SCIENCE CHINA
Mathematics

• ARTICLES •

October 2016 Vol. 59 No. 10: 1869–1908
doi: 10.1007/s11425-016-0015-x

Existence, uniqueness and stability of pyramidal traveling fronts in reaction-diffusion systems

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Received January 8, 2016; accepted March 27, 2016; published online August 2, 2016

Abstract In the one-dimensional space, traveling wave solutions of parabolic differential equations have been widely studied and well characterized. Recently, the mathematical study on higher-dimensional traveling fronts has attracted a lot of attention and many new types of nonplanar traveling waves have been observed for scalar reaction-diffusion equations with various nonlinearities. In this paper, by using the comparison argument and constructing appropriate super- and subsolutions, we study the existence, uniqueness and stability of three-dimensional traveling fronts of pyramidal shape for monotone bistable systems of reaction-diffusion equations in $\mathbb{R}^3$. The pyramidal traveling fronts are characterized as either a combination of planar traveling fronts on the lateral surfaces or a combination of two-dimensional V-form waves on the edges of the pyramid. In particular, our results are applicable to some important models in biology, such as Lotka-Volterra competition-diffusion systems with or without spatio-temporal delays, and reaction-diffusion systems of multiple obligate mutualists.

Keywords reaction-diffusion systems, bistability, pyramidal traveling fronts, existence, uniqueness, stability

MSC(2010) 35K57, 35B35, 35B40

Citation: Wang Z C, Li W T, Ruan S G. Existence, uniqueness and stability of pyramidal traveling fronts in reaction-diffusion systems. Sci China Math, 2016, 59: 1869–1908, doi: 10.1007/s11425-016-0015-x

1 Introduction

In the one-dimensional space, traveling wave solutions of parabolic differential equations have been widely studied and well characterized, see for example, Conley and Gardner [7], Fife and McLeod [13, 14], Gardner [16], Liang and Zhao [36], Mischaikow and Hutson [39], Tsai [51], and Volpert et al. [52]. In high-dimensional spaces, however, because propagating wave fronts may change shape and evolve to new nonplanar traveling waves, it is still interesting but extremely difficult and challenging to find and characterize possible nonplanar traveling waves. From the dynamical point of view, the characterization of nonplanar traveling waves is essential for a complete understanding of the structure of global attractors, which usually determine the long-time behavior of solutions of reaction-diffusion equations under consideration.

Recently, the mathematical study of higher-dimensional traveling fronts has attracted a lot of attention and many new types of nonplanar traveling waves have been observed for the following *scalar* reaction-diffusion equation with various nonlinearities:

$$\frac{\partial}{\partial t}u(\boldsymbol{x},t) = d\Delta u(\boldsymbol{x},t) + f(u(\boldsymbol{x},t)), \quad \boldsymbol{x} \in \mathbb{R}^m, \quad t > 0. \tag{1.1}$$

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For the combustion nonlinearity, Bonnet and Hamel [2], Hamel et al. [23] and Wang and Bu [55] have studied V-form curved fronts of (1.1) with $m = 2$. For the Fisher-KPP case, nonplanar traveling wave solutions of (1.1) with $m \geqslant 2$ have been studied by Brazhnik and Tyson [3], Hamel and Roquejoffre [27] and Huang [31]. For the unbalanced bistable case (specially for Allen-Cahn equation), V-form front solutions of (1.1) with $m = 2$ have been studied by Hamel et al. [24, 25], Ninomiya and Taniguchi [42, 43] and Gui [19], cylindrically symmetric traveling fronts of (1.1) with $m \geqslant 3$ have been studied by Hamel et al. [24, 25], and traveling fronts with pyramidal shapes of (1.1) with $m \geqslant 3$ have been studied by Taniguchi [47–50] and Kurokawa and Taniguchi [34]. Wang and Wu [57] and Sheng et al. [45] extended the arguments of Ninomiya and Taniguchi [42, 43] and Taniguchi [47, 48] and established two-dimensional V-shaped traveling fronts and pyramidal traveling wave fronts, respectively, for bistable reaction-diffusion equations with time-periodic nonlinearity (see [8]); namely, (1.1) with a nonlinearity $f(u,t)$ such that $f(\cdot,\cdot) = f(\cdot,\cdot+T)$ for some $T > 0$. In particular, Sheng et al. [46] have studied the multidimensional stability of V-form traveling fronts in the Allen-Cahn equation. Multidimensional stability of planar traveling waves in reaction-diffusion equations has been studied by Xin [58], Levermore and Xin [35], Kapitula [33], and Zeng [59, 60].

Note that the nonplanar traveling waves obtained in the above mentioned studies are connected and convex. It needs to be pointed out that the balanced bistable case (specially $f(u) = u(1-u^2)$), which is more interesting and complex, has been studied by Chen et al. [6] and del Pino et al. [9, 10]. Chen et al. [6] have studied the existence and qualitative properties of cylindrically symmetric traveling waves with paraboloid like interfaces of (1.1), which are also connected and convex. In [9], del Pino et al. have showed a new stationary wave when dimension $m \geqslant 9$, which is a counterexample to De Giorgi's conjecture. In [10], del Pino et al. have proved that there exist traveling wave solutions whose traveling fronts are non-connected, multi-component surfaces, and that there are solutions whose fronts are non-convex when $m \geqslant 3$. Other related studies can be found by Bu and Wang [4], Chapuisat [5], El Smaily et al. [11], Fife [12], Hamel [22], Hamel and Nadirashvili [26], Hamel and Roquejoffre [27], Morita and Ninomiya [40] and Wang [54].

In contrast to the scalar equations, the study on nonplanar traveling waves of *systems* of reaction-diffusion equations mainly focuses on two-dimensional V-form curved fronts. Haragus and Scheel [28–30] have studied almost planar waves (V-form waves) in reaction-diffusion systems by using bifurcation theory. Here "almost planar" means that the interface region is close to the hyperplanes (the angle of the interface is close to π). By developing the arguments of Ninomiya and Taniguchi [42, 43], Wang [53] has established the existence and stability of two-dimensional V-form curved fronts for the following systems with $m = 2$,

$$\frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{D}\Delta \boldsymbol{u} + \boldsymbol{F}(\boldsymbol{u}(\boldsymbol{x},t)), \quad \boldsymbol{x} \in \mathbb{R}^m, \quad t > 0, \quad \boldsymbol{u}(\boldsymbol{x},t) \in \mathbb{R}^N, \quad N > 1, \tag{1.2}$$

under the following hypotheses:

(H1) $\boldsymbol{D} = \mathrm{diag}(D_1, D_2, \ldots, D_N)$ is a diagonal matrix of order N with $D_i > 0$.

(H2) $\boldsymbol{F}$ has two stable equilibrium points $\boldsymbol{E}^- \ll \boldsymbol{E}^+$, i.e., $\boldsymbol{F}(\boldsymbol{E}^\pm) = \boldsymbol{0}$, where $\boldsymbol{0} = \{0, \ldots, 0\}$, and all eigenvalues of $\boldsymbol{F}'(\boldsymbol{E}^\pm)$ have negative real parts.

(H3) There exist two vectors $\boldsymbol{R}^\pm = (r_1^\pm, \ldots, r_N^\pm)$ with $r_i^\pm > 0$ $(i = 1, \ldots, N)$ and two positive numbers $\lambda^\pm$ such that $\boldsymbol{F}'(\boldsymbol{E}^+)\boldsymbol{R}^+ \leqslant -\lambda^+\boldsymbol{R}^+$ and $\boldsymbol{F}'(\boldsymbol{E}^-)\boldsymbol{R}^- \leqslant -\lambda^-\boldsymbol{R}^-$.

(H4) The reaction term $\boldsymbol{F}(\boldsymbol{u}) = (F^1(\boldsymbol{u}), \ldots, F^N(\boldsymbol{u}))$ is defined on an open domain $\Omega \subset \mathbb{R}^N$, is of class C^1 in $\boldsymbol{u}$, and satisfies the following conditions:

$$\frac{\partial F^i}{\partial u_j}(\boldsymbol{u}) \geqslant 0 \quad \text{for all } \ \boldsymbol{u} \in [\boldsymbol{E}^-, \boldsymbol{E}^+] \subset \Omega \ \text{ and for all } \ 1 \leqslant i \neq j \leqslant N.$$

Furthermore, there exist non-negative constants L_{ij}^- and L_{ij}^+ such that

$$\frac{\partial F^i}{\partial u_j}(\boldsymbol{u}) + L_{ij}^-\{u_i - E_i^-\}^- + L_{ij}^+\{E_i^+ - u_i\}^- \geqslant 0, \quad i \neq j,$$

for $\boldsymbol{u} \in [\widehat{\boldsymbol{E}}^-, \widehat{\boldsymbol{E}}^+] \subset \Omega$, where $\widehat{\boldsymbol{E}}^- \ll \boldsymbol{E}^- \ll \boldsymbol{E}^+ \ll \widehat{\boldsymbol{E}}^+$ and for any $a \in \mathbb{R}$,

$$\{a\}^- = \begin{cases} 0, & \text{if } \ a \geqslant 0, \\ -a, & \text{if } \ a < 0. \end{cases}$$

(H5) System (1.2) admits a planar traveling wave front

$$\boldsymbol{U}(\boldsymbol{e} \cdot \boldsymbol{x} + ct) = (U_1(\boldsymbol{e} \cdot \boldsymbol{x} + ct), \ldots, U_N(\boldsymbol{e} \cdot \boldsymbol{x} + ct))$$

satisfying the following ordinary differential equations:

$$\begin{cases} D_i U_i'' - cU_i' + F^i(\boldsymbol{U}) = 0, \\ \boldsymbol{U}(\pm\infty) := \lim\limits_{\xi \to \pm\infty} \boldsymbol{U}(\xi) = \boldsymbol{E}^{\pm}, \\ U_i' > 0 \quad \text{on} \quad \mathbb{R} \quad \text{for} \quad i = 1, \ldots, N, \end{cases}$$

where $\xi = \boldsymbol{e} \cdot \boldsymbol{x} + ct$ with $\boldsymbol{e} \in \mathbb{R}^m$ and $|\boldsymbol{e}| = 1$, $c > 0$ is the wave speed.

Here the real vector-valued function $\boldsymbol{u}(\boldsymbol{x}, t) = (u_1(\boldsymbol{x}, t), \ldots, u_N(\boldsymbol{x}, t))$ is unknown and $\boldsymbol{F}'(\boldsymbol{E})$ denotes the Jacobian matrix of $\boldsymbol{F}$ at $\boldsymbol{E} \in \mathbb{R}^N$. For two vectors $\boldsymbol{c} = (c_1, \ldots, c_N)$ and $\boldsymbol{d} = (d_1, \ldots, d_N)$, the symbol $\boldsymbol{c} \ll \boldsymbol{d}$ means $c_i < d_i$ for each $i \in \{1, \ldots, N\}$ and $\boldsymbol{c} \leqslant \boldsymbol{d}$ means $c_i \leqslant d_i$ for each $i \in \{1, \ldots, N\}$. The interval $[\boldsymbol{c}, \boldsymbol{d}]$ denotes the set of $\boldsymbol{q} \in \mathbb{R}^N$ with $\boldsymbol{c} \leqslant \boldsymbol{q} \leqslant \boldsymbol{d}$. For some comments on the assumptions (H1)–(H5) we refer to Wang [53]. In general, the assumptions (H1)–(H4) do not ensure that System (1.2) admits a traveling planar wave front connecting the equilibria $\boldsymbol{E}^-$ and $\boldsymbol{E}^+$. Therefore, the assumption on the existence of planar traveling wave solutions in (H5) is standard. A further assumption is that the wave speed $c > 0$. It should be pointed out that to determine the sign of the wave speed c for a given reaction-diffusion system is a very difficult job. Nevertheless, some sufficient conditions can be given for the positivity of the wave speed c for some particular cases (see Wang [53] and Alcahrani et al. [1] for some examples).

It follows from Volpert et al. [52, Chapter 3] that there exist positive constants C_1 and β_1 such that

$$|U_i(\pm\xi) - E_i^{\pm}| + |U_i'(\pm\xi)| + |U_i''(\pm\xi)| \leqslant C_1 \mathrm{e}^{-\beta_1 |\xi|} \quad \text{for} \quad \xi \geqslant 0 \quad \text{and} \quad i = 1, \ldots, N. \tag{1.3}$$

Contrasting to the results of Haragus and Scheel [28–30], which are valid only for sufficiently small $s-c > 0$ (namely, when the curved wave speed s is sufficiently close to the planar wave speed c), the results of Wang [53] hold for any $s > c > 0$. In particular, the results are applicable to some important biological models with $m = 2$ (see [53, Section 5] for details), such as Lotka-Volterra competition-diffusion systems with or without spatio-temporal delays, and reaction-diffusion systems of multiple obligate mutualists.

Recently, Ni and Taniguchi [41] have established the existence of pyramidal traveling wave solutions for competition-diffusion systems in $\mathbb{R}^m$ ($m \geqslant 3$), which covers the classical Lotka-Volterra competition-diffusion system with two components. Note that such pyramidal traveling wave solutions in $\mathbb{R}^3$ are indeed three-dimensional traveling wave solutions with pyramidal structures and are neither cylindrically symmetric nor reducible to two-dimensional traveling wave solutions. Also notice that traveling wave solutions with pyramidal shape for the Allen-Cahn equation (a single equation) are first constructed by Taniguchi [47] in $\boldsymbol{x} \in \mathbb{R}^3$. His method is to use the super- and subsolutions technique and the comparison principle, which is similar to that of Ninomiya and Taniguchi [42]. To construct a suitable supersolution, a key technique is to construct an appropriate mollified pyramid above a pyramid in $\mathbb{R}^3$. Kurokawa and Taniguchi [34] have extended the argument of Taniguchi [47] and established pyramidal traveling fronts for the Allen-Cahn equation in $\mathbb{R}^m$ ($m \geqslant 4$). Taniguchi [48] has studied the uniqueness and asymptotic stability of pyramidal traveling fronts established in Taniguchi [47]. For a given admissible pyramid it has been proved that a pyramidal traveling front is uniquely determined and that it is asymptotically stable under the condition that given perturbations decay at infinity. Furthermore, the pyramidal traveling front is characterized as a combination of planar traveling fronts on the lateral surfaces and as a combination of two-dimensional V-form traveling fronts on the edges, respectively. Recently, Sheng et al. [45] have developed the arguments of Taniguchi [47,48] and studied periodic pyramidal traveling fronts for bistable reaction-diffusion equations with time-periodic nonlinearity. More recently, Taniguchi [49,50] has constructed generalized pyramidal traveling fronts with convex polyhedral shapes.

Though the existence of pyramidal traveling fronts for competition-diffusion systems has been established by Ni and Taniguchi [41], the uniqueness and stability of pyramidal traveling fronts still remain open. The purpose of this paper is to extend the arguments of Taniguchi [47, 48] for a scalar equation to study the existence, uniqueness and stability of traveling waves of pyramidal shapes for the reaction-diffusion system (1.2) in $\mathbb{R}^3$ under the assumptions (H1)–(H5). The main method is also to use the super- and subsolutions technique and the comparison principle. We would like to point out that even though the main strategy of the current paper is similar to that in [47, 48], it needs new techniques and many modifications to obtain the expected results due to the presence of nonlinear coupling in the system which is a nontrivial work. First, because we are treating a coupled *system* of reaction-diffusion equations (not a single equation), we have to use the planar traveling wave fronts of the system to modify the super- and subsolutions of Taniguchi [47, 48] so that they can be applied to the system. To reach this aim, we define two monotone vector-valued functions $\boldsymbol{P}(\cdot)$ and $\boldsymbol{Q}(\cdot)$ and incorporate them into the resulting super- and subsolutions. Of course, the functions $\boldsymbol{P}(\cdot)$ and $\boldsymbol{Q}(\cdot)$ have been used by the first author in [53]. Second, as seen in the following, the super- and subsolutions constructed later cannot be bounded from above by $\boldsymbol{E}^+$ and from below by $\boldsymbol{E}^-$, which results in the comparison principle on $[\boldsymbol{E}^-,\boldsymbol{E}^+]$ (see the first part of the condition (H4)) being invalid for the supersolutions and subsolutions. This is very different from the case for a single equation. Therefore, we construct an auxiliary system (2.1) to help our analysis for the below (1.4), which is the traveling wave system corresponding to the original reaction-diffusion system (1.2). The auxiliary system (2.1) with nonlinearity $\boldsymbol{G}(\boldsymbol{u})$, which has been constructed by Wang [53], admits the comparison principle on an interval $[\widehat{\boldsymbol{E}}^-,\widehat{\boldsymbol{E}}^+]$ larger than $[\boldsymbol{E}^-,\boldsymbol{E}^+]$. In particular, $\boldsymbol{G}(\boldsymbol{u})\equiv\boldsymbol{F}(\boldsymbol{u})$ for $\boldsymbol{u}\in[\boldsymbol{E}^-,\boldsymbol{E}^+]$, and a solution of System (2.1) with nonlinearity $\boldsymbol{G}(\boldsymbol{u})$ bounded in $[\boldsymbol{E}^-,\boldsymbol{E}^+]$ is also a solution of System (1.2) with nonlinearity $\boldsymbol{F}(\boldsymbol{u})$. Third, we prove the asymptotic stability of the pyramidal traveling front established in Section 3 by considering two cases, $\boldsymbol{u}^0\geqslant\boldsymbol{v}^-$ and $\boldsymbol{u}^0\leqslant\boldsymbol{v}^-$, respectively. See below for the definitions of $\boldsymbol{u}^0$ and $\boldsymbol{v}^-$. Note that we prove for the later case by using an argument similar to that in [43,53], which is different from that in [48], where an estimate from below for the solutions of the initial value problem is required.

In the following we state our main result in this paper. Throughout this paper, we always assume that the assumptions (H1)–(H5) hold and let $m=3$ and $c>0$. For any $\boldsymbol{e}_1,\boldsymbol{e}_2,\ldots,\boldsymbol{e}_k\in\mathbb{R}^N$, define

$$\bigwedge_{j=1}^k\boldsymbol{e}_j=\Big(\min_{1\leqslant j\leqslant k}e_{j1},\ldots,\min_{1\leqslant j\leqslant k}e_{jN}\Big)\quad\text{and}\quad\bigvee_{j=1}^k\boldsymbol{e}_j=\Big(\max_{1\leqslant j\leqslant k}e_{j1},\ldots,\max_{1\leqslant j\leqslant k}e_{jN}\Big),$$

where $k\in\mathbb{N}$. For $\boldsymbol{c}=(c_1,\ldots,c_N)$, denote $|\boldsymbol{c}|=\sqrt{\sum_{i=1}^N c_i^2}$. For any bounded $\boldsymbol{u}\in C(\mathbb{R}^3,\mathbb{R}^N)$, define

$$\|\boldsymbol{u}\|_{C(\mathbb{R}^3)}=\sup_{\boldsymbol{x}\in\mathbb{R}^3}|\boldsymbol{u}(\boldsymbol{x})|.$$

Fix $s>c$. We assume that solutions travel towards the $-x_3$ direction without loss of generality. Take

$$\boldsymbol{u}(\boldsymbol{x},t)=\boldsymbol{v}(\boldsymbol{x}',x_3+st,t),\quad \boldsymbol{x}'=(x_1,x_2),\quad \boldsymbol{x}=(\boldsymbol{x}',x_3).$$

Then we have

$$\frac{\partial\boldsymbol{v}}{\partial t}=\boldsymbol{D}\Delta\boldsymbol{v}-s\frac{\partial\boldsymbol{v}}{\partial x_3}+\boldsymbol{F}(\boldsymbol{v}),\quad \boldsymbol{x}\in\mathbb{R}^3,\quad t>0,\tag{1.4}$$

$$\boldsymbol{v}(\boldsymbol{x},0)=\boldsymbol{v}^0(\boldsymbol{x}),\quad \boldsymbol{x}\in\mathbb{R}^3.\tag{1.5}$$

We seek for $\boldsymbol{V}(\boldsymbol{x})$ with

$$\mathcal{L}[\boldsymbol{V}]:=-\boldsymbol{D}\Delta\boldsymbol{V}+s\frac{\partial\boldsymbol{V}}{\partial x_3}-\boldsymbol{F}(\boldsymbol{V})=\boldsymbol{0},\quad \boldsymbol{x}\in\mathbb{R}^3.\tag{1.6}$$

Let $n \geqslant 3$ be a given integer and $m_* = \frac{\sqrt{s^2-c^2}}{c}$. Let $\{\boldsymbol{A}_j = (A_j, B_j)\}_{j=1}^n$ be a set of unit vectors in $\mathbb{R}^2$ such that

$$A_j B_{j+1} - A_{j+1} B_j > 0, \quad j = 1, 2, \ldots, n-1, \quad A_n B_1 - A_1 B_n > 0.$$

Now $(m_* \boldsymbol{A}_j, 1) \in \mathbb{R}^3$ is the normal vector of $\{\boldsymbol{x} \in \mathbb{R}^3 \mid -x_3 = m_*(\boldsymbol{A}_j, \boldsymbol{x}')\}$. Set

$$h_j(\boldsymbol{x}') = m_*(\boldsymbol{A}_j, \boldsymbol{x}') \quad \text{and} \quad h(\boldsymbol{x}') = \max_{1 \leqslant j \leqslant n} h_j(\boldsymbol{x}') = m_* \max_{1 \leqslant j \leqslant n} (\boldsymbol{A}_j, \boldsymbol{x}')$$

for $\boldsymbol{x}' \in \mathbb{R}^2$. We can obtain that $h(\boldsymbol{x}') \geqslant 0$ for $\boldsymbol{x}' \in \mathbb{R}^2$ and $\lim_{R \to \infty} \inf_{|\boldsymbol{x}'| \geqslant R} h(\boldsymbol{x}') = \infty$. We call $\{\boldsymbol{x} = (\boldsymbol{x}', x_3) \in \mathbb{R}^3 \mid -x_3 = h(\boldsymbol{x}')\}$ *a three-dimensional pyramid* in $\mathbb{R}^3$. Letting

$$\Omega_j = \{\boldsymbol{x}' \in \mathbb{R}^2 \mid h(\boldsymbol{x}') = h_j(\boldsymbol{x}')\}$$

for $j = 1, \ldots, n$, we have $\mathbb{R}^2 = \bigcup_{j=1}^n \Omega_j$. Denote the boundary of Ω_j by $\partial \Omega_j$. Let

$$E = \bigcup_{j=1}^n \partial \Omega_j.$$

Now we set

$$S_j = \{\boldsymbol{x} \in \mathbb{R}^3 | -x_3 = h_j(\boldsymbol{x}') \text{ for } \boldsymbol{x}' \in \Omega_j\}$$

for $j = 1, \ldots, n$, and call $\bigcup_{j=1}^n S_j \subset \mathbb{R}^3$ the *lateral surface of a pyramid*. Denote

$$\Gamma_j = S_j \cap S_{j+1}, \quad \Gamma_n = S_n \cap S_1, \quad j = 1, \ldots, n-1.$$

Then $\Gamma := \bigcup_{j=1}^n \Gamma_j$ represents the set of all edges of a pyramid. Define

$$\boldsymbol{v}^-(\boldsymbol{x}) = \boldsymbol{U}\left(\frac{c}{s}(x_3 + h(\boldsymbol{x}'))\right) = \max_{1 \leqslant j \leqslant n} \boldsymbol{U}\left(\frac{c}{s}(x_3 + h_j(\boldsymbol{x}'))\right)$$

and

$$D(\gamma) = \left\{\boldsymbol{x} \in \mathbb{R}^3 \mid \mathrm{dist}\left(\boldsymbol{x}, \bigcup_{j=1}^n \Gamma_j\right) > \gamma\right\}$$

for $\gamma > 0$. We note that the above setting on a pyramid comes from [47]. The following theorem is the main result of this paper.

Theorem 1.1. *Assume that* (H1)–(H5) *hold. Then for each* $s > c > 0$*, there exists a solution* $\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{V}(\boldsymbol{x}', x_3 + st)$ *of* (1.2) *satisfying* (1.6), $\boldsymbol{V}(\boldsymbol{x}) > \boldsymbol{v}^-(\boldsymbol{x})$ *and*

$$\lim_{\gamma \to \infty} \sup_{\boldsymbol{x} \in D(\gamma)} |\boldsymbol{V}(\boldsymbol{x}) - \boldsymbol{v}^-(\boldsymbol{x})| = 0. \tag{1.7}$$

Furthermore, for any $\boldsymbol{u}^0 \in C(\mathbb{R}^3, \mathbb{R}^N)$ *with* $\boldsymbol{u}^0(\boldsymbol{x}) \in [\boldsymbol{E}^-, \boldsymbol{E}^+]$ *for* $\boldsymbol{x} \in \mathbb{R}^3$ *and*

$$\lim_{\gamma \to \infty} \sup_{\boldsymbol{x} \in D(\gamma)} |\boldsymbol{u}^0(\boldsymbol{x}) - \boldsymbol{V}(\boldsymbol{x})| = 0, \tag{1.8}$$

the solution $\boldsymbol{u}(\boldsymbol{x}, t; \boldsymbol{u}^0)$ *of* (1.2) *with the initial value* $\boldsymbol{u}^0$ *satisfies*

$$\lim_{t \to \infty} \|\boldsymbol{u}(\cdot, \cdot, t; \boldsymbol{u}^0) - \boldsymbol{V}(\cdot, \cdot + st)\|_{C(\mathbb{R}^3)} = 0. \tag{1.9}$$

Following Theorem 1.1, we can see that the function $\boldsymbol{V}$ satisfying (1.6) and (1.7) is unique. Following (1.7), we know that the nonplanar traveling wave $\boldsymbol{V}$ has pyramidal structures and is characterized as a combination of planar traveling fronts on the lateral surface. In the following, we call $\boldsymbol{V}(\boldsymbol{x}', x_3 + st)$ a *pyramidal traveling front* of (1.2). In the end of Section 4 (see Corollary 4.18), we further characterize the pyramidal traveling fronts as a combination of two-dimensional V-form waves on the edges of the pyramid. Note that when $N = 1$, namely, when System (1.2) reduces to a scalar equation, the result of Theorem 1.1 has been obtained by Taniguchi [47, 48].

The rest of this paper is organized as follows. In Section 2, we give some preliminaries which are needed in the following sections. Theorem 1.1 will be proved in Sections 3 and 4. More specifically, we show the existence of a pyramidal traveling front $\boldsymbol{V}$ of (1.2) in Section 3 and prove the asymptotic stability of the front $\boldsymbol{V}$ in Section 4. In Section 5, we apply Theorem 1.1 to three important models in biology, namely, a two-species Lotka-Volterra reaction-diffusion competition system, a two-species competition system with spatio-temporal delays, and a reaction-diffusion systems of multiple obligate mutualists. Finally in Section 6, we present some discussions of this work.

2 Preliminaries

Associated with System (1.4)–(1.5), consider the following initial value problem:

$$\frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{D}\Delta \boldsymbol{u} - s\frac{\partial \boldsymbol{u}}{\partial x_3} + \boldsymbol{G}(\boldsymbol{u}(\boldsymbol{x}, t)), \quad \boldsymbol{x} \in \mathbb{R}^3, \quad t > 0, \tag{2.1}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}^0 \in C(\mathbb{R}^3, \mathbb{R}^N) \cap L^\infty(\mathbb{R}^3, \mathbb{R}^N), \tag{2.2}$$

where $\boldsymbol{G}(\boldsymbol{u}) = (G^1(\boldsymbol{u}), \dots, G^N(\boldsymbol{u}))$ with $G^i(\boldsymbol{u}) = F^i(\boldsymbol{u}) + H^i_-(\boldsymbol{u}) + H^i_+(\boldsymbol{u})$ and

$$H^i_-(\boldsymbol{u}) = \sum_{1 \leqslant j \leqslant N, j \neq i} L^-_{ij}\{u_i - E^-_i\}^-(u_j - E^-_j),$$

$$H^i_+(\boldsymbol{u}) = \sum_{1 \leqslant j \leqslant N, j \neq i} L^+_{ij}\{E^+_i - u_i\}^-(u_j - E^+_j)$$

for $i = 1, \dots, N$. It is obvious that $\boldsymbol{G}(\boldsymbol{u}) = \boldsymbol{F}(\boldsymbol{u})$ for $\boldsymbol{u} \in [\boldsymbol{E}^-, \boldsymbol{E}^+]$.

In this section, we establish a comparison theorem for the auxiliary system (2.1) and give the relationship between solutions of (1.4)–(1.5) and solutions of (2.1)–(2.2). Then we obtain a mollified pyramid which was constructed by Taniguchi [47].

Definition 2.1. A continuous vector-valued function $\boldsymbol{u}$ is called a *supersolution* (*subsolution*) of (2.1) on $\mathbb{R}^3 \times \mathbb{R}_+$ if $u_i(\cdot, t) \in C^2(\mathbb{R}^3)$ for $t \in (0, \infty)$, $u_i(\boldsymbol{x}, \cdot) \in C^1(0, +\infty)$ for $\boldsymbol{x} \in \mathbb{R}^3$, and $\boldsymbol{u}$ satisfies that

$$\mathcal{N}[\boldsymbol{u}] := \frac{\partial \boldsymbol{u}}{\partial t} - \boldsymbol{D}\Delta \boldsymbol{u} + s\frac{\partial \boldsymbol{u}}{\partial x_3} - \boldsymbol{G}(\boldsymbol{u}) \geqslant \boldsymbol{0} \quad (\leqslant \boldsymbol{0})$$

for all $\boldsymbol{x} \in \mathbb{R}^3$ and $t \in (0, \infty)$.

Following Wang [53], we have the following theorem and corollaries.

Theorem 2.2. *Assume that* (H1)–(H4) *hold. Suppose that* $\boldsymbol{u}^+$ *and* $\boldsymbol{u}^-$ *are supersolution and subsolution of* (2.1) *on* $\mathbb{R}^3 \times \mathbb{R}_+$, *respectively, and satisfy* $\boldsymbol{u}^\pm(\boldsymbol{x}, t) \in [\widehat{\boldsymbol{E}}^-, \widehat{\boldsymbol{E}}^+]$ *and* $\boldsymbol{u}^-(\boldsymbol{x}, 0) \leqslant \boldsymbol{u}^+(\boldsymbol{x}, 0)$ *for any* $\boldsymbol{x} \in \mathbb{R}^3$ *and* $t \geqslant 0$. *Then one has* $\boldsymbol{u}^-(\boldsymbol{x}, t) \leqslant \boldsymbol{u}^+(\boldsymbol{x}, t)$ *for any* $\boldsymbol{x} \in \mathbb{R}^3$ *and* $t > 0$.

Corollary 2.3. *Assume that* (H1)–(H4) *hold. Suppose that* $\boldsymbol{u}^+$ *and* $\boldsymbol{u}^-$ *are supersolution and subsolution of* (2.1) *on* $\mathbb{R}^3 \times \mathbb{R}_+$, *respectively, and satisfy* $\boldsymbol{u}^+(\boldsymbol{x}, t) \in [\boldsymbol{E}^-, \widehat{\boldsymbol{E}}^+]$, $\boldsymbol{u}^-(\boldsymbol{x}, t) \in [\widehat{\boldsymbol{E}}^-, \boldsymbol{E}^+]$ *and* $\boldsymbol{u}^-(\boldsymbol{x}, 0) \leqslant \boldsymbol{u}^+(\boldsymbol{x}, 0)$ *for any* $\boldsymbol{x} \in \mathbb{R}^3$ *and* $t \geqslant 0$. *Then for any* $\boldsymbol{v}^0 \in X$ *with* $\boldsymbol{v}^0(\boldsymbol{x}) \in [\boldsymbol{E}^-, \boldsymbol{E}^+]$ *and* $\boldsymbol{u}^-(\boldsymbol{x}, 0) \leqslant \boldsymbol{v}^0(\boldsymbol{x}) \leqslant \boldsymbol{u}^+(\boldsymbol{x}, 0)$ *for any* $\boldsymbol{x} \in \mathbb{R}^3$, *the solution* $\boldsymbol{v}(\boldsymbol{x}, t; \boldsymbol{v}^0)$ *of* (1.4)–(1.5) *satisfies* $\boldsymbol{u}^-(\boldsymbol{x}, t) \leqslant \boldsymbol{v}(\boldsymbol{x}, t; \boldsymbol{v}^0) \leqslant \boldsymbol{u}^+(\boldsymbol{x}, t)$ *and* $\boldsymbol{E}^- \leqslant \boldsymbol{v}(\boldsymbol{x}, t; \boldsymbol{v}^0) \leqslant \boldsymbol{E}^+$ *for any* $\boldsymbol{x} \in \mathbb{R}^3$ *and* $t > 0$.

