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Exponential mean-square stability of the improved split-step theta methods for non-autonomous stochastic differential equations

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Abstract We consider the mean-square stability of the so-called improved split-step theta method for stochastic differential equations. First, we study the mean-square stability of the method for linear test equations with real parameters. When $\theta \geqslant 3/2$, the improved split-step theta methods can reproduce the mean-square stability of the linear test equations for any step sizes $h > 0$. Then, under a coupled condition on the drift and diffusion coefficients, we consider exponential mean-square stability of the method for nonlinear non-autonomous stochastic differential equations. Finally, the obtained results are supported by numerical experiments.

Keywords stochastic differential equations, mean-square stability, improved split-step theta methods, exponential mean-square stability

MSC(2010) 65C20, 65L20, 60H35

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1 Introduction

We consider the numerical solution of non-autonomous stochastic differential equations (SDEs) in the Itô's sense

$$\begin{cases} dx(t) = f(t, x(t))dt + g(t, x(t))dW(t), \quad t \geqslant 0, \\ x(0) = x_0, \end{cases} \tag{1.1}$$

where $x_0 \in \mathbb{R}^d$, $f : \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}^d$, $g : \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}^{d\times m}$ and $W(t)$ is m-dimensional Brownian motion defined on the complete probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$. $\{\mathcal{F}_t\}_{t\geqslant 0}$ is increasing and satisfies the usual condition. Many real-world phenomena can be modeled by these SDEs in many fields such as economic biology [19], medicine, finance [13] and engineering [20]. However, many SDEs arising in these applications cannot be solved analytically, hence one needs to develop effective numerical methods to solve them. In the recent twenty years, many numerical methods have been constructed (see [7, 12, 21]). Stability analysis of these numerical methods for SDEs has recently received a great deal of attention. For example, mean-square stability [2, 4, 10, 18] and asymptotic stability [3], and almost sure stability [3, 12] have been investigated. Especially, the exponential stability, which is an important topic in the stability analysis of SDEs, has been researched in [8, 10, 14]. It not only can guarantee that errors introduced in one time step will decay exponentially in future time steps, but also implies asymptotic stability. Furthermore, it was shown, by the Chebyshev inequality and the Borel-Cantelli lemma, that exponential mean-square stability implies almost sure stability. There are many results about Milstein type methods such as [9,11,15,16,22,23], but the results of exponential mean-square stability of the Milstein type methods are very few. In this paper, we study exponential mean-square stability of the improved split-step theta methods (ISST) mentioned by Yue [24],

$$\begin{cases} Y_n = y_n + \theta h f(t_n + \theta h, Y_n), \\ y_{n+1} = y_n + h f(t_n + \theta h, Y_n) + g(t_n + \theta h, Y_n)\Delta W_n + \displaystyle\sum_{j_1,j_2=1}^{m} L^{j_1} g_{j_2}(t_n + \theta h, Y_n) I_{j_1,j_2}^{t_n,t_{n+1}}, \end{cases} \tag{1.2}$$

where

$$L^{j_1} = \sum_{k=1}^{d} g_{k,j_1} \frac{\partial}{\partial x^k}, \quad g = (g_1, g_2, \ldots, g_m), \quad I_{j_1,j_2}^{t_n,t_{n+1}} = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_2} dW_{s_1}^{j_1} dW_{s_2}^{j_2}$$

and $j_1, j_2 = 1, 2, \ldots, m$. If the diffusion matrix g fulfills the so-called commutativity condition (see [12] for more details)

$$L^{j_1} g_{k,j_2} = L^{j_2} g_{k,j_1}, \quad j_1, j_2 = 1, \ldots, m, \quad k = 1, \ldots, d \tag{1.3}$$

by using

$$I_{j_1,j_2}^{t_n,t_{n+1}} + I_{j_2,j_1}^{t_n,t_{n+1}} = \Delta W_n^{j_1} \Delta W_n^{j_2}$$

for $j_1 \neq j_2$, we obtain the simple improved split-step theta methods

$$\begin{cases} Y_n = y_n + \theta h f(t_n + \theta h, Y_n), \\ y_{n+1} = y_n + h f(t_n + \theta h, Y_n) + g(t_n + \theta h, Y_n)\Delta W_n \\ \qquad + \dfrac{1}{2} \displaystyle\sum_{j_1,j_2=1}^{m} L^{j_1} g_{j_2}(t_n + \theta h, Y_n)(\Delta W_n^{j_1} \Delta W_n^{j_2} - \delta_{j_1,j_2} h), \end{cases} \tag{1.4}$$

where if $j_1 = j_2$ then $\delta_{j_1,j_2} = 1$, or else $\delta_{j_1,j_2} = 0$.

