of second-order in L_2 but of 3/2-order in H_1, and the convergence rate for the pressure p in L^2 is of first-order. These are consistent with the results for Navier-Stokes equations [7].

In Figure 2, we plot the errors vs. δt of the scheme (3.16)–(3.17). We observe that the convergence rates for all measured quantities are of second-order. Since the computational procedure and complexity of this scheme is essentially the same as that of (3.1)–(3.2), we have that, this scheme is recommended in practice.

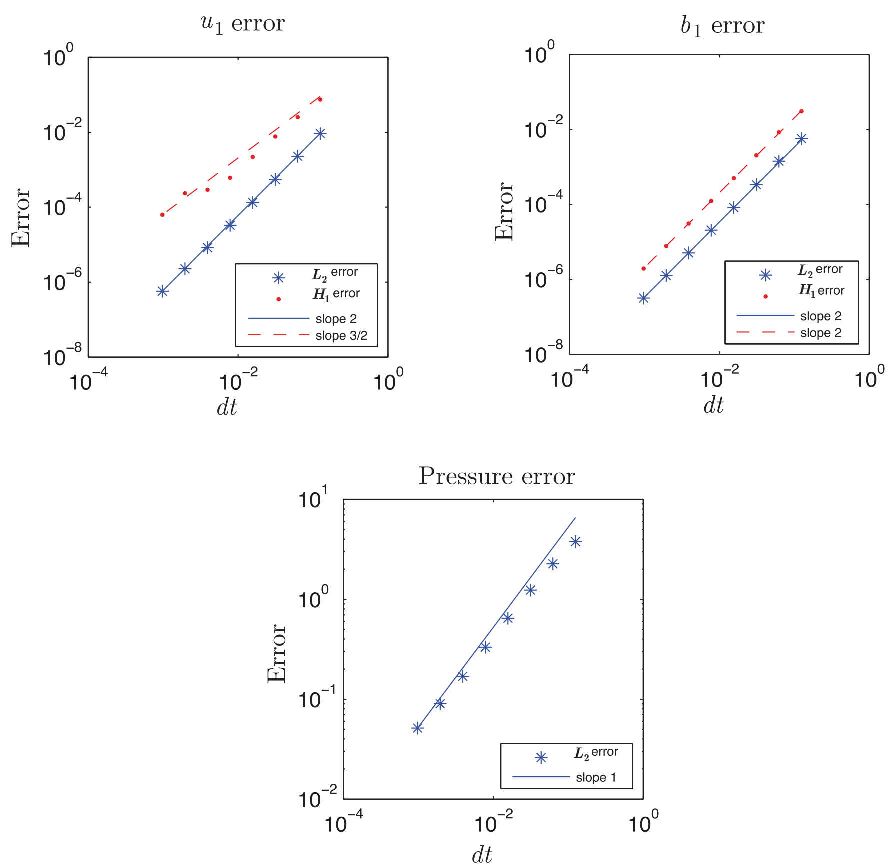

Figure 1 Errors of u_1, b_1, p at $T = 1$ with the scheme (3.1)–(3.2)

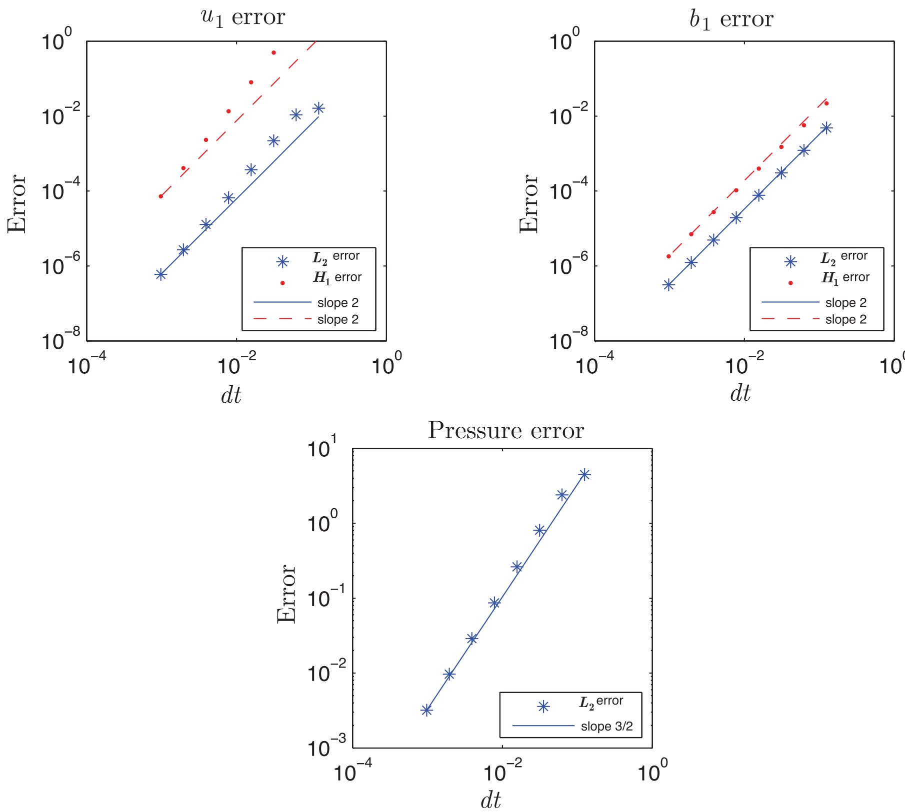

Figure 2 Errors of u_1, b_1, p at $T = 1$ with the scheme (3.16)–(3.17)