Corollary 2.4. *Assume that* (H1)–(H4) *hold. If* $\boldsymbol{v}_1$ *and* $\boldsymbol{v}_2$ *are a pair of supersolution and subsolution of* (1.4) *on* $\mathbb{R}^3 \times \mathbb{R}_+$ *with* $\boldsymbol{E}^+ \geqslant \boldsymbol{v}_1(\cdot, 0) \geqslant \boldsymbol{v}_2(\cdot, 0) \geqslant \boldsymbol{E}^-$ *on* $\mathbb{R}^3$, *then* $\boldsymbol{v}_1(\boldsymbol{x}, t) \geqslant \boldsymbol{v}_2(\boldsymbol{x}, t)$ *on* $\mathbb{R}^3 \times \mathbb{R}^+$.

The following lemma can be proved as in [53, Theorem 2.2] via using the results of Martin and Smith [38].

Lemma 2.5. *Assume that* $\boldsymbol{u}^\pm \in C(\mathbb{R}^3 \times [0, \infty), \mathbb{R}^N)$ *solve the following linear system*:

$$\frac{\partial}{\partial t}\boldsymbol{u}^\pm = \boldsymbol{D}\Delta \boldsymbol{u}^\pm - s\frac{\partial}{\partial x_3}\boldsymbol{u}^\pm + \boldsymbol{H}^\pm(\boldsymbol{x}, t)\boldsymbol{u}^\pm(\boldsymbol{x}, t),$$

$$\boldsymbol{u}^{\pm}(0)=\boldsymbol{u}^{\pm,0}\in C(\mathbb{R}^3,\mathbb{R}^N)\cap L^{\infty}(\mathbb{R}^3,\mathbb{R}^N),$$

where $\boldsymbol{H}^{\pm}(\boldsymbol{x},t)=(h_{ij}^{\pm}(\boldsymbol{x},t))_{N\times N}$, *in which* $h_{ij}^{\pm}(\boldsymbol{x},t)\in C(\mathbb{R}^3\times\mathbb{R}_+,\mathbb{R})\cap L^{\infty}(\mathbb{R}^3\times\mathbb{R}_+,\mathbb{R})$ *are matrix-valued functions and satisfy* $h_{ij}^{\pm}(\boldsymbol{x},t)\geqslant 0$ *on* $\mathbb{R}^3\times\mathbb{R}_+$ *for* $i\neq j$. *If* $\boldsymbol{H}^{+}(\boldsymbol{x},t)\geqslant\boldsymbol{H}^{-}(\boldsymbol{x},t)$ *and* $\boldsymbol{u}^{+,0}(\boldsymbol{x})\geqslant\boldsymbol{u}^{-,0}(\boldsymbol{x})\geqslant\boldsymbol{0}$ *for any* $\boldsymbol{x}\in\mathbb{R}^3$ *and* $t\geqslant 0$, *then* $\boldsymbol{u}^{+}(\boldsymbol{x},t)\geqslant\boldsymbol{u}^{-}(\boldsymbol{x},t)$ *for* $\boldsymbol{x}\in\mathbb{R}^3$ *and* $t\geqslant 0$.

Let $\widetilde{\rho}(r)\in C^{\infty}[0,\infty)$ be a function with the following properties:

$$\begin{aligned}&\widetilde{\rho}(r)>0,\quad \widetilde{\rho}_r(r)\leqslant 0\ \text{ for } r\geqslant 0,\\ &\widetilde{\rho}(r)=1,\quad \text{if } r>0 \text{ is small enough},\\ &\widetilde{\rho}(r)=\mathrm{e}^{-r},\quad \text{if } r>0 \text{ is large enough, say } r>R_0,\\ &\int_{\mathbb{R}^2}\widetilde{\rho}(|\boldsymbol{x}'|)d\boldsymbol{x}'=1.\end{aligned}$$

Assume $R_0>1$ without loss of generality. We have $\int_{\mathbb{R}^2}\widetilde{\rho}(|\boldsymbol{x}'|)d\boldsymbol{x}'=2\pi\int_0^{\infty}r\widetilde{\rho}(r)dr$.

Put $\rho(\boldsymbol{x}')=\widetilde{\rho}(|\boldsymbol{x}'|)$. Then $\rho:\mathbb{R}^2\to\mathbb{R}$ belongs to $C^{\infty}(\mathbb{R}^2)$ and satisfies $\int_{\mathbb{R}^2}\rho(\boldsymbol{x}')d\boldsymbol{x}'=1$ and $(\rho*h_j)(\boldsymbol{x}')=h_j(\boldsymbol{x}')$ for $\boldsymbol{x}'\in\mathbb{R}^2$ and $j=1,\ldots,n$. Here the convolution $\rho*h_j$ of ρ and h_j is defined by

$$(\rho*h_j)(\boldsymbol{x}')=\int_{\mathbb{R}^2}\rho(\boldsymbol{y}')h_j(\boldsymbol{x}'-\boldsymbol{y}')d\boldsymbol{y}'.$$

For all non-negative integers j_1 and j_2 with $0\leqslant j_1+j_2\leqslant 3$, we have

$$|D_{x_1}^{j_1}D_{x_2}^{j_2}\rho(\boldsymbol{x}')|\leqslant M_1\rho(\boldsymbol{x}')\quad\text{ for all }\quad \boldsymbol{x}'\in\mathbb{R}^2,$$

where $D_{x_i}^{j_i}=\frac{\partial^{j_i}}{\partial x_i^{j_i}}$, $M_1>0$ is a constant.

Define $\varphi=\rho*h$, namely,

$$\varphi(\boldsymbol{x}')=\int_{\mathbb{R}^2}\rho(\boldsymbol{x}'-\boldsymbol{y}')h(\boldsymbol{y}')d\boldsymbol{y}'=\int_{\mathbb{R}^2}\rho(\boldsymbol{y}')h(\boldsymbol{x}'-\boldsymbol{y}')d\boldsymbol{y}' \tag{2.3}$$

for $\boldsymbol{x}'\in\mathbb{R}^2$. We call $-x_3=\varphi(\boldsymbol{x}')$ a *mollified pyramid* for a pyramid $-x_3=h(\boldsymbol{x}')$. Set

$$S(\boldsymbol{x}')=\frac{s}{\sqrt{1+|\nabla\varphi(\boldsymbol{x}')|^2}}-c, \tag{2.4}$$

where $\nabla\varphi(\boldsymbol{x}')=(\frac{\partial\varphi}{\partial x_1},\frac{\partial\varphi}{\partial x_2})$. The following lemmas come from [47, 48].

Lemma 2.6. *Let* φ *and* S *be given by* (2.3) *and* (2.4), *respectively. For any pair of fixed integers* $j_1\geqslant 0$ *and* $j_2\geqslant 0$, *one has* $\sup_{\boldsymbol{x}'\in\mathbb{R}^2}|D_{x_1}^{j_1}D_{x_2}^{j_2}\varphi(\boldsymbol{x}')|<\infty$. *In addition, one has*

$$\begin{aligned}&h(\boldsymbol{x}')<\varphi(\boldsymbol{x}')\leqslant h(\boldsymbol{x}')+2\pi m_*\int_0^{\infty}r^2\tilde{\rho}(r)dr,\\ &|\nabla\varphi(\boldsymbol{x}')|<m_*,\quad 0<S(\boldsymbol{x}')\leqslant s-c,\quad \forall\,\boldsymbol{x}'\in\mathbb{R}^2\end{aligned}$$

and

$$\begin{aligned}&\lim_{\lambda\to\infty}\sup\{S(\boldsymbol{x}')\mid \boldsymbol{x}'\in\mathbb{R}^2,\mathrm{dist}(\boldsymbol{x}',E)\geqslant\lambda\}=0,\\ &\lim_{\lambda\to\infty}\sup\{\varphi(\boldsymbol{x}')-h(\boldsymbol{x}')\mid \boldsymbol{x}'\in\mathbb{R}^2,\mathrm{dist}(\boldsymbol{x}',E)\geqslant\lambda\}=0.\end{aligned}$$

Lemma 2.7. *There exist positive constants* ν_1 *and* ν_2 *such that*

$$0<\nu_1=\inf_{\boldsymbol{x}'\in\mathbb{R}^2}\frac{\varphi(\boldsymbol{x}')-h(\boldsymbol{x}')}{S(\boldsymbol{x}')}\leqslant\sup_{\boldsymbol{x}'\in\mathbb{R}^2}\frac{\varphi(\boldsymbol{x}')-h(\boldsymbol{x}')}{S(\boldsymbol{x}')}=\nu_2<\infty.$$

In addition, for every pair of integers $j_1\geqslant 0$ *and* $j_2\geqslant 0$ *with* $2\leqslant j_1+j_2\leqslant 3$, *one has*

$$\sup_{\boldsymbol{x}'\in\mathbb{R}^2}\frac{D_{x_1}^{j_1}D_{x_2}^{j_2}\varphi(\boldsymbol{x}')}{S(\boldsymbol{x}')}<\infty.$$

3 Existence of pyramidal traveling fronts

In this section, we establish the existence of pyramidal traveling fronts for System (1.2) in $\mathbb{R}^3$. The main method is to construct a suitable supersolution $\boldsymbol{v}^+$ of (2.1) with $\boldsymbol{v}^+ > \boldsymbol{v}^-$ and then take a limit for the solution $\boldsymbol{v}(\boldsymbol{x}, t; \boldsymbol{v}^-)$ of (1.4)–(1.5) with $\boldsymbol{v}^0 = \boldsymbol{v}^-$ as $t \to +\infty$. The limit function is just the desired front $\boldsymbol{V}$. By Corollary 2.3, we have $\boldsymbol{v}^-(\boldsymbol{x}) \leqslant \boldsymbol{V}(\boldsymbol{x}) \leqslant \boldsymbol{v}^+(\mathrm{x})$ on $\mathbb{R}^3$. The construction of the supersolution $\boldsymbol{v}^+$ is a combination of the arguments in [47, 53]. In addition, we construct a subsolution $\widehat{\boldsymbol{v}}(\boldsymbol{x})$ of (2.1), which will be used to establish the stability of the pyramidal traveling front $\boldsymbol{V}$ in the next section.

For $\alpha \in (0,1)$, set $\frac{1}{\alpha}h(\alpha \boldsymbol{x}') = h(\boldsymbol{x}')$. Define $z_3 = \alpha x_3$, $\boldsymbol{z}' = \alpha \boldsymbol{x}'$, $\boldsymbol{z} = \alpha \boldsymbol{x}$, and

$$\mu(\boldsymbol{x}) := \frac{x_3 + \frac{1}{\alpha}\varphi(\alpha \boldsymbol{x}')}{\sqrt{1 + |\nabla\varphi(\alpha \boldsymbol{x}')|^2}} = \frac{1}{\alpha}\frac{z_3 + \varphi(\boldsymbol{z}')}{\sqrt{1 + |\nabla\varphi(\boldsymbol{z}')|^2}}. \tag{3.1}$$

Then we have $\mu_{x_3} = \frac{1}{\sqrt{1+|\nabla\varphi(\boldsymbol{z}')|^2}}$, $\mu_{x_3x_3} = 0$, and

$$\mu_{x_i} := (\sqrt{1 + |\nabla\varphi(\boldsymbol{z}')|^2})^{-1}\varphi_{z_i} - \alpha\mu X_i(\boldsymbol{z}'), \quad \mu_{x_ix_i} = \alpha Y_i(\boldsymbol{z}') - \alpha^2\mu Z_i(\boldsymbol{z}'),$$

where

$$\begin{aligned}
X_i(\boldsymbol{z}') &= \sqrt{1 + |\nabla\varphi(\boldsymbol{z}')|^2}\frac{\partial}{\partial z_i}(\sqrt{1 + |\nabla\varphi(\boldsymbol{z}')|^2})^{-1},\\
Y_i(\boldsymbol{z}') &= \frac{\partial}{\partial z_i}((\sqrt{1 + |\nabla\varphi(\boldsymbol{z}')|^2})^{-1}\varphi_{z_i}) - \frac{X_i(\boldsymbol{z}')}{\sqrt{1 + |\nabla\varphi(\boldsymbol{z}')|^2}}\varphi_{z_i},\\
Z_i(\boldsymbol{z}') &= \frac{\partial X_i}{\partial z_i} - X_i^2(\boldsymbol{z}'),
\end{aligned}$$

and $i = 1, 2$. Set

$$\sigma(\boldsymbol{x}') = \varepsilon S(\alpha \boldsymbol{x}'),$$

where ε and α are positive constants, which will be determined later. Then we have

$$\sigma_{x_j}(\boldsymbol{x}') = \alpha\varepsilon S_{z_j}(\boldsymbol{z}') \quad \text{and} \quad \sigma_{x_jx_j}(\boldsymbol{x}') = \alpha^2\varepsilon S_{z_jz_j}(\boldsymbol{z}'), \quad j = 1, 2.$$

Take $\eta^\pm > 0$ small enough so that $\eta^-\boldsymbol{R}^- \ll \boldsymbol{R}^+$ and $\eta^+\boldsymbol{R}^+ \ll \boldsymbol{R}^-$. Let $\boldsymbol{P}^- := \eta^-\boldsymbol{R}^-$, $\boldsymbol{P}^+ := \boldsymbol{R}^+$, $\boldsymbol{Q}^- := \boldsymbol{R}^-$ and $\mathbf{Q}^+ := \eta^+\boldsymbol{R}^+$. The assumptions (H2) and (H3) imply that there exist constant matrixes $\boldsymbol{A}^\pm = (\mu_{ij}^\pm)$ such that $\frac{\partial F^i}{\partial u_j}(\boldsymbol{E}^\pm) < \mu_{ij}^\pm$ for all $i, j = 1, \dots, N$, $\boldsymbol{A}^+\boldsymbol{P}^+ \ll -\frac{1}{2}\lambda^+\boldsymbol{P}^+$, $\boldsymbol{A}^+\boldsymbol{Q}^+ \ll -\frac{1}{2}\lambda^+\boldsymbol{Q}^+$, $\boldsymbol{A}^-\boldsymbol{P}^- \ll -\frac{1}{2}\lambda^-\boldsymbol{P}^-$, $\boldsymbol{A}^-\boldsymbol{Q}^- \ll -\frac{1}{2}\lambda^-\boldsymbol{Q}^-$. Define

$$\omega(\zeta) := \frac{1}{2}\left(1 + \tanh\frac{\zeta}{2}\right), \quad \zeta \in \mathbb{R}.$$

Let $\boldsymbol{P}^\pm = (p_1^\pm, \dots, p_N^\pm)$ and $\boldsymbol{Q}^\pm = (q_1^\pm, \dots, q_N^\pm)$. Define positive vector-valued functions

$$\boldsymbol{P}(\zeta) := (P_1(\zeta), \dots, P_N(\zeta)) \quad \text{and} \quad \boldsymbol{Q}(\zeta) := (Q_1(\zeta), \dots, Q_N(\zeta))$$

by $P_i(\zeta) = \omega(\zeta)p_i^+ + (1 - \omega(\zeta))p_i^-$ and $Q_i(\zeta) = \omega(\zeta)q_i^+ + (1 - \omega(\zeta))q_i^-$, where $i = 1, \dots, N$. It is easy to see that $\mathbf{P}(\zeta)$ and $\boldsymbol{Q}(\zeta)$ satisfy the following:

$$\begin{aligned}
&p_i^- \leqslant P_i(\cdot) \leqslant p_i^+ \quad \text{and} \quad P_i'(\cdot) > 0 \quad \text{on } \mathbb{R}, \quad p^0 := \max_{1\leqslant i\leqslant N} p_i^+ > 0, \quad p_0 := \min_{1\leqslant i\leqslant N} p_i^- > 0,\\
&|\boldsymbol{P}(\pm\zeta) - \boldsymbol{P}^\pm| + |\boldsymbol{P}'(\pm\zeta)| + |\boldsymbol{P}''(\pm\zeta)| \leqslant K_1\mathrm{e}^{-\zeta} \quad \text{for} \quad \zeta \geqslant 0 \quad \text{and} \quad \text{some} \quad K_1 > 0,
\end{aligned}$$

and

$$\begin{aligned}
&q_i^+ \leqslant Q_i(\cdot) \leqslant q_i^- \quad \text{and} \quad Q_i'(\cdot) < 0 \quad \text{on } \mathbb{R}, \quad q^0 := \max_{1\leqslant i\leqslant N} q_i^- > 0, \quad q_0 := \min_{1\leqslant i\leqslant N} q_i^+ > 0,\\
&|\boldsymbol{Q}(\pm\zeta) - \boldsymbol{Q}^\pm| + |\boldsymbol{Q}'(\pm\zeta)| + |\boldsymbol{Q}''(\pm\zeta)| \leqslant K_2\mathrm{e}^{-\zeta} \quad \text{for} \quad \zeta \geqslant 0 \quad \text{and} \quad \text{some} \quad K_2 > 0.
\end{aligned}$$

Recall that $\boldsymbol{v}^-(\boldsymbol{x})$ is a subsolution of (1.4). In particular, $\frac{\partial}{\partial x_3}v_i^-(\boldsymbol{x}) > 0$ for any $\boldsymbol{x} \in \mathbb{R}^3$, $i = 1, \dots, N$.

Lemma 3.1. *Assume that* (H1)–(H5) *hold. There exist a positive constant* $\varepsilon^+ < 1$ *and a positive function* $\alpha^+(\varepsilon)$ *such that for* $0 < \varepsilon < \varepsilon^+$ *and* $0 < \alpha < \alpha^+(\varepsilon)$,

$$\boldsymbol{v}^+(\boldsymbol{x};\varepsilon,\alpha) := \boldsymbol{U}(\mu(\boldsymbol{x})) + \boldsymbol{P}(\mu(\boldsymbol{x}))\sigma(\boldsymbol{x}')$$

is a supersolution of (1.6). *Furthermore,*

$$\lim_{\gamma\to\infty} \sup_{\boldsymbol{x}\in D(\gamma)} |\boldsymbol{v}^+(\boldsymbol{x};\varepsilon,\alpha) - \boldsymbol{v}^-(\boldsymbol{x})| \leqslant p^0\varepsilon, \tag{3.2}$$

$$\boldsymbol{v}^-(\boldsymbol{x}) < \boldsymbol{v}^+(\boldsymbol{x};\varepsilon,\alpha) \quad \text{for} \quad \boldsymbol{x}\in\mathbb{R}^3, \tag{3.3}$$

$$\frac{\partial}{\partial x_3} v_i^+(\boldsymbol{x};\varepsilon,\alpha) > 0 \quad \text{for} \quad \boldsymbol{x}\in\mathbb{R}^3, \quad i = 1,2,\ldots,N. \tag{3.4}$$

Proof. Firstly, we show that v^+ is a supersolution of (1.6). Note that $\boldsymbol{v}^+(\boldsymbol{x}';\varepsilon,\alpha) := \boldsymbol{U}(\mu(\boldsymbol{x})) + \boldsymbol{P}(\mu(\boldsymbol{x}))\sigma(\boldsymbol{x}') > \boldsymbol{E}^-$ and $\{v_i^+ - E_i^-\}^- \equiv 0$. Therefore, $H_-^i(\boldsymbol{v}^+) \equiv 0$. Consequently, we have

$$\begin{aligned}
\mathcal{N}_i[\boldsymbol{v}^+] &= -D_i\Delta v_i^+ + s\frac{\partial}{\partial x_3}v_i^+ - F^i(\boldsymbol{v}^+) - H_+^i(\boldsymbol{v}^+)\\
&= -D_i\sum_{j=1}^{2}[(U_i'(\mu)\mu_{x_j})_{x_j} + (p_i'(\mu)\mu_{x_j}\sigma(\boldsymbol{x}') + p_i(\mu)\sigma_{x_j}(\boldsymbol{x}'))_{x_j}]\\
&\quad - \frac{D_i(U_i''(\mu) + p_i''(\mu)\sigma(\boldsymbol{x}'))}{1+|\nabla\varphi(\alpha\boldsymbol{x}')|^2} + \frac{s(U_i'(\mu) + p_i'(\mu)\sigma(\boldsymbol{x}'))}{\sqrt{1+|\varphi(\alpha\boldsymbol{x}')|^2}}\\
&\quad - F^i(\boldsymbol{U}(\mu) + \boldsymbol{P}(\mu)\sigma(\boldsymbol{x}')) - H_+^i(\boldsymbol{U}(\mu) + \boldsymbol{P}(\mu)\sigma(\boldsymbol{x}'))\\
&= D_i\left(1 - \sum_{j=1}^{2}\mu_{x_j}^2 - \frac{1}{1+|\nabla\varphi(\alpha\boldsymbol{x}')|^2}\right)(U_i''(\mu) + p_i''(\mu)\sigma(\boldsymbol{x}'))\\
&\quad - D_i\sum_{j=1}^{2}\mu_{x_jx_j}(U_i'(\mu) + p_i'(\mu)\sigma(\boldsymbol{x}')) - 2D_i\sum_{j=1}^{2}p_i'(\mu)\mu_{x_j}\sigma_{x_j}(\boldsymbol{x}')\\
&\quad - D_i\sum_{j=1}^{2}p_i(\mu)\sigma_{x_jx_j}(\boldsymbol{x}') - D_ip_i''(\mu)\sigma(\boldsymbol{x}')\\
&\quad + \left(\frac{s}{\sqrt{1+|\varphi(\alpha\boldsymbol{x}')|^2}} - c\right)U_i'(\mu) + \frac{s}{\sqrt{1+|\varphi(\alpha\boldsymbol{x}')|^2}}p_i'(\mu)\sigma(\boldsymbol{x}')\\
&\quad + F^i(\boldsymbol{U}(\mu)) - F^i(\boldsymbol{U}(\mu) + \boldsymbol{P}(\mu)\sigma(\boldsymbol{x}')) - H_+^i(\boldsymbol{U}(\mu) + \boldsymbol{P}(\mu)\sigma(\boldsymbol{x}')).
\end{aligned}$$

Let

$$\begin{aligned}
I_{i1} &= D_i\left(1 - \mu_{x_1}^2 - \mu_{x_2}^2 - \frac{1}{1+|\nabla\varphi(\alpha\boldsymbol{x}')|^2}\right)(U_i''(\mu) + p_i''(\mu)\sigma(\boldsymbol{x}'))\\
&= \left[\frac{2\alpha D_i\mu(X_1\varphi_{z_1} + X_2\varphi_{z_2})}{\sqrt{1+|\nabla\varphi(\alpha\boldsymbol{x}')|^2}} - \alpha^2\mu^2(X_1^2 + X_2^2)\right](U_i''(\mu) + p_i''(\mu)\sigma(\boldsymbol{x}')),\\
I_{i2} &= -D_i(U_i'(\mu) + p_i'(\mu)\sigma(\boldsymbol{x}'))\sum_{j=1}^{2}\mu_{x_jx_j} - 2D_ip_i'(\mu)\sum_{j=1}^{2}\mu_{x_j}\sigma_{x_j}(\boldsymbol{x}')\\
&= -\alpha D_i(U_i'(\mu) + p_i'(\mu)\sigma(\boldsymbol{x}'))\left[\sum_{j=1}^{2}Y_j(\boldsymbol{z}') - \alpha\mu\sum_{j=1}^{2}Z_j(\boldsymbol{z}')\right]\\
&\quad - 2\alpha\varepsilon D_ip_i'(\mu)\sum_{j=1}^{2}\left(\frac{\varphi_{z_j}}{\sqrt{1+|\nabla\varphi(\boldsymbol{z}')|^2}} - \alpha\mu X_j(\boldsymbol{z}')\right)S_{z_j},\\
I_{i3} &= -D_ip_i''(\mu)\sigma(\boldsymbol{x}') + \frac{sp_i'(\mu)\sigma(\boldsymbol{x}')}{\sqrt{1+|\nabla\varphi(\alpha\boldsymbol{x}')|^2}} = -\varepsilon D_ip_i''(\mu)S(\boldsymbol{z}') + \frac{\varepsilon sp_i'(\mu)S(\boldsymbol{z}')}{\sqrt{1+|\nabla\varphi(\boldsymbol{z}')|^2}},
\end{aligned}$$

$$I_{i4} = -D_i \sum_{j=1}^{2} p_i(\mu)\sigma_{x_j x_j}(\boldsymbol{x}') = -\alpha^2 \varepsilon D_i p_i(\mu) \sum_{j=1}^{2} S_{z_j z_j},$$
$$I_{i5} = \left(\frac{s}{\sqrt{1+|\varphi(\alpha \boldsymbol{x}')|^2}} - c\right) U_i'(\mu) = S(\boldsymbol{z}')U_i'(\mu),$$
$$I_{i6} = F^i(\boldsymbol{U}(\mu)) - F^i(\boldsymbol{U}(\mu) + \boldsymbol{P}(\mu)\sigma(\boldsymbol{x}')),$$

and

$$I_{i7} = H_+^i(\boldsymbol{U}(\mu) + \boldsymbol{P}(\mu)\sigma(\boldsymbol{x}')).$$

By Lemmas 2.5 and 2.6 and direct calculations, we have

$$\sup_{\boldsymbol{x}'\in\mathbb{R}^2}\left|\frac{I_{i1}(\boldsymbol{x}')}{S(\alpha\boldsymbol{x}')}\right| \leqslant C_{i1}\alpha, \quad \sup_{\boldsymbol{x}'\in\mathbb{R}^2}\left|\frac{I_{i2}(\boldsymbol{x}')}{S(\alpha\boldsymbol{x}')}\right| \leqslant C_{i2}\alpha \quad \text{and} \quad \sup_{\boldsymbol{x}'\in\mathbb{R}^2}\left|\frac{I_{i4}(\boldsymbol{x}')}{S(\alpha\boldsymbol{x}')}\right| \leqslant C_{i4}\alpha^2$$

for $0<\alpha<1$ and $0<\varepsilon<1$, where C_{i1}, C_{i2} and C_{i4} are positive constants independent of α and ε, $i=1,\dots,N$.

For $\boldsymbol{v}\in\mathbb{R}^N$ and $r>0$, we define $B_r(\boldsymbol{v}) := \{\boldsymbol{u}\in\mathbb{R}^N : |\boldsymbol{u}-\boldsymbol{v}|<r\}$. Now by the definition of $\mu_{ij}^{\pm}$, there exist a sufficiently small positive constant

$$\epsilon_0 < \min\left\{\frac{p_0}{4}, \frac{q_0}{4}, \frac{1}{4}\min_{1\leqslant i\leqslant N}(E_i^+ - E_i^-)\right\}$$

and a positive constant κ such that

$$\begin{aligned}
&\frac{\partial F^i}{\partial u_j}(\boldsymbol{u}) \leqslant \mu_{ij}^{\pm} \quad \text{for all} \quad \boldsymbol{u}\in B_{4\epsilon_0}(\boldsymbol{E}^{\pm}) \subset [\widehat{\boldsymbol{E}}^-, \widehat{\boldsymbol{E}}^+] \quad \text{and for all} \quad i,j=1,\dots,N,\\
&\sum_{j=1}^{N}\mu_{ij}^{\pm} r_j \leqslant -\kappa r_i \quad \text{for} \quad \boldsymbol{r}=(r_1,\dots,r_N)\in\mathbb{R}_+^N\cap B_{2\epsilon_0}(\boldsymbol{P}^{\pm})\\
&\quad \text{or} \;\; \boldsymbol{r}=(r_1,\dots,r_N)\in\mathbb{R}_+^N\cap B_{2\epsilon_0}(\boldsymbol{Q}^{\pm}).
\end{aligned} \tag{3.5}$$

Take $D=\max_{1\leqslant i\leqslant N} D_i$, $L^-=\max_{1\leqslant i,j\leqslant N} L_{ij}^-$ and $L^+=\max_{1\leqslant i,j\leqslant N} L_{ij}^+$. Using the fact that $\boldsymbol{U}(x)\to\boldsymbol{E}^{\pm}$ as $x\to\pm\infty$ and the properties of $\boldsymbol{P}(x)$ and $\boldsymbol{Q}(x)$, there exists a sufficiently large constant $M>0$ such that

$$\begin{aligned}
&|\boldsymbol{U}(x)-\boldsymbol{E}^+|<\epsilon_0 \quad \text{and} \quad |\boldsymbol{U}(x)-\boldsymbol{E}^+|<\frac{\kappa(p_0-\epsilon_0)}{8p^0N(L^++1)} \quad \text{for} \quad x>M,\\
&|\boldsymbol{U}(x)-\boldsymbol{E}^-|<\epsilon_0 \quad \text{and} \quad |\boldsymbol{U}(x)-\boldsymbol{E}^-|<\frac{\kappa(q_0-\epsilon_0)}{8q^0N(L^-+1)} \quad \text{for} \quad x<-M,\\
&|\boldsymbol{P}(x)-\boldsymbol{P}^+|\leqslant\epsilon_0 \quad \text{and} \quad |\boldsymbol{P}(x)-\boldsymbol{P}^+|\leqslant\frac{\kappa(p_0-\epsilon_0)}{8p^0N(L^++1)} \quad \text{for} \quad x>M,\\
&|\boldsymbol{P}(x)-\boldsymbol{P}^-|\leqslant\epsilon_0 \quad \text{for} \quad x\leqslant -M, \quad |\boldsymbol{Q}(x)-\boldsymbol{Q}^+|\leqslant\epsilon_0 \quad \text{for} \quad x>M,\\
&|\boldsymbol{Q}(x)-\boldsymbol{Q}^-|\leqslant\frac{\kappa(q_0-\epsilon_0)}{8q^0N(L^-+1)} \quad \text{and} \quad |\boldsymbol{Q}(x)-\boldsymbol{Q}^-|\leqslant\epsilon_0 \quad \text{for} \quad x<-M,\\
&|P_i'(x)|<\frac{1}{8s}\kappa(p_0-\epsilon_0) \quad \text{and} \quad |P_i''(x)|<\frac{1}{8D}\kappa(p_0-\epsilon_0) \quad \text{for} \quad |x|>M,\\
&|Q_i'(x)|<\frac{1}{8s}\kappa(q_0-\epsilon_0) \quad \text{and} \quad |Q_i''(x)|<\frac{1}{8D}\kappa(q_0-\epsilon_0) \quad \text{for} \quad |x|>M.
\end{aligned} \tag{3.6}$$

For $\varepsilon\in(0,\epsilon_0/(Np^0s))$, we have $\boldsymbol{v}^+(\boldsymbol{x};\varepsilon,\alpha)\in[\widehat{\boldsymbol{E}}^-,\widehat{\boldsymbol{E}}^+]$ for $\boldsymbol{x}\in\mathbb{R}^3$. Furthermore, take

$$\varepsilon < \min\left\{\frac{1}{2p^0sN}\min_{1\leqslant i\leqslant N}\{E_i^+ - U_i(M)\}, \frac{\kappa(p_0-\epsilon_0)}{8sN(L^++1)(p^0)^2}\right\}. \tag{3.7}$$

For $|\mu(\boldsymbol{x})|>M$ and $\boldsymbol{z}'\in\mathbb{R}^2$, we have

$$|I_{i3}| \leqslant \varepsilon D_i|p_i''(\mu)|S(\boldsymbol{z}') + \varepsilon s|p_i'(\mu)|S(\boldsymbol{z}') \leqslant \frac{1}{4}\varepsilon\kappa(p_0-\epsilon_0)S(\boldsymbol{z}').$$