The rest of this paper is organized as follows. In Section 2, we consider the mean-square stability of the liner SDEs. In Section 3, we investigate the exponential mean-square stability of the nonlinear non-autonomous SDEs. In Section 4, we give some numerical results to support our theorems. In Section 5, we present the conclusion of this paper.

2 Linear mean-square stability

Throughout this paper, unless otherwise specified, we use the notation

$$\|x\| := (|x_1|^2 + \cdots + |x_d|^2), \quad \langle x, y\rangle := x_1 y_1 + \cdots + x_d y_d$$

and denote the expected x by $\mathbb{E}[x]$ for all $x, y \in \mathbb{R}^d$. In this section, we study the stability properties of the method (1.2) for the linear stochastic equations. First, we introduce the linear test SDEs

$$\begin{cases} dx(t) = \alpha x(t) dt + \mu x(t) dW(t), \quad t \geqslant 0, \\ x(0) = x_0, \end{cases} \tag{2.1}$$

where $\alpha, \mu \in \mathbb{R}$, and α and μ are constants. From [1,14], the zero solution $x(t)$ to (2.1) is said to be mean-square stable if

$$\lim_{t\to\infty} \mathbb{E}[\|x(t)\|^2] = 0.$$

It is known in [5,17] that mean-square stability for (2.1) is equivalent to

$$\alpha + \frac{1}{2}\mu^2 < 0. \tag{2.2}$$

Applying the method (1.2) to the SDEs (2.1), we have the following discrete schemes:

$$\begin{cases} Y_n = y_n + \theta h\alpha Y_n, \\ y_{n+1} = y_n + h\alpha Y_n + \mu Y_n \Delta W_n + \dfrac{1}{2}\mu^2 Y_n(\Delta W_n^2 - h). \end{cases} \tag{2.3}$$

For simplicity, (2.3) can also be written in the form

$$\begin{cases} Y_n = y_n + \theta h\alpha Y_n, \\ y_{n+1} = y_n + h\alpha Y_n + \mu\sqrt{h} Y_n \xi_n + \dfrac{1}{2}\mu^2 Y_n h(\xi_n^2 - 1), \end{cases} \tag{2.4}$$

where each ξ_n is an independent normal $(0,1)$ random variable. By gathering (2.4), we obtain that

$$y_{n+1} = (p + q\xi_n + r\xi_n^2)y_n, \tag{2.5}$$

where

$$p = \frac{(1 - \theta h\alpha + h\alpha - \frac{1}{2}\mu^2 h)}{(1 - \alpha\theta h)}, \quad q = \frac{\mu\sqrt{h}}{(1 - \alpha\theta h)}, \quad r = \frac{\frac{1}{2}\mu^2 h}{(1 - \alpha\theta h)}$$

and $1 - \alpha\theta h \neq 0$.

Now, with these ready, we state the first theorem.

Theorem 2.1. *Applying* (1.2) *to* (2.1) *and defining*

$$h^* := 2\left|\frac{\alpha + \frac{1}{2}\mu^2}{(1 - 2\theta)\alpha^2 + \frac{1}{2}\mu^4}\right|, \tag{2.6}$$

then we have the following statements.

(i) *Under the condition that* $0 \leqslant \theta \leqslant 1/2$, *if the test SDEs are not mean-square stable, then the ISST method* (1.2) *is not mean-square stable for any step sizes* $h > 0$; *if the test SDEs are mean-square stable, then the ISST method* (1.2) *is mean-square stable for all step sizes* $0 < h < h^*$.