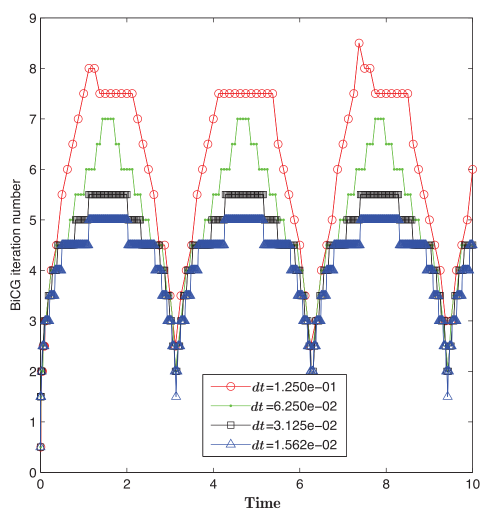

Figure 3 Evolution of BiCGSTAB iteration numbers

Next, we examine the convergence performance of the preconditioned BiCGSTAB for solving (5.9). In Figure 3, the iteration numbers of BiCGSTAB for various δt are plotted for a tolerance of

$$\tau = 10^{-10}.$$

We observe that convergences are achieved with a few iterations for all cases, and as δt decreases, the number of iterations required decreases.

7 Concluding remarks

In this paper, we developed several semi-discretized schemes for the MHD equations. These schemes are based on the standard and rotational pressure-correction schemes for the Navier-Stokes equations, and enjoy the following properties:

- There are unconditionally energy stable. Hence, a suitable time adaptive strategy can be used.
- They lead to, at each time step, a coupled linear system for the velocity and magnetic field, and a Poisson equation for the pressure. The coupled linear system is positive definite so it can be efficiently solved by one's favorite method, in particular, by a preconditioned BiCGSTAB method with the block diagonal preconditioner.
- The schemes do not enforce explicitly the divergence-free condition for the magnetic field, or in other words, do not involve a projection step as in [2]. However, it is shown that our schemes lead to divergence-free magnetic field if the initial magnetic field is divergence-free.

Hence, these schemes are very effective and easy to implement, and we believe that they can be very useful in numerical simulation of MHD flows.

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References

1 Badia S, Codina R, Planas R. On an unconditionally convergent stabilized finite element approximation of resistive magnetohydrodynamics. J Comput Phys, 2013, 234: 399–416
2 Badia S, Planas R, Gutiérrez-Santacreu J V. Unconditionally stable operator splitting algorithms for the incompressible magnetohydrodynamics system discretized by a stabilized finite element formulation based on projections. Internat J Numer Methods Engrg, 2013, 93: 302–328
3 Chen F, Shen J. Efficient spectral-Galerkin methods for systems of coupled second-order equations and their applications. J Comput Phys, 2012, 231: 5016–5028
4 Duvaut G, Lions J-L. Inéquations en thermoélasticité et magnétohydrodynamique. Arch Ration Mech Anal, 1972, 46: 241–279
5 Gerbeau J F. A stabilized finite element method for the incompressible magnetohydrodynamic equations. Numer Math, 2000, 87: 83–111
6 Golub G H, Van Loan C F. Matrix Computations. Maryland: JHU Press, 2012
7 Guermond J-L, Minev P, Shen J. An overview of projection methods for incompressible flows. Comput Methods Appl Mech Engrg, 2006, 195: 6011–6045
8 Guermond J-L, Quartapelle L. On the approximation of the unsteady Navier-Stokes equations by finite element projection methods. Numer Math, 1998, 80: 207–238
9 Guermond J-L, Shen J. On the error estimates of rotational pressure-correction projection methods. Math Comp, 2004, 73: 1719–1737
10 Gunzburger M D, Meir A J, Peterson J S. On the existence, uniqueness, and finite element approximation of solutions of the equations of stationary, incompressible magnetohydrodynamics. Math Comp, 1991, 56: 523–563
11 Liu J G, Pego R. Stable discretization of magnetohydrodynamics in bounded domains. Commun Math Sci, 2010, 8: 235–251
12 Schötzau D. Mixed finite element methods for stationary incompressible magneto-hydrodynamics. Numer Math, 2004, 96: 771–800
13 Sermange M, Temam R. Some mathematical questions related to the MHD equations. Comm Pure Appl Math, 1983, 36: 635–664
14 Shen J. Efficient spectral-Galerkin method I: Direct solvers for second- and fourth-order equations by using Legendre polynomials. SIAM J Sci Comput, 1994, 15: 1489–1505
15 Timmermans L J P, Minev P D, Van De Vosse F N. An approximate projection scheme for incompressible flow using spectral elements. Int J Numer Methods Fluids, 1996, 22: 673–688
16 van Kan J. A second-order accurate pressure-correction scheme for viscous incompressible flow. SIAM J Sci Stat Comput, 1986, 7: 870–891