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Transmission eigenvalue problem for inhomogeneous absorbing media with mixed boundary condition

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Abstract Consider the transmission eigenvalue problem for the wave scattering by a dielectric inhomogeneous absorbing obstacle lying on a perfect conducting surface. After excluding the purely imaginary transmission eigenvalues, we prove that the transmission eigenvalues exist and form a discrete set for inhomogeneous non-absorbing media, by using analytic Fredholm theory. Moreover, we derive the Faber-Krahn type inequalities revealing the lower bounds on real transmission eigenvalues in terms of the media parameters. Then, for inhomogeneous media with small absorption, we prove that the transmission eigenvalues also exist and form a discrete set by using perturbation theory. Finally, for homogeneous media, we present possible components of the eigenvalue-free zone quantitatively, giving the geometric understanding on this problem.

Keywords transmission eigenvalues, inhomogeneity, absorption, Helmholtz equation, inverse scattering

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1 Introduction

The transmission eigenvalue problems (TEVs) are boundary value problems for a coupled differential system. Motivated by the determination of media property using these eigenvalues in inverse scattering problems [9, 10, 14], the analysis for these new problems, such as the discreteness of eigenvalues, the distribution and asymptotic behavior, has become an important mathematical research area. These eigenvalue problems are not trivial since the corresponding operators are neither elliptic nor self-adjoint, and consequently, cannot be covered by any existing standard spectrum theory for differential operators. From the algebraic points of view, they are essentially the quadric eigenvalue problems. The efficient computational schemes for catching several small eigenvalues in finite dimensional space for non-absorbing media with constant background and Dirichlet boundary condition have been established in [17, 20].

The theoretical motivation for studying the TEVs lies on the fact that they are related to nonstandard spectral problems for several classes of differential operators. For interior transmission eigenvalue problems arising in inverse scattering, they cannot be unified in a general framework due to their own special structure, since the physical configurations for wave scattering are very complicated. For example, if the electromagnetic waves are applied, we need to consider the differential operator related to the Maxwell equations; whereas the Helmholtz equation should be considered for scattering of the acoustic waves. Even if for scattering of acoustic plane waves, the scattering process also depends on the interior

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structures of the media such as inhomogeneity and absorption, as well as the boundary property of the interested media. All these practical configurations lead to many smart eigenvalue problems for differential systems, which constitute the interesting research areas for spectral theory of differential operators in pure mathematics.

The motivation for the researches on TEVs from the applied areas comes from its potential applications for inverse scattering problems. Firstly, it has been shown that real TEVs can be determined from the corresponding scattered data such as the far field of the scattered wave [3]. Secondly, the scattered wave contains the information about the obstacle. Therefore, TEVs carry information about obstacle properties and can be used to quantify the presence of abnormalities inside homogeneous media. Moreover, by using this information, we can test the integrity of materials which arises frequently in many nondestructive testing scenarios. Due to the above reasons, TEVs play an essential role in some recently developed reconstruction algorithms for inverse scattering problems, such as the linear sampling method and the factorization method [2, 16]. To carry out this process, of particular interest is the spectrum property associated with this generalized eigenvalue problem, corresponding to different scattering models.

Roughly speaking, the scattering models are determined by the obstacle structure as well as its boundary state. In the case that an inhomogeneous non-absorbing obstacle embedded in a homogeneous non-absorbing media, the TEVs have been well studied. By using analytic Fredholm theory, one can show for several cases that the corresponding TEVs form at most a discrete set with infinity as the only possible accumulation point [11, 12]. For scalar isotropic media, Päivärinta and Sylvester [18] obtained the existence of a finite number of TEVs, provided that the index of refraction be large enough. Then Kirsch [15] extended this result to anisotropic media for both the scalar case and Maxwell's equations. The difficult case for a medium with cavities is studied in [4]. At the same time, Cakoni and Haddar [7] presented a general proof for the existence of TEVs for a wide class of scattering problems. In [22], Yang and Monk considered the interior transmission problem for a bounded isotropic non-absorbing dielectric medium lying on an infinite conducting surface. They established the Fredholm property and showed that TEVs exist and form a discrete set. In [21], the discreteness and the continuity of the spectral projections on the medium contrast is established for standard interior transmission model by proving that interior transmission operator has upper triangular compact resolvent. It is noteworthy to mention that there are still many open questions meriting further investigations [8]. In fact, up to now most of the researches on TEVs have only considered the configurations with non-absorbing homogeneous background and non-absorbing obstacle. This restriction was given to avoid certain mathematical difficulties in dealing with non-self-adjoint operators. The study on TEVs for absorbing media was initiated in [5]. For a constant index of refraction, they also gain regions in the complex plane where the TEVs cannot exist and obtain *a-priori* estimates for real TEVs. Recently, it has been investigated for the simulation of the radiation of an antenna situated on a large metallic structure [1].