(ii) *Under the condition that* $\theta > 1/2$ *and*

$$\alpha^2 < \frac{\mu^4}{2(2\theta - 1)},$$

if the test SDEs are not mean-square stable, then the ISST method (1.2) *is not mean-square stable for any step sizes* $h > 0$; *if the test SDEs are mean-square stable, then the ISST method* (1.2) *is mean-square stable for all step sizes* $0 < h < h^*$.

(iii) *Under the condition that* $\theta > 1/2$ *and*

$$\alpha^2 \geqslant \frac{\mu^4}{2(2\theta - 1)},$$

if the test SDEs are not mean-square stable, then the ISST method (1.2) *is not mean-square stable for all step sizes* $0 < h < h^*$; *if the test SDEs are mean-square stable, then the ISST method* (1.2) *is mean-square stable for all step sizes* $h > 0$.

Proof. (i) By (2.5), we have that

$$\mathbb{E}[|y_{n+1}|^2] = \mathbb{E}[(p + q\xi_n + r\xi_n^2)^2]\mathbb{E}[|y_n|^2].$$

From [12], we know that $\mathbb{E}[\xi_n] = 0$, $\mathbb{E}[\xi_n^2] = 1$, $\mathbb{E}[\xi_n^3] = 0$, and $\mathbb{E}[\xi_n^4] = 3$. Then we obtain that

$$\mathbb{E}[|y_{n+1}|^2] = ((p + r)^2 + q^2 + 2r^2)\mathbb{E}[|y_n|^2]. \tag{2.7}$$

Applying that the test SDEs are not mean-square stable, we obtain

$$\alpha + \frac{1}{2}\mu^2 \geqslant 0.$$

Using (2.7) leads us to

$$\begin{aligned}
&(p+r)^2+q^2+2r^2\\
&=\left(\frac{1-h\alpha\theta+h\alpha}{1-h\alpha\theta}\right)^2+\left(\frac{\mu h^{\frac{1}{2}}}{1-h\alpha\theta}\right)^2+2\left(\frac{\frac{1}{2}\mu^2 h}{1-h\alpha\theta}\right)^2\\
&=\frac{(1-h\alpha\theta)^2+2h(\alpha+\frac{\mu^2}{2})+h^2((1-2\theta)\alpha^2+\frac{\mu^4}{2})}{(1-h\alpha\theta)^2}\\
&\geqslant 1, \qquad (2.8)
\end{aligned}$$

i.e., the ISST method (1.2) is not mean-square stable. If the test SDEs are mean-square stable, namely, $\alpha+\frac{1}{2}\mu^2<0$, then we deduce that, under the condition that $0\leqslant\theta\leqslant 1/2$ and $h<h^*$, the ISST method (1.2) is mean-square stable if and only if

$$(p+r)^2+q^2+2r^2<1.$$

(ii) Under the condition that $\theta>1/2$ and $\alpha^2<\frac{\mu^4}{2(2\theta-1)}$, in a similar way as proving (i), we can obtain the corresponding conclusion.

(iii) Under the condition that $\theta>1/2$ and $\alpha^2\geqslant\frac{\mu^4}{2(2\theta-1)}$, if the test SDEs are not mean-square stable, by using (2.8), then we deduce that

$$(p+r)^2+q^2+2r^2\geqslant 1$$

for any step sizes $h<h^*$. Obviously, we obtain that the ISST method (1.2) is not mean-square stable. Similarly, under the condition that $\theta>1/2$ and $\alpha^2\geqslant\frac{\mu^4}{2(2\theta-1)}$, if the test SDEs are stable, by using (2.8), then we deduce that

$$(p+r)^2+q^2+2r^2<1$$

for any step sizes $h>0$, i.e., the ISST method (1.2) is mean-square stable. □

Remark 2.2. From Theorem 2.1 and the inequality (2.2), under the condition

$$\theta\geqslant\frac{\mu^4}{4\alpha^2}+\frac{1}{2},$$

if the test problem (2.1) is mean-square stable, then the method (2.2) is mean-square stable for any step sizes $h>0$. Furthermore, by the inequality (2.2), when $\theta\geqslant 3/2$, the improved split-step theta methods can reproduce the mean-square stability of the linear test equations for any step sizes $h>0$.