Since

$$\begin{aligned}I_{i6} &= -\bigg(\sum_{j=1}^{N}\frac{\partial}{\partial u_j}F^i(\boldsymbol{U}(\mu)+\theta\varepsilon\boldsymbol{P}(\mu)S(\boldsymbol{z}'))p_j(\mu)\bigg)\varepsilon S(\boldsymbol{z}')\\ &\geqslant -\bigg(\sum_{j=1}^{N}\mu_{ij}^{+}p_j(\mu)\bigg)\sigma(\boldsymbol{z}') \geqslant \kappa p_i(\mu)\varepsilon S(\boldsymbol{z}')\\ &\geqslant \kappa(p_i^{+}-\epsilon_0)\varepsilon S(\boldsymbol{z}')\end{aligned}$$

and

$$\begin{aligned}I_{i7} &= \sum_{j=1,\dots,N;j\neq i} L_{ij}^{+}\{E_i^{+}-U_i(\mu)-\varepsilon p_i(\mu)S(\boldsymbol{z}')\}^{-}(U_j(\mu)+\varepsilon p_j(\mu)S(\boldsymbol{z}')-E_j^{+})\\ &\leqslant \sum_{j=1,\dots,N;j\neq i}\varepsilon^2 L_{ij}^{+}p_i(\mu)p_j(\mu)S^2(\boldsymbol{z}')\\ &\leqslant \frac{1}{8}\kappa(p_0-\epsilon_0)\varepsilon S(\boldsymbol{z}')\end{aligned}$$

for $\mu > M$ and $\boldsymbol{z}' \in \mathbb{R}^2$ due to (3.5)–(3.7), we have

$$\begin{aligned}\mathcal{N}_i[\boldsymbol{v}^{+}] &= I_{i1}+I_{i2}+I_{i3}+I_{i4}+I_{i5}+I_{i6}-I_{i7}\\ &\geqslant S(\boldsymbol{z}')\bigg[-C_{i1}\alpha - C_{i2}\alpha - \frac{1}{4}\kappa(p_0-\epsilon_0)\varepsilon\\ &\quad - C_{i4}\alpha^2 + \kappa(p_i^{+}-\epsilon_0)\varepsilon - \frac{1}{8}\kappa(p_0-\epsilon_0)\varepsilon\bigg]\\ &> \bigg[-(C_{i1}+C_{i2}+\alpha C_{i4})\alpha + \frac{1}{2}\kappa(p_i^{+}-\epsilon_0)\varepsilon\bigg]S(\boldsymbol{z}') > 0\end{aligned}$$

for $\mu(\boldsymbol{x}) > M$ and $\boldsymbol{z}' \in \mathbb{R}^2$ provided that

$$\alpha < \min_{1\leqslant i\leqslant N}\bigg\{\frac{\kappa(p_i^{+}-\epsilon_0)}{2(C_{i1}+C_{i2}+C_{i4})}\bigg\}\varepsilon.$$

By (3.7), we have that $I_{i7}(\boldsymbol{x}) = 0$ when $\mu(\boldsymbol{x}) \leqslant M$. Then using an argument similar to that for $\mu(\boldsymbol{x}) > M$, we have $\mathcal{N}_i[\boldsymbol{v}^{+}] > 0$ for $\mu(\boldsymbol{x}) < -M$ and $\boldsymbol{z}' \in \mathbb{R}^2$ provided that

$$\alpha < \min_{1\leqslant i\leqslant N}\bigg\{\frac{\kappa(p_i^{-}-\epsilon_0)}{2(C_{i1}+C_{i2}+C_{i4})}\bigg\}\varepsilon.$$

Let

$$M_{ij} := \sup_{\boldsymbol{u}\in[\widehat{\boldsymbol{E}}^{-},\widehat{\boldsymbol{E}}^{+}]}\bigg|\frac{\partial}{\partial u_j}F^i(\boldsymbol{u})\bigg| \tag{3.8}$$

and $C_{i6} = \sum_{j=1}^{N} M_{ij}p_j^{+}$, $i,j = 1,\dots,N$. Then $|I_{i6}| \leqslant C_{i6}\varepsilon S(\boldsymbol{z}')$ for all $\boldsymbol{z}' \in \mathbb{R}^2$. Take a constant $C_{i3} > 0$ such that $|I_{i3}| \leqslant C_{i3}\varepsilon S(\boldsymbol{z}')$ for all $\boldsymbol{z}' \in \mathbb{R}^2$, $i = 1,\dots,N$. Let

$$p^{*} := \min_{|x|\leqslant M, 1\leqslant i\leqslant N} U_i'(x) > 0.$$

For $|\mu(\boldsymbol{x})| \leqslant M$ and $\boldsymbol{z}' \in \mathbb{R}^2$, we have

$$\begin{aligned}\mathcal{N}_i[v_i^{+}] &= I_{i1}+I_{i2}+I_{i3}+I_{i4}+I_{i5}+I_{i6}\\ &\geqslant S(\boldsymbol{z}')[-C_{i1}\alpha - C_{i2}\alpha - C_{i3}\varepsilon - C_{i4}\alpha^2 + p^{*} - C_{i6}\varepsilon] > 0.\end{aligned}$$

Up to now, we have showed that v^{+} is a supersolution of (1.6) provided that

$$\varepsilon < \min_{1\leqslant i\leqslant N}\frac{p^{*}}{2(C_{i3}+C_{i6})} \quad \text{and} \quad \alpha < \min_{1\leqslant i\leqslant N}\frac{\min\{p^{*},\kappa(p_0-\epsilon_0)\varepsilon\}}{2(C_{i1}+C_{i2}+C_{i4})}.$$

Now we prove the inequality (3.3). It suffices to prove

$$U_i\left(\frac{c}{s}(x_3+h_j(\boldsymbol{x}'))\right)<v_i^+(\boldsymbol{x};\varepsilon,\alpha)\quad \text{for all}\quad i=1,\dots,N\quad\text{and}\quad j=1,\dots,n.$$

When $\mu(\boldsymbol{x})\geqslant\frac{c}{s}(x_3+h_j(\boldsymbol{x}'))$, it is easy to get

$$U_i\left(\frac{c}{s}(x_3+h_j(\boldsymbol{x}'))\right)\leqslant U_i(\mu(\boldsymbol{x}))<v_i^+(\boldsymbol{x};\varepsilon,\alpha).$$

Assume that

$$\mu(\boldsymbol{x})<\frac{c}{s}(x_3+h_j(\boldsymbol{x}')).$$

By the definition of μ, we have

$$\frac{c}{s}(x_3+h_j(\boldsymbol{x}'))>\frac{x_3+\frac{1}{\alpha}\varphi(\alpha\boldsymbol{x}')}{\sqrt{1+|\nabla\varphi(\alpha\boldsymbol{x}')|^2}}=\frac{x_3+h_j(\boldsymbol{x}')+\frac{1}{\alpha}\varphi(\alpha\boldsymbol{x}')-h_j(\boldsymbol{x}')}{\sqrt{1+|\nabla\varphi(\alpha\boldsymbol{x}')|^2}}.$$

It follows that

$$(x_3+h_j(\boldsymbol{x}'))\left(\frac{s}{\sqrt{1+|\nabla\varphi(\boldsymbol{z}')|^2}}-c\right)<\frac{s}{\alpha}\frac{h_j(\boldsymbol{z}')-\ \varphi(\boldsymbol{z}')}{\sqrt{1+|\nabla\varphi(\boldsymbol{z}')|^2}},$$

namely,

$$x_3+h_j(\boldsymbol{x}')<\frac{s}{\alpha}\frac{1}{\sqrt{1+|\nabla\varphi(\boldsymbol{z}')|^2}}\frac{h_j(\boldsymbol{z}')-\ \varphi(\boldsymbol{z}')}{S(\boldsymbol{z}')}.$$

By the definition of ν_1, we have

$$x_3+h_j(\boldsymbol{x}')<\frac{s}{\alpha}\frac{1}{\sqrt{1+|\nabla\varphi(\boldsymbol{z}')|^2}}\frac{h_j(\boldsymbol{z}')-\ \varphi(\boldsymbol{z}')}{S(\boldsymbol{z}')}\leqslant-\frac{c\nu_1}{\alpha}.$$

Since

$$h_j(\boldsymbol{x}')=\frac{1}{\alpha}h_j(\boldsymbol{z}')\leqslant\frac{1}{\alpha}h(\boldsymbol{z}')\leqslant\frac{1}{\alpha}\varphi(\boldsymbol{z}'),$$

we have

$$\begin{aligned}
&v_i^+(\boldsymbol{x};\varepsilon,\alpha)-U_i\left(\frac{c}{s}(x_3+h_j(\boldsymbol{x}'))\right)\\
&\quad=U_i\left(\frac{x_3+\frac{1}{\alpha}\varphi(\alpha\boldsymbol{x}')}{\sqrt{1+|\nabla\varphi(\alpha\boldsymbol{x}')|^2}}\right)+\varepsilon p_i\left(\frac{x_3+\frac{1}{\alpha}\varphi(\alpha\boldsymbol{x}')}{\sqrt{1+|\nabla\varphi(\alpha\boldsymbol{x}')|^2}}\right)S(\alpha\boldsymbol{x}')\\
&\qquad-U_i\left(\frac{c}{s}(x_3+h_j(\boldsymbol{x}'))\right)\\
&\quad\geqslant U_i\left(\frac{x_3+h_j(\boldsymbol{x}')}{\sqrt{1+|\nabla\varphi(\alpha\boldsymbol{x}')|^2}}\right)+\varepsilon p_i\left(\frac{x_3+\frac{1}{\alpha}\varphi(\alpha\boldsymbol{x}')}{\sqrt{1+|\nabla\varphi(\alpha\boldsymbol{x}')|^2}}\right)S(\alpha\boldsymbol{x}')\\
&\qquad-U_i\left(\frac{c}{s}(x_3+h_j(\boldsymbol{x}'))\right).
\end{aligned}$$

Since

$$\begin{aligned}
&U_i\left(\frac{x_3+h_j(\boldsymbol{x}')}{\sqrt{1+|\nabla\varphi(\alpha\boldsymbol{x}')|^2}}\right)-U_i\left(\frac{c}{s}(x_3+h_j(\boldsymbol{x}'))\right)\\
&\quad=(x_3+h_j(\boldsymbol{x}'))\frac{1}{s}S(\alpha\boldsymbol{x}')\\
&\qquad\times\int_0^1U_i'\left((x_3+h_j(\boldsymbol{x}'))\left(\frac{\theta}{\sqrt{1+|\nabla\varphi(\alpha\boldsymbol{x}')|^2}}+\frac{c}{s}(1-\theta)\right)\right)d\theta,
\end{aligned}$$

we have

$$v_i^+(\boldsymbol{x};\varepsilon,\alpha)-U_i\left(\frac{c}{s}(x_3+h_j(\boldsymbol{x}'))\right)$$

$$\geqslant \varepsilon p_0 S(\alpha \boldsymbol{x}') + (x_3 + h_j(\boldsymbol{x}'))\frac{1}{s}S(\alpha \boldsymbol{x}') \times \int_0^1 U_i'\bigg((x_3 + h_j(\boldsymbol{x}'))\bigg(\frac{\theta}{\sqrt{1+|\nabla\varphi(\alpha\boldsymbol{x}')|^2}} + \frac{c}{s}(1-\theta)\bigg)\bigg)d\theta.$$

Note that

$$\frac{c}{s} \leqslant \frac{\theta}{\sqrt{1+|\nabla\varphi(\alpha\boldsymbol{x}')|^2}} + \frac{c}{s}(1-\theta) \leqslant 1 \quad \text{and} \quad x_3 + h_j(\boldsymbol{x}') < 0.$$

Consequently, we have

$$\begin{aligned}
& v_i^+(\boldsymbol{x};\varepsilon,\alpha) - U_i\bigg(\frac{c}{s}(x_3 + h_j(\boldsymbol{x}'))\bigg) \\
&\geqslant \varepsilon p_0 S(\alpha \boldsymbol{x}') + (x_3 + h_j(\boldsymbol{x}'))\frac{1}{s}S(\alpha \boldsymbol{x}') \\
&\quad \times \int_0^1 U_i'\bigg((x_3 + h_j(\boldsymbol{x}'))\bigg(\frac{\theta}{\sqrt{1+|\nabla\varphi(\alpha\boldsymbol{x}')|^2}} + \frac{c}{s}(1-\theta)\bigg)\bigg)d\theta \\
&\geqslant \frac{S(\alpha\boldsymbol{x}')}{s}[(x_3 + h_j(\boldsymbol{x}'))C_1 \mathrm{e}^{-\frac{c\beta_1}{s}|x_3 + h_j(\boldsymbol{x}')|} + s\varepsilon p_0] \\
&\geqslant \frac{S(\alpha\boldsymbol{x}')}{s}\bigg[-\frac{sC_1}{c\beta_1}\sup_{x > \frac{c^2\nu_1\beta_1}{s\alpha}} x\mathrm{e}^{-|x|} + s\varepsilon p_0\bigg] \\
&\geqslant \frac{S(\alpha\boldsymbol{x}')}{s}\bigg[-\frac{sC_1}{c\beta_1}\frac{c^2\nu_1\beta_1}{s\alpha}\mathrm{e}^{-\frac{c^2\nu_1\beta_1}{s\alpha}} + s\varepsilon p_0\bigg] \\
&= \frac{S(\alpha\boldsymbol{x}')}{s}\bigg[-\frac{c\nu_1 C_1}{\alpha}\mathrm{e}^{-\frac{c^2\nu_1\beta_1}{s\alpha}} + s\varepsilon p_0\bigg] \\
&\geqslant 0
\end{aligned}$$

provided that $\alpha < \alpha^*(\varepsilon)$, where $0 < \alpha^*(\varepsilon) < \frac{s}{c^2\nu_1\beta_1}$ satisfies

$$s\varepsilon p_0 - \frac{c\nu_1 C_1}{\alpha}\mathrm{e}^{-\frac{c^2\nu_1\beta_1}{s\alpha}} > 0 \quad \text{for} \quad \alpha < \alpha^*(\varepsilon).$$

Now we prove (3.2). It is sufficient to show that

$$\lim_{\gamma\to\infty}\sup_{\boldsymbol{x}\in D(\gamma)}\bigg|U_j(\mu(\boldsymbol{x})) - U_j\bigg(\frac{c}{s}(x_3 + h(\boldsymbol{x}'))\bigg)\bigg| = 0$$

for all $j = 1,\dots,N$. Assume the contrary for some $l \in \{1,\dots,N\}$. Then there exist a positive constant ϵ' and sequences $\{\gamma_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ and $\{\boldsymbol{x}_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^3$ such that

$$\lim_{k\to\infty}\gamma_k = \infty, \quad \boldsymbol{x}_k \in D(\gamma_k) \tag{3.9}$$

and

$$\bigg|U_l(\mu(\boldsymbol{x}_k)) - U_l\bigg(\frac{c}{s}(x_{k,3} + h(\boldsymbol{x}_k'))\bigg)\bigg| \geqslant \epsilon', \tag{3.10}$$

where $\boldsymbol{x}_k' = (x_{k,1}, x_{k,2})$. It follows that

$$\mu(\boldsymbol{x}_k) = \frac{1}{\alpha}\frac{z_{k,3} + \varphi(\boldsymbol{z}_k')}{\sqrt{1+|\nabla\varphi(\boldsymbol{z}_k')|^2}} = \frac{x_{k,3} + h(\boldsymbol{x}_k') + \frac{1}{\alpha}(\varphi(\boldsymbol{z}_k') - h(\boldsymbol{z}_k'))}{\sqrt{1+|\nabla\varphi(\boldsymbol{z}_k')|^2}}.$$

If $\lim_{k\to\infty}\mathrm{dist}(\boldsymbol{x}_k', E) = \infty$, by Lemma 2.6 we have $\lim_{k\to\infty}|\varphi(\boldsymbol{x}_k') - h(\boldsymbol{x}_k')| = 0$ and $\lim_{k\to\infty}S(\boldsymbol{x}_k') = 0$. If further $x_{k,3} + h(\boldsymbol{x}_k') \to \pm\infty$ as $k \to +\infty$, then $\mu(\boldsymbol{x}_k) \to \pm\infty$, which again contradicts (3.10). If $x_{k,3} + h(\boldsymbol{x}_k')$ are bounded for $k \in \mathbb{N}$, we have

$$\lim_{k\to\infty}\bigg|\mu(\boldsymbol{x}_k) - \frac{c}{s}(x_{k,3} + h(\boldsymbol{x}_k'))\bigg| = 0.$$

This contradicts (3.10) once more. If $\mathrm{dist}(\boldsymbol{x}'_k, E)$ keeps finite uniformly in k, then (3.9) implies that $\lim_{k\to\infty}(x_{k,3}+h(\boldsymbol{x}'_k)) = \pm\infty$ and $\lim_{k\to\infty}\mu(\boldsymbol{x}_k) = \pm\infty$, respectively. This contradicts (3.10). Thus, we have proved (3.2).

Finally, we take

$$\varepsilon^+ = \min\left\{1, \frac{\epsilon_0}{Np^0 s}, \min_{1\leqslant i\leqslant N}\frac{p^*}{2(C_{i3}+C_{i6})}, \min_{1\leqslant i\leqslant N}\frac{E_i^+ - U_i(M)}{2Np^0 s}, \frac{\kappa(p_0-\epsilon_0)}{8sN(L^+ +1)(p^0)^2}\right\}$$

and

$$\alpha^+(\varepsilon) := \min\left\{1, \min_{1\leqslant i\leqslant N}\frac{\min\{p^*, \kappa(p_0-\epsilon_0)\varepsilon\}}{2(C_{i1}+C_{i2}+C_{i4})}, \alpha^*(\varepsilon)\right\}.$$

This completes the proof. □

Take $\psi(\vartheta) := -\frac{1}{m_*\beta_2}\ln(1+\exp(-\beta_2\vartheta))$. There exist some constants $C'_i > 0$ $(i = 2,3,4)$ such that

$$\begin{aligned}
&\max\left\{\left|\psi(\vartheta) - \frac{\vartheta}{m_*}\right|, \left|\psi'(\vartheta) - \frac{1}{m_*}\right|\right\} \leqslant C'_2\,\mathrm{sech}(\beta_2\vartheta), \quad \text{for } \vartheta \leqslant 0,\\
&\max\{|\psi(\vartheta)|, |\psi'(\vartheta)|\} \leqslant C'_2\,\mathrm{sech}(\beta_2\vartheta), \quad \text{for } \vartheta \geqslant 0,\\
&\max\{|\psi''(\vartheta)|, |\psi'''(\vartheta)|\} \leqslant C'_2\,\mathrm{sech}(\beta_2\vartheta), \quad \text{for } \vartheta \in \mathbb{R},\\
&c - \frac{s\psi(\vartheta)}{\sqrt{1+\psi'(\vartheta)^2}} \geqslant C'_3\min\{1, \exp(\beta_2\vartheta)\}, \quad \text{for } \vartheta \in \mathbb{R},\\
&0 \leqslant \frac{s}{\sqrt{1+\psi'(\vartheta)^2}} - cm_* \leqslant C'_4\min\{1, \exp(\beta_2\vartheta)\}, \quad \text{for } \vartheta \in \mathbb{R}.
\end{aligned} \tag{3.11}$$

We notice that (3.11) follows directly from [43] (see also [53]).

To establish the existence of pyramidal traveling fronts for System (1.2), we still need the following lemma which was proved in [53].

Lemma 3.2. *Assume that* (H1)–(H5) *hold. There exist a positive constant ε^- and a positive function $\alpha^-(\varepsilon)$ so that, for $0 < \varepsilon < \varepsilon^-$ and $0 < \alpha < \alpha^-(\varepsilon)$,*

$$\widehat{\boldsymbol{v}}(x,z;\varepsilon,\alpha) := \boldsymbol{U}\left(\frac{x+\psi(\alpha z)/\alpha}{\sqrt{1+\psi'(\alpha z)^2}}\right) - \varepsilon\boldsymbol{Q}\left(\frac{x+\psi(\alpha z)/\alpha}{\sqrt{1+\psi'(\alpha z)^2}}\right)\mathrm{sech}(\beta_2\alpha z)$$

is a subsolution to the following system:

$$\frac{\partial\boldsymbol{u}}{\partial t} = \boldsymbol{D}\left(\frac{\partial^2}{\partial x^2}\boldsymbol{u} + \frac{\partial^2}{\partial z^2}\boldsymbol{u}\right) - s\frac{\partial}{\partial z}\boldsymbol{u} + \boldsymbol{G}(\boldsymbol{u}(x,z,t)),$$

where $x, z \in \mathbb{R}$. In addition, we have $\frac{\partial}{\partial x}\widehat{v}_i > 0$, $i = 1,\dots,N$.

We note that

$$\varepsilon^- = \min\left\{\frac{\epsilon_0}{q^0}, \frac{1}{2q^0}\min_{1\leqslant i\leqslant N}\{U_i(-M) - E_i^-\}, \frac{\kappa(q_0-\epsilon_0)}{8N(L^- +1)(q^0)^2}, \min_{1\leqslant i\leqslant N}\frac{C'_3 p^*}{2(C'_{i3}+C'_{i5})}\inf_{\vartheta\in\mathbb{R}}\frac{\min\{1,\exp(\vartheta)\}}{\mathrm{sech}(\vartheta)}\right\}$$

and

$$\alpha^-(\varepsilon) := \min\left\{1, \min_{1\leqslant i\leqslant N}\frac{C'_3 p^*}{2(C'_{i1}+C'_{i2})}\inf_{\vartheta\in\mathbb{R}}\frac{\min\{1,\exp(\vartheta)\}}{\mathrm{sech}(\vartheta)}, \min_{1\leqslant i\leqslant N}\frac{\kappa(q_0-\epsilon_0)\varepsilon}{2(C'_{i1}+C'_{i2})}\right\},$$

where ϵ_0, q^0, q_0 and M are defined as before, $C'_{i1}, C'_{i2}, C'_{i3}, C'_{i5}$ and C'_3 are positive constants. Thus, it is obvious that

$$\widehat{\boldsymbol{v}}^j(\boldsymbol{x};\varepsilon,\alpha) := \boldsymbol{U}\left(\frac{\frac{h_j(\boldsymbol{x}')}{m_*} + \frac{\psi(\alpha x_3)}{\alpha}}{\sqrt{1+\psi'(\alpha x_3)^2}}\right) - \varepsilon\boldsymbol{Q}\left(\frac{\frac{h_j(\boldsymbol{x}')}{m_*} + \frac{\psi(\alpha x_3)}{\alpha}}{\sqrt{1+\psi'(\alpha x_3)^2}}\right)\mathrm{sech}(\beta_2\alpha x_3)$$

is a subsolution of (2.1) on $t > 0$ and $\boldsymbol{x} \in \mathbb{R}^3$. Consequently, we have that

$$\widetilde{\boldsymbol{v}}(\boldsymbol{x};\varepsilon,\alpha) := \bigvee_{j=1}^n \widehat{\boldsymbol{v}}^j(\boldsymbol{x};\varepsilon,\alpha) = \boldsymbol{U}\left(\frac{h(\boldsymbol{x}')/m_* + \psi(\alpha x_3)/\alpha}{\sqrt{1+\psi'(\alpha x_3)^2}}\right)$$

$$- \varepsilon \boldsymbol{Q}\left(\frac{h(\boldsymbol{x}')/m_* + \psi(\alpha x_3)/\alpha}{\sqrt{1+\psi'(\alpha x_3)^2}}\right)\text{sech}(\beta_2 \alpha x_3)$$

is a subsolution of (2.1) on $t > 0$.

In the following, we show the existence of pyramidal traveling fronts of (1.2). By the parabolic estimate, we know that there exists $K > 0$ such that solutions $\boldsymbol{v}(\boldsymbol{x}, t; \boldsymbol{v}^0)$ of (1.4)–(1.5) with $\boldsymbol{v}^0(\boldsymbol{x}) \in [\boldsymbol{E}^-, \boldsymbol{E}^+]$ satisfy $\|\boldsymbol{v}(\cdot, t; \boldsymbol{v}^0)\|_{C^3(\mathbb{R}^3)} < K$ for any $t \geqslant 1$. Since $\boldsymbol{v}^-$ is a subsolution of (1.6), we have $\boldsymbol{v}(\boldsymbol{x}, t_1; \boldsymbol{v}^-) \leqslant \boldsymbol{v}(\boldsymbol{x}, t_2; \boldsymbol{v}^-)$ for all $\boldsymbol{x} \in \mathbb{R}^3$ and $0 < t_1 \leqslant t_2$. Consequently, define

$$\boldsymbol{V}(\boldsymbol{x}) := \lim_{t\to\infty} \boldsymbol{v}(\boldsymbol{x}, t; \boldsymbol{v}^-) \tag{3.12}$$

for all $\boldsymbol{x} \in \mathbb{R}^3$. It follows that $\boldsymbol{v}(\cdot, t\,; \boldsymbol{v}^-)$ converges monotonically to $\boldsymbol{V}(\cdot)$ under the norm $\|\cdot\|_{C^2_{\text{loc}}(\mathbb{R}^3)}$ as $t \to \infty$. Since $\boldsymbol{V}(\boldsymbol{x}) \leqslant \boldsymbol{v}^+(\boldsymbol{x}; \varepsilon, \alpha)$ for any $\boldsymbol{x} \in \mathbb{R}^3$, by the arbitrariness of ε and α, we have

$$\lim_{\gamma\to\infty} \sup_{\boldsymbol{x}\in D(\gamma)} |\boldsymbol{V}(\boldsymbol{x}) - \boldsymbol{v}^-(\boldsymbol{x})| = 0. \tag{3.13}$$

Furthermore, following an argument in [44] we know that $\boldsymbol{V}(\cdot)$ defined by (3.12) satisfies (1.6).

We thus have proved the following theorem on the existence of pyramidal traveling fronts for System (1.2).

Theorem 3.3. *Assume that* (H1)–(H5) *hold. For any* $s > c$, (1.2) *admits a pyramidal traveling front* $\boldsymbol{V}$ *satisfying* (1.6), (3.13) *and* $\boldsymbol{v}^-(\boldsymbol{x}) < \boldsymbol{V}(\boldsymbol{x}) < \boldsymbol{v}^+(\boldsymbol{x}; \varepsilon, \alpha)$ *for any* $\boldsymbol{x} \in \mathbb{R}^3$, *where* $0 < \varepsilon < \varepsilon^+$ *and* $0 < \alpha < \alpha^+(\varepsilon)$. *Moreover, one has* $\frac{\partial}{\partial x_3} V_i(\boldsymbol{x}) > 0$ *for* $\boldsymbol{x} \in \mathbb{R}^3$, $i = 1, \ldots, N$.

In view of the monotonicity of $\boldsymbol{v}^-(\boldsymbol{x})$ in the variable x_3, we conclude that $\frac{\partial}{\partial x_3} V_i(\boldsymbol{x}) \geqslant 0$ for all $\boldsymbol{x} \in \mathbb{R}^3$. Then the strong maximum principle implies the strict inequality.

4 Stability and uniqueness of traveling curved fronts

In this section we develop the arguments of Taniguchi [48] and Wang [53] to establish the stability and uniqueness of the pyramidal traveling front $\boldsymbol{V}$ obtained in Section 3. We first prove that (1.9) holds for $\boldsymbol{u}_0 \geqslant \boldsymbol{v}^-$ and $\boldsymbol{u}_0 \leqslant \boldsymbol{v}^-$, respectively. See Theorems 4.13 and 4.17. We then characterize the pyramidal traveling front as a combination of two-dimensional V-form fronts on the edges of the pyramid.

Consider the following Cauchy problem:

$$\begin{cases} \dfrac{\partial}{\partial t}\widetilde{\boldsymbol{w}}(\xi,\eta,t) - \boldsymbol{D}\dfrac{\partial^2}{\partial \xi^2}\widetilde{\boldsymbol{w}}(\xi,\eta,t) - \boldsymbol{D}\dfrac{\partial^2}{\partial \eta^2}\widetilde{\boldsymbol{w}}(\xi,\eta,t) + \bar{s}\dfrac{\partial}{\partial \eta}\widetilde{\boldsymbol{w}}(\xi,\eta,t) - \boldsymbol{F}(\widetilde{\boldsymbol{w}}) = \boldsymbol{0}, \\ \widetilde{\boldsymbol{w}}(\xi,\eta,0) = \widetilde{\boldsymbol{w}}^0(\xi,\eta), \end{cases} \tag{4.1}$$

where $(\xi, \eta) \in \mathbb{R}^2$, $t > 0$, $i = 1, \ldots, N$; $\widetilde{\boldsymbol{w}}(\xi, \eta, t) = (\widetilde{\boldsymbol{w}}_1(\xi, \eta, t), \ldots, \widetilde{\boldsymbol{w}}_N(\xi, \eta, t))$. The following theorem was established by Wang [53].

Theorem 4.1. *Assume that* (H1)–(H5) *hold. Then for each* $\bar{s} > c$, *there exists a steady state* $\boldsymbol{\Phi}(\xi, \eta; \bar{s})$ *of* (4.1) *satisfying* $\boldsymbol{\Phi}(\xi, \eta; \bar{s}) > \tilde{\boldsymbol{v}}^-(\xi, \eta)$ *and*

$$\lim_{R\to\infty} \sup_{\xi^2+\eta^2>R^2} |\boldsymbol{\Phi}(\xi,\eta) - \tilde{\boldsymbol{v}}^-(\xi,\eta)| = 0,$$

where

$$\tilde{\boldsymbol{v}}^-(\xi,\eta) = \boldsymbol{U}\left(\frac{c}{\bar{s}}\left(\eta + \frac{\sqrt{\bar{s}^2 - c^2}}{c}|\xi|\right)\right).$$

Moreover, for any $\widetilde{\boldsymbol{w}}^0 \in C(\mathbb{R}^2, \mathbb{R}^N)$ *with* $\widetilde{\boldsymbol{w}}^0(\xi, \eta) \in [\boldsymbol{E}^-, \boldsymbol{E}^+]$ *for* $(\xi, \eta) \in \mathbb{R}^2$ *and*

$$\lim_{R\to\infty} \sup_{\xi^2+\eta^2>R^2} |\widetilde{\boldsymbol{w}}^0(\xi,\eta) - \tilde{\boldsymbol{v}}^-(\xi,\eta)| = 0,$$

the solution $\widetilde{\boldsymbol{w}}(\xi, \eta, t; \widetilde{\boldsymbol{w}}^0)$ *of* (4.1) *with the initial value* $\widetilde{\boldsymbol{w}}^0$ *satisfies*

$$\lim_{t\to\infty} \|\widetilde{\boldsymbol{w}}(\cdot, t; \widetilde{\boldsymbol{w}}^0) - \boldsymbol{\Phi}(\cdot)\|_{C(\mathbb{R}^2)} = 0.$$

For any subset $\mathcal{D} \subset \mathbb{R}^3$ we denote the characteristic function of $\mathcal{D}$ by $\chi_{\mathcal{D}}$, namely, $\chi_{\mathcal{D}}(\boldsymbol{x}) = 1$ for $\boldsymbol{x} \in \mathcal{D}$ and $\chi_{\mathcal{D}}(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \notin \mathcal{D}$. Let $h_{ij}(\boldsymbol{x}, t) \in C(\mathbb{R}^3 \times \mathbb{R}_+)$ $(i, j = 1, \dots, N)$ be given continuous functions satisfying

$$0 \leqslant h_{ij}(\boldsymbol{x}, t) \leqslant M_{ij}, \quad i \neq j, \quad \sup_{\boldsymbol{x} \in \mathbb{R}^3, t>0} |h_{ij}(\boldsymbol{x}, t)| \leqslant M_{ij}, \quad i = j, \tag{4.2}$$

where M_{ij} is defined by (3.8). Consider the following linear system:

$$\begin{cases} \dfrac{\partial}{\partial t} w_i - D_i \displaystyle\sum_{k=1}^{3} \dfrac{\partial^2}{\partial x_k^2} w_i + s \dfrac{\partial}{\partial x_3} w_i - \displaystyle\sum_{j=1}^{N} h_{ij}(\boldsymbol{x}, t) w_j = 0, \quad \boldsymbol{x} \in \mathbb{R}^3, \quad t > 0, \\ w_i(\boldsymbol{x}, 0) = w_i^0(\boldsymbol{x}) \in C(\mathbb{R}^3, \mathbb{R}^N) \cap L^\infty(\mathbb{R}^3, \mathbb{R}^N), \quad \boldsymbol{x} \in \mathbb{R}^3, \quad i = 1, \dots, N. \end{cases} \tag{4.3}$$

Lemma 4.2. *Let $\boldsymbol{w}(\boldsymbol{x}, t) := (w_1(\boldsymbol{x}, t), \dots, w_N(\boldsymbol{x}, t))$ be a solution of (4.3). Then there exist positive constants $\tilde{A}$, $\tilde{B}$ and λ_0 such that*

$$\max_{1 \leqslant i \leqslant N} \sup_{\boldsymbol{x} \in \mathbb{R}^3} w_i(\boldsymbol{x}, t) \leqslant \mathrm{e}^{\lambda_0 t} \max\Big\{0, \max_{1 \leqslant i \leqslant N} \sup_{\boldsymbol{x} \in \mathbb{R}^3} w_i^0(\boldsymbol{x})\Big\}, \quad \forall\, t > 0,$$
$$\mathrm{e}^{\lambda_0 t} \min\Big\{0, \min_{1 \leqslant i \leqslant N} \inf_{\boldsymbol{x} \in \mathbb{R}^3} w_0(\boldsymbol{x})\Big\} \leqslant \min_{1 \leqslant i \leqslant N} \inf_{\boldsymbol{x} \in \mathbb{R}^3} w_i(\boldsymbol{x}, t), \quad \forall\, t > 0,$$
$$\max_{1 \leqslant i \leqslant N} \sup_{\boldsymbol{x} \in \mathbb{R}^3} |w_i(\boldsymbol{x}, t)| \leqslant \mathrm{e}^{\lambda_0 t} \max_{1 \leqslant i \leqslant N} \|w_i^0\|_{L^\infty(\mathbb{R}^3)}, \quad \forall\, t > 0$$

and for any $\gamma > 0$,

$$\begin{aligned} \sup_{1 \leqslant i \leqslant N} \sup_{\boldsymbol{x} \in D(2\gamma)} |w_i(\boldsymbol{x}, t)| \leqslant{}& \mathrm{e}^{\lambda_0 t} \frac{3\pi \tilde{A}}{\tilde{B}} \int_{\frac{\sqrt{3}\gamma}{3\sqrt{t}}}^{+\infty} \exp(-\tilde{B} r^2) dr \max_{1 \leqslant i \leqslant N} \sup_{\boldsymbol{x} \in D(\gamma)^c} |w_i^0(\boldsymbol{x})| \\ &+ \frac{\pi \sqrt{\pi} \tilde{A}}{\tilde{B} \sqrt{\tilde{B}}} \mathrm{e}^{\lambda_0 t} \sup_{1 \leqslant i \leqslant N} \sup_{\boldsymbol{x} \in D(\gamma)} |w_i^0(\boldsymbol{x})|, \quad \forall\, t > 0, \end{aligned} \tag{4.4}$$

where $D(\gamma)^c = \{\boldsymbol{x} \mid \boldsymbol{x} \in \mathbb{R}^3, \boldsymbol{x} \notin D(\gamma)\}$. In particular, one has

$$\sup_{1 \leqslant i \leqslant N} |w_i(\boldsymbol{x}_0, t)| \leqslant \mathrm{e}^{\lambda_0 t} \frac{3\pi \tilde{A}}{\tilde{B}} \int_{\frac{R}{\sqrt{t}}}^{+\infty} \exp(-\tilde{B} r^2) dr \max_{1 \leqslant i \leqslant N} \sup_{\boldsymbol{x} \in \mathbb{R}^3} |w_i^0(\boldsymbol{x})|, \quad \forall\, t > 0 \tag{4.5}$$

for any $\boldsymbol{x}_0 \in \mathbb{R}^3$ and $R > 0$, provided that $w_i^0(\boldsymbol{x}) = 0$ for any $i = 1, \dots, N$ and $\boldsymbol{x} \in B(\boldsymbol{x}_0, \sqrt{3}R) := \{\boldsymbol{x} \in \mathbb{R}^3 \mid |\boldsymbol{x} - \boldsymbol{x}_0| < \sqrt{3}R\}$.