Motivated by existing works mentioned above, here we consider an interior transmission problem arising in a non-destructive testing application from inverse scattering models. In this case, the mixed boundary condition should be introduced due to the presence of a perfect electric conducting surface. Moreover, we assume the general case that both the obstacle and the background media are inhomogeneous absorbing media. Such extensions of the transmission model make our problem more difficult and enable the corresponding theoretical results applicable to simulate more realistic physical settings. The essential difficulty here is, after establishing the variational formulation of the problem, that the existing schemes proving the existence of infinite number of eigenvalues is invalid due to the inhomogeneity of the media. We need a more elegant technique for inhomogeneous media to prove the existence of an infinite number of eigenvalues. More precisely, we firstly establish the discreteness of the TEVs for non-absorbing media from Fredholm theory, by decomposing the variational form into an operator equation with several components on Sobolev spaces. Then the existence of an infinite set of TEVs for nonabsorbing inhomogeneous media is shown by rewriting the variational formulation as a generalized eigenvalue problem and studying the corresponding operators involved. Since we consider the general inhomogeneous background here, the technique applied in [6] to prove the existence of an infinite set of TEVs for homogeneous background media must be improved essentially. We present quantitative indices for the inhomogeneity of the media to ensure the existence of an infinite set of transmission eigenvalues. Then we make use of the stability of eigenvalues for closed operators under small perturbations, as those analyzed in [13], to prove the existence of (complex) TEVs. We would like to point out that, the technique applied in [21] may be a possible candidate for dealing with our problem with mixed boundary condition here. However, since some coercivity condition on the refraction index near the obstacle boundary are assumed there, essential changes such as the domain for interior transmission operator and the restriction on refraction index are required for our model with mixed boundary condition.

This paper is organized as follows. In Section 2, we formulate the interior transmission problem by a coupled homogeneous system for the Helmholtz equation, and give its equivalent form. Then in Section 3 we show that the TEVs form at most a discrete set based on the analytic Fredholm theory and give the Faber-Krahn type inequality for non-absorbing media. In addition, we also show the existence of an infinite number of eigenvalues for this setting. Then the existence and discreteness of the (complex) TEVs for the inhomogeneous media with small absorption are generated from the perturbation theory of compact operators. In Section 4, we give a geometric description of the components of eigenvalue-free zone for homogeneous absorbing media. Finally, in Section 5, we give some conclusions and present some possible works in the future.

2 Problem setting and its variational form

Let $D \subset \mathbb{R}^d$ with $d = 2, 3$ be a bounded domain with a piecewise smooth Lipschitz boundary $\Gamma = \Gamma_m \cup \Gamma_\alpha$, where Γ_m is the interface between the perfect conducting substrate D_m and D, and Γ_α is the interface between the background dielectric medium D_α and D. Suppose that $D_m = \{(x, z) : z < h(x)\}$ with $(x, z) = (x_1, x_2)$ for $d = 2$ and $(x, z) = (x_1, x_2, x_3)$ for $d = 3$. Denote by ν the unit normal outward vector to ∂D, see Figure 1 for the geometric configuration.