3 Exponential mean-square stability

In this section, we will consider exponential mean-square stability of the method (1.2) for the nonlinear stochastic differential equations. Next, we first state the following lemma.

Lemma 3.1. *If there exist a negative constant β and a positive constant $\hat{h}$ such that for all $(t,x)\in\mathbb{R}_+\times\mathbb{R}^d$,*

$$\langle x,f(t,x)\rangle+\frac{1}{2}\|g(t,x)\|^2+\frac{\hat{h}}{4}\sum_{j_1,j_2=1}^{m}\|L^{j_1}g_{j_2}(t,x)\|^2\leqslant\beta\|x\|^2, \qquad (3.1)$$

then the solution $x(t)$ to SDEs (1.1) *satisfies*

$$\mathbb{E}[\|x(t)\|^2]\leqslant\exp(2\beta t)\mathbb{E}[\|x_0\|^2]. \qquad (3.2)$$

Proof. From (3.1), we obtain that

$$\langle x,f(t,x)\rangle+\frac{1}{2}\|g(t,x)\|^2\leqslant\beta\|x\|^2.$$

From [14, Theorem 4.4.4], it is easy to obtain the conclusion. □

Remark 3.2. The coupled condition (3.1) can admit highly nonlinear diffusion coefficients such as

$$dx(t) = (-x(t) - x^3(t) - x^5(t))dt + x^2(t)dW(t).$$

By Lemma 3.1, (3.1) is a sufficient condition for exponential mean-square stability of the exact solution. Subsequently, we will prove that it is also a sufficient condition for exponential mean-square stability of the method (1.2) under certain conditions.

Theorem 3.3. *Assume that SDEs (1.1) satisfy (3.1). Then we have the following statements:*

(i) *If $\theta \geqslant 1/2$ and $\beta < 0$, then the ISST method (1.2) is mean-square contractive for all $0 < h \leqslant \hat{h}$, i.e.,*

$$\mathbb{E}[\|y_{n+1}\|^2] \leqslant \mathbb{E}[\|y_n\|^2].$$

(ii) *If $\theta > 1/2$ and $\beta < 0$, then the ISST method (1.2) is exponentially mean-square stable for all $0 < h \leqslant \hat{h}$, i.e.,*

$$\mathbb{E}[\|y_{n+1}\|^2] \leqslant \exp\left(\frac{2\beta h(2\theta - 1)}{2\theta - 1 - 2\beta h\theta^2}\right)\mathbb{E}[\|y_n\|^2].$$

(iii) *If $0 \leqslant \theta \leqslant 1/2$, $\beta < 0$ and there exists a constant γ such that*

$$\|f(t, x)\|^2 \leqslant \gamma\|x\|^2, \tag{3.3}$$

then there exists a constant h_0 such that the ISST method (1.2) is exponentially mean-square stable for $h \in (0, h_0)$, i.e.,

$$\mathbb{E}[\|y_{n+1}\|^2] \leqslant \exp\left(\frac{h((1-2\theta)h\gamma + 2\beta)}{(1 + \theta h\sqrt{\gamma})^2}\right)\mathbb{E}[\|y_n\|^2].$$

Proof. (i) From (1.2), we derive that

$$\langle y_n, f(t_n + \theta h, Y_n)\rangle = \langle Y_n, f(t_n + \theta h, Y_n)\rangle - \theta h\langle f(t_n + \theta h, Y_n), f(t_n + \theta h, Y_n)\rangle \tag{3.4}$$