Proof. Define $\widehat{\boldsymbol{w}}(\boldsymbol{x}, t) = (\widehat{w}_1(\boldsymbol{x}, t), \dots, \widehat{w}_N(\boldsymbol{x}, t))$ by $w_i(\boldsymbol{x}, t) = \mathrm{e}^{\lambda_0' t} \widehat{w}_i(\boldsymbol{x}, t)$, where $\lambda_0' := \sum_{i=1}^N \sum_{j=1}^N M_{ij}$ and M_{ij} is defined by (3.8). Then we have

$$\begin{cases} \dfrac{\partial}{\partial t} \widehat{w}_i - D_i \displaystyle\sum_{k=1}^{3} \dfrac{\partial^2}{\partial x_k^2} \widehat{w}_i + s \dfrac{\partial}{\partial x_3} \widehat{w}_i + (\lambda_0' - h_{ii}(\boldsymbol{x}, t)) \widehat{w}_i - \displaystyle\sum_{j=1, j \neq i}^{N} h_{ij}(\boldsymbol{x}, t) \widehat{w}_j = 0, \\ \widehat{w}_i(\boldsymbol{x}, 0) = w_i^0(\boldsymbol{x}), \end{cases} \tag{4.6}$$

where $\boldsymbol{x} \in \mathbb{R}^3$, $t > 0$, $i = 1, \dots, N$. It is easy to show that the constant-valued function $\overline{\boldsymbol{w}}(\boldsymbol{x}, t) = (\overline{w}_1(\boldsymbol{x}, t), \dots, \overline{w}_N(\boldsymbol{x}, t))$ defined by

$$\overline{w}_i(\boldsymbol{x}, t) \equiv \max\Big\{0, \max_{1 \leqslant i \leqslant N} \sup_{\boldsymbol{x} \in \mathbb{R}^3} w_i^0(\boldsymbol{x})\Big\}, \quad \boldsymbol{x} \in \mathbb{R}^3, \quad t \geqslant 0$$

is a supersolution of (4.6). Similarly, the function $\underline{\boldsymbol{w}}(\boldsymbol{x}, t) = (\underline{w}_1(\boldsymbol{x}, t), \dots, \underline{w}_N(\boldsymbol{x}, t))$ defined by

$$\underline{w}_i(\boldsymbol{x}, t) \equiv \min\Big\{0, \min_{1 \leqslant i \leqslant N} \inf_{\boldsymbol{x} \in \mathbb{R}^3} w_i^0(\boldsymbol{x})\Big\}, \quad \boldsymbol{x} \in \mathbb{R}^3, \quad t > 0$$

is a subsolution of (4.6). By Lemma 2.5, we have

$$\min\Big\{0, \min_{1 \leqslant i \leqslant N} \inf_{\boldsymbol{x} \in \mathbb{R}^3} w_i^0(\boldsymbol{x})\Big\} \leqslant \widehat{w}_i(\boldsymbol{x}, t) \leqslant \max\Big\{0, \max_{1 \leqslant i \leqslant N} \sup_{\boldsymbol{x} \in \mathbb{R}^3} w_i^0(\boldsymbol{x})\Big\}$$

for $\boldsymbol{x} \in \mathbb{R}^3$, $t > 0$ and $i = 1, \dots, N$. Therefore, for any $\boldsymbol{x} \in \mathbb{R}^3$ and $t > 0$ we have

$$\mathrm{e}^{\lambda_0' t} \min\Big\{0, \min_{1\leqslant i\leqslant N} \inf_{\boldsymbol{x}\in\mathbb{R}^3} w_i^0(\boldsymbol{x})\Big\} \leqslant w_i(\boldsymbol{x}, t) \leqslant \mathrm{e}^{\lambda_0' t} \max\Big\{0, \max_{1\leqslant i\leqslant N} \sup_{\boldsymbol{x}\in\mathbb{R}^3} w_i^0(\boldsymbol{x})\Big\},$$

where $i = 1, \dots, N$. We have proved the first three inequalities in the lemma for any $\lambda_0 \geqslant \lambda_0'$. We will determine an exact $\lambda_0 > 0$ below.

Now we prove inequality (4.4). Consider the initial-value problems

$$\begin{cases} \dfrac{\partial}{\partial t} w_i^+ - D_i \displaystyle\sum_{k=1}^{3} \dfrac{\partial^2}{\partial x_k^2} w_i^+ + s \dfrac{\partial}{\partial x_3} w_i^+ + (\lambda_0' - h_{ii}(\boldsymbol{x}, t)) w_i^+ - \displaystyle\sum_{j=1, j\neq i}^{N} h_{ij}(\boldsymbol{x}, t) w_j^+ = 0, \\ w_i^+(\boldsymbol{x}, 0) = \max\{0, w_i^0(\boldsymbol{x})\} \end{cases}$$

and

$$\begin{cases} \dfrac{\partial}{\partial t} w_i^- - D_i \displaystyle\sum_{k=1}^{3} \dfrac{\partial^2}{\partial x_k^2} w_i^- + s \dfrac{\partial}{\partial x_3} w_i^- + (\lambda_0' - h_{ii}(\boldsymbol{x}, t)) w_i^- - \displaystyle\sum_{j=1, j\neq i}^{N} h_{ij}(\boldsymbol{x}, t) w_j^- = 0, \\ w_i^-(\boldsymbol{x}, 0) = \min\{0, w_i^0(\boldsymbol{x})\}, \end{cases}$$

where $\boldsymbol{x} \in \mathbb{R}^3$, $t > 0$ and $i = 1, \dots, N$. It is easy to show that $w_i^+(\boldsymbol{x}, t) \geqslant 0$, $w_i^-(\boldsymbol{x}, t) \leqslant 0$ and $w_i^-(\boldsymbol{x}, t) \leqslant \widehat{w}_i(\boldsymbol{x}, t) \leqslant w_i^+(\boldsymbol{x}, t)$ for $\boldsymbol{x} \in \mathbb{R}^3$, $t > 0$ and $i = 1, \dots, N$. Consider

$$\begin{cases} \dfrac{\partial}{\partial t} \widetilde{w}_i - D_i \displaystyle\sum_{k=1}^{3} \dfrac{\partial^2}{\partial x_k^2} \widetilde{w}_i + s \dfrac{\partial}{\partial x_3} \widetilde{w}_i + (\lambda_0' - M_{ii}) \widetilde{w}_i - \displaystyle\sum_{j=1, j\neq i}^{N} M_{ij} \widetilde{w}_j = 0, \quad \boldsymbol{x} \in \mathbb{R}^3, \quad t > 0, \\ \widetilde{w}_i(\boldsymbol{x}, 0) = |w_i^0(\boldsymbol{x})|, \quad \boldsymbol{x} \in \mathbb{R}^3. \end{cases} \tag{4.7}$$

By virtue of $\widetilde{w}_i(\boldsymbol{x}, 0) \geqslant w_i^+(\boldsymbol{x}, 0)$, it follows from Lemma 2.5 that $\widetilde{w}_i(\boldsymbol{x}, t) \geqslant w_i^+(\boldsymbol{x}, t)$ for $\boldsymbol{x} \in \mathbb{R}^3$, $t > 0$ and $1 \leqslant i \leqslant N$. Similarly, we have $\widetilde{w}_i(\boldsymbol{x}, t) \geqslant -w_i^-(\boldsymbol{x}, t)$ for $\boldsymbol{x} \in \mathbb{R}^3$, $t > 0$ and $1 \leqslant i \leqslant N$. Consequently, we obtain

$$|\widehat{w}_i(\boldsymbol{x}, t)| \leqslant \widetilde{w}_i(\boldsymbol{x}, t), \quad \forall\, \boldsymbol{x} \in \mathbb{R}^3, \quad t > 0, \quad 1 \leqslant i \leqslant N.$$

Following [15, Chapter 9, Theorems 2 and 3], we know that there exists a smooth $N \times N$ matrix-valued function $\boldsymbol{\Psi}(\boldsymbol{x}, \boldsymbol{y}, t, s)$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3$ and $0 \leqslant s < t \leqslant 2$ such that

$$\widetilde{\boldsymbol{w}}(\boldsymbol{x}, t) = \int_{\mathbb{R}^3} \boldsymbol{\Psi}(\boldsymbol{x}, \boldsymbol{y}, t, 0) \widetilde{\boldsymbol{w}}(\boldsymbol{y}, 0) d\boldsymbol{y}.$$

Since the coefficients in (4.7) are constants, the matrix-valued function $\boldsymbol{\Psi}(\boldsymbol{x}, \boldsymbol{y}, t, s)$ can be rewritten into $\boldsymbol{\Psi}(\boldsymbol{x} - \boldsymbol{y}, t - s)$, see [15, Subsection 9.2]. It follows that

$$\widetilde{\boldsymbol{w}}(\boldsymbol{x}, t) = \int_{\mathbb{R}^3} \boldsymbol{\Psi}(\boldsymbol{x} - \boldsymbol{y}, t) \widetilde{\boldsymbol{w}}(\boldsymbol{y}, 0) d\boldsymbol{y}$$

for $\boldsymbol{x} \in \mathbb{R}^3$ and $0 < t \leqslant 2$. Consequently, by the uniqueness of solutions we have

$$\begin{aligned} \widetilde{\boldsymbol{w}}(\boldsymbol{x}, t) = & \int_{\mathbb{R}^3} \boldsymbol{\Psi}(\boldsymbol{x} - \boldsymbol{y}_1, 1) d\boldsymbol{y}_1 \int_{\mathbb{R}^3} \boldsymbol{\Psi}(\boldsymbol{y}_1 - \boldsymbol{y}_2, 1) d\boldsymbol{y}_2 \\ & \cdots \int_{\mathbb{R}^3} \boldsymbol{\Psi}(\boldsymbol{y}_{k-1} - \boldsymbol{y}_k, 1) d\boldsymbol{y}_k \int_{\mathbb{R}^3} \boldsymbol{\Psi}(\boldsymbol{y}_k - \boldsymbol{y}, t - k) \widetilde{\boldsymbol{w}}(\boldsymbol{y}, 0) d\boldsymbol{y} \end{aligned}$$

for any $t > 0$, where $k = \max\{[t - 1], 0\}$ and $[s] = \max\{n : n \in \mathbb{Z}, n \leqslant s\}$ for any $s \in \mathbb{R}$. Therefore, we have

$$\begin{aligned} \widetilde{w}_i(\boldsymbol{x}, t) = & \int_{\mathbb{R}^3} \boldsymbol{\Psi}_i(\boldsymbol{x} - \boldsymbol{y}_1, 1) d\boldsymbol{y}_1 \int_{\mathbb{R}^3} \boldsymbol{\Psi}(\boldsymbol{y}_1 - \boldsymbol{y}_2, 1) d\boldsymbol{y}_2 \cdots \int_{\mathbb{R}^3} \boldsymbol{\Psi}(\boldsymbol{y}_{k-1} - \boldsymbol{y}_k, 1) d\boldsymbol{y}_k \\ & \times \int_{\mathbb{R}^3} \boldsymbol{\Psi}(\boldsymbol{y}_k - \boldsymbol{y}, t - k)(\chi_{D(\gamma)}(\boldsymbol{y}) \widetilde{\boldsymbol{w}}(\boldsymbol{y}, 0) + \chi_{D(\gamma)^c}(\boldsymbol{y}) \widetilde{\boldsymbol{w}}(\boldsymbol{y}, 0)) d\boldsymbol{y} \end{aligned}$$

for any $t>0$ and $i\in\{1,2,\ldots,N\}$. Consequently, we have

$$\begin{aligned}\widetilde{w}_i(\boldsymbol{x},t)\leqslant{}&\int_{\mathbb{R}^3}\sum_{j=1}^N|\Psi_{ij}(\boldsymbol{x}-\boldsymbol{y}_1,1)|d\boldsymbol{y}_1\int_{\mathbb{R}^3}\sum_{l,j=1}^N|\Psi_{lj}(\boldsymbol{y}_1-\boldsymbol{y}_2,1)|d\boldsymbol{y}_2\\&\cdots\int_{\mathbb{R}^3}\sum_{l,j=1}^N|\Psi_{lj}(\boldsymbol{y}_{k-1}-\boldsymbol{y}_k,1)|d\boldsymbol{y}_k\\&\times\int_{\mathbb{R}^3}\sum_{l,j=1}^N|\Psi_{lj}(\boldsymbol{y}_k-\boldsymbol{y},t-k)|\chi_{D(\gamma)}(\boldsymbol{y})d\boldsymbol{y}\max_{1\leqslant j\leqslant N}\sup_{\boldsymbol{y}\in D(\gamma)}|w_j^0(\boldsymbol{y})|\\&+\int_{\mathbb{R}^3}\sum_{j=1}^N|\Psi_{ij}(\boldsymbol{x}-\boldsymbol{y}_1,1)|d\boldsymbol{y}_1\int_{\mathbb{R}^3}\sum_{l,j=1}^N|\Psi_{lj}(\boldsymbol{y}_1-\boldsymbol{y}_2,1)|d\boldsymbol{y}_2\\&\cdots\int_{\mathbb{R}^3}\sum_{l,j=1}^N|\Psi_{lj}(\boldsymbol{y}_{k-1}-\boldsymbol{y}_k,1)|d\boldsymbol{y}_k\\&\times\int_{\mathbb{R}^3}\sum_{l,j=1}^N|\Psi_{lj}(\boldsymbol{y}_k-\boldsymbol{y},t-k)|\chi_{D(\gamma)^c}(\boldsymbol{y})d\boldsymbol{y}\max_{1\leqslant j\leqslant N}\sup_{\boldsymbol{y}\in D(\gamma)^c}|w_j^0(\boldsymbol{y})|.\end{aligned}\tag{4.8}$$

By [15, Chapter 9], there exist positive numbers $\tilde{A}\geqslant 1$ and $\tilde{B}\leqslant 1$ such that

$$\sum_{1\leqslant l,j\leqslant N}|\Psi_{lj}(\boldsymbol{x}-\boldsymbol{y},t-s)|\leqslant\tilde{A}(t-s)^{-\frac{3}{2}}\exp\bigg(-\tilde{B}\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{t-s}\bigg)$$

for any $0\leqslant s<t\leqslant 2$. Since $\gamma\leqslant|\mathrm{dist}(\boldsymbol{x},\Gamma)-\mathrm{dist}(\boldsymbol{y},\Gamma)|\leqslant|\boldsymbol{x}-\boldsymbol{y}|$ for any $\boldsymbol{x}\in D(2\gamma)$ and $\boldsymbol{y}\in D(\gamma)^c$, we have that

$$\begin{aligned}&\int_{\mathbb{R}^3}\sum_{j=1}^N|\Psi_{ij}(\boldsymbol{x}-\boldsymbol{y}_1,1)|d\boldsymbol{y}_1\int_{\mathbb{R}^3}\sum_{l,j=1}^N|\Psi_{lj}(\boldsymbol{y}_1-\boldsymbol{y}_2,1)|d\boldsymbol{y}_2\\&\quad\cdots\int_{\mathbb{R}^3}\sum_{l,j=1}^N|\Psi_{lj}(\boldsymbol{y}_{k-1}-\boldsymbol{y}_k,1)|d\boldsymbol{y}_k\int_{\mathbb{R}^3}\sum_{l,j=1}^N|\Psi_{lj}(\boldsymbol{y}_k-\boldsymbol{y},t-k)|\chi_{D(\gamma)^c}(\boldsymbol{y})d\boldsymbol{y}\\&\quad\leqslant\tilde{A}^{k+1}\int_{\mathbb{R}^3}\exp(-\tilde{B}|\boldsymbol{x}-\boldsymbol{y}_1|^2)d\boldsymbol{y}_1\int_{\mathbb{R}^3}\exp(-\tilde{B}|\boldsymbol{y}_1-\boldsymbol{y}_2|^2)d\boldsymbol{y}_2\\&\qquad\cdots\int_{\mathbb{R}^3}\exp(-\tilde{B}|\boldsymbol{y}_{k-1}-\boldsymbol{y}_k|^2)d\boldsymbol{y}_k\int_{\mathbb{R}^3}(t-k)^{-\frac{3}{2}}\exp\bigg(-\tilde{B}\frac{|\boldsymbol{y}_k-\boldsymbol{y}|^2}{t-k}\bigg)\chi_{D(\gamma)^c}(\boldsymbol{y})d\boldsymbol{y}\\&\quad=\tilde{A}^{k+1}\int_{\mathbb{R}^3}\exp\bigg(-\frac{\tilde{B}}{2}|\boldsymbol{x}-\boldsymbol{y}_1|^2\bigg)d\boldsymbol{y}_1\int_{\mathbb{R}^3}\exp\bigg(-\frac{\tilde{B}}{2}|\boldsymbol{y}_1-\boldsymbol{y}_2|^2\bigg)d\boldsymbol{y}_2\\&\qquad\cdots\int_{\mathbb{R}^3}\exp\bigg(-\frac{\tilde{B}}{2}|\boldsymbol{y}_{k-1}-\boldsymbol{y}_y|^2\bigg)d\boldsymbol{y}_k\int_{\mathbb{R}^3}(t-k)^{-\frac{3}{2}}\exp\bigg(-\frac{\tilde{B}}{2}|\boldsymbol{x}-\boldsymbol{y}_1|^2\bigg)\\&\qquad\cdots\exp\bigg(-\tilde{B}\frac{|\boldsymbol{y}_k-\boldsymbol{y}|^2}{t-k}\bigg)\chi_{D(\gamma)^c}(\boldsymbol{y})d\boldsymbol{y}\\&\quad\leqslant\tilde{A}^{k+1}\bigg(\frac{t}{t-k}\bigg)^{\frac{3}{2}}\bigg(\int_{\mathbb{R}^3}\exp\bigg(-\frac{\tilde{B}|\boldsymbol{z}|^2}{2}\bigg)d\boldsymbol{z}\bigg)^k\int_{\mathbb{R}^3}t^{-\frac{3}{2}}\exp\bigg(-\tilde{B}\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{t}\bigg)\chi_{D(\gamma)^c}(\boldsymbol{y})d\boldsymbol{y}\\&\quad\leqslant\tilde{A}^{k+1}(1+t)^{\frac{3}{2}}\bigg(\int_{\mathbb{R}^3}\exp\bigg(-\frac{\tilde{B}|\boldsymbol{z}|^2}{2}\bigg)d\boldsymbol{z}\bigg)^k\int_{\mathbb{R}^3}t^{-\frac{3}{2}}\exp\bigg(-\tilde{B}\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{t}\bigg)\chi_{D(\gamma)^c}(\boldsymbol{y})d\boldsymbol{y}\\&\quad\leqslant\mathrm{e}^{2t}\tilde{A}^{k+1}\bigg(\frac{2\sqrt{2}\pi\sqrt{\pi}}{\tilde{B}\sqrt{\tilde{B}}}\bigg)^k\int_{\mathbb{R}^3\setminus B(\boldsymbol{x},\gamma)}t^{-\frac{3}{2}}\exp\bigg(-\tilde{B}\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{t}\bigg)d\boldsymbol{y}\\&\quad\leqslant 3\mathrm{e}^{2t}\tilde{A}^{k+1}\bigg(\frac{2\sqrt{2}\pi\sqrt{\pi}}{\tilde{B}\sqrt{\tilde{B}}}\bigg)^k\int_{\boldsymbol{y}\in\mathbb{R}^3,|y_1|\geqslant\frac{\sqrt{3}\gamma}{3}}t^{-\frac{3}{2}}\exp\bigg(-\tilde{B}\frac{|\boldsymbol{y}|^2}{t}\bigg)d\boldsymbol{y}\end{aligned}$$

$$= 3e^{2t}\frac{\tilde{A}\pi}{\tilde{B}}\bigg(\frac{2\sqrt{2}\pi\sqrt{\pi}\tilde{A}}{\tilde{B}\sqrt{\tilde{B}}}\bigg)^t\int_{\frac{\sqrt{3}\gamma}{3\sqrt{t}}}^{+\infty}\exp(-\tilde{B}r^2)dr$$

for $\boldsymbol{x}\in D(2\gamma)$, where we have used the facts that $k=0$ for $t\in(0,2)$, $t-k<2$, and $k\leqslant t$ and $(1+t)^{\frac{3}{2}}<e^{2t}$ for $t>0$. In addition, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3}\sum_{j=1}^N|\Psi_{ij}(\boldsymbol{x}-\boldsymbol{y}_1,1)|d\boldsymbol{y}_1\int_{\mathbb{R}^3}\sum_{l,j=1}^N|\Psi_{lj}(\boldsymbol{y}_1-\boldsymbol{y}_2,1)|d\boldsymbol{y}_2\\
&\quad\cdots\int_{\mathbb{R}^3}\sum_{l,j=1}^N|\Psi_{lj}(\boldsymbol{y}_{k-1}-\boldsymbol{y}_k,1)|d\boldsymbol{y}_k\int_{\mathbb{R}^3}\sum_{l,j=1}^N|\Psi_{lj}(\boldsymbol{y}_k-\boldsymbol{y},t-k)|\chi_{D(\gamma)}(\boldsymbol{y})d\boldsymbol{y}\\
&\quad\leqslant\tilde{A}^{k+1}(1+t)^{\frac{3}{2}}\bigg(\int_{\mathbb{R}^3}\exp\bigg(-\frac{\tilde{B}}{2}|\boldsymbol{z}|^2\bigg)d\boldsymbol{z}\bigg)^k\int_{\mathbb{R}^3}t^{-\frac{3}{2}}\exp\bigg(-\tilde{B}\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{t}\bigg)\chi_{D(\gamma)}(\boldsymbol{y})d\boldsymbol{y}\\
&\quad\leqslant e^{2t}\tilde{A}^{k+1}\bigg(\frac{2\sqrt{2}\pi\sqrt{\pi}}{\tilde{B}\sqrt{\tilde{B}}}\bigg)^k\int_{\mathbb{R}^3}t^{-\frac{3}{2}}\exp\bigg(-\tilde{B}\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{t}\bigg)d\boldsymbol{y}\\
&\quad\leqslant e^{2t}\frac{\tilde{A}\pi\sqrt{\pi}}{\tilde{B}\sqrt{\tilde{B}}}\bigg(\frac{2\sqrt{2}\pi\sqrt{\pi}\tilde{A}}{\tilde{B}\sqrt{\tilde{B}}}\bigg)^t.
\end{aligned}$$

Thus, letting $\lambda_0:=\lambda_0'+2+\ln(\frac{2\sqrt{2}\pi\sqrt{\pi}\tilde{A}}{\tilde{B}\sqrt{\tilde{B}}})$ yields the inequality (4.4). Note that the constants $\tilde{A}$ and $\tilde{B}$ are independent of γ.

To prove the inequality (4.5), we need only to replace $\boldsymbol{x}$ and $D(\gamma)$ with $\boldsymbol{x}_0$ and $B(\boldsymbol{x}_0,\sqrt{3}R)$ in (4.8). This completes the proof. □

Remark 4.3. The positive constants $\tilde{A}$, $\tilde{B}$ and λ_0 in Lemma 4.2 are independent of the functions $h_{ij}(x,t)\in C(\mathbb{R}^3\times\mathbb{R}_+)$ $(i,j=1,\dots,N)$ satisfying (4.2).

As in [48], in the following we show that the pyramidal traveling front $\boldsymbol{V}$ converges to two-dimensional V-form fronts on edges of the pyramid at infinity. For each j $(1\leqslant j\leqslant n)$ we consider a plane perpendicular to an edge $\Gamma_j=S_j\cap S_{j+1}$. Then the cross section of $-x_3=\max\{h_j(\boldsymbol{x}'),h_{j+1}(\boldsymbol{x}')\}$ in this plane has a V-form front. Let $\boldsymbol{V}^j$ be the two-dimensional V-form front as in Theorem 4.1 corresponding to the cross section $-x_3=\max\{h_j(\boldsymbol{x}'),h_{j+1}(\boldsymbol{x}')\}$. We first determine the exact formulation of $\boldsymbol{V}^j$.

Let $A_{n+1}:=A_1$ and $B_{n+1}:=B_1$. Define

$$p_j:=A_jB_{j+1}-A_{j+1}B_j>0,\quad q_j:=\sqrt{(A_{j+1}-A_j)^2+(B_{j+1}-B_j)^2}>0,\quad 1\leqslant j\leqslant n.$$

Take

$$\nu_j=\frac{1}{\sqrt{1+m_*^2}}\{m_*A_j,m_*B_j,1\},\quad j=1,\dots,n+1.$$

The direction of Γ_j is given by

$$\nu_{j+1}\times\nu_j=\frac{1}{\sqrt{m_*^2p_j^2+q_j^2}}\left\{\begin{array}{c}B_{j+1}-B_j\\ A_j-A_{j+1}\\ m_*(A_{j+1}B_j-A_jB_{j+1})\end{array}\right\},$$

and the traveling direction of the two-dimensional V-form wave $\boldsymbol{V}^j$ is given by

$$(\nu_{j+1}\times\nu_j)\times\frac{\nu_{j+1}-\nu_j}{|\nu_{j+1}-\nu_j|}=\frac{1}{q_j\sqrt{m_*^2p_j^2+q_j^2}}\left\{\begin{array}{c}m_*(B_{j+1}-B_j)p_j\\ m_*(A_j-A_{j+1})p_j\\ q_j^2\end{array}\right\}.$$

Let s_j be the speed of $\boldsymbol{V}^j$ and $2\theta_j$ $(0<\theta_j<\pi/2)$ be the angle between S_j and S_{j+1}. Then we get

$$s_j\sin\theta_j=c,\quad\sin\theta_j=\sqrt{m_*^2p_j^2+q_j^2}(q_j\sqrt{1+m_*^2})^{-1},\quad s_j=sq_j(\sqrt{m_*^2p_j^2+q_j^2})^{-1}.$$

The speed of $\boldsymbol{V}^j$ toward the x_3-axis equals

$$s_j\sqrt{m_*^2p_j^2+q_j^2}/q_j = c\sqrt{1+m_*^2} = s,$$

which coincides with the speed of $\boldsymbol{V}$. Let

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \boldsymbol{R}_j \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}, \quad \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = \boldsymbol{R}_j^{\mathrm{T}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

where $\boldsymbol{R}_j^{\mathrm{T}}$ is the transposed matrix of $\boldsymbol{R}_j$. Here we take

$$\boldsymbol{R}_j = \begin{pmatrix} \frac{A_{j+1}-A_j}{q_j} & \frac{m_*(B_{j+1}-B_j)p_j}{q_j\sqrt{m_*^2p_j^2+q_j^2}} & \frac{B_{j+1}-B_j}{\sqrt{m_*^2p_j^2+q_j^2}} \\ \frac{B_{j+1}-B_j}{q_j} & \frac{m_*(A_j-A_{j+1})p_j}{q_j\sqrt{m_*^2p_j^2+q_j^2}} & \frac{A_j-A_{j+1}}{\sqrt{m_*^2p_j^2+q_j^2}} \\ 0 & \frac{q_j}{\sqrt{m_*^2p_j^2+q_j^2}} & -\frac{m_*p_j}{\sqrt{m_*^2p_j^2+q_j^2}} \end{pmatrix}.$$

Define $\boldsymbol{V}^j$ as $\boldsymbol{V}^j(\boldsymbol{x}) := \boldsymbol{\Phi}(\xi,\eta;s_j)$. Direct calculations show that

$$-D_i\frac{\partial^2}{\partial\xi^2}\Phi_i - D_i\frac{\partial^2}{\partial\eta^2}\Phi_i + s_j\frac{\partial}{\partial\eta}\Phi_i - F^i(\boldsymbol{\Phi}) = 0, \quad \forall\,(\xi,\eta)\in\mathbb{R}^2, \quad i=1,\dots,N.$$

Hence, for each j $(1\leqslant j\leqslant n)$, $\boldsymbol{V}^j(\boldsymbol{x})$ satisfies (1.6). We call $\boldsymbol{V}^j$ a *planar V-form front* corresponding to an edge Γ_j.

Set

$$Q_j := \{\boldsymbol{x}\in\mathbb{R}^3 \mid \mathrm{dist}(\boldsymbol{x},\Gamma) = \mathrm{dist}(\boldsymbol{x},\Gamma_j)\}, \quad 1\leqslant j\leqslant n.$$

Then we have $\mathbb{R}^3 = \bigcup_{j=1}^n Q_j$. Define

$$\hat{\boldsymbol{V}}(\boldsymbol{x}) := \bigvee_{1\leqslant j\leqslant n} \boldsymbol{V}^j(\boldsymbol{x}).$$

We have that $\hat{\boldsymbol{V}}(\boldsymbol{x})$ is strictly monotone increasing in x_3 due to the strict monotonicity of $\boldsymbol{V}^j(\boldsymbol{x})$ in x_3. In addition, $\hat{\boldsymbol{V}}(\boldsymbol{x})$ has the following properties.