Assume that there are absorption and inhomogeneity in both the obstacle and the background medium. For this configuration, the interior transmission eigenvalue problem corresponding to the acoustic wave scattering (also TE mode for electromagnetic wave scattering) in $\mathbb{R}^d$ reads as: Find nonzero (w, v) in a suitable function space for some $k \in \mathbb{C}$ such that

$$\Delta w + k^2\left(\epsilon_1(x) + \mathrm{i}\frac{\gamma_1(x)}{k}\right)w = 0, \quad \text{in } D, \tag{2.1}$$

$$\Delta v + k^2\left(\epsilon_0(x) + \mathrm{i}\frac{\gamma_0(x)}{k}\right)v = 0, \quad \text{in } D, \tag{2.2}$$

$$\frac{\partial w}{\partial \nu} - \frac{\partial v}{\partial \nu} = 0, \quad \text{on } \Gamma_\alpha, \tag{2.3}$$

$$w - v = 0, \quad \text{on } \Gamma_\alpha, \tag{2.4}$$

$$w = v = 0, \quad \text{on } \Gamma_m. \tag{2.5}$$

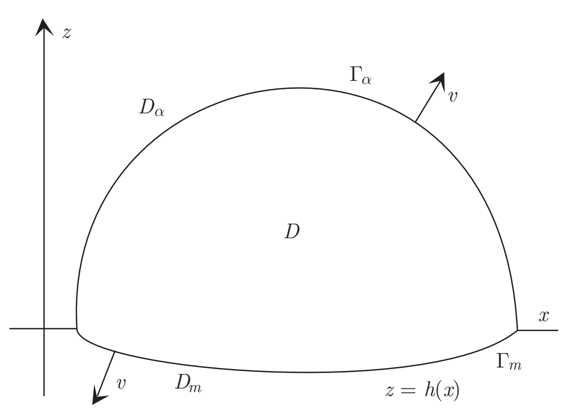

Figure 1 Geometric configuration of the problem

Here the real coefficients $(\epsilon_j(x), \gamma_j(x))$ for $j = 0, 1$ are assumed to satisfy

$$\epsilon_j(x), \gamma_j(x) \in L^\infty(D), \quad \epsilon_j(x) \geqslant \theta_j > 0, \quad \gamma_j(x) \geqslant 0 \quad \text{a.e. in } D.$$

For complex-valued functions $u, v \in L^2(D)$, define $(u, v)_{L^2(D)} := \int_D u\bar{v}dx$ and $\|u\|$ represents the $L^2(D)$ norm. Except for the classical spaces $H^1(D)$ and $H_0^1(D)$, we also need the Sobolev spaces (see [22] for 2-dimensional case)

$$\begin{aligned}
&H(\mathrm{div}, D) = \{u \in (L^2(D))^d : \nabla \cdot u \in L^2(D)\},\\
&H_0(\mathrm{div}, D) = \{u \in H(\mathrm{div}, D) : \nu \cdot u = 0 \text{ on } \Gamma\},\\
&H_{0\alpha}(\mathrm{div}, D) = \{u \in H(\mathrm{div}, D) : \nu \cdot u = 0 \text{ on } \Gamma_\alpha\}
\end{aligned}$$

and

$$\begin{aligned}
&\tilde{H}(D) = \{u \in H^1(D) : \nabla u \in H(\mathrm{div}, D)\},\\
&\tilde{H}_0(D) = \{u \in H_0^1(D) : \nabla u \in H_0(\mathrm{div}, D)\},\\
&\tilde{H}_{0\alpha}(D) = \{u \in H_0^1(D) : \nabla u \in H_{0\alpha}(\mathrm{div}, D)\}
\end{aligned}$$

equipped with the scalar product $(u, v)_{\tilde{H}(D)} := (u, v)_{L^2(D)} + (\nabla u, \nabla v)_{L^2(D)} + (\Delta u, \Delta v)_{L^2(D)}$ for $\tilde{H}(D)$, $\tilde{H}_0(D)$, $\tilde{H}_{0\alpha}(D)$, which are the Hilbert spaces, and $\tilde{H}_0(D)$ is equivalent to $H_0^2(D)$. Then the TEV problem can be restated as: Find the values of $k \in \mathbb{C}$ such that there exists nonzero function pair $(w, v) \in (L^2(D))^2$ satisfying $w - v \in \tilde{H}_{0\alpha}(D)$ and

$$\begin{cases}
\Delta w + k^2\left(\epsilon_1(x) + \mathrm{i}\dfrac{\gamma_1(x)}{k}\right)w = 0, & \text{in } D,\\
\Delta v + k^2\left(\epsilon_0(x) + \mathrm{i}\dfrac{\gamma_0(x)}{k}\right)v = 0, & \text{in } D,\\
w = v = 0, & \text{on } \Gamma_m.
\end{cases}$$

The boundary conditions (2.3) and (2.4) are incorporated in the requirement $w - v \in \tilde{H}_{0\alpha}(D)$. It is already known based on the analytic Fredholm theory that, for lossless media without the conducting surface, the set of TEVs is at most discrete with $+\infty$ as the only possible accumulation point [12, 18, 19]. We will prove that such a conclusion is also true under some hypotheses for our new model where there exist both absorption and inhomogeneity in the background media.