and

$$\begin{aligned}\|y_{n+1}\|^2 = {} & \|y_n\|^2 + h^2\|f(t_n + \theta h, Y_n)\|^2 + \|g(t_n + \theta h, Y_n)\Delta W_n\|^2 \\ & + \left\|\sum_{j_1,j_2=1}^{m} L^{j_1} g_{j_2}(t_n + \theta h, Y_n) I_{j_1,j_2}^{t_n,t_{n+1}}\right\|^2 + 2\langle y_n, hf(t_n + \theta h, Y_n)\rangle \\ & + 2\langle y_n, g(t_n + \theta h, Y_n)\Delta W_n\rangle + 2\left\langle y_n, \sum_{j_1,j_2=1}^{m} L^{j_1} g_{j_2}(t_n + \theta h, Y_n) I_{j_1,j_2}^{t_n,t_{n+1}}\right\rangle \\ & + 2\langle hf(t_n + \theta h, Y_n), g(t_n + \theta h, Y_n)\Delta W_n\rangle \\ & + 2\left\langle hf(t_n + \theta h, Y_n), \sum_{j_1,j_2=1}^{m} L^{j_1} g_{j_2}(t_n + \theta h, Y_n) I_{j_1,j_2}^{t_n,t_{n+1}}\right\rangle \\ & + 2\left\langle g(t_n + \theta h, Y_n)\Delta W_n, \sum_{j_1,j_2=1}^{m} L^{j_1} g_{j_2}(t_n + \theta h, Y_n) I_{j_1,j_2}^{t_n,t_{n+1}}\right\rangle. \end{aligned} \tag{3.5}$$

Substituting (3.4) into (3.5), we obtain

$$\begin{aligned}\|y_{n+1}\|^2 \leqslant {} & \|y_n\|^2 + (1 - 2\theta)h^2\|f(t_n + \theta h, Y_n)\|^2 + \|g(t_n + \theta h, Y_n)\Delta W_n\|^2 \\ & + \left\|\sum_{j_1,j_2=1}^{m} L^{j_1} g_{j_2}(t_n + \theta h, Y_n) I_{j_1,j_2}^{t_n,t_{n+1}}\right\|^2 \\ & + 2h\langle Y_n, f(t_n + \theta h, Y_n)\rangle + 2\langle y_n, g(t_n + \theta h, Y_n)\Delta W_n\rangle \\ & + 2\left\langle y_n, \sum_{j_1,j_2=1}^{m} L^{j_1} g_{j_2}(t_n + \theta h, Y_n) I_{j_1,j_2}^{t_n,t_{n+1}}\right\rangle \end{aligned}$$

$$
\begin{aligned}
&+2\langle hf(t_n+\theta h,Y_n),g(t_n+\theta h,Y_n)\Delta W_n\rangle\\
&+2\left\langle hf(t_n+\theta h,Y_n),\sum_{j_1,j_2=1}^{m}L^{j_1}g_{j_2}(t_n+\theta h,Y_n)I_{j_1,j_2}^{t_n,t_{n+1}}\right\rangle\\
&+2\left\langle g(t_n+\theta h,Y_n)\Delta W_n,\sum_{j_1,j_2=1}^{m}L^{j_1}g_{j_2}(t_n+\theta h,Y_n)I_{j_1,j_2}^{t_n,t_{n+1}}\right\rangle. \qquad (3.6)
\end{aligned}
$$

From [12], we know that

$$
\begin{aligned}
&\mathbb{E}\left[\left\|\sum_{j_1,j_2=1}^{m}L^{j_1}g_{j_2}(t_n+\theta h,Y_n)I_{j_1,j_2}^{t_n,t_{n+1}}\right\|^2\right]=\frac{h^2}{2}\sum_{j_1,j_2=1}^{m}\mathbb{E}[\|L^{j_1}g_{j_2}(t_n+\theta h,Y_n)\|^2],\\
&\mathbb{E}[\Delta W_n]=0,\quad \mathbb{E}[\Delta W_n I_{j_1,j_2}^{t_n,t_{n+1}}]=0,\quad \mathbb{E}[I_{j_1,j_2}^{t_n,t_{n+1}}]=0.
\end{aligned}
$$

Taking expectations on both sides of (3.6) and using (3.1), we obtain

$$
\begin{aligned}
\mathbb{E}[\|y_{n+1}\|^2]&\leqslant \mathbb{E}[\|y_n\|^2]+h\mathbb{E}[\|g(t_n+\theta h,Y_n)\|^2]+\frac{h^2}{2}\sum_{j_1,j_2=1}^{m}\mathbb{E}[\|L^{j_1}g_{j_2}(t_n+\theta h,Y_n)\|^2]\\
&\quad+2h\mathbb{E}\langle Y_n,f(t_n+\theta h,Y_n)\rangle+(1-2\theta)h^2\mathbb{E}[\|f(t_n+\theta h,Y_n)\|^2]\\
&\leqslant \mathbb{E}[\|y_n\|^2]+2\beta h\mathbb{E}[\|Y_n\|^2]+(1-2\theta)h^2\mathbb{E}[\|f(t_n+\theta h,Y_n)\|^2]. \qquad (3.7)
\end{aligned}
$$