Lemma 4.4. *$\hat{\boldsymbol{V}}(\boldsymbol{x})$ satisfies $\boldsymbol{v}^-(\boldsymbol{x}) < \hat{\boldsymbol{V}}(\boldsymbol{x}) < \boldsymbol{V}(\boldsymbol{x})$ for $\boldsymbol{x}\in\mathbb{R}^3$ and*

$$\lim_{\gamma\to\infty}\sup_{\boldsymbol{x}\in D(\gamma)} |\hat{\boldsymbol{V}}(\boldsymbol{x}) - \boldsymbol{v}^-(\boldsymbol{x})| = 0. \tag{4.9}$$

Proof. By Theorem 4.1 we have

$$\boldsymbol{U}\left(\frac{c}{s}(x_3+h_j(\boldsymbol{x}'))\right) \vee \boldsymbol{U}\left(\frac{c}{s}(x_3+h_{j+1}(\boldsymbol{x}'))\right) < \boldsymbol{V}^j(\boldsymbol{x}), \quad \boldsymbol{x}\in\mathbb{R}^3.$$

It follows that $\boldsymbol{v}^-(\boldsymbol{x}) = \boldsymbol{U}(\frac{c}{s}(x_3+h(\boldsymbol{x}'))) < \hat{\boldsymbol{V}}(\boldsymbol{x})$ for $\boldsymbol{x}\in\mathbb{R}^3$. In addition, by

$$\boldsymbol{U}\left(\frac{c}{s}(x_3+h_j(\boldsymbol{x}'))\right) \vee \boldsymbol{U}\left(\frac{c}{s}(x_3+h_{j+1}(\boldsymbol{x}'))\right) < \boldsymbol{V}^j(\boldsymbol{x}),$$

we get $\boldsymbol{V}^j(\boldsymbol{x}) \leqslant \boldsymbol{V}(\boldsymbol{x})$ for $\boldsymbol{x}\in\mathbb{R}^3$. Therefore, we have $\hat{\boldsymbol{V}}(\boldsymbol{x}) \leqslant \boldsymbol{V}(\boldsymbol{x})$ for $\boldsymbol{x}\in\mathbb{R}^3$. Finally, (4.9) follows from (3.13). This completes the proof. □

Assume that $\boldsymbol{v}^0\in[\boldsymbol{E}^-,\boldsymbol{E}^+]$ satisfies (1.8). Let

$$\boldsymbol{v}(\boldsymbol{x},t;\boldsymbol{v}^0) = (v_1(\boldsymbol{x},t;\boldsymbol{v}^0),\dots,v_N(\boldsymbol{x},t;\boldsymbol{v}^0))$$

be the solution of (1.4) and (1.5). By Lemma 4.2, we have

$$\begin{aligned}&\max_{1\leqslant i\leqslant N}\sup_{\boldsymbol{x}\in D(2\gamma)}|v_i(\boldsymbol{x},t;\boldsymbol{v}^0)-V_i(\boldsymbol{x})|\\&\quad\leqslant \mathrm{e}^{\lambda_0 t}\frac{3\pi\tilde{A}}{\tilde{B}}\int_{\frac{\sqrt{3}\gamma}{3\sqrt{t}}}^{+\infty}\exp(-\tilde{B}r^2)dr\max_{1\leqslant i\leqslant N}\sup_{\boldsymbol{x}\in D(\gamma)^c}|v_i^0(\boldsymbol{x})-V_i(\boldsymbol{x})|\\&\qquad+\frac{\pi\sqrt{\pi}\tilde{A}}{\tilde{B}\sqrt{\tilde{B}}}\mathrm{e}^{\lambda_0 t}\sup_{1\leqslant i\leqslant N}\sup_{\boldsymbol{x}\in D(\gamma)}|v_i^0(\boldsymbol{x})-V_i(\boldsymbol{x})|\end{aligned}\tag{4.10}$$

for any $\gamma>0$ and $t>0$. It follows that

$$\lim_{\gamma\to\infty}\max_{1\leqslant i\leqslant N}\sup_{\boldsymbol{x}\in D(\gamma)}|v_i(\boldsymbol{x},t;\boldsymbol{v}^0)-V_i(\boldsymbol{x})|=0\quad\text{for any fixed}\quad t>0,$$

which implies

$$\lim_{\gamma\to\infty}\sup_{\boldsymbol{x}\in D(\gamma)}|\boldsymbol{v}(\boldsymbol{x},t;\boldsymbol{v}^0)-\boldsymbol{v}^-(\boldsymbol{x})|=0\quad\text{for any fixed}\quad t>0,\tag{4.11}$$

$$\lim_{\gamma\to\infty}\max_{1\leqslant j\leqslant n}\sup_{\boldsymbol{x}\in D(\gamma),\boldsymbol{x}\in Q_j}|\boldsymbol{v}(\boldsymbol{x},t;\boldsymbol{v}^0)-\boldsymbol{V}^j(\boldsymbol{x})|=0\quad\text{for any fixed}\quad t>0\tag{4.12}$$

and

$$\lim_{\gamma\to\infty}\sup_{\boldsymbol{x}\in D(\gamma)}|\boldsymbol{v}(\boldsymbol{x},t;\boldsymbol{v}^0)-\hat{\boldsymbol{V}}(\boldsymbol{x})|=0\quad\text{for any fixed}\quad t>0.\tag{4.13}$$

Now we state a proposition which plays a key role in the following estimates.

Proposition 4.5. *Assume that* $\boldsymbol{v}^0\in[\boldsymbol{E}^-,\boldsymbol{E}^+]$ *satisfies* (1.8). *For any given* $\varepsilon_1>0$, *one can choose* $T^*>0$ *large enough such that*

$$\lim_{R\to\infty}\max_{1\leqslant j\leqslant n}\sup_{|\boldsymbol{x}|\geqslant R,\boldsymbol{x}\in Q_j}|\boldsymbol{v}(\boldsymbol{x},t;\boldsymbol{v}^0)-\boldsymbol{V}^j(\boldsymbol{x})|<\varepsilon_1\quad\textit{for any fixed}\quad t\geqslant T^*.\tag{4.14}$$

Proof. Set

$$I_j:=\Omega_j\cap\Omega_{j+1}=\left\{r\begin{pmatrix}A_j+A_{j+1}\\B_j+B_{j+1}\end{pmatrix}\,\middle|\,r\geqslant 0\right\},\quad 1\leqslant j\leqslant n-1,$$

$$I_n:=\Omega_n\cap\Omega_1=\left\{r\begin{pmatrix}A_n+A_1\\B_n+B_1\end{pmatrix}\,\middle|\,r\geqslant 0\right\}.$$

Then I_j is the projection of Γ_j onto the x_1-x_2 plane and $\bigcup_{j=1}^n I_j$ is the projection of Γ onto the x_1-x_2 plane.

Fix $j\in\{1,\ldots,n\}$. Without loss of generality, we assume that $\boldsymbol{x}\in Q_j$ as $|\boldsymbol{x}|\to\infty$. Since $(\partial/\partial x_1)^2+(\partial/\partial x_2)^2$ is invariant under rotations on the x_1-x_2 plane, we assume $\Omega_j\cap\Omega_{j+1}=\{(0,x_2,0)\mid x_2\geqslant 0\}$, $(A_j,B_j)=(A,B)$ and $(A_{j+1},B_{j+1})=(-A,B)$, where $A>0$, $B>0$ and $A^2+B^2=1$. Two planes S_{j+1} and S_j are $-x_3=m_*(-Ax_1+Bx_2)$ and $-x_3=m_*(Ax_1+Bx_2)$, respectively. The common line Γ_j of them is $x_1=0$, $-x_3=m_*Bx_2$. The projection of Q_j onto the x_1-x_2 plane is given by $\{x_2\geqslant a|x_1|,x_1\geqslant 0\}\cup\{x_2\geqslant b|x_1|,x_1\leqslant 0\}$ for some $a>0$ and $b>0$.

By the assumption on $\boldsymbol{v}^0$, we have

$$\lim_{\gamma\to\infty}\sup_{\boldsymbol{x}\in D(\gamma),\boldsymbol{x}\in Q_j}\left|\boldsymbol{v}^0(\boldsymbol{x})-\boldsymbol{U}\left(\frac{c}{s}(x_3+m_*Bx_2+m_*A|x_1|)\right)\right|=0.$$

The unit normal vector of the common line Γ_j directing downwards and lying on the plane $\{x_1=0\}$ is given by

$$\frac{1}{\sqrt{1+m_*^2B^2}}\begin{pmatrix}0\\m_*B\\-1\end{pmatrix}.$$

Since $2\theta_j$ is the angle between S_j and S_{j+1} $(0<\theta_j<\pi/2)$, we have that $\sin\theta_j=\sqrt{1+m_*^2B^2}/\sqrt{1+m_*^2}$. In this case, we make a change of variables as follows:

$$\begin{pmatrix}\xi\\ \eta\\ \zeta\end{pmatrix}=\begin{pmatrix}-1 & 0 & 0\\ 0 & \frac{m_*B}{\sqrt{1+m_*^2B^2}} & \frac{1}{\sqrt{1+m_*^2B^2}}\\ 0 & \frac{1}{\sqrt{1+m_*^2B^2}} & -\frac{m_*B}{\sqrt{1+m_*^2B^2}}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}.$$

Then we have

$$\boldsymbol{U}\bigg(\frac{c}{s_j}\bigg(\eta+\frac{\sqrt{s_j^2-c^2}}{c}|\xi|\bigg)\bigg)=\boldsymbol{U}\bigg(\frac{c}{s}(x_3+m_*Bx_2+m_*A|x_1|)\bigg),$$

where $s_j=\frac{s}{\sqrt{1+m_*^2B^2}}$. It is obvious that

$$\boldsymbol{V}^j(\boldsymbol{x})=\boldsymbol{\Phi}(\xi,\eta;s_j)=\boldsymbol{\Phi}\bigg(-x_1,\frac{x_3+m_*Bx_2}{\sqrt{1+m_*^2B^2}};s_j\bigg)$$

is a solution of (1.6). Let $\tilde{\boldsymbol{W}}(\xi,\eta,t)=(\tilde{W}_1(\xi,\eta,t;\tilde{\boldsymbol{W}}^0),\dots,\tilde{W}_N(\xi,\eta,t;\tilde{\boldsymbol{W}}^0))$ be the solution of

$$\begin{cases}\frac{\partial}{\partial t}\tilde{\boldsymbol{W}}-\boldsymbol{D}\frac{\partial^2}{\partial\xi^2}\tilde{\boldsymbol{W}}-\boldsymbol{D}\frac{\partial^2}{\partial\eta^2}\tilde{\boldsymbol{W}}+s_j\frac{\partial}{\partial\eta}\tilde{\boldsymbol{W}}-\boldsymbol{F}(\tilde{\boldsymbol{W}})=0, \quad (\xi,\eta)\in\mathbb{R}^2, \quad t>0,\\ \tilde{\boldsymbol{W}}(\xi,\eta,0)=\tilde{\boldsymbol{W}}^0(\xi,\eta), \quad (\xi,\eta)\in\mathbb{R}^2.\end{cases} \tag{4.15}$$

Taking $\boldsymbol{W}^0(\boldsymbol{x})=\tilde{\boldsymbol{W}}^0(-x_1,\frac{x_3+m_*Bx_2}{\sqrt{1+m_*^2B^2}})$, we have that $\boldsymbol{W}(\boldsymbol{x},t;\boldsymbol{W}^0)=\tilde{\boldsymbol{W}}(\xi,\eta,t;\tilde{\boldsymbol{W}}^0)$ satisfies

$$\begin{cases}\frac{\partial}{\partial t}\boldsymbol{W}-\boldsymbol{D}\Delta\boldsymbol{W}+s\frac{\partial}{\partial x_3}\boldsymbol{W}-\boldsymbol{F}(\boldsymbol{W})=\boldsymbol{0}, \quad \boldsymbol{x}\in\mathbb{R}^3, \quad t>0,\\ \boldsymbol{W}(\boldsymbol{x},0)=\boldsymbol{W}^0(\boldsymbol{x}), \quad \boldsymbol{x}\in\mathbb{R}^3.\end{cases} \tag{4.16}$$

Utilizing (4.9) and the assumption on $\boldsymbol{v}^0$, we have

$$\lim_{\gamma\to\infty}\sup_{\boldsymbol{x}\in D(\gamma)\cap Q_j}|\boldsymbol{v}^0(\boldsymbol{x})-\boldsymbol{V}^j(\boldsymbol{x})|=0.$$

Choose functions $g_i(\cdot)\in C(\mathbb{R})\cap L^\infty(\mathbb{R})$ $(i=1,\dots,N)$ with

$$\begin{aligned}&g_i(\gamma)=\sup_{\boldsymbol{x}\in D(\gamma)\cap Q_j}|v_i^0(\boldsymbol{x})-V_i^j(\boldsymbol{x})| \quad \text{for} \quad \gamma\geqslant 1,\\ &\sup_{\boldsymbol{x}\in D(\gamma)\cap Q_j}|v_i^0(\boldsymbol{x})-V_i^j(\boldsymbol{x})|\leqslant g_i(\gamma)\leqslant E_i^+-E_i^-+1+\|v_i^0-E_i^-\|_{L^\infty(\mathbb{R}^3)} \quad \text{for} \quad 0<\gamma<1,\\ &g_i'(\gamma)\leqslant 0 \quad \text{for} \quad 0<\gamma<1,\\ &g_i(\gamma)=g_i(-\gamma) \quad \text{for} \quad \gamma\in\mathbb{R}.\end{aligned}$$

It is obvious that $g_i(\gamma)$ is monotone nonincreasing in $\gamma>0$ and satisfies $\lim_{\gamma\to\infty}g_i(\gamma)=0$. Since

$$\mathrm{dist}(\boldsymbol{x},\Gamma)=\mathrm{dist}(\boldsymbol{x},\Gamma_j)=\frac{\sqrt{(1+m_*^2B^2)x_1^2+(x_3+m_*Bx_2)^2}}{\sqrt{1+m_*^2B^2}} \quad \text{for} \quad \boldsymbol{x}\in Q_j,$$

we have, for $\boldsymbol{x}\in Q_j$, that

$$|v_i^0(\boldsymbol{x})-V_i^j(\boldsymbol{x})|\leqslant g_i(\mathrm{dist}(\boldsymbol{x},\Gamma))=g_i\bigg(\frac{\sqrt{(1+m_*^2B^2)x_1^2+(x_3+m_*Bx_2)^2}}{\sqrt{1+m_*^2B^2}}\bigg). \tag{4.17}$$

We study (4.15) for $\tilde{\boldsymbol{W}}^{\pm,0}(\xi,\eta)=(\tilde{W}_1^{\pm,0}(\xi,\eta),\dots,\tilde{W}_N^{\pm,0}(\xi,\eta))$ with

$$\tilde{W}_i^{+,0}(\xi,\eta):=\min\{\Phi_i(\xi,\eta;\bar{s})+g_i(\sqrt{\xi^2+\eta^2}),E_i^+\}$$

and

$$\tilde{W}_i^{-,0}(\xi,\eta) := \max\{\Phi_i(\xi,\eta;\bar{s}) - g_i(\sqrt{\xi^2+\eta^2}), E_i^-\},$$

which is equivalent to studying (4.16) for $\boldsymbol{W}^{\pm,0}(\boldsymbol{x}) = (W_1^{\pm,0}(\boldsymbol{x}),\dots,W_N^{\pm,0}(\boldsymbol{x}))$ with

$$W_i^{+,0}(\boldsymbol{x}) := \min\left\{V_i^j(\boldsymbol{x}) + g_i\left(\sqrt{x_1^2 + \frac{1}{1+m_*^2B^2}(x_3+m_*Bx_2)^2}\right), E_i^+\right\}$$

and

$$W_i^{-,0}(\boldsymbol{x}) := \max\left\{V_i^j(\boldsymbol{x}) - g_i\left(\sqrt{x_1^2 + \frac{1}{1+m_*^2B^2}(x_3+m_*Bx_2)^2}\right), E_i^-\right\},$$

respectively. Then we have

$$\lim_{R\to\infty}\sup_{\xi^2+\eta^2>R^2}|\tilde{W}_i^{\pm,0}(\xi,\eta) - \Phi_i(\xi,\eta;s_j)| = 0, \quad i=1,\dots,N.$$

For $s_j = \frac{s}{\sqrt{1+m_*^2B^2}}$, applying Theorem 4.1 we have

$$\lim_{t\to\infty}\|\tilde{\boldsymbol{W}}(\xi,\eta,t;\tilde{\boldsymbol{W}}^{\pm,0}) - \boldsymbol{\Phi}(\xi,\eta;s_j)\|_{C(\mathbb{R}^2)} = 0,$$

which implies that $\lim_{t\to\infty}\|\boldsymbol{W}(\boldsymbol{x},t;\boldsymbol{W}^{\pm,0}) - \boldsymbol{V}^j(\boldsymbol{x})\|_{C(\mathbb{R}^3)} = 0$. Take $T_j>0$ large enough such that

$$\sup_{t\geqslant T_j}\|\boldsymbol{W}(\cdot,t;\boldsymbol{W}^{\pm,0}) - \boldsymbol{V}^j(\cdot)\|_{C(\mathbb{R}^3)} < \frac{\varepsilon_1}{2}. \tag{4.18}$$

Put $\boldsymbol{v}^\pm(\boldsymbol{x},t) = \boldsymbol{v}(\boldsymbol{x},t;\boldsymbol{v}^0) - \boldsymbol{W}(\boldsymbol{x},t;\boldsymbol{W}^{\pm,0})$. Then $\boldsymbol{v}^\pm$ satisfy

$$\begin{aligned}&\left(\frac{\partial}{\partial t} - D_i\frac{\partial^2}{\partial x_1^2} - D_i\frac{\partial^2}{\partial x_2^2} - D_i\frac{\partial^2}{\partial x_3^2} + s\frac{\partial}{\partial x_3}\right)v_i^\pm(\boldsymbol{x},t)\\ &\quad+\sum_{k=1}^N\left(\int_0^1\frac{\partial}{\partial u_k}F^i(\theta\boldsymbol{v}(\boldsymbol{x},t) + (1-\theta)\boldsymbol{W}(\boldsymbol{x},t;\boldsymbol{W}^{\pm,0}))d\theta\right)v_k^\pm(\boldsymbol{x},t) = 0, \quad \boldsymbol{x}\in\mathbb{R}^3, \quad t>0,\\ &v_i^\pm(\boldsymbol{x},0) = v_i^0(\boldsymbol{x}) - W_i^{\pm,0}(\boldsymbol{x}), \quad \boldsymbol{x}\in\mathbb{R}^3,\end{aligned}$$

respectively. In particular, from (4.17) we have $\boldsymbol{v}^+(\boldsymbol{x},0)\leqslant\boldsymbol{0}$ and $\boldsymbol{v}^-(\boldsymbol{x},0)\geqslant\boldsymbol{0}$ for $\boldsymbol{x}\in Q_j$. Let $\hat{\boldsymbol{v}}^\pm(\boldsymbol{x},t)$ be defined by

$$\begin{aligned}&\left(\frac{\partial}{\partial t} - D_i\frac{\partial^2}{\partial x_1^2} - D_i\frac{\partial^2}{\partial x_2^2} - D_i\frac{\partial^2}{\partial x_3^2} + s\frac{\partial}{\partial x_3}\right)\hat{v}_i^\pm(\boldsymbol{x},t)\\ &\quad+\sum_{k=1}^N\left(\int_0^1\frac{\partial}{\partial u_k}F^i(\theta\boldsymbol{v}(\boldsymbol{x},t) + (1-\theta)\boldsymbol{W}(\boldsymbol{x},t;\boldsymbol{W}^{\pm,0}))d\theta\right)\hat{v}_k^\pm(\boldsymbol{x},t) = 0, \quad \boldsymbol{x}\in\mathbb{R}^3, \quad t>0,\\ &\hat{v}_i^+(\boldsymbol{x},0) = \max\{v_i^+(\boldsymbol{x},0),0\} \quad \text{and} \quad \hat{v}_i^-(\boldsymbol{x},0) = \max\{-v_i^-(\boldsymbol{x},0),0\}, \quad \boldsymbol{x}\in\mathbb{R}^3.\end{aligned}$$

It is easy to see that $\hat{\boldsymbol{v}}^+(\boldsymbol{x},0)\geqslant\boldsymbol{v}^+(\boldsymbol{x},0)$ and $-\hat{\boldsymbol{v}}^-(\boldsymbol{x},0)\leqslant\boldsymbol{v}^-(\boldsymbol{x},0)$ for $\boldsymbol{x}\in\mathbb{R}^3$. By the comparison principle we obtain

$$\boldsymbol{v}^+(\boldsymbol{x},t)\leqslant\hat{\boldsymbol{v}}^+(\boldsymbol{x},t), \quad -\hat{\boldsymbol{v}}^-(\boldsymbol{x},t)\leqslant\boldsymbol{v}^-(\boldsymbol{x},t), \quad \forall\,\boldsymbol{x}\in\mathbb{R}^3, \quad t>0. \tag{4.19}$$

Notice that $|\hat{v}_i^\pm(\boldsymbol{x},0)|\leqslant 2(E_i^+-E_i^-)+1$ for $\boldsymbol{x}\in\mathbb{R}^3$ and $\hat{v}_i^\pm(\boldsymbol{x},0)=0$ for $\boldsymbol{x}\in Q_j$, where $i=1,\dots,N$. Applying the inequality (4.5) to $\hat{\boldsymbol{v}}^\pm(\boldsymbol{x},t)$, one has

$$0\leqslant\hat{v}_i^\pm(\boldsymbol{x},t)\leqslant\left(2\max_{1\leqslant i\leqslant N}(E_i^+-E_i^-)+1\right)e^{\lambda_0 t}\frac{3\pi\tilde{A}}{\tilde{B}}\int_{\frac{R}{\sqrt{t}}}^{+\infty}\exp(-\tilde{B}r^2)dr, \quad \forall\,t>0,$$

if $\boldsymbol{x} \in Q_j$ and $\sqrt{3}R < \mathrm{dist}(\boldsymbol{x}, \partial Q_j)$ for $i = 1, \dots, N$. It follows that

$$\lim_{R\to\infty} \sup_{\boldsymbol{x}\in Q_j, \mathrm{dist}(\boldsymbol{x},\partial Q_j)\geqslant R} \hat{v}_i^{\pm}(\boldsymbol{x}, t) = 0, \quad i = 1, \dots, N$$

for any fixed $t > 0$. Applying this equality, (4.18) and (4.19) to $\boldsymbol{v}(\boldsymbol{x}, t; \boldsymbol{v}^0) = \boldsymbol{v}^{\pm}(\boldsymbol{x}, t) + \boldsymbol{W}(\boldsymbol{x}, t; \boldsymbol{W}^0)$, for given $t \geqslant T_j$ we can take a constant $R_j > 0$ large enough such that

$$\sup_{\boldsymbol{x}\in Q_j,\ \mathrm{dist}(\boldsymbol{x},\partial Q_j)\geqslant R_j} |\boldsymbol{v}(\boldsymbol{x}, t; \boldsymbol{v}^0) - \boldsymbol{V}^j(\boldsymbol{x})| < \varepsilon_1. \tag{4.20}$$

Thus we have obtained the estimates on Q_j for given j.

Set

$$T^* := \max\{T_1, \dots, T_n\}.$$

Fix $t \geqslant T^*$. Let $\hat{R} := \max\{R_1, R_2, \dots, R_n\}$. From the definitions of Γ and Q_j we get

$$\lim_{R\to\infty} \inf_{|\boldsymbol{x}|\geqslant R, \mathrm{dist}(\boldsymbol{x},\partial Q_j)\leqslant \hat{R}} \mathrm{dist}(\boldsymbol{x}, \Gamma) = \infty \quad \text{for all} \quad 1 \leqslant j \leqslant n.$$

Using (4.12), we have

$$\lim_{R\to\infty} \max_{1\leqslant j\leqslant n} \sup_{|\boldsymbol{x}|\geqslant R, \boldsymbol{x}\in Q_j, \mathrm{dist}(\boldsymbol{x},\partial Q_j)\leqslant \hat{R}} |\boldsymbol{v}(\boldsymbol{x}, t; \boldsymbol{v}^0) - \boldsymbol{V}^j(\boldsymbol{x})| = 0.$$

By this estimate and (4.20), we obtain (4.14). The proof is completed. □

Lemma 4.6. *Assume that $\boldsymbol{v}^0 \in [\boldsymbol{E}^-, \boldsymbol{E}^+]$ satisfies* (1.8). *Let $\boldsymbol{V}$ be as in Theorem* 3.3. *For any given $\varepsilon_1 > 0$, one can choose $T^* > 0$ large enough such that*

$$\lim_{R\to\infty} \sup_{|\boldsymbol{x}|\geqslant R} |\boldsymbol{v}(\boldsymbol{x}, t; \boldsymbol{v}^0) - \boldsymbol{V}(\boldsymbol{x})| < \varepsilon_1 \quad \textit{for any fixed} \quad t \geqslant T^*. \tag{4.21}$$

In particular, one has

$$\lim_{R\to\infty} \sup_{|\boldsymbol{x}|\geqslant R} |\boldsymbol{V}(\boldsymbol{x}) - \hat{\boldsymbol{V}}(\boldsymbol{x})| = 0. \tag{4.22}$$

Proof. By taking $\boldsymbol{v}^0 = \boldsymbol{V}$ in Proposition 4.5, for any $\varepsilon_1 > 0$ we have

$$\lim_{R\to\infty} \max_{1\leqslant j\leqslant n} \sup_{|\boldsymbol{x}|\geqslant R, \boldsymbol{x}\in Q_j} |\boldsymbol{V}(\boldsymbol{x}) - \boldsymbol{V}^j(\boldsymbol{x})| < \varepsilon_1.$$

Due to the arbitrariness of $\varepsilon_1 > 0$, we obtain the equalities (4.22) and

$$\lim_{R\to\infty} \max_{1\leqslant j\leqslant n} \sup_{|\boldsymbol{x}|\geqslant R, \boldsymbol{x}\in Q_j} |\boldsymbol{V}(\boldsymbol{x}) - \boldsymbol{V}^j(\boldsymbol{x})| = 0.$$

Furthermore, using the last equality and Proposition 4.5, we can obtain (4.21). This completes the proof. □

The equality (4.22) shows that the pyramidal traveling front $\boldsymbol{V}$ converges to two-dimensional V-form fronts $\boldsymbol{\Phi}$ near the edges.

Lemma 4.7. *Let $\boldsymbol{V}$ be as in Theorem* 3.3. *Then it satisfies*

$$\lim_{R\to\infty} \sup_{|x_3+h(\boldsymbol{x}')|\geqslant R} \left| \frac{\partial}{\partial x_3} \boldsymbol{V}(\boldsymbol{x}) \right| = 0.$$

In addition, for any $\delta \in (0, \epsilon_0)$ we have

$$\min_{1\leqslant i\leqslant N} \inf_{E_i^-+\delta\leqslant V_i^j(\boldsymbol{x})\leqslant E_i^+-\delta} \frac{\partial}{\partial x_3} V_i^j(\boldsymbol{x}) > 0, \quad 1 \leqslant j \leqslant n,$$

and

$$\min_{1\leqslant i\leqslant N} \inf_{E_i^-+\delta\leqslant V_i(\boldsymbol{x})\leqslant E_i^+-\delta} \frac{\partial}{\partial x_3} V_i(\boldsymbol{x}) > 0. \tag{4.23}$$

Proof. Note that

$$\frac{\partial}{\partial x_3}V_i^j(\boldsymbol{x}) := \frac{q_j}{\sqrt{m_*^2 p_j^2 + q_j^2}}\frac{\partial}{\partial \eta}\Phi_i(\xi,\eta;s_j),$$

where $\xi = ((A_{j+1}-A_j)x_1 + (B_{j+1}-B_j)x_2)/q_j$ and

$$\eta = (m_*(B_{j+1}-B_j)p_j x_1 + m_*(A_j - A_{j+1})p_j x_2 + q_j^2 x_3)/(q_j\sqrt{m_*^2 p_j^2 + q_j^2}).$$

It follows from Wang [53, Lemma 4.2] that $\min_{1\leqslant i\leqslant N}\min_{E_i^- +\delta\leqslant V_i^j(\boldsymbol{x})\leqslant E_i^+ -\delta}\frac{\partial}{\partial x_3}V_i^j(\boldsymbol{x}) > 0$.

Now we show that (4.23) holds. Since $\frac{\partial}{\partial x_3}V_i > 0$ in $\mathbb{R}^3$, $\frac{\partial}{\partial x_3}V_i$ has a positive minimum on any compact subset of $\mathbb{R}^3$. Thus we need only to study $\frac{\partial}{\partial x_3}V_i$ as $|\boldsymbol{x}|\to\infty$. Fix $i\in\{1,\dots,N\}$. Let

$$\widetilde{\Omega}_i = \{\boldsymbol{x}\in\mathbb{R}^3 \mid E_i^- + \delta \leqslant V_i(\boldsymbol{x}) \leqslant E_i^+ - \delta\}.$$

By (4.22) and (3.13) we have

$$\lim_{R\to+\infty}\sup_{\boldsymbol{x}\in B(Q_j,2),|\boldsymbol{x}|\geqslant R}|\boldsymbol{V}(\boldsymbol{x}) - \boldsymbol{V}^j(\boldsymbol{x})| = 0,$$

where $B(Q_j,2) := \{\boldsymbol{x}\in\mathbb{R}^3 \mid \mathrm{dist}(\boldsymbol{x},Q_j)\leqslant 2\}$, $j\in\{1,\dots,n\}$. Then there exists $\hat{R}_j > 0$ such that

$$\sup_{\boldsymbol{x}\in B(Q_j,2),|\boldsymbol{x}|\geqslant \hat{R}_j}|\boldsymbol{V}(\boldsymbol{x}) - \boldsymbol{V}^j(\boldsymbol{x})| < \frac{\delta}{2}.$$

Consequently, we have $E_i^- + \frac{\delta}{2} \leqslant V_i^j(\boldsymbol{x}) < E_i^+ - \frac{\delta}{2}$ for $\boldsymbol{x}\in B(Q_j,2)\cap\widetilde{\Omega}_i$ with $|\boldsymbol{x}|\geqslant\hat{R}_j$. For any $\boldsymbol{x}^0\in Q_j$, we have

$$\lim_{R\to+\infty}\sup_{\boldsymbol{x}^0\in Q_j,|\boldsymbol{x}^0|\geqslant R}\|F^i(\boldsymbol{V}(\cdot)) - F^i(\boldsymbol{V}^j(\cdot))\|_{L^p(B(\boldsymbol{x}^0,2))} = 0,$$

where $p>3$, $B(\boldsymbol{x}^0,r) := \{\boldsymbol{x}\in\mathbb{R}^3 \mid |\boldsymbol{x}-\boldsymbol{x}^0| < r\}$. Applying the interior Schauder estimate of [17, Theorem 9.11] to

$$-D_i\Delta(V_i - V_i^j) + s\frac{\partial}{\partial x_3}(V_i - V_i^j) = F^i(\boldsymbol{V}) - F^i(\boldsymbol{V}^j) \quad \text{in} \quad B(\boldsymbol{x}^0,2), \quad \forall\,\boldsymbol{x}^0\in Q_j,$$

we obtain

$$\lim_{R\to+\infty}\sup_{\boldsymbol{x}^0\in Q_j,|\boldsymbol{x}^0|\geqslant R}\|V_i(\cdot) - V_i^j(\cdot)\|_{W^{2,p}(B(\boldsymbol{x}^0,1))} = 0.$$

Therefore, we have

$$\lim_{R\to+\infty}\sup_{\boldsymbol{x}\in Q_j,|\boldsymbol{x}|\geqslant R}\left|\frac{\partial}{\partial x_3}V_i(\boldsymbol{x}) - \frac{\partial}{\partial x_3}V_i^j(\boldsymbol{x})\right| = 0.$$

Thus, by virtue of the estimate on $\boldsymbol{V}^j$ there exists $\tilde{R}_j > \hat{R}_j$ such that

$$\min_{\boldsymbol{x}\in\widetilde{\Omega}_i\cap Q_j,|\boldsymbol{x}|\geqslant\tilde{R}_j}\frac{\partial}{\partial x_3}V_i(\boldsymbol{x}) > 0.$$

Applying the above arguments to all $j=1,\dots,n$ and $i=1,\dots,N$, we obtain (4.23).