Firstly, we write (2.1)–(2.2) as an equivalent quadratic eigenvalue problem for $u := w - v \in \tilde{H}_{0\alpha}(D)$ for a fourth order differential operator. In fact, (2.1)–(2.2) generates

$$\Delta u + k^2\left(\epsilon_0 + \mathrm{i}\frac{\gamma_0}{k}\right)u = -k^2 w\left(\epsilon_c + \mathrm{i}\frac{\gamma_c}{k}\right) \quad \text{in } D \tag{2.6}$$

with $\epsilon_c(x) := \epsilon_1(x) - \epsilon_0(x)$ and $\gamma_c(x) := \gamma_1(x) - \gamma_0(x)$. Dividing both sides of (2.6) by $k^2\epsilon_c + \mathrm{i}k\gamma_c$ and applying the equation for w generate

$$[\Delta + k^2\epsilon_1(x) + \mathrm{i}k\gamma_1(x)]\frac{1}{k^2\epsilon_c(x) + \mathrm{i}k\gamma_c(x)}[\Delta + k^2\epsilon_0(x) + \mathrm{i}k\gamma_0(x)]u = 0 \quad \text{in } D. \tag{2.7}$$

On the other hand, we have from (2.6) and the boundary condition $w = 0$ in (2.5) that

$$\frac{1}{k^2\epsilon_c + \mathrm{i}k\gamma_c}(\Delta + k^2\epsilon_0 + \mathrm{i}k\gamma_0)u = 0 \quad \text{on } \Gamma_m, \tag{2.8}$$

which will turn out to be a natural boundary condition for u.

In addition, we note that $u = w - v \in \tilde{H}_{0\alpha}(D)$ implies that

$$u = 0 \quad \text{on} \quad \Gamma = \Gamma_m \cup \Gamma_\alpha, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma_\alpha.$$

By defining

$$P_0(x) := k\epsilon_0(x) + \mathrm{i}\gamma_0(x), \quad P_1(x) := k\epsilon_1(x) + \mathrm{i}\gamma_1(x), \quad P_c(x) := k\epsilon_c(x) + \mathrm{i}\gamma_c(x),$$

we finally conclude that the transmission eigenvalues $k \in \mathbb{C}$ are those such that there exists non-zero solution $u \in \tilde{H}_{0\alpha}(D)$ to the problem

$$\begin{cases} [\Delta + kP_1(x)]\left(\dfrac{1}{kP_c(x)}[\Delta + kP_0(x)]u\right) = 0 \quad \text{in } D, \\ u = 0 \quad \text{on} \quad \Gamma = \Gamma_m \cup \Gamma_\alpha, \\ \dfrac{1}{kP_c(x)}(\Delta + kP_0(x))u = 0 \quad \text{on } \Gamma_m, \quad \dfrac{\partial u}{\partial v} = 0 \quad \text{on } \Gamma_\alpha. \end{cases} \tag{2.9}$$

Using the boundary conditions, the variational form of the above interior TEV problem is to find $k \in \mathbb{C}$ and the corresponding nonzero function $u \in \tilde{H}_{0\alpha}(D)$ satisfying

$$\int_D \frac{1}{P_c(x)}[\Delta u + kP_0(x)u][\Delta\bar{\psi} + kP_1(x)\bar{\psi}]dx = 0 \quad \text{for all} \quad \psi \in \tilde{H}_{0\alpha}(D), \tag{2.10}$$

where $^-$ denotes the complex conjugate. In a similar fashion, the TEV problem can also be written as: Find $k \in \mathbb{C}$ and the corresponding nonzero function $u \in \tilde{H}_{0\alpha}(D)$ satisfying

$$\int_D \frac{-1}{P_c(x)}(\Delta u + kP_1(x)u)(\Delta\bar{\psi} + kP_0(x)\bar{\psi})dx = 0 \quad \text{for all} \quad \psi \in \tilde{H}_{0\alpha}(D). \tag{2.11}$$

Similar to the proof in [5, 6, 19], we can prove the following equivalence result under our physical configuration.