Furthermore, using $\theta\geqslant 1/2$ and $\beta<0$ leads us to

$$\mathbb{E}[\|y_{n+1}\|^2]\leqslant \mathbb{E}[\|y_n\|^2].$$

(ii) Solving the ISST method (1.2) yields

$$hf\big(t_n+\theta h,Y_n\big)=\frac{Y_n-y_n}{\theta}. \qquad (3.8)$$

Substituting (3.8) into (3.7), we have

$$
\begin{aligned}
\mathbb{E}[\|y_{n+1}\|^2]&\leqslant \frac{(1-\theta)^2}{\theta^2}\mathbb{E}[\|y_n\|^2]+\left(\frac{(1-2\theta)}{\theta^2}+2\beta h\right)\mathbb{E}[\|Y_n\|^2]\\
&\quad+\left(\frac{2\theta-1}{\theta^2}\right)\mathbb{E}[\langle 2Y_n,y_n\rangle]. \qquad (3.9)
\end{aligned}
$$

Applying the inequality

$$\langle 2Y_n,y_n\rangle\leqslant\left(\frac{2\theta-1-2\beta\theta^2h}{2\theta-1}\right)\|Y_n\|^2+\left(\frac{2\theta-1}{2\theta-1-2\beta\theta^2h}\right)\|y_n\|^2, \qquad (3.10)$$

$\theta>1/2$ and $\beta<0$, we obtain

$$
\begin{aligned}
\mathbb{E}[\|y_{n+1}\|^2]&\leqslant\left(1+\frac{2\beta h(2\theta-1)}{2\theta-1-2\beta\theta^2h}\right)\mathbb{E}[\|y_n\|^2]\\
&\leqslant \exp\left(\frac{2\beta h(2\theta-1)}{2\theta-1-2\beta h\theta^2}\right)\mathbb{E}[\|y_n\|^2]. \qquad (3.11)
\end{aligned}
$$

Hence, we obtain that the ISST method is exponentially mean-square stable for all $0<h\leqslant\hat{h}$.

(iii) Using (1.2) and (3.3), under $0\leqslant\theta\leqslant 1/2$, we have that

$$
\begin{aligned}
\mathbb{E}[\|y_{n+1}\|^2]&\leqslant \mathbb{E}[\|y_n\|^2]+2\beta h\mathbb{E}[\|Y_n\|^2]+(1-2\theta)h^2\mathbb{E}[\|f(t_n+\theta h,Y_n)\|^2]\\
&\leqslant \mathbb{E}[\|y_n\|^2]+h((1-2\theta)\gamma h+2\beta)\mathbb{E}[\|Y_n\|^2]. \qquad (3.12)
\end{aligned}
$$

Defining

$$h_0 = \begin{cases} \leqslant \hat{h}, & \theta = \dfrac{1}{2}, \\ \min\left\{\hat{h}, \dfrac{-2\beta}{(1-2\theta)\gamma}\right\}, & \theta \in \left[0, \dfrac{1}{2}\right), \end{cases}$$

and combining (1.2) and (3.3) gives

$$\begin{aligned} \|y_n\| &\leqslant \|Y_n\| + \theta h \|f(t_n + \theta h, Y_n)\| \\ &\leqslant (1 + \theta h \sqrt{\gamma}) \|Y_n\|, \end{aligned}$$

and

$$\|Y_n\|^2 \geqslant \left(\frac{1}{1 + \theta h \sqrt{\gamma}}\right)^2 \|y_n\|^2. \tag{3.13}$$