Obviously, the assumption $|x_3 + h(\boldsymbol{x}')|\to\infty$ implies $\mathrm{dist}(\boldsymbol{x},\Gamma)\to\infty$. It follows that

$$\lim_{R\to+\infty}\sup_{x_3+h(\boldsymbol{x}')\geqslant R}|\boldsymbol{V}(\boldsymbol{x}) - \boldsymbol{E}^+|\to 0 \quad \text{and} \quad \lim_{R\to+\infty}\sup_{x_3+h(\boldsymbol{x}')\leqslant -R}|\boldsymbol{V}(\boldsymbol{x}) - \boldsymbol{E}^-|\to 0,$$

which yields $\lim_{R\to\infty}\sup_{|x_3+h(\boldsymbol{x}')|\geqslant R}|F^i(\boldsymbol{V}(\boldsymbol{x}))| = 0$. Applying the interior Schauder estimate to

$$-D_i\Delta V_i + s\frac{\partial}{\partial x_3}V_i = F^i(\boldsymbol{V}) \quad \text{in} \quad B(\bar{\boldsymbol{x}},2), \quad \forall\,\bar{\boldsymbol{x}}\in\mathbb{R}^3,$$

we have

$$\lim_{R\to\infty}\sup_{1\leqslant i\leqslant N}\sup\{\|V_i\|_{W^{2,p}(B(\bar{\boldsymbol{x}},1))}\mid \bar{\boldsymbol{x}}\in\mathbb{R}^3, |\bar{x}_3+h(\bar{\boldsymbol{x}}')|\geqslant R\}=0$$

for $p>3$. Therefore, we have

$$\lim_{R\to\infty}\sup_{|x_3+h(\boldsymbol{x}')|\geqslant R}\left|\frac{\partial}{\partial x_3}V_i(\boldsymbol{x})\right|=0,\quad 1\leqslant i\leqslant N.$$

This completes the proof. □

Lemma 4.8. *Assume that $\delta\in(0,\epsilon_0)$. For any $\boldsymbol{x}\in\mathbb{R}^3$ with*

$$E_i^-+\delta\leqslant \hat{V}_i(\boldsymbol{x})=\max_{1\leqslant j\leqslant n}V_i^j(\boldsymbol{x})\leqslant E_i^+-\delta,$$

we have

$$\inf_{0<\varrho<\varrho_0}\frac{\hat{V}_i(\boldsymbol{x}',x_3+\varrho)-\hat{V}_i(\boldsymbol{x})}{\varrho}\geqslant \min_{1\leqslant j\leqslant n}\min_{1\leqslant i\leqslant N}\inf_{E_i^-+\frac{\delta}{2}\leqslant V_i^j(\boldsymbol{x})\leqslant E_i^+-\frac{\delta}{2}}\frac{\partial}{\partial x_3}V_i^j(\boldsymbol{x})>0,$$

where ϱ_0 is a positive constant depending on δ and is independent of $\boldsymbol{x}$.

Proof. Fix $i\in\{1,\dots,N\}$. By the uniform continuity of $\hat{\boldsymbol{V}}$, there exists $\varrho_0>0$ such that

$$E_i^-+\frac{\delta}{2}\leqslant \hat{V}_i(\boldsymbol{x}',x_3+\varrho)\leqslant E_i^+-\frac{\delta}{2}\quad\text{for}\quad \varrho\in(0,\varrho_0),$$

if $\boldsymbol{x}$ satisfies $E_i^-+\delta\leqslant\hat{V}_i(\boldsymbol{x})\leqslant E_i^+-\delta$. For any $\boldsymbol{x}^0=(x_1^0,x_2^0,x_3^0)\in\mathbb{R}^3$ with $E_i^-+\delta\leqslant\hat{V}_i(\boldsymbol{x}^0)\leqslant E_i^+-\delta$, there exists $j_0\in\{1,\dots,n\}$ such that $\hat{V}_i(\boldsymbol{x}^0)=V_i^{j_0}(\boldsymbol{x}^0)$. Then we have

$$\begin{aligned}\hat{V}_i(x_1^0,x_2^0,x_3^0+\varrho)-\hat{V}_i(\boldsymbol{x}^0)&=\hat{V}_i(x_1^0,x_2^0,x_3^0+\varrho)-V_i^{j_0}(\boldsymbol{x}^0)\\&\geqslant V_i^{j_0}(x_1^0,x_2^0,x_3^0+\varrho)-V_i^{j_0}(\boldsymbol{x}^0)\\&\geqslant \varrho\min_{E_i^-+\frac{\delta}{2}\leqslant V_i^{j_0}(\boldsymbol{x})\leqslant E_i^+-\frac{\delta}{2}}\frac{\partial}{\partial x_3}V_i^{j_0}(\boldsymbol{x})\\&\geqslant \varrho\min_{1\leqslant j\leqslant n}\min_{1\leqslant i\leqslant N}\inf_{E_i^-+\frac{\delta}{2}\leqslant V_i^j(\boldsymbol{x})\leqslant E_i^+-\frac{\delta}{2}}\frac{\partial}{\partial x_3}V_i^j(\boldsymbol{x}).\end{aligned}$$

Finally, the arbitrariness of ϱ and $\boldsymbol{x}^0$ yields the expected result. This completes the proof. □

For $M>0$ defined in Lemma 3.1, it is not difficult to show that

$$E_i^+>\check{E}_i^+:=\sup_{\frac{c}{s}(x_3+h(\boldsymbol{x}'))\leqslant M}V_i(\boldsymbol{x})\geqslant U_i(M)$$

and

$$E_i^-<U_i\bigg(-M+\frac{c}{s}m_0\bigg)\leqslant\check{E}_i^-:=\inf_{\frac{c}{s}(x_3+h(\boldsymbol{x}'))\geqslant -M}V_i(\boldsymbol{x})$$

for $i=1,\dots,N$, where $m_0=2\pi m_*\int_0^\infty r^2\tilde{\rho}(r)dr$. In particular, $\check{E}_i^\pm$ are independent of $\varepsilon>0$ and $\alpha>0$. By Lemmas 4.7 and 4.8, there exists $\beta_3>0$ so that

$$\min_{1\leqslant i\leqslant N}\inf_{|\frac{c}{s}(x_3+h(\boldsymbol{x}'))|\leqslant M}\frac{\partial}{\partial x_3}V_i(\boldsymbol{x})>\beta_3\quad\text{and}\quad\min_{1\leqslant i\leqslant N}\inf_{|\mu(\boldsymbol{x})|\leqslant M}\frac{\partial}{\partial x_3}v_i^+(\boldsymbol{x})>\beta_3.$$

Lemma 4.9. *There exist a positive constant ρ sufficiently large and a positive constant β small enough such that, for any $\delta>0$ with*

$$\delta<\delta^*:=\min\bigg\{\frac{\epsilon_0}{Np^0},\min_{1\leqslant i\leqslant N}\bigg\{\frac{1}{2p^0}(E_i^+-\check{E}_i^+)\bigg\},\frac{\kappa(p_0-\epsilon_0)}{8N(L^++1)(p^0)^2}\bigg\},$$

$\boldsymbol{W}^+$ *defined by*

$$\boldsymbol{W}^+(\boldsymbol{x},t;\delta) = \boldsymbol{V}(\boldsymbol{x}', x_3 + \rho\delta(1-\mathrm{e}^{-\beta t})) + \delta \boldsymbol{P}(\varsigma^+)\mathrm{e}^{-\beta t}$$

is a supersolution of (2.1)*, and for any* $\delta > 0$ *with*

$$\delta < \min\left\{\frac{\epsilon_0}{Nq^0}, \min_{1\leqslant i\leqslant N}\left\{\frac{1}{2q^0}(U_i(-M)-E_i^-)\right\}, \frac{\kappa(q_0-\epsilon_0)}{8N(L^-+1)(q^0)^2}\right\},$$

$\boldsymbol{W}^-$ *defined by*

$$\boldsymbol{W}^-(\boldsymbol{x},t;\delta) = \boldsymbol{V}(\boldsymbol{x}', x_3 - \rho\delta(1-\mathrm{e}^{-\beta t})) - \delta \boldsymbol{Q}(\varsigma^-)\mathrm{e}^{-\beta t}$$

is a subsolution of (2.1)*, where*

$$\varsigma^\pm = \frac{x_3 \pm \rho\delta(1-\mathrm{e}^{-\beta t}) + \varphi(\boldsymbol{x}')}{\sqrt{1+|\nabla\varphi(\boldsymbol{x}')|^2}}.$$

Proof. It follows from Lemma 2.5 that $h(\boldsymbol{x}') \leqslant \varphi(\boldsymbol{x}') \leqslant m_0 + h(\boldsymbol{x}')$ for all $\boldsymbol{x}' \in \mathbb{R}^2$. It is easy to verify that there exist constants $C_i^+ > 0$ such that

$$\left|P_i''(\mu)(\mu_{x_1}^2+\mu_{x_2}^2) + P_i'(\mu)(\mu_{x_1x_1}+\mu_{x_2x_2}) + P_i''(\mu)\frac{1}{1+|\nabla\varphi(\alpha\boldsymbol{x}')|^2}\right| \leqslant C_i^+, \tag{4.24}$$

$$\left|Q_i''(\mu)(\mu_{x_1}^2+\mu_{x_2}^2) + Q_i'(\mu)(\mu_{x_1x_1}+\mu_{x_2x_2}) + Q_i''(\mu)\frac{1}{1+|\nabla\varphi(\alpha\boldsymbol{x}')|^2}\right| \leqslant C_i^+ \tag{4.25}$$

for any $\alpha \in (0,1]$ and $\boldsymbol{x} \in \mathbb{R}^3$, where μ is defined by (3.1) and $i = 1,\ldots,N$. In addition, we can take $M > 0$ large enough in Lemma 3.1 so that

$$\left|P_i''(\mu)(\mu_{x_1}^2+\mu_{x_2}^2) + P_i'(\mu)(\mu_{x_1x_1}+\mu_{x_2x_2}) + \frac{P_i''(\mu)}{1+|\nabla\varphi(\alpha\boldsymbol{x}')|^2}\right| < \frac{1}{4D}\kappa(p_0-\epsilon_0), \tag{4.26}$$

$$\left|Q_i''(\mu)(\mu_{x_1}^2+\mu_{x_2}^2) + Q_i'(\mu)(\mu_{x_1x_1}+\mu_{x_2x_2}) + \frac{Q_i''(\mu)}{1+|\nabla\varphi(\alpha\boldsymbol{x}')|^2}\right| < \frac{1}{4D}\kappa(q_0-\epsilon_0) \tag{4.27}$$

for any $\alpha \in (0,1]$ and $\mu > M$ or $\alpha \in (0,1]$ and $\mu < -M + \frac{c}{s}m_0 < 0$, $i = 1,\ldots,N$.

We omit the rest of the proof, which is similar to that of [53, Lemma 4.2]. This completes the proof. □

Lemma 4.10. *There exists a positive constant* ρ *sufficiently large and a positive constant* β *small enough such that, for any* $\delta > 0$ *with*

$$\delta < \delta^* \leqslant \min\left\{\frac{\epsilon_0}{Np^0}, \min_{1\leqslant i\leqslant N}\left\{\frac{1}{2p^0}(E_i^+-U_i(M))\right\}, \frac{\kappa(p_0-\epsilon_0)}{8N(p^0)^2(L^++1)}\right\},$$

$\boldsymbol{w}^+$ *defined by*

$$\boldsymbol{w}^+(\boldsymbol{x},t;\delta) = \boldsymbol{v}^+(\boldsymbol{x}', x_3 + \rho\delta(1-\mathrm{e}^{-\beta t});\varepsilon,\alpha) + \delta \boldsymbol{P}(\tau)\mathrm{e}^{-\beta t}$$

is a supersolution of (2.1)*, where*

$$\tau = \frac{x_3 + \rho\delta(1-\mathrm{e}^{-\beta t}) + \varphi(\alpha\boldsymbol{x}')/\alpha}{\sqrt{1+|\nabla\varphi(\alpha\boldsymbol{x}')|^2}}.$$

The proof of the lemma is similar to that of Lemma 4.9. Following [53, Lemma 4.4], we obtain the following lemma.

Lemma 4.11. *There exists a positive constant* ρ *sufficiently large and a positive constant* β *small enough such that, for any* $\delta > 0$ *with*

$$\delta < \min\left\{\frac{\epsilon_0}{q^0}, \min_{1\leqslant i\leqslant N}\left\{\frac{1}{2q^0}(U_i(-M)-E_i^-)\right\}, \frac{\kappa(q_0-\epsilon_0)}{8N(q^0)^2(L^-+1)}\right\},$$

$\widehat{\boldsymbol{w}}_j$*'s defined by*

$$\widehat{\boldsymbol{w}}_j(\boldsymbol{x},t;\delta) = \boldsymbol{U}(\widehat{\varrho}) - \varepsilon \boldsymbol{Q}(\widehat{\varrho})\mathrm{sech}(\alpha x_3) - \delta \boldsymbol{Q}(\widehat{\varrho})\mathrm{e}^{-\beta t}$$

are subsolutions of (2.1), $j = 1, \dots, n$, *where*

$$\widehat{\varrho} = \frac{h_j(\boldsymbol{x}')/m_* - \rho\delta(1 - \mathrm{e}^{-\beta t}) + \psi(\alpha x_3)/\alpha}{\sqrt{1 + \psi'^2(\alpha x_3)}}.$$

Following this lemma, we know that

$$\widetilde{\boldsymbol{w}}(\boldsymbol{x}, t; \delta) := \bigvee_{j=1}^{n} \widehat{\boldsymbol{w}}_j(\boldsymbol{x}, t; \delta) = \boldsymbol{U}(\widetilde{\varrho}) - \varepsilon \boldsymbol{Q}(\widetilde{\varrho})\mathrm{sech}(\alpha x_3) - \delta \boldsymbol{Q}(\widetilde{\varrho})\mathrm{e}^{-\beta t}$$

is also a subsolution of (2.1), where

$$\widetilde{\varrho} = \frac{h(\boldsymbol{x}')/m_* - \rho\delta(1 - \mathrm{e}^{-\beta t}) + \psi(\alpha x_3)/\alpha}{\sqrt{1 + \psi'^2(\alpha x_3)}}.$$

In the following we prove (1.9) for the case $\boldsymbol{u}^0 = \boldsymbol{v}^0$ with $\boldsymbol{v}^0 \geqslant \boldsymbol{v}^-$. We further restrict that $\varepsilon_0^+ < \min\{\varepsilon^+, \frac{\delta^* p_0}{4(s+1)p^0}\}$ in Lemma 3.1. Then for $\varepsilon \in (0, \varepsilon_0^+)$ and $\alpha \in (0, \alpha^+(\varepsilon))$, let $\boldsymbol{v}^+(\boldsymbol{x}; \varepsilon, \alpha)$ be as in Lemma 3.1. Define

$$\boldsymbol{V}^*(\boldsymbol{x}) := \lim_{t\to\infty} \boldsymbol{v}(\boldsymbol{x}, t; \boldsymbol{v}_*^+), \quad \forall\, \boldsymbol{x} \in \mathbb{R}^3,$$

where $\boldsymbol{v}_*^+(\boldsymbol{x}; \varepsilon, \alpha) = \boldsymbol{v}^+(\boldsymbol{x}; \varepsilon, \alpha) \wedge \boldsymbol{E}^+$. Since $\boldsymbol{v}^+(\boldsymbol{x}; \varepsilon, \alpha)$ is a supersolution of (2.1), $\boldsymbol{v}_*^+(\boldsymbol{x})$ is a supersolution of (1.4). Consequently, we have that $\boldsymbol{v}(\boldsymbol{x}, t; \boldsymbol{v}_*^+) \leqslant \boldsymbol{v}_*^+(\boldsymbol{x})$ for any $\boldsymbol{x} \in \mathbb{R}^3$ and $t > 0$. Then proceeding the similar argument as to $\boldsymbol{V}(\boldsymbol{x})$, we have that $\boldsymbol{V}^*(\boldsymbol{x})$ is C^2 in $\boldsymbol{x}$ and satisfies (1.6). Clearly,

$$\boldsymbol{V}(\boldsymbol{x}) \leqslant \boldsymbol{V}^*(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^3.$$

Lemma 4.12. *For* $\boldsymbol{x} \in \mathbb{R}^3$, $\boldsymbol{V}^*(\boldsymbol{x}) \equiv \boldsymbol{V}(\boldsymbol{x})$ *holds.*

Proof. Assume the contrary. Namely, $\boldsymbol{V}^*(\boldsymbol{x}) \not\equiv \boldsymbol{V}(\boldsymbol{x})$. Take $\delta \in (\frac{\delta^*}{2}, \delta^*)$. By the definition of $\boldsymbol{V}^*(x)$, there exists a sufficiently large $\lambda > 0$ such that

$$\boldsymbol{v}_*^+(\boldsymbol{x}) \leqslant \boldsymbol{V}(\boldsymbol{x}', x_3 + \lambda) + \delta \boldsymbol{P}\left(\frac{x_3 + \lambda + \varphi(\boldsymbol{x}')}{\sqrt{1 + |\nabla\varphi(\boldsymbol{x}')|^2}}\right), \quad \forall\, \boldsymbol{x} \in \mathbb{R}^3. \tag{4.28}$$

Due to Lemma 4.9, we know that the function $\boldsymbol{W}^+(\boldsymbol{x}', x_3 + \lambda, t; \delta)$ is a supersolution of (2.1) on $t \geqslant 0$. Thus by Corollary 2.3 we have

$$\boldsymbol{v}(\boldsymbol{x}, t; \boldsymbol{v}_*^+) \leqslant \boldsymbol{W}^+(\boldsymbol{x}', x_3 + \lambda, t; \delta) \tag{4.29}$$

for $\boldsymbol{x} \in \mathbb{R}^3$ and $t > 0$. Letting $t \to \infty$ we get

$$\boldsymbol{V}^*(\boldsymbol{x}) \leqslant \boldsymbol{V}(\boldsymbol{x}', x_3 + \lambda + \rho\delta) \quad \text{for} \quad \boldsymbol{x} \in \mathbb{R}^3. \tag{4.30}$$

Here we first show that

$$\lim_{R\to\infty} \sup_{|\boldsymbol{x}|\geqslant R} |\boldsymbol{V}^*(\boldsymbol{x}) - \hat{\boldsymbol{V}}(\boldsymbol{x})| = 0. \tag{4.31}$$

It follows from (4.30) that $\lim_{R\to\infty} \sup_{|x_3+h(\boldsymbol{x}')|\geqslant R} |\boldsymbol{V}^*(\boldsymbol{x}) - \hat{\boldsymbol{V}}(\boldsymbol{x})| = 0$. For $\boldsymbol{x} \in \mathbb{R}^3$ with $|x_3 + h(\boldsymbol{x}')| \leqslant R^*$ for some sufficiently large $R^* > 0$, there must be $\mathrm{dist}(\boldsymbol{x}', E) \to \infty$ if $\mathrm{dist}(\boldsymbol{x}, \Gamma) \to \infty$. Then by $\boldsymbol{V}^* \leqslant \boldsymbol{v}^+$ and Lemmas 2.6 and 2.7 we have

$$\lim_{\gamma\to\infty} \sup_{|x_3+h(\boldsymbol{x}')|\leqslant R^*, \boldsymbol{x}\in D(\gamma)} |\boldsymbol{V}^*(\boldsymbol{x}) - \boldsymbol{v}^-(\boldsymbol{x})| = 0.$$

Combining the above arguments, we obtain $\lim_{\gamma\to+\infty} \sup_{\boldsymbol{x}\in D(\gamma)} |\boldsymbol{V}^* - \boldsymbol{v}^-| = 0$. Applying Proposition 4.5 to $\boldsymbol{V}^*$, we obtain (4.31).

Define

$$\Lambda := \inf\{\lambda \in \mathbb{R} \mid \boldsymbol{V}^*(\boldsymbol{x}) \leqslant \boldsymbol{V}(\boldsymbol{x}', x_3 + \lambda), \forall\, \boldsymbol{x} \in \mathbb{R}^3\}.$$

Then $\Lambda \geqslant 0$ and $\boldsymbol{V}^*(\boldsymbol{x}) \leqslant \boldsymbol{V}(\boldsymbol{x}', x_3+\Lambda)$ for $\boldsymbol{x} \in \mathbb{R}^3$. The assumption $\boldsymbol{V}^*(\boldsymbol{x}) \not\equiv \boldsymbol{V}(\boldsymbol{x})$ yields $\Lambda > 0$. By the strong maximum principle of elliptic equations we have that either $V_i^*(\boldsymbol{x}) \equiv V_i(\boldsymbol{x}', x_3+\Lambda)$ for all $\boldsymbol{x} \in \mathbb{R}^3$ and some $i \in \{1,\ldots,N\}$ or $\boldsymbol{V}^*(\boldsymbol{x}) \ll \boldsymbol{V}(\boldsymbol{x}', x_3+\Lambda)$ for any $\boldsymbol{x} \in \mathbb{R}^3$. We conclude that the former is impossible. In fact, take a sequence $\{\boldsymbol{x}'_m \in \mathbb{R}^2\}_{m\in\mathbb{N}}$ satisfying $h(\boldsymbol{x}'_m) \to +\infty$ and $\mathrm{dist}(\boldsymbol{x}'_m, E) \to +\infty$. Then by $\boldsymbol{v}^- \leqslant \boldsymbol{V} \leqslant \boldsymbol{V}^* \leqslant \boldsymbol{v}^+$, we have

$$\lim_{m\to+\infty} \boldsymbol{V}^*(\boldsymbol{x}'_m, -h(\boldsymbol{x}'_m)) = \boldsymbol{U}(0) \quad \text{and} \quad \liminf_{m\to+\infty} \boldsymbol{V}(\boldsymbol{x}'_m, -h(\boldsymbol{x}'_m)+\Lambda) \geqslant \boldsymbol{U}\left(\frac{c}{s}\Lambda\right),$$

which contradicts $V_i^*(\boldsymbol{x}) \equiv V_i(\boldsymbol{x}', x_3+\Lambda)$.

Now we assume that

$$\boldsymbol{V}^*(\boldsymbol{x}', x_3) \ll \boldsymbol{V}(\boldsymbol{x}', x_3+\Lambda), \quad \forall \boldsymbol{x} \in \mathbb{R}^3.$$

By Lemma 4.7, we can take $R_* > 0$ sufficiently large satisfying

$$2\rho \sup_{|x_3+h(\boldsymbol{x}')|\geqslant R_*-\rho\delta^*} \left|\frac{\partial}{\partial x_3}\boldsymbol{V}(\boldsymbol{x}', x_3+\Lambda)\right| < p_0.$$

Define

$$\mathcal{D} := \{\boldsymbol{x} \in \mathbb{R}^3 \mid |x_3+h(\boldsymbol{x}')| \leqslant R_*\}.$$

We choose a constant $\epsilon_1 > 0$ sufficiently small satisfying $0 < \epsilon_1 < \min\{\frac{\delta^*}{2}, \frac{\Lambda}{4\rho}\}$. Utilizing Lemma 4.8, for $\boldsymbol{x} \in \mathcal{D}$ we have

$$\begin{aligned}&\hat{V}_i\left(\boldsymbol{x}', x_3+\frac{\Lambda}{2}\right) - \hat{V}_i\left(\boldsymbol{x}', x_3+\frac{\Lambda}{4}\right)\\ &\quad\geqslant \min\left\{\varrho_0, \frac{\Lambda}{4}\right\} \min_{1\leqslant j\leqslant n}\min_{1\leqslant i\leqslant N} \inf_{E_i^-+\frac{\delta_0}{2}\leqslant V_i^j(\boldsymbol{x})\leqslant E_i^+-\frac{\delta_0}{2}} \frac{\partial}{\partial x_3}V_i^j(\boldsymbol{x}) > 0,\end{aligned}$$

where

$$\delta_0 = \min_{1\leqslant i\leqslant N}\min\left\{\frac{\delta^*}{2}, E_i^+ - \max_{1\leqslant j\leqslant n}\sup_{\boldsymbol{x}\in\mathcal{D}} V_i^j\left(\boldsymbol{x}', x_3+\frac{\Lambda}{2}\right), \min_{1\leqslant j\leqslant n}\inf_{\boldsymbol{x}\in\mathcal{D}} V_i^j(\boldsymbol{x}) - E_i^-\right\},$$

and ϱ_0 is defined in Lemma 4.10 associated with δ_0. Thus, it follows that

$$\begin{aligned}&\inf_{\boldsymbol{x}\in\mathcal{D}}(\hat{V}_i(\boldsymbol{x}', x_3+\Lambda-2\rho\epsilon_1) - \hat{V}_i(\boldsymbol{x}))\\ &\quad> \min\left\{\varrho_0, \frac{\Lambda}{4}\right\} \min_{1\leqslant j\leqslant n}\min_{1\leqslant i\leqslant N} \inf_{E_i^-+\frac{\delta_0}{2}\leqslant V_i^j(\boldsymbol{x})\leqslant E_i^+-\frac{\delta_0}{2}} \frac{\partial}{\partial x_3}V_i^j(\boldsymbol{x}) > 0.\end{aligned}$$

Applying Lemma 4.7 and (4.31), we have that there exists $R_0 > 0$ such that

$$\boldsymbol{V}^*(\boldsymbol{x}) < \boldsymbol{V}\left(\boldsymbol{x}', x_3+\frac{\Lambda}{2}\right) \leqslant \boldsymbol{V}(\boldsymbol{x}', x_3+\Lambda-2\rho\epsilon_1) \quad \text{for} \quad \boldsymbol{x} \in \mathcal{D} \quad \text{with} \quad |\boldsymbol{x}| > R_0.$$

Since $\mathcal{D} \cap B(\boldsymbol{0}; R_0)$ is compact, we have $\boldsymbol{V}^*(\boldsymbol{x}) < \boldsymbol{V}(\boldsymbol{x}', x_3+\Lambda-2\rho\epsilon_1)$ in $\mathcal{D} \cap B(\boldsymbol{0}; R_0)$ for sufficiently small ϵ_1. Thus,

$$\boldsymbol{V}^*(\boldsymbol{x}) < \boldsymbol{V}(\boldsymbol{x}', x_3+\Lambda-2\rho\epsilon_1) \quad \text{in} \quad \mathcal{D}.$$

In $\mathbb{R}^3 \setminus \mathcal{D}$, we have

$$V_i(\boldsymbol{x}', x_3+\Lambda) - V_i(\boldsymbol{x}', x_3+\Lambda-2\rho\epsilon_1) = 2\rho\epsilon_1 \int_0^1 \frac{\partial}{\partial x_3} V_i(\boldsymbol{x}', x_3+\Lambda+2\theta\rho\epsilon_1)d\theta \leqslant \epsilon_1 p_0$$

for $i = 1,\ldots,N$. Combining both cases, we have $\boldsymbol{V}^*(\boldsymbol{x}) \leqslant \boldsymbol{V}(\boldsymbol{x}', x_3+\Lambda-2\rho\epsilon_1) + \epsilon_1\boldsymbol{P}^-$ in $\mathbb{R}^3$. By Lemma 4.11 we know that $\boldsymbol{W}^+(\boldsymbol{x}', x_3+\Lambda-2\rho\epsilon_1, t; \epsilon_1)$ is a supersolution of (2.1). Thus $\boldsymbol{V}^*(\boldsymbol{x}) \leqslant \boldsymbol{W}^+(\boldsymbol{x}', x_3+\Lambda-2\rho\epsilon_1, t; \epsilon_1)$ for $\boldsymbol{x} \in \mathbb{R}^3$ and $t > 0$. Letting $t \to \infty$ yields $\boldsymbol{V}^*(\boldsymbol{x}) \leqslant \boldsymbol{V}(\boldsymbol{x}', x_3+\Lambda-\rho\epsilon_1)$ for $\boldsymbol{x} \in \mathbb{R}^3$. This contradicts the definition of Λ. Thus $\Lambda = 0$ follows and we have proved that $\boldsymbol{V}^*(\boldsymbol{x}) \equiv \boldsymbol{V}(\boldsymbol{x})$. The proof is completed. □

Theorem 4.13. *Assume that* (H1)–(H5) *hold. Let* $\boldsymbol{V}(\boldsymbol{x}', x_3+st)$ *be a pyramidal traveling front of* (1.2) *with speed* $s > c$ *established in Section* 3. *Assume that* $\boldsymbol{v}^0 \in C(\mathbb{R}^3, \mathbb{R}^N)$ *satisfying* $\boldsymbol{v}^0(\boldsymbol{x}) \in [\boldsymbol{E}^-, \boldsymbol{E}^+]$ *for* $\boldsymbol{x} \in \mathbb{R}^3$, $\boldsymbol{v}^0(\boldsymbol{x}) \geqslant \boldsymbol{v}^-(\boldsymbol{x})$ *for* $\boldsymbol{x} \in \mathbb{R}^3$ *and*

$$\lim_{\gamma\to\infty} \sup_{\boldsymbol{x}\in D(\gamma)} |\boldsymbol{v}^0(\boldsymbol{x}) - \boldsymbol{V}(\boldsymbol{x})| = 0. \tag{4.32}$$

Then the solution $\boldsymbol{v}(\boldsymbol{x}, t; \boldsymbol{v}^0)$ *of* (1.4) *with the initial value* $\boldsymbol{v}^0$ *satisfies*

$$\lim_{t\to\infty} \|\boldsymbol{v}(\cdot, t; \boldsymbol{v}^0) - \boldsymbol{V}(\cdot)\|_{C(\mathbb{R}^3)} = 0. \tag{4.33}$$

Proof. Let $\delta \in (0, \frac{\delta^*}{2})$ be given arbitrarily. Take $\varepsilon \in (0, \min\{\varepsilon_0^+, \frac{\delta^*}{4s}\})$. Utilizing (4.21), we take $\alpha \in (0, \alpha^+(\varepsilon))$ such that

$$\boldsymbol{v}(\boldsymbol{x}, 1; \boldsymbol{v}^0) \leqslant \boldsymbol{v}^+(\boldsymbol{x}; \varepsilon, \alpha) + \delta p_0 \boldsymbol{I} \quad \text{for} \quad \boldsymbol{x} \in \mathbb{R}^3,$$

where $\boldsymbol{I}$ is the $N \times N$ identical matrix. By using an argument similar to that in Taniguchi [48], we have that

$$\lim_{t\to\infty} \|\boldsymbol{v}(\cdot, t; \boldsymbol{v}^-) - \boldsymbol{V}(\cdot)\|_{L^\infty(\mathbb{R}^3)} = 0 \quad \text{and} \quad \lim_{t\to\infty} \|\boldsymbol{v}(\cdot, t; \boldsymbol{v}_*^+) - \boldsymbol{V}(\cdot)\|_{L^\infty(\mathbb{R}^3)} = 0.$$

Take $\hat{t} > 0$ large enough such that

$$\boldsymbol{v}(\boldsymbol{x}, t; \boldsymbol{v}^-) \leqslant \boldsymbol{v}(\boldsymbol{x}, t; \boldsymbol{v}_*^+) < \boldsymbol{V}(\boldsymbol{x}) + \delta p_0 \boldsymbol{I} \quad \text{for} \quad \boldsymbol{x} \in \mathbb{R}^3 \quad \text{and} \quad t \geqslant \hat{t}. \tag{4.34}$$

Let ρ and β be as in Lemma 4.12 and note that ρ and β are independent of δ. We have that $\boldsymbol{w}^+(\boldsymbol{x}, t; \delta)$ is a supersolution of (1.3). Then there exists $\tilde{t} > 0$ large enough so that

$$\boldsymbol{v}(\boldsymbol{x}, t+1; \boldsymbol{v}^0) < \boldsymbol{v}^+(\boldsymbol{x}', x_3 + \rho\delta) + \delta \mathrm{e}^{-\lambda_0 \hat{t}} p_0 \boldsymbol{I}$$

for any $t \geqslant \tilde{t}$. Let $\boldsymbol{v}_*^{+,\delta}(\boldsymbol{x}) = \boldsymbol{v}^+(\boldsymbol{x}', x_3 + \rho\delta) \wedge \boldsymbol{E}^+$. Then

$$\boldsymbol{v}(\boldsymbol{x}, \tilde{t}+1; \boldsymbol{v}^0) < \boldsymbol{v}_*^{+,\delta}(\boldsymbol{x}) + \delta \mathrm{e}^{-\lambda_0 \hat{t}} p_0 \boldsymbol{I}.$$