Next, using (3.13) and $h \in (0, h_0)$ leads us to

$$\begin{aligned} \mathbb{E}[\|y_{n+1}\|^2] &\leqslant \left(1 + \frac{h((1-2\theta)h\gamma + 2\beta)}{(1+\theta h\sqrt{\gamma})^2}\right)\mathbb{E}[\|y_n\|^2] \\ &\leqslant \exp\left(\frac{h((1-2\theta)h\gamma + 2\beta)}{(1+\theta h\sqrt{\gamma})^2}\right)\mathbb{E}[\|y_n\|^2], \end{aligned}$$

i.e., the method is exponentially mean-square stable. □

4 Numerical results

In this section, some numerical experiments are presented to support conclusions obtained in the previous sections. We simulate the numerical solution in the idea of [6], and the mean-square numerical solution is estimated by averaging 1,000 sample paths throughout this section. More precisely, it is obtained by

$$\frac{1}{1000} \sum_{i=1}^{1000} \|y_n^i\|^2,$$

where y_n^i denotes the i-th numerical solution at $t = nh$.

Example 4.1. We consider the linear test equation (see [15])

$$\begin{cases} dx(t) = -x(t)dt + x(t)dW(t), \\ x(0) = 1, \qquad t \geqslant 0, \end{cases} \tag{4.1}$$

with an exact solution

$$x(t) = \exp\left(-\frac{3}{2}t + W(t)\right).$$

At the same time, noting the fact that the coefficients of (4.1) satisfy (2.2), we get that (4.1) is mean-square stable. By Remark 2.2, when $\theta \geqslant 3/4$, the ISST method is mean-square stable for any step sizes $h > 0$. To show the influence of parameter θ and step size h on mean-square stability of the ISST method, we choose the fixed parameter $\theta = 0.75$ in Figure 1 and vary different step sizes $h = 1$, $h = 0.5$ and $h = 0.25$ on the interval $[0, 15]$. Meanwhile, we fix the step size $h = 0.1$ for different values of $\theta = 1$, $\theta = 0.85$ and $\theta = 0.75$. The mean-square of numerical solutions are plotted in Figure 1.

Example 4.2. We consider the following two-dimensional SDEs (see [3]):

$$\begin{cases} dx_1(t) = (-5x_1(t) - 2x_1^3(t))dt + (1.5x_1(t) + 0.5x_2(t))dW(t), \\ dx_2(t) = (2x_1(t) - 5x_2(t) - x_2^3(t))dt + (-0.5x_1(t) - 1.5x_2(t))dW(t), \end{cases} \tag{4.2}$$

with $x_1(0) = 3$, $x_2(0) = 4$ and $t \geqslant 0$.

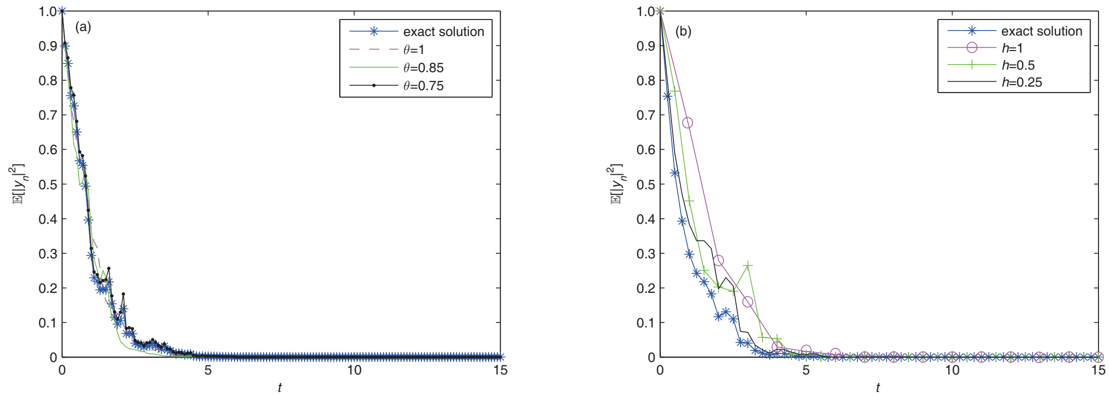

Figure 1 Mean square of numerical solution with different values of θ (a) and step sizes h (b) for Example 4.1