Lemma 4.2 implies that $\boldsymbol{v}(\boldsymbol{x}, \tilde{t} + \hat{t} + 1; \boldsymbol{v}^0) \leqslant \boldsymbol{v}(\boldsymbol{x}, \hat{t}; \boldsymbol{v}_*^{+,\delta}) + \delta p_0 \boldsymbol{I}$ for $\boldsymbol{x} \in \mathbb{R}^3$. Using (4.34), we have $\boldsymbol{v}(\boldsymbol{x}, \tilde{t}+\hat{t}+1; \boldsymbol{v}^0) \leqslant \boldsymbol{V}(\boldsymbol{x}', x_3+\rho\delta)+2\delta p_0 \boldsymbol{I}$ for $\boldsymbol{x} \in \mathbb{R}^3$. By Lemma 4.9, it follows that $\boldsymbol{v}(\boldsymbol{x}, t+\tilde{t}+\hat{t}+1; \boldsymbol{v}^0) \leqslant \boldsymbol{W}^+(\boldsymbol{x}', x_3 + \rho\delta, t; 2\delta)$ for $t > 0$. Therefore, we have

$$\boldsymbol{V}(\boldsymbol{x}) \leqslant \boldsymbol{v}(\boldsymbol{x}, t; \boldsymbol{v}^0) \leqslant \boldsymbol{V}(\boldsymbol{x}', x_3 + \rho\delta + 2\rho\delta) + 2\delta p^0 \leqslant \boldsymbol{V}(\boldsymbol{x}) + M^* \delta \boldsymbol{I}$$

for $t > t_\delta := \tilde{t} + \hat{t} + 1$, where $M^* > 0$ is a constant and is independent of δ. Due to the arbitrariness of δ, we have completed the proof. □

Now we consider the case that the initial value $\boldsymbol{u}^0 = \boldsymbol{v}^0$ satisfies $\boldsymbol{v}^0 \leqslant \boldsymbol{v}^-$. Define

$$\delta_* = \min\left\{\frac{\epsilon_0}{Nq^0}, \min_{1\leqslant i\leqslant N} \frac{1}{4q^0}(U_i(-M) - E_i^-), \frac{\kappa(q_0 - \epsilon_0)}{8N(L^- + 1)(q^0)^2}\right\}.$$

Take $0 < \varepsilon < \min\{\varepsilon_0^-, \frac{1}{2}\delta_*, \frac{\epsilon_0 q_0}{2(q^0)^2}\}$. Define

$$\overline{\boldsymbol{v}}(\boldsymbol{x}) := \boldsymbol{V}(\boldsymbol{x}', x_3 - M') \quad \text{and} \quad \boldsymbol{v}_-(\boldsymbol{x}) := \widetilde{\boldsymbol{v}}(\boldsymbol{x}; \varepsilon, \alpha) \vee \overline{\boldsymbol{v}}(\boldsymbol{x}),$$

where $M' > 0$ is a constant specified later. Recall that $\widetilde{\boldsymbol{w}}$ is defined in Lemma 4.11. Set

$$\overline{\boldsymbol{w}}(\boldsymbol{x}, t; \delta) := \boldsymbol{W}^-(\boldsymbol{x}', x_3 - M', t; \delta) \quad \text{and} \quad \boldsymbol{w}_-(\boldsymbol{x}, t; \delta) := \widetilde{\boldsymbol{w}}(\boldsymbol{x}, t; \delta) \vee \overline{\boldsymbol{w}}(\boldsymbol{x}, t; \delta).$$

Lemma 4.14. *For any positive constant δ and any initial function $\boldsymbol{v}^0$ satisfying*

$$\lim_{\gamma\to\infty}\sup_{\boldsymbol{x}\in D(\gamma)}|\boldsymbol{v}^0-\boldsymbol{v}^-|=0 \quad \text{and} \quad \boldsymbol{v}^0(\boldsymbol{x})\in[\boldsymbol{E}^-,\boldsymbol{E}^+] \quad \text{for} \quad \boldsymbol{x}\in\mathbb{R}^3,$$

there exist positive constants $\varepsilon<\min\{\varepsilon^-,\frac{1}{4}\delta_*,\frac{\epsilon_0 q_0}{2(q^0)^2}\}$, $\alpha<\alpha^-(\varepsilon)$, T' and M' such that

$$\boldsymbol{v}_-(\boldsymbol{x})-\delta\boldsymbol{Q}^+\leqslant\boldsymbol{v}(\boldsymbol{x},T';\boldsymbol{v}^0) \quad \text{for} \quad \boldsymbol{x}\in\mathbb{R}^3.$$

Proof. Clearly, $\boldsymbol{E}^-\leqslant\boldsymbol{v}(\boldsymbol{x},t;\boldsymbol{v}^0)\leqslant\boldsymbol{E}^+$. Applying Proposition 4.5 with $\varepsilon_1=\frac{\delta q_0}{4}$ and Lemma 4.7, we have

$$\lim_{R\to\infty}\sup_{|\boldsymbol{x}|>R}|\boldsymbol{v}(\boldsymbol{x},T^*+1;\boldsymbol{v}^0)-\boldsymbol{V}(\boldsymbol{x})|\leqslant\frac{\delta q_0}{4}, \tag{4.35}$$

where T^* is determined in Proposition 4.5. Fix $T'=T^*+1$. By (4.35) we can choose a large constant M' such that

$$\overline{\boldsymbol{v}}(\boldsymbol{x})-\delta\boldsymbol{Q}^+=\boldsymbol{V}(\boldsymbol{x}',x_3-M')-\delta\boldsymbol{Q}^+\leqslant\boldsymbol{v}(\boldsymbol{x},T';\boldsymbol{v}^0) \quad \text{for} \quad \boldsymbol{x}\in\mathbb{R}^3.$$

From (4.35) there exists a positive constant R_1 such that

$$\boldsymbol{V}(\boldsymbol{x})-\frac{\delta}{4}\boldsymbol{Q}^+\leqslant\boldsymbol{v}(\boldsymbol{x},T';\boldsymbol{v}^0) \quad \text{for} \quad |\boldsymbol{x}|>R_1.$$

Note that

$$\widetilde{\boldsymbol{v}}(\boldsymbol{x};\varepsilon,\alpha)\leqslant\boldsymbol{U}\bigg(\frac{h(\boldsymbol{x}')/m_*+\psi(\alpha x_3)/\alpha}{\sqrt{1+\psi'(\alpha x_3)^2}}\bigg)-\varepsilon\boldsymbol{Q}^+\text{sech}(\beta_2\alpha x_3).$$

Since $\frac{\psi(\alpha x_3)}{\alpha}=-\frac{1}{m_*\alpha\beta_2}\ln(1+\exp(-\beta_2\alpha x_3))\leqslant\frac{x_3}{m_*}$, we have

$$\widetilde{\boldsymbol{v}}(\boldsymbol{x};\varepsilon,\alpha)\leqslant\boldsymbol{U}\bigg(\frac{x_3+h(\boldsymbol{x}')}{m_*\sqrt{1+\psi'(\alpha x_3)^2}}\bigg)-\varepsilon\boldsymbol{Q}^+\text{sech}(\beta_2\alpha x_3).$$

It is not difficult to show that there exists $R_1'>0$ such that

$$\boldsymbol{U}\bigg(\frac{x_3+h(\boldsymbol{x}')}{m_*\sqrt{1+\psi'(\alpha x_3)^2}}\bigg)-\frac{\delta}{2}\boldsymbol{Q}^+\leqslant\boldsymbol{v}^-(\boldsymbol{x})$$

for $\boldsymbol{x}\in\mathbb{R}^3$ with $|x_3+h(\boldsymbol{x}')|>R_1'$. Since $\frac{1}{m_*\sqrt{1+\psi'(\alpha x_3)^2}}\to\frac{c}{s}$ as $\alpha x_3\to-\infty$, there exists $R_2>0$ such that

$$\boldsymbol{U}\bigg(\frac{x_3+h(\boldsymbol{x}')}{m_*\sqrt{1+\psi'(\alpha x_3)^2}}\bigg)-\frac{\delta}{2}\boldsymbol{Q}^+\leqslant\boldsymbol{v}^-(\boldsymbol{x})$$

for $\boldsymbol{x}\in\mathbb{R}^3$ with $|x_3+h(\boldsymbol{x}')|\leqslant R_1'$ and $\alpha x_3<-R_2$. Note that R_2 is independent of $\alpha\in(0,1)$. For $-R_2/\alpha\leqslant x_3\leqslant R_1'$ and $|x_3+h(\boldsymbol{x}')|\leqslant R_1'$, it follows from the definition of ψ that there exists a small positive constant α such that

$$\begin{aligned}
&\frac{1}{\sqrt{1+\psi'(\alpha x_3)^2}}\bigg(\frac{1}{\alpha}\psi(\alpha x_3)+\frac{1}{m_*}h(\boldsymbol{x}')\bigg)\\
&\quad=\frac{1}{\sqrt{1+\psi'(\alpha x_3)^2}}\bigg(\frac{x_3}{m_*}-\frac{1}{\alpha m_*\beta_2}\ln(1+\mathrm{e}^{\alpha\beta_2x_3})+\frac{1}{m_*}h(\boldsymbol{x}')\bigg)\\
&\quad\leqslant\frac{1}{\sqrt{1+\psi'(\alpha x_3)^2}}\bigg(\frac{R_1'}{m_*}-\frac{1}{\alpha m_*\beta_2}\ln(1+\mathrm{e}^{-\beta_2R_2})\bigg)\\
&\quad\leqslant U_i^{-1}\bigg(E_i^-+\frac{\delta}{2}q_i^+\bigg).
\end{aligned}$$

Take R_1' and R_2 large enough so that

$$\{\boldsymbol{x}\in\mathbb{R}^3\mid|\boldsymbol{x}|\leqslant R_1\}\subset\{\boldsymbol{x}\in\mathbb{R}^3\mid-R_2/\alpha\leqslant x_3\leqslant R_1',|x_3+h(\boldsymbol{x}')|\leqslant R_1'\}.$$

Therefore,

$$\widetilde{\boldsymbol{v}}(\boldsymbol{x};\varepsilon,\alpha)-\delta\boldsymbol{Q}^{+}\leqslant\boldsymbol{v}(\boldsymbol{x},T';\boldsymbol{v}^{0})\quad\text{for}\quad\boldsymbol{x}\in\mathbb{R}^{3}.$$

Finally, it is clear that

$$\boldsymbol{v}_{-}(\boldsymbol{x})-\delta\boldsymbol{Q}^{+}=(\widetilde{\boldsymbol{v}}(\boldsymbol{x};\varepsilon,\alpha)-\delta\boldsymbol{Q}^{+})\vee(\overline{\boldsymbol{v}}(\boldsymbol{x})-\delta\boldsymbol{Q}^{+}).$$

This completes the proof. □

Take $0<\delta<\delta_*$. For $\boldsymbol{x}\in\mathbb{R}^3$, define

$$\boldsymbol{v}_{-}^{\delta}(\boldsymbol{x}):=\widetilde{\boldsymbol{v}}_{-}^{\delta}(\boldsymbol{x})\vee\overline{\boldsymbol{v}}(\boldsymbol{x}',x_3-m_*\rho\delta),$$

where $\widetilde{\boldsymbol{v}}_{-}^{\delta}(\boldsymbol{x})=\boldsymbol{U}(\breve{\varrho}(\boldsymbol{x}))-\varepsilon\boldsymbol{Q}(\breve{\varrho}(\boldsymbol{x}))\mathrm{sech}(\alpha x_3)$ with

$$\breve{\varrho}(\boldsymbol{x}):=\frac{h(\boldsymbol{x}')/m_*-\rho\delta+\psi(\alpha x_3)/\alpha}{\sqrt{1+\psi'(\alpha x_3)^2}}.$$

In view of $\boldsymbol{w}_{-}(\boldsymbol{x},t;\delta)\leqslant\boldsymbol{v}(\boldsymbol{x},t+T';\boldsymbol{v}^0)$, taking $t\to\infty$ we have

$$\boldsymbol{v}_{-}^{\delta}(\boldsymbol{x})\leqslant\liminf_{t\to+\infty}\boldsymbol{v}(\boldsymbol{x},t;\boldsymbol{v}^0). \tag{4.36}$$

Lemma 4.15. *We have*

$$\lim_{R\to\infty}\inf_{|\boldsymbol{x}|\geqslant R}(\boldsymbol{v}_{-}^{\delta}(\boldsymbol{x})-\boldsymbol{v}^{-}(\boldsymbol{x}',x_3-m_*\rho\delta))\geqslant\boldsymbol{0}. \tag{4.37}$$

Proof. It is clear that $|\boldsymbol{v}_{-}^{\delta}(\boldsymbol{x})-\boldsymbol{V}(\boldsymbol{x}',x_3-m_*\rho\delta)|\to0$ as $x_3\to+\infty$ uniformly for $\boldsymbol{x}'\in\mathbb{R}^2$. In addition, one can show that

$$\lim_{R\to\infty}\sup_{|x_3+h(\boldsymbol{x}')|\geqslant R}|\boldsymbol{v}_{-}^{\delta}(\boldsymbol{x})-\boldsymbol{V}(\boldsymbol{x}',x_3-m_*\rho\delta)|=0.$$

It remains to consider $|x_3+h(\boldsymbol{x}')|<X_2$ for some $X_2>0$ sufficiently large and $x_3<X_1$. To ensure that $|\boldsymbol{x}|\to+\infty$, there must be $x_3\to-\infty$. By the definition of $\widetilde{\boldsymbol{v}}_{-}^{\delta}(\boldsymbol{x})$ we have

$$\lim_{R\to\infty}\sup_{|x_3+h(\boldsymbol{x}')|<X_2,x_3\leqslant-R}|\widetilde{\boldsymbol{v}}_{-}^{\delta}(\boldsymbol{x})-\boldsymbol{v}^{-}(\boldsymbol{x}',x_3-m_*\rho\delta)|=0.$$

Since $\boldsymbol{v}^{-}(\boldsymbol{x}',x_3-m_*\rho\delta)\leqslant\boldsymbol{V}(\boldsymbol{x}',x_3-m_*\rho\delta)$, it follows that (4.37) holds. The proof is completed. □

Lemma 4.16. *The limit of $\boldsymbol{v}(\boldsymbol{x},t;\boldsymbol{v}_{-}^{\delta})$ as $t\to\infty$ exists and the limit function*

$$\boldsymbol{V}_*^{\delta}(\boldsymbol{x}):=\lim_{t\to\infty}\boldsymbol{v}(\boldsymbol{x},t;\boldsymbol{v}_{-}^{\delta})$$

satisfies $\mathcal{L}[\boldsymbol{V}_*^{\delta}]=\boldsymbol{0}$, $\boldsymbol{v}_{-}^{\delta}\leqslant\boldsymbol{V}_*^{\delta}\leqslant\boldsymbol{V}$ and $\boldsymbol{V}_*^{\delta}(\boldsymbol{x})\geqslant\boldsymbol{V}(\boldsymbol{x}',x_3-m_*\rho\delta)$ on $\mathbb{R}^3$.

Proof. Take $\boldsymbol{v}_{-}^{*}(\boldsymbol{x})=\boldsymbol{v}_{-}^{\delta}(\boldsymbol{x})\vee\boldsymbol{v}^{-}(\boldsymbol{x})$. Then $\boldsymbol{v}_{-}^{\delta}\leqslant\boldsymbol{v}_{-}^{*}$. By the comparison principle, we have $\boldsymbol{v}(\boldsymbol{x},t;\boldsymbol{v}_{-}^{\delta})\leqslant\boldsymbol{v}(\boldsymbol{x},t;\boldsymbol{v}_{-}^{*})$. It follows from Theorem 4.13 that

$$\lim_{t\to\infty}\sup_{\boldsymbol{x}\in\mathbb{R}^3}|\boldsymbol{v}(\cdot,t;\boldsymbol{v}_{-}^{*})-\boldsymbol{V}(\cdot)|=0.$$

Since $\boldsymbol{v}_{-}^{\delta}$ is a subsolution of (1.4), the solution $\boldsymbol{v}(\boldsymbol{x},t;\boldsymbol{v}_{-}^{\delta})$ is increasing in t and the limiting function $\boldsymbol{V}_*^{\delta}$ exists with

$$\mathcal{L}[\boldsymbol{V}_*^{\delta}]=\boldsymbol{0}\quad\text{and}\quad\boldsymbol{v}_{-}^{\delta}\leqslant\boldsymbol{V}_*^{\delta}\leqslant\boldsymbol{V}.$$

By (4.36), we get $\lim_{R\to\infty}\inf_{|\boldsymbol{x}|\geqslant R}(\boldsymbol{v}_{-}^{\delta}(\boldsymbol{x})-\boldsymbol{v}^{-}(\boldsymbol{x}',x_3-m_*\rho\delta))\geqslant\boldsymbol{0}$. Applying Proposition 4.5 we further have

$$\lim_{R\to\infty}\inf_{|\boldsymbol{x}|\geqslant R}(\boldsymbol{V}_*^{\delta}(\boldsymbol{x})-\boldsymbol{V}(\boldsymbol{x}',x_3-m_*\rho\delta))\geqslant\boldsymbol{0}. \tag{4.38}$$

We prove $\boldsymbol{V}_*^\delta(\boldsymbol{x}) \geqslant \boldsymbol{V}(\boldsymbol{x}', x_3 - m_*\rho\delta)$ for all $\boldsymbol{x} \in \mathbb{R}^3$ by contradiction. Take

$$\Lambda^* = \min\{\lambda > 0 \mid \boldsymbol{V}_*^\delta(\boldsymbol{x}) \geqslant \boldsymbol{V}(\boldsymbol{x}', x_3 - \lambda) \text{ for } \boldsymbol{x} \in \mathbb{R}^3\}$$

and assume $\Lambda^* > m_*\rho\delta$. By (4.38), we have $V_i(\cdot, \cdot - \Lambda^*) \not\equiv V_{*,i}^\delta(\cdot,\cdot)$ for all $i = 1, \ldots, N$. Furthermore, the strong maximum principle implies that

$$\boldsymbol{V}(\boldsymbol{x}', x_3 - \Lambda^*) \ll \boldsymbol{V}_*^\delta(\boldsymbol{x}) \quad \text{for} \quad \boldsymbol{x} \in \mathbb{R}^3. \tag{4.39}$$

Note that $\lim_{R\to\infty} \sup_{|x_3+h(\boldsymbol{x}')|\geqslant R} |\frac{\partial}{\partial x_3}\boldsymbol{V}(\boldsymbol{x})| = 0$. Take $R_* > 0$ large enough so that

$$2\rho \sup_{|x_3+h(\boldsymbol{x}')|>R_*-\rho\delta_*} \left|\frac{\partial}{\partial x_3}\boldsymbol{V}(\boldsymbol{x}', x_3 - \Lambda^*)\right| < q_0.$$

By $\boldsymbol{v}_-^\delta(\boldsymbol{x}) \leqslant \boldsymbol{V}_*^\delta(\boldsymbol{x})$ and $\boldsymbol{V}(\boldsymbol{x}', x_3 - \Lambda^*) < \boldsymbol{V}(\boldsymbol{x}', x_3 - m_*\rho\delta)$ for $\boldsymbol{x} \in \mathbb{R}^3$ and (4.38), we can choose $0 < h^* < \min\{\frac{\delta_*}{2}, \frac{\Lambda^*-m_*\rho\delta}{2\rho}\}$ small enough such that

$$\boldsymbol{V}(\boldsymbol{x}', x_3 - \Lambda^* + 2\rho h^*) < \boldsymbol{V}_*^\delta(\boldsymbol{x}) \quad \text{in} \quad \mathcal{D}', \tag{4.40}$$

where

$$\mathcal{D}' := \{(\boldsymbol{x}) : |x_3 + h(\boldsymbol{x}')| \leqslant R_*\}.$$

In $\mathbb{R}^3\backslash\mathcal{D}'$, we have

$$\begin{aligned}
&\boldsymbol{V}(\boldsymbol{x}', x_3 - \Lambda^* + 2\rho h^*) - \boldsymbol{V}(\boldsymbol{x}', x_3 - \Lambda^*)\\
&\quad = 2\rho h^* \int_0^1 \frac{\partial}{\partial x_3}\boldsymbol{V}(\boldsymbol{x}', x_3 - \Lambda^* + 2\theta\rho h^*)d\theta \leqslant h^*\mathbf{Q}^+,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\boldsymbol{W}^-(\boldsymbol{x}', x_3 - \Lambda^* + 2\rho h^*, 0; h^*)\\
&\quad \leqslant \boldsymbol{V}(\boldsymbol{x}', x_3 - \Lambda^* + 2\rho h^*) - h^*\boldsymbol{Q}^+ \leqslant \boldsymbol{V}(\boldsymbol{x}', x_3 - \Lambda^*) \quad \text{in} \quad \mathbb{R}^3\backslash\mathcal{D}'.
\end{aligned} \tag{4.41}$$

Combining (4.39)–(4.41), we have $\boldsymbol{W}^-(\boldsymbol{x}', x_3 - \Lambda^* + 2\rho h^*, 0; h^*) \leqslant \boldsymbol{V}_*^\delta(\boldsymbol{x})$ in $\mathbb{R}^3$. Since $\boldsymbol{W}^-(\boldsymbol{x}', x_3 - \Lambda^* + 2\rho h^*, t; h^*)$ is a subsolution of (2.1), Corollary 2.3 yields that

$$\boldsymbol{W}^-(\boldsymbol{x}', x_3 - \Lambda^* + 2\rho h^*, t; h^*) \leqslant \boldsymbol{V}_*^\delta(\boldsymbol{x}) \quad \text{in} \quad \mathbb{R}^3 \times [0, \infty).$$

Letting $t \to \infty$ in the last inequality, we get $\boldsymbol{V}(\boldsymbol{x}', x_3 - \Lambda^* + \rho h^*) \leqslant \boldsymbol{V}_*^\delta(\boldsymbol{x})$ in $\mathbb{R}^3$, which contradicts the definition of Λ^*. This completes the proof. □

Theorem 4.17. *Assume that* (H1)–(H5) *hold. If* $\boldsymbol{v}^0(\boldsymbol{x})$ *satisfies* $\boldsymbol{v}^0(\boldsymbol{x}) \leqslant \boldsymbol{v}^-(\boldsymbol{x})$ *and* $\boldsymbol{v}^0(\boldsymbol{x}) \in [\boldsymbol{E}^-, \boldsymbol{E}^+]$ *for* $\boldsymbol{x} \in \mathbb{R}^3$ *and*

$$\lim_{\gamma\to\infty} \sup_{\boldsymbol{x}\in D(\gamma)} |\boldsymbol{v}^0(\boldsymbol{x}) - \boldsymbol{v}^-(\boldsymbol{x})| = 0,$$

then the solution $\boldsymbol{v}(\boldsymbol{x}, t; \boldsymbol{v}^0)$ *of* (1.4)–(1.5) *satisfies*

$$\lim_{t\to\infty} \|\boldsymbol{v}(\cdot, t; \boldsymbol{v}^0) - \boldsymbol{V}(\cdot)\|_{C(\mathbb{R}^3)} = 0.$$

Proof. Given any $\delta < \frac{\delta_*}{4}$, by $\boldsymbol{v}^0(\boldsymbol{x}) \leqslant \boldsymbol{v}^-(\boldsymbol{x})$ we have $\boldsymbol{v}(\boldsymbol{x}, t; \boldsymbol{v}^0) \leqslant \boldsymbol{V}(\boldsymbol{x})$ for any $\boldsymbol{x} \in \mathbb{R}^3$ and $t > 0$. Since $\lim_{t\to+\infty} \boldsymbol{v}(\boldsymbol{x}, t; \boldsymbol{v}_-^\delta) = \boldsymbol{V}_*^\delta(\boldsymbol{x}) \geqslant \boldsymbol{V}(\boldsymbol{x}', x_3 - m_*\rho\delta)$, there exists $\hat{t} > 0$ such that $\boldsymbol{v}(\boldsymbol{x}, t; \boldsymbol{v}_-^\delta) \geqslant \boldsymbol{V}(\boldsymbol{x}', x_3 - m_*\rho\delta) - \delta q_0\boldsymbol{I}$ for $t \geqslant \hat{t}$. It follows from Lemma 4.14 that

$$\boldsymbol{v}(\boldsymbol{x}, T'; \boldsymbol{v}^0) \geqslant \boldsymbol{v}_-(\boldsymbol{x}) - \delta\boldsymbol{Q}^+.$$

Then by (4.36) there exists $t' > 0$ so that

$$\boldsymbol{v}(\boldsymbol{x}, t + T'; \boldsymbol{v}^0) \geqslant \boldsymbol{v}_-^\delta(\boldsymbol{x}) - \delta q_0 \mathrm{e}^{-\lambda_0\hat{t}}\boldsymbol{I} \quad \text{for} \quad t \geqslant t'.$$

By Lemma 4.2, we have $\boldsymbol{v}(\boldsymbol{x},\hat{t}+t'+T';\boldsymbol{v}^0)\geqslant \boldsymbol{v}(\boldsymbol{x},\hat{t};\boldsymbol{v}_-^\delta)-\delta q_0\boldsymbol{I}$. Therefore, we have

$$\boldsymbol{v}(\boldsymbol{x},\hat{t}+t'+T';\boldsymbol{v}^0)\geqslant \boldsymbol{V}(\boldsymbol{x}',x_3-m_*\rho\delta)-2\delta q_0\boldsymbol{I}$$

for $\boldsymbol{x}\in\mathbb{R}^3$. By Lemma 4.11, we have

$$\boldsymbol{v}(\boldsymbol{x},t+\hat{t}+t'+T';\boldsymbol{v}^0)\geqslant \boldsymbol{W}^-(\boldsymbol{x}',x_3-m_*\rho\delta,t;2\delta)$$

for $t>0$. Then

$$\boldsymbol{V}(\boldsymbol{x})\geqslant \boldsymbol{v}(\boldsymbol{x},t+\hat{t}+t'+T';\boldsymbol{v}^0)\geqslant \boldsymbol{V}(\boldsymbol{x}',x_3-m_*\rho\delta-2\rho\delta)-2\delta q^0\boldsymbol{I}\mathrm{e}^{-\beta t},\quad t>0.$$

It follows that for any $t>T_\delta:=\hat{t}+t'+T'$,

$$\boldsymbol{v}(\boldsymbol{x},t;\boldsymbol{v}^0)\geqslant \boldsymbol{V}(\boldsymbol{x})-2\delta q^0\boldsymbol{I}-2M''\rho\delta\boldsymbol{I}-M''m_*\rho\delta\boldsymbol{I},$$

where $M''=\sup_{\boldsymbol{x}\in\mathbb{R}^3}|\frac{\partial}{\partial x_3}\boldsymbol{V}(\boldsymbol{x})|$. From the arbitrariness of $\delta>0$, we have that $\boldsymbol{v}(\cdot,t;\boldsymbol{v}^0)$ converges to $\boldsymbol{V}(\cdot)$ as $t\to\infty$ in $\|\cdot\|_{C(\mathbb{R}^3)}$. The proof is completed. □

Proof of Theorem 1.1. Take $\boldsymbol{v}^0(\boldsymbol{x})=\boldsymbol{u}^0(x)$. Let

$$\boldsymbol{v}_+^0(\boldsymbol{x})=\boldsymbol{v}^-(\boldsymbol{x})\vee\boldsymbol{v}^0(\boldsymbol{x})\quad\text{and}\quad \boldsymbol{v}_-^0(\boldsymbol{x})=\boldsymbol{v}^-(\boldsymbol{x})\wedge\boldsymbol{v}^0(\boldsymbol{x}).$$

Then $\boldsymbol{E}^-\leqslant\boldsymbol{v}_-^0\leqslant\boldsymbol{v}^-\leqslant\boldsymbol{v}_+^0\leqslant\boldsymbol{E}^+$, $\boldsymbol{E}^-\leqslant\boldsymbol{v}_-^0\leqslant\boldsymbol{v}^0\leqslant\boldsymbol{v}_+^0\leqslant\boldsymbol{E}^+$ and

$$\lim_{\gamma\to\infty}\sup_{\boldsymbol{x}\in D(\gamma)}|\boldsymbol{v}_\pm^0(\boldsymbol{x})-\boldsymbol{v}^-(\boldsymbol{x})|=0.$$

Note that $\boldsymbol{u}(\boldsymbol{x},t;\boldsymbol{u}^0)=\boldsymbol{v}(\boldsymbol{x}',x_3+st,t;\boldsymbol{v}^0)$. By the comparison principle and using Theorems 4.13 and 4.17, we complete the proof. □

The following corollary shows that a three-dimensional pyramidal traveling front is uniquely determined as a combination of two-dimensional V-form fronts.

Corollary 4.18. *Assume that* (H1)–(H5) *hold. Let* $\boldsymbol{V}$ *be the three-dimensional pyramidal traveling front associated with the pyramid* $-x_3=h(\boldsymbol{x}')$. *If* (1.6) *has a solution* $\boldsymbol{W}$ *with*

$$\lim_{\gamma\to\infty}\sup_{\boldsymbol{x}\in D(\gamma)}|\boldsymbol{W}(\boldsymbol{x})-\hat{\boldsymbol{V}}(\boldsymbol{x})|=0,$$

then $\boldsymbol{W}\equiv\boldsymbol{V}$.

5 Applications

In this section, we apply the results of this paper to three important models in biology.

5.1 Two species Lotka-Volterra competition-diffusion systems

Consider a Lotka-Volterra competition-diffusion system with two components

$$\begin{cases}\dfrac{\partial}{\partial t}u_1=\Delta u_1+u_1(\boldsymbol{x},t)[1-u_1(\boldsymbol{x},t)-k_1u_2(\boldsymbol{x},t)],\\ \dfrac{\partial}{\partial t}u_2=d\Delta u_2+ru_2(\boldsymbol{x},t)[1-u_2(\boldsymbol{x},t)-k_2u_1(\boldsymbol{x},t)],\end{cases}\quad \boldsymbol{x}\in\mathbb{R}^3,\quad t>0,\tag{5.1}$$

where k_1, k_2, r and d are positive constants. The variables $u_1(\boldsymbol{x},t)$ and $u_2(\boldsymbol{x},t)$ stand for the population densities of two competing species, respectively. Assume that $k_1>1$ and $k_2>1$. Note that (5.1) is normalized so that it has the equilibrium solutions $(u_1,u_2)=(1,0),(0,1)$, denoted by $\boldsymbol{E}_u=(1,0)$ and $\boldsymbol{E}_v=(0,1)$. It is well known that (5.1) admits a planar traveling wave solution $\boldsymbol{\Phi}(\boldsymbol{x}\cdot\boldsymbol{e}+ct):=$

$(\phi_1(\boldsymbol{x}\cdot\boldsymbol{e}+ct),\phi_2(\boldsymbol{x}\cdot\boldsymbol{e}+ct))$ with wave speed $c\in\mathbb{R}$ connecting $\boldsymbol{E}_u=(1,0)$ and $\boldsymbol{E}_v=(0,1)$ (see [20,21,32] and the references therein), where $\boldsymbol{e}\in\mathbb{R}^3$ and $|\boldsymbol{e}|=1$. In particular, the traveling wave solution $\boldsymbol{\Phi}(\xi)=(\phi_1(\xi),\phi_2(\xi))$ is unique up to translation. It should be pointed out that to determine the sign of the wave speed c for (5.1) is a difficult job. Recently, some sufficient conditions have been obtained for the positivity of the wave speed c (see [1,20]).

Put $u_2^*=1-u_2$. Then (5.1) reduces to (for the sake of simplicity, we drop the symbol $*$)

$$\begin{cases}\dfrac{\partial}{\partial t}u_1=\Delta u_1+u_1(\boldsymbol{x},t)[1-k_1-u_1(\boldsymbol{x},t)+k_1u_2(\boldsymbol{x},t)],\\ \dfrac{\partial}{\partial t}u_2=d\Delta u_2+r(1-u_2(\boldsymbol{x},t))[k_2u_1(\boldsymbol{x},t)-u_2(\boldsymbol{x},t)],\end{cases}\quad \boldsymbol{x}\in\mathbb{R}^3,\quad t>0. \tag{5.2}$$

Correspondingly, the equilibria $\boldsymbol{E}_u=(1,0)$ and $\boldsymbol{E}_v=(0,1)$ become $\boldsymbol{E}^1=(1,1)$ and $\boldsymbol{E}^0=(0,0)$, respectively. In addition, (5.2) admits a unique traveling wave solution

$$\boldsymbol{\Psi}(\boldsymbol{x}\cdot\boldsymbol{e}+ct):=(\psi_1(\boldsymbol{x}\cdot\boldsymbol{e}+ct),\psi_2(\boldsymbol{x}\cdot\boldsymbol{e}+ct))$$

connecting $\boldsymbol{E}^0=(0,0)$ and $\boldsymbol{E}^1=(1,1)$. It is easy to verify that (H1)–(H4) hold (see the arguments of Example 1 in [53, Section 5]). Furthermore, we assume that the planar wave speed $c>0$. Then (H5) holds.