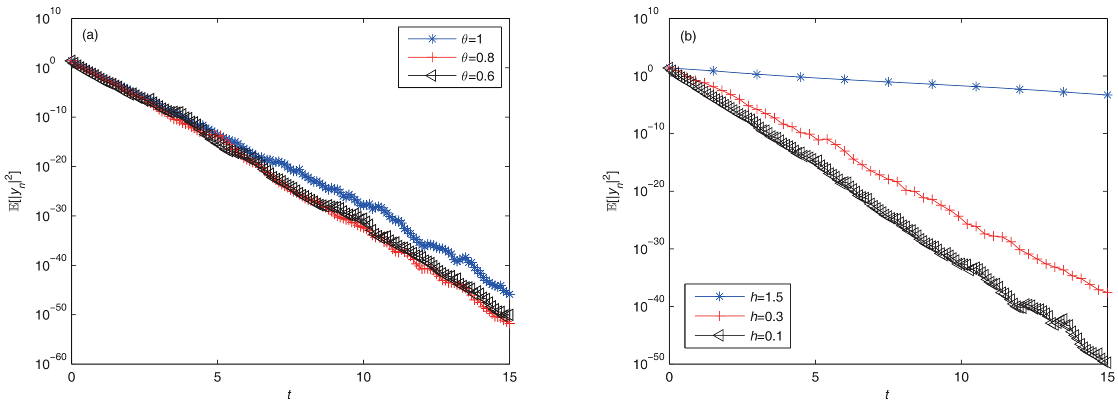

Figure 2 Mean square of numerical solution with different values of θ (a) and step sizes h (b) for Example 4.2

Let

$$x=(x_1(t),x_2(t))^{\mathrm{T}},\quad f(x)=(-5x_1(t)-2x_1^3(t),2x_1(t)-5x_2(t)-x_2^3(t))^{\mathrm{T}}$$

and

$$g(x)=(1.5x_1(t)+0.5x_2(t),-0.5x_1(t)-1.5x_2(t))^{\mathrm{T}}.$$

We may obtain that

$$\begin{aligned}\langle x,f(x)\rangle+\frac{1}{2}\|g(x)\|^2&=-\frac{15}{4}x_1^2-2x_1^4+\frac{7}{2}x_1x_2-\frac{15}{4}x_2^2-x_2^4\\&\leqslant -2x_1^2-2x_1^4-2x_2^2-x_2^4\\&\leqslant -2(x_1^2+x_2^2),\\\|L^1g_1(t,x)\|^2&=4(x_1^2+x_2^2),\end{aligned}$$

and the coefficients of Equation (4.2) satisfy Condition (3.1). From Inequality (3.1), it is also easy to compute the maximum $\hat{h}=2$ and

$$\beta=h-2,\quad h<\hat{h}=2.$$

By Lemma 3.1 and [14, Theorem 4.4.4], the exact solutions to (4.2) are exponential mean-square stable. Choosing the fixed step size $h=0.1$ with different values of $\theta=1$, $\theta=0.8$ and $\theta=0.6$, and the fixed value $\theta=0.6$ for different step sizes $h=1.5$, $h=0.3$, $h=0.1$ on the interval $[0,15]$, we apply the

ISST method (1.2) and generate 10^3 numerical sample paths. The mean-square of numerical solutions are shown in Figure 2.

The mean-square of numerical solution of Example 4.1 tends to zero as illustrated in Figure 1. Furthermore, we can see that, by Figure 2, the mean-square of numerical solutions of Example 4.2 exponentially tends to zero. By the two examples, the results obtained coincide with the theoretical results.

5 Conclusion

In this work, we carried out the mean-square stability analysis on the improved split-step theta method for SDEs under a local Lipschitz condition and a coupled condition on the drift and diffusion coefficients. Different from most of the existing exponential mean-square stability results for SDEs, our results can be applied to equations of which the diffusion coefficient is highly nonlinear. Both theoretical analysis and numerical tests show that the improved split-step theta method is efficient for the numerical solution of SDEs. In the future, we will further extend these results to SDEs driven by fractional Brownian motion.

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