Fix $s>c>0$. Let $h_j(\boldsymbol{x}')$ $(j=1,\dots,n)$, $h(\boldsymbol{x}')$ and $D(\gamma)$ be defined in Section 1. It follows from Theorem 1.1 that there exists a solution $\boldsymbol{u}(\boldsymbol{x},t)=\boldsymbol{V}(\boldsymbol{x}',x_3+st)=(V_1(\boldsymbol{x}',x_3+st),V_2(\boldsymbol{x}',x_3+st))$ of (5.2) satisfying $\boldsymbol{V}(\boldsymbol{x})>\boldsymbol{\Psi}^-(\boldsymbol{x})$ for $\boldsymbol{x}\in\mathbb{R}^3$ and

$$\lim_{\gamma\to\infty}\sup_{\boldsymbol{x}\in D(\gamma)}|\boldsymbol{V}(\boldsymbol{x})-\boldsymbol{\Psi}^-(\boldsymbol{x})|=0,$$

where

$$\boldsymbol{\Psi}^-(\boldsymbol{x})=\boldsymbol{\Psi}\left(\frac{c}{s}(x_3+h(\boldsymbol{x}'))\right)=\left(\psi_1\left(\frac{c}{s}(x_3+h(\boldsymbol{x}'))\right),\psi_2\left(\frac{c}{s}(x_3+h(\boldsymbol{x}'))\right)\right).$$

Moreover, for any $\boldsymbol{u}^0(\boldsymbol{x})\in C(\mathbb{R}^3,[\boldsymbol{E}^0,\boldsymbol{E}^1])$ satisfying

$$\lim_{\gamma\to\infty}\sup_{\boldsymbol{x}\in D(\gamma)}|\boldsymbol{u}^0(\boldsymbol{x})-\boldsymbol{V}(\boldsymbol{x})|=0,$$

the solution $\boldsymbol{u}(\boldsymbol{x},t;\boldsymbol{u}^0)$ of (5.2) with the initial value $\boldsymbol{u}^0$ satisfies

$$\lim_{t\to\infty}\|\boldsymbol{u}(\cdot,\cdot,t;\boldsymbol{u}^0)-\boldsymbol{V}(\cdot,\cdot+st)\|_{C(\mathbb{R}^2\times\mathbb{R})}=0.$$

Returning to System (5.1), we know that there exists a solution

$$\boldsymbol{u}(\boldsymbol{x},t)=\boldsymbol{U}(\boldsymbol{x}',x_3+st)=(U_1(\boldsymbol{x}',x_3+st),\ U_2(\boldsymbol{x}',x_3+st))$$

of (5.2) satisfying $U_1(\boldsymbol{x})>\phi_1^-(\boldsymbol{x})$ and $U_2(\boldsymbol{x})<\phi_2^-(\boldsymbol{x})$ for $\boldsymbol{x}\in\mathbb{R}^3$, and

$$\lim_{\gamma\to\infty}\sup_{\boldsymbol{x}\in D(\gamma)}|\boldsymbol{U}(\boldsymbol{x})-\boldsymbol{\Phi}^-(\boldsymbol{x})|=0,$$

where

$$\boldsymbol{\Phi}^-(\boldsymbol{x})=(\phi_1^-(\boldsymbol{x}),\phi_2^-(\boldsymbol{x}))=\left(\phi_1\left(\frac{c}{s}(x_3+h(\boldsymbol{x}'))\right),\phi_2\left(\frac{c}{s}(x_3+h(\boldsymbol{x}'))\right)\right).$$

Furthermore, for any $\boldsymbol{u}^0(\boldsymbol{x})\in C(\mathbb{R}^3,[\boldsymbol{E}^0,\boldsymbol{E}^1])$ satisfying

$$\lim_{\gamma\to\infty}\sup_{\boldsymbol{x}\in D(\gamma)}|\boldsymbol{u}^0(\boldsymbol{x})-\boldsymbol{U}(\boldsymbol{x})|=0,$$

the solution $\boldsymbol{u}(\boldsymbol{x},t;\boldsymbol{u}^0)$ of (5.1) with the initial value $\boldsymbol{u}^0$ satisfies

$$\lim_{t\to\infty}\|\boldsymbol{u}(\cdot,\cdot,t;\boldsymbol{u}^0)-\boldsymbol{U}(\cdot,\cdot+st)\|_{C(\mathbb{R}^2\times\mathbb{R})}=0.$$

5.2 Lotka-Volterra competition-diffusion systems with spatio-temporal delays

Consider a Lotka-Volterra competition-diffusion system with spatio-temporal delays

$$\begin{cases} \dfrac{\partial}{\partial t}u_1 = \Delta u_1 + u_1(\boldsymbol{x},t)[1 - u_1(\boldsymbol{x},t) - k_1(g_1 * u_2)(\boldsymbol{x},t)], \\ \dfrac{\partial}{\partial t}u_2 = d\Delta u_2 + ru_2(\boldsymbol{x},t)[1 - u_2(\boldsymbol{x},t) - k_2(g_2 * u_1)(\boldsymbol{x},t)], \end{cases} \tag{5.3}$$

where $\boldsymbol{x} \in \mathbb{R}^3$, $t > 0$, $g_1(\boldsymbol{x},t) = \frac{1}{\tau_1}\mathrm{e}^{-\frac{1}{\tau_1}t}\frac{1}{(4\pi dt)^{\frac{3}{2}}}\mathrm{e}^{-\frac{|\boldsymbol{x}|^2}{4dt}}$, $g_2(\boldsymbol{x},t) = \frac{1}{\tau_2}\mathrm{e}^{-\frac{1}{\tau_2}t}\frac{1}{(4\pi t)^{\frac{3}{2}}}\mathrm{e}^{-\frac{|\boldsymbol{x}|^2}{4t}}$, $\tau_i > 0$ and

$$\begin{cases} (g_1 * u_2)(\boldsymbol{x},t) = \displaystyle\int_{-\infty}^{t}\int_{\mathbb{R}^3} g_1(\boldsymbol{x}-\boldsymbol{y},t-s)u_2(\boldsymbol{y},s)d\boldsymbol{y}ds, \\ (g_2 * u_1)(\boldsymbol{x},t) = \displaystyle\int_{-\infty}^{t}\int_{\mathbb{R}^3} g_2(\boldsymbol{x}-\boldsymbol{y},t-s)u_1(\boldsymbol{y},s)d\boldsymbol{y}ds, \end{cases}$$

which has been studied by Gourley and Ruan [18] and Lin and Li [37]. The coefficients k_1, k_2, r and d are assumed to be the same as in Subsection 5.1. After changes of variables (see Example 2 in [53, Section 5]), (5.3) reduces to the following system:

$$\begin{cases} \dfrac{\partial}{\partial t}\hat{u}_1 = \Delta\hat{u}_1 + \hat{u}_1(\boldsymbol{x},t)[1 - k_1 - \hat{u}_1(\boldsymbol{x},t) + k_1\hat{u}_3(\boldsymbol{x},t)], \\ \dfrac{\partial}{\partial t}\hat{u}_2 = d\Delta\hat{u}_2 + r(1-\hat{u}_2(\boldsymbol{x},t))[k_2\hat{u}_4(\boldsymbol{x},t) - \hat{u}_2(\boldsymbol{x},t)], \\ \dfrac{\partial}{\partial t}\hat{u}_3 = d\Delta\hat{u}_3 + \gamma_1(\hat{u}_2 - \hat{u}_3), \\ \dfrac{\partial}{\partial t}\hat{u}_4 = \Delta\hat{u}_4 + \gamma_2(\hat{u}_1 - \hat{u}_4). \end{cases} \tag{5.4}$$

The equilibria of (5.4) corresponding to $\boldsymbol{E}_u = (1,0)$ and $\boldsymbol{E}_v = (0,1)$ of (5.3) are $\boldsymbol{E}_1 = (1,1,1,1)$ and $\boldsymbol{E}_0 = (0,0,0,0)$. It is not difficult to show that (H1)–(H4) hold for (5.4) (see also [53]). Following Lin and Li [37], we know that (5.4) admits a traveling wave front $\boldsymbol{\Psi}(\xi) = (\psi_1(\xi),\psi_2(\xi),\psi_3(\xi),\psi_4(\xi))$ with $\xi = \boldsymbol{x}\cdot\boldsymbol{e} + ct$ satisfying $\psi_i'(\xi) > 0$ for $\xi \in \mathbb{R}$, $\boldsymbol{\Psi}(\xi) \to \boldsymbol{E}_0$ as $\xi \to -\infty$ and $\boldsymbol{\Psi}(\xi) \to \boldsymbol{E}_1$ as $\xi \to +\infty$, where $\boldsymbol{e} \in \mathbb{R}^3$ and $|\boldsymbol{e}| = 1$.

Assume $c > 0$. Then (H5) holds. For any $s > c > 0$, let $h_j(\boldsymbol{x}')$ $(j = 1,\dots,n)$, $h(\boldsymbol{x}')$ and $D(\gamma)$ be defined in Subsection 5.1. Denote

$$\boldsymbol{\Psi}^-(\boldsymbol{x}) = \boldsymbol{\Psi}\left(\frac{c}{s}(x_3 + h(\boldsymbol{x}'))\right) = \left(\psi_1\left(\frac{c}{s}(x_3 + h(\boldsymbol{x}'))\right),\dots,\psi_4\left(\frac{c}{s}(x_3 + h(\boldsymbol{x}'))\right)\right).$$

By Theorem 1.1, for any $s > c$ System (5.4) admits a pyramidal traveling front

$$\boldsymbol{V}(\boldsymbol{x}', x_3 + st) := (V_1(\boldsymbol{x}', x_3 + st),\dots,V_4(\boldsymbol{x}', x_3 + st))$$

satisfying

$$\begin{cases} \Delta V_1(\boldsymbol{x}) - s\dfrac{\partial}{\partial x_3}V_1(\boldsymbol{x}) + V_1(\boldsymbol{x})[1 - k_1 - V_1(\boldsymbol{x}) + k_1V_3(\boldsymbol{x})] = 0, \\ d\Delta V_2(\boldsymbol{x}) - s\dfrac{\partial}{\partial x_3}V_2(\boldsymbol{x}) + r(1 - V_2(\boldsymbol{x}))[k_2V_4(\boldsymbol{x}) - V_2(\boldsymbol{x})] = 0, \\ d\Delta V_3(\boldsymbol{x}) - s\dfrac{\partial}{\partial x_3}V_3(\boldsymbol{x}) + \gamma_1[V_2(\boldsymbol{x}) - V_3(\boldsymbol{x})] = 0, \\ \Delta V_4(\boldsymbol{x}) - s\dfrac{\partial}{\partial x_3}V_4(\boldsymbol{x}) + \gamma_2[V_1(\boldsymbol{x}) - V_4(\boldsymbol{x})] = 0 \end{cases}$$

for any $\boldsymbol{x} \in \mathbb{R}^3$ and $\lim_{\gamma\to\infty}\sup_{\boldsymbol{x}\in D(\gamma)}|\boldsymbol{V}(\boldsymbol{x}) - \boldsymbol{\Psi}^-(\boldsymbol{x})| = 0$. Moreover, for any

$$\hat{\boldsymbol{u}}^0(\boldsymbol{x}) := (\hat{u}_1^0(\boldsymbol{x}),\dots,\hat{u}_4^0(\boldsymbol{x}))$$

with $\hat{u}_i^0 \in C(\mathbb{R}^3, [0,1])$ and

$$\lim_{\gamma\to\infty} \sup_{\boldsymbol{x}\in D(\gamma)} |\hat{\boldsymbol{u}}^0(\boldsymbol{x}) - \boldsymbol{\Psi}^-(\boldsymbol{x})| = 0,$$

the solution $\hat{\boldsymbol{u}}(\boldsymbol{x}, t; \hat{\boldsymbol{u}}^0)$ of (5.4) with the initial value $\hat{\boldsymbol{u}}^0$ satisfies

$$\lim_{t\to\infty} \|\hat{\boldsymbol{u}}(\cdot,\cdot,t;\hat{\boldsymbol{u}}^0) - \boldsymbol{V}(\cdot,\cdot+st)\|_{C(\mathbb{R}^2\times\mathbb{R})} = 0.$$

It is not difficult to find that $\boldsymbol{\Phi}(\xi) = (\phi_1(\xi), \phi_2(\xi))$ with $\phi_1(\xi) = \psi_1(\xi)$ and $\phi_2(\xi) = 1 - \psi_2(\xi)$ is a planar traveling wave front of (5.3) (see [37, 53]). In addition,

$$\boldsymbol{U}(\boldsymbol{x}', x_3 + st) := (U_1(\boldsymbol{x}', x_3 + st), U_2(\boldsymbol{x}', x_3 + st))$$

with $U_1(\boldsymbol{x}', x_3 + st) = V_1(\boldsymbol{x}', x_3 + st)$ and $U_2(\boldsymbol{x}', x_3 + st) = 1 - V_2(\boldsymbol{x}', x_3 + st)$ is a pyramidal traveling front of (5.3) satisfying

$$\begin{cases} \Delta U_1(\boldsymbol{x}) - s\dfrac{\partial}{\partial x_3}U_1(\boldsymbol{x}) + U_1(\boldsymbol{x})[1 - U_1(\boldsymbol{x}) - k_1(g_1 \odot U_2)(\boldsymbol{x})] = 0,\\ d\Delta U_2(\boldsymbol{x}) - s\dfrac{\partial}{\partial x_3}U_2(\boldsymbol{x}) + rU_2(\boldsymbol{x})[1 - U_2(\boldsymbol{x}) - k_2(g_2 \odot U_1)(\boldsymbol{x})] = 0, \end{cases}$$

and $\lim_{\gamma\to\infty}\sup_{\boldsymbol{x}\in D(\gamma)} |\boldsymbol{U}(\boldsymbol{x}) - \boldsymbol{\Phi}^-(\boldsymbol{x})| = 0$, where

$$\boldsymbol{\Phi}^-(\boldsymbol{x}) = \boldsymbol{\Phi}\left(\frac{c}{s}(x_3 + h(\boldsymbol{x}'))\right) = \left(\phi_1\left(\frac{c}{s}(x_3 + h(\boldsymbol{x}'))\right), \phi_2\left(\frac{c}{s}(x_3 + h(\boldsymbol{x}'))\right)\right),$$

$$(g_1 \odot U_2)(\boldsymbol{x}) = \int_0^\infty \int_{\mathbb{R}^3} \frac{1}{\tau_1}\mathrm{e}^{-\frac{s}{\tau_1}} \frac{1}{(4\pi ds)^{\frac{3}{2}}}\mathrm{e}^{-\frac{x^2+y^2+z^2}{4ds}} U_2(x_1 - x, x_2 - y, x_3 - z - cs)dxdydz,$$

$$(g_2 \odot U_1)(\boldsymbol{x}) = \int_0^\infty \int_{\mathbb{R}^2} \frac{1}{\tau_2}\mathrm{e}^{-\frac{s}{\tau_2}} \frac{1}{(4\pi s)^{\frac{3}{2}}}\mathrm{e}^{-\frac{x^2+y^2+z^2}{4s}} U_1(x_1 - x, x_2 - y, x_3 - z - cs)dxdydz.$$

For (5.3), give an initial value $\boldsymbol{u}^0(\boldsymbol{x}, \theta) = (u_1^0(\boldsymbol{x}, \theta), u_2^0(\boldsymbol{x}, \theta))$ with

$$u_i^0(\boldsymbol{x}, \theta) \in C(\mathbb{R}^3 \times (-\infty, 0], [0,1]) \quad \text{and} \quad \lim_{\gamma\to\infty} \sup_{\boldsymbol{x}\in D(\gamma)} |\boldsymbol{u}^0(\boldsymbol{x}) - \boldsymbol{\Phi}^-(\boldsymbol{x})| = 0.$$

Furthermore, let $u_3^0(\boldsymbol{x}) = (g_1 * u_2^0)(\boldsymbol{x}, 0)$, $u_4^0(\boldsymbol{x}) = (g_2 * u_1^0)(\boldsymbol{x}, 0)$ and

$$\tilde{\boldsymbol{u}}^0(\boldsymbol{x}) = (u_1^0(\boldsymbol{x}, 0), u_2^0(\boldsymbol{x}, 0), u_3^0(\boldsymbol{x}), u_4^0(\boldsymbol{x})).$$

Let $\hat{\boldsymbol{u}}(\boldsymbol{x}, t; \tilde{\boldsymbol{u}}^0)$ be the solution of (5.4) with the initial value $\tilde{\boldsymbol{u}}^0$. Then by Lin and Li [37, Theorem 3.3], we have that $\boldsymbol{u}(\boldsymbol{x}, t; \boldsymbol{u}^0) = (u_1(\boldsymbol{x}, t; \boldsymbol{u}^0), u_2(\boldsymbol{x}, t; \boldsymbol{u}^0))$ defined by

$$u_1(\boldsymbol{x}, t; \boldsymbol{u}^0) = \hat{u}_1(\boldsymbol{x}, t; \tilde{\mathbf{u}}^0), \quad u_2(\boldsymbol{x}, t; \boldsymbol{u}^0) = 1 - \hat{u}_2(\boldsymbol{x}, t; \tilde{\mathbf{u}}^0) \quad \text{for} \quad t > 0,$$

and

$$u_i(\boldsymbol{x}, \theta; \boldsymbol{u}^0) = u_i^0(\boldsymbol{x}, \theta) \quad \text{for} \quad \theta \leqslant 0, \quad i = 1, 2$$

is a classical solution of (5.3) with the initial value $\boldsymbol{u}^0$. Following the previous arguments, we have

$$\lim_{t\to\infty} \|\boldsymbol{u}(\cdot,\cdot,t;\boldsymbol{u}^0) - \boldsymbol{U}(\cdot,\cdot+st)\|_{C(\mathbb{R}^2\times\mathbb{R})} = 0.$$

5.3 Reaction-diffusion systems with multiple obligate mutualists

Consider a system of m obligate mutualists

$$\frac{\partial}{\partial t}u_i(x, y, t) = D_i\Delta u_i + u_i\left(-(m-2) - u_i + \frac{(1 + (m-1)\beta)\sum_{1\leqslant j\leqslant m, j\neq i} u_j}{1 + \beta\sum_{1\leqslant j\leqslant m, j\neq i} u_j}\right), \tag{5.5}$$

where $D_i > 0$, $\beta > \frac{m-2}{m-1}$, $i = 1, 2, \ldots, m$ and $m \geqslant 3$. This system has been studied by Mischaikow and Hutson [39] (see also the references therein). The system exactly admits three equilibria $\boldsymbol{E}_0 = (0, \ldots, 0)$, $\boldsymbol{E}_\theta = \frac{m-2}{(m-1)\beta}(1, \ldots, 1)$, $\boldsymbol{E}_1 = (1, \ldots, 1)$. As showed by Wang [53], (5.5) satisfies (H1)–(H4) by replacing $\boldsymbol{E}^-$ and $\boldsymbol{E}^+$ with $\boldsymbol{E}_0$ and $\boldsymbol{E}_1$, respectively. It follows from [39] that (5.5) admits a traveling wave front $\boldsymbol{\Phi}(\xi) = (\phi_1(\xi), \ldots, \phi_n(\xi))$ connecting $\boldsymbol{E}_0$ and $\boldsymbol{E}_1$, where $\xi = \boldsymbol{x} \cdot \boldsymbol{e} + ct$, $\boldsymbol{e} \in \mathbb{R}^3$ and $|\boldsymbol{e}| = 1$. In addition, the traveling wave front is unique up to translation and satisfies $\phi_i'(\xi) > 0$ for $\xi \in \mathbb{R}$. Assume that $c > 0$ (in fact, when $D_1 = \cdots = D_n$ and $\frac{1}{6} + \frac{1}{2\beta} - \frac{m}{\beta^2(m-1)} + \frac{m}{\beta^3(m-1)^2}\ln(1 + \beta(m-1)) > 0$, there holds $c > 0$). Then Theorem 1.1 is applicable to (5.5).

6 Discussion

In the recent years, great attention has been paid to the study of multidimensional traveling fronts for scalar reaction-diffusion equations and various new types of nonplanar traveling waves have been observed, such as V-formed curved fronts for two-dimensional spaces (see [2,19,23–25,42,43,46,55,57]), cylindrically symmetric traveling fronts (see [24, 25]) and traveling fronts with pyramidal shapes (see [34, 45, 47–50]) in higher-dimensional spaces.

For systems of reaction-diffusion equations, most results are on two-dimensional V-form curved fronts (see [28–30, 53]). For Lotka-Volterra competition-diffusion systems in higher-dimensional spaces, Ni and Taniguchi [41] established the existence of pyramidal traveling wave solutions. In this article, by extending the arguments of [47,48] for a scalar equation and using the approaches of [53] for a system, we studied the existence, uniqueness and stability of traveling waves of pyramidal shapes for reaction-diffusion systems in the three-dimensional space $\mathbb{R}^3$ and applied the theoretical results to some biological models, such as competition-diffusion systems with or without spatio-temporal delays and reaction-diffusion systems of multiple obligate mutualists.

Recently, Wang et al. [56] have established the existence of axisymmetric traveling fronts in Lotka-Volterra competition-diffusion systems in the three-dimensional space $\mathbb{R}^3$, i.e., traveling fronts which are axially symmetric with respect to the x_3-axis. However, we were unable to prove the uniqueness and stability of such axisymmetric traveling fronts. It will be interesting to study the existence, uniqueness and stability of axisymmetric traveling fronts and other types of nonplanar traveling fronts for reaction-diffusion systems in higher-dimensional spaces.

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant Nos. 11371179 and 11271172) and National Science Foundation of USA (Grant No. DMS-1412454). The authors thank the two anonymous reviewers and the handling editor for their helpful comments and suggestions.

References

1 Alcahrani E O, Davidson F A, Dodds N. Travelling waves in near-degenerate bistable competition models. Math Model Nat Phenom, 2010, 5: 13–35

2 Bonnet A, Hamel F. Existence of non-planar solutions of a simple model of premixed Bunsen flames. SIAM J Math Anal, 1999, 31: 80–118

3 Brazhnik P K, Tyson J J. On traveling wave solutions of Fisher's equation in two spatial dimensions. SIAM J Appl Math, 1999, 60: 371–391

4 Bu Z-H, Wang Z-C. Curved fronts of monostable reaction-advection-diffusion equations in space-time periodic media. Commun Pure Appl Anal, 2016, 15: 139–160

5 Chapuisat G. Existence and nonexistence of curved front solution of a biological equation. J Differential Equations, 2007, 236: 237–279

6 Chen X, Guo J-S, Hamel F, et al. Traveling waves with paraboloid like interfaces for balanced bistable dynamics. Ann Inst H Poincaré Anal Non Linéaire, 2007, 24: 369–393

7 Conley C, Gardner R. An application of the generalized Morse index to traveling wave solutions of a competitive reaction-diffusion model. Indiana Univ Math J, 1984, 33: 319–343

8 Daners D, McLeod P K. Abstract Evolution Equations, Periodic Problems and Applications. Harlow: Longman Scientific and Technical, 1992

9 del Pino M, Kowalczyk M, Wei J. A counterexample to a conjecture by De Giorgi in large dimensions. C R Math Acad Sci Paris, 2008, 346: 1261–1266
10 del Pino M, Kowalczyk M, Wei J. Traveling waves with multiple and non-convex fronts for a bistable semilinear parabolic equation. Comm Pure Appl Math, 2013, 66: 481–547
11 El Smaily M, Hamel F, Huang R. Two-dimensional curved fronts in a periodic shear flow. Nonlinear Anal, 2011, 74: 6469–6486
12 Fife P C. Dynamics of Internal Layers and Diffusive Interfaces. Philadelphia: SIAM, 1988
13 Fife P C, McLeod J B. The approach of solutions of nonlinear diffusion equations to traveling wave solutions. Arch Ration Mech Anal, 1977, 65: 355–361
14 Fife P C, McLeod J B. A phase plane discussion of convergence to traveling fronts for nonlinear diffusions. Arch Ration Mech Anal, 1981, 75: 281–314
15 Friedman A. Partial Differential Equations of Parabolic Type. New Jersey: Prentice-Hall, 1964
16 Gardner R A. Existence and stability of traveling wave solutions of competition model: A degree theoretical approach. J Differential Equations, 1982, 44: 343–364
17 Gilbarg D, Trudinger N S. Elliptic Partial Differential Equations of Second Order. Berlin: Springer-Verlag, 1983
18 Gourley S A, Ruan S. Convergence and travelling fronts in functional differential equations with nonlocal terms: A competition model. SIAM J Math Anal, 2003, 35: 806–822
19 Gui C. Symmetry of traveling wave solutions to the Allen-Cahn equation in $\mathbb{R}^2$. Arch Ration Mech Anal, 2011, 203: 1037–1065
20 Guo J-S, Lin Y-C. The sign of the wave speed for the Lotka-Volterra competition-diffusion system. Commun Pure Appl Anal, 2013, 12: 2083–2090
21 Guo J-S, Wu C-H. Recent developments on wave propagation in 2-species competition systems. Discrete Contin Dyn Syst Ser B, 2012, 17: 2713–2724
22 Hamel F. Bistable transition fronts in $\mathbb{R}^N$. Adv Math, 2016, 289: 279–344
23 Hamel F, Monneau R, Roquejoffre J-M. Stability of travelling waves in a model for conical flames in two space dimensions. Ann Sci École Norm Sup (4), 2004, 37: 469–506
24 Hamel F, Monneau R, Roquejoffre J-M. Existence and qualitative properties of multidimensional conical bistable fronts. Discrete Contin Dyn Syst, 2005, 13: 1069–1096
25 Hamel F, Monneau R, Roquejoffre J-M. Asymptotic properties and classification of bistable fronts with Lipschitz level sets. Discrete Contin Dyn Syst, 2006, 14: 75–92
26 Hamel F, Nadirashvili N. Travelling fronts and entire solutions of the Fisher-KPP equation in $\mathbb{R}^N$. Arch Ration Mech Anal, 2001, 157: 91–163
27 Hamel F, Roquejoffre J-M. Heteroclinic connections for multidimensional bistable reaction-diffusion equations. Discrete Contin Dyn Syst Ser S, 2011, 4: 101–123
28 Haragus M, Scheel A. Corner defects in almost planar interface propagation. Ann Inst H Poincaré Anal Non Linéaire, 2006, 23: 283–329
29 Haragus M, Scheel A. Almost planar waves in anisotropic media. Comm Partial Differential Equations, 2006, 31: 791–815
30 Haragus M, Scheel A. A bifurcation approach to non-planar traveling waves in reaction-diffusion systems. GAMM-Mitt, 2007, 30: 75–95
31 Huang R. Stability of travelling fronts of the Fisher-KPP equation in $\mathbb{R}^N$. NoDEA Nonlinear Differential Equations Appl, 2008, 15: 599–622
32 Kan-on Y. Parameter dependence of propagation speed of travelling waves for competition-diffusion equations. SIAM J Math Anal, 1995, 26: 340–363
33 Kapitula T. Multidimensional stability of planar travelling waves. Trans Amer Math Soc, 1997, 349: 257–269
34 Kurokawa Y, Taniguchi M. Multi-dimensional pyramidal travelling fronts in the Allen-Cahn equations. Proc Roy Soc Edinburgh Sect A, 2011, 141: 1031–1054
35 Levermore C D, Xin J X. Multidimensional stability of traveling waves in a bistable reaction-diffusion equation II. Comm Partial Differential Equations, 1992, 17: 1901–1924
36 Liang X, Zhao X-Q. Asymptotic speeds of spread and traveling waves for monotone semiflows with applications. Comm Pure Appl Math, 2007, 60: 1–40
37 Lin G, Li W-T. Bistable wavefronts in a diffusive and competitive Lotka-Volterra type system with nonlocal delays. J Differential Equations, 2008, 244: 487–513
38 Martin R H, Smith H L. Abstract functional differential equations and reaction-diffusion systems. Trans Amer Math Soc, 1990, 321: 1–44
39 Mischaikow K, Hutson V. Travelling waves for mutualist species. SIAM J Math Anal, 1993, 24: 987–1008
40 Morita Y, Ninomiya H. Monostable-type traveling waves of bistable reaction-diffusion equations in the multi-dimensional space. Bull Inst Math Acad Sin, 2008, 3: 567–584

41 Ni W M, Taniguchi M. Traveling fronts of pyramidal shapes in competition-diffusion systems. Netw Heterog Media, 2013, 8: 379–395

42 Ninomiya H, Taniguchi M. Existence and global stability of traveling curved fronts in the Allen-Cahn equations. J Differential Equations, 2005, 213: 204–233

43 Ninomiya H, Taniguchi M. Global stability of traveling curved fronts in the Allen-Cahn equations. Discrete Contin Dyn Syst, 2006, 15: 829–832

44 Sattinger D H. Monotone methods in nonlinear elliptic and parabolic boundary value problems. Indiana Univ Math J, 1972, 21: 979–1000

45 Sheng W-J, Li W-T, Wang Z-C. Periodic pyramidal traveling fronts of bistable reaction-diffusion equations with time-periodic nonlinearity. J Differential Equations, 2012, 252: 2388–2424

46 Sheng W-J, Li W-T, Wang Z-C. Multidimensional stability of V-shaped traveling fronts in the Allen-Cahn equation. Sci China Math, 2013, 56: 1969–1982

47 Taniguchi M. Traveling fronts of pyramidal shapes in the Allen-Cahn equations. SIAM J Math Anal, 2007, 39: 319–344

48 Taniguchi M. The uniqueness and asymptotic stability of pyramidal traveling fronts in the Allen-Cahn equations. J Differential Equations, 2009, 246: 2103–2130

49 Taniguchi M. Multi-dimensional traveling fronts in bistable reaction-diffusion equations. Discrete Contin Dyn Syst, 2012, 32: 1011–1046

50 Taniguchi M. An $(N-1)$-dimensional convex compact set gives an N-dimensional traveling front in the Allen-Cahn equation. SIAM J Math Anal, 2015, 47: 455–476

51 Tsai J-C. Global exponential stability of traveling waves in monotone bistable systems. Discrete Contin Dyn Syst, 2008, 21: 601–623

52 Volpert A I, Volpert V A, Volpert V A. Travelling Wave Solutions of Parabolic Systems. Providence: American Mathematical Society, 1994

53 Wang Z-C. Traveling curved fronts in monotone bistable systems. Discrete Contin Dyn Syst, 2012, 32: 2339–2374

54 Wang Z-C. Cylindrically symmetric traveling fronts in periodic reaction-diffusion equations with bistable nonlinearity. Proc Roy Soc Edinburgh Sect A, 2015, 145: 1053–1090

55 Wang Z-C, Bu Z-H. Nonplanar traveling fronts in reaction-diffusion equations with combustion and degenerate Fisher-KPP nonlinearities. J Differential Equations, 2016, 260: 6405–6450

56 Wang Z-C, Niu H-L, Ruan S. On the existence of axisymmetric traveling fronts in Lotka-Volterra competition-diffusion systems in $\mathbb{R}^3$. Discrete Contin Dyn Syst Ser B, 2016, in press

57 Wang Z-C, Wu J. Periodic traveling curved fronts in reaction-diffusion equation with bistable time-periodic nonlinearity. J Differential Equations, 2011, 250: 3196–3229

58 Xin J X. Multidimensional stability of traveling waves in a bistable reaction-diffusion equation I. Comm Partial Differential Equations, 1992, 17: 1889–1899

59 Zeng H. Stability of planar travelling waves for bistable reaction-diffusion equations in multiple dimensions. Appl Anal, 2014, 93: 653–664

60 Zeng H. Multidimensional stability of traveling fronts in monostable reaction-diffusion equations with complex perturbations. Sci China Math, 2014, 57: 353–366