Lemma 2.1. *If $w, v \in L^2(D)$ satisfy that $w - v \in \tilde{H}(D)$ and w, v satisfy* (2.1)–(2.5)*, then $u := w - v \in \tilde{H}_{0\alpha}(D)$ satisfies* (2.9)*. Conversely, if $u \in \tilde{H}_{0\alpha}(D)$ is a solution of* (2.9)*, then*

$$w := -\frac{1}{k^2\epsilon_c + \mathrm{i}k\gamma_c}(\Delta + k^2\epsilon_0 + \mathrm{i}k\gamma_0)u \in L^2(D) \quad \textit{and} \quad v := w - u \in L^2(D)$$

satisfy (2.1)–(2.5).

Proof. It is clear that if w, v satisfy (2.1)–(2.5) then $u := w - v \in \tilde{H}_{0\alpha}(D)$ satisfies (2.9). On the contrary, if $u \in \tilde{H}_{0\alpha}(D)$ solves (2.9), then $w := -\frac{1}{k^2\epsilon_c+\mathrm{i}k\gamma_c}(\Delta + k^2\epsilon_0 + \mathrm{i}k\gamma_0)u \in L^2(D)$ meets (2.1) from the equation in (2.9). Since $v := w - u = -\frac{1}{k^2\epsilon_c+\mathrm{i}k\gamma_c}(\Delta + k^2\epsilon_1 + \mathrm{i}k\gamma_1)u \in L^2(D)$ and

$$[\Delta + k^2\epsilon_0(x) + \mathrm{i}k\gamma_0(x)]\frac{1}{k^2\epsilon_c(x) + \mathrm{i}k\gamma_c(x)}[\Delta + k^2\epsilon_1(x) + \mathrm{i}k\gamma_1(x)]u = 0 \quad \text{in} \quad D$$

from the analogous derivation to (2.7), the fact that v solves (2.2)–(2.4) is obvious due to $u = w - v \in \tilde{H}_{0\alpha}(D)$. From the boundary condition in (2.9) and the definition of w, we known that $w = 0$ on Γ_m and therefore $v = w - u = 0$ on Γ_m from the boundary condition for u. The proof is complete. □

3 Eigenvalues for inhomogeneous absorbing media

In the following, we will fix our analysis on 2-dimensional case with $d = 2$. However, it is easy to check that the arguments also hold for 3-dimensional case with some obvious modifications.

Based on the equivalence result given in Lemma 2.1, we can analyze TEVs of system (2.1)–(2.5) in terms of (2.9). The first result describing the transmission eigenvalue distribution is that we can exclude the pure imaginary values if $\epsilon_c\gamma_c > 0$.

Theorem 3.1. *The interior transmission problem* (2.1)–(2.5) *for absorbing media does not have any pure imaginary eigenvalue $k = \mathrm{i}\tau$ for any $\tau > 0$ if $\epsilon_c(x)\gamma_c(x) > 0$ in D. Moreover, for nonabsorbing media (i.e., $\gamma_1(x) \equiv \gamma_0(x) \equiv 0$), there is no imaginary eigenvalues $k = \mathrm{i}\tau$ for any $\tau \in \mathbb{R}$ if either $\epsilon_c(x) > 0$ or $\epsilon_c(x) < 0$ in D.*

Proof. Consider $k = \mathrm{i}\tau$ for real $\tau > 0$. Taking $\psi(x) = u(x)$ in (2.10), we obtain that

$$\begin{aligned} 0 &= \int_D \frac{1}{\tau\epsilon_c + \gamma_c}[\Delta u - (\tau^2\epsilon_0 + \tau\gamma_0)u][\Delta\bar{u} - (\tau^2\epsilon_1 + \tau\gamma_1)\bar{u}]dx \\ &= \int_D \frac{1}{\tau\epsilon_c + \gamma_c}|\Delta u - (\tau^2\epsilon_0 + \tau\gamma_0)u|^2 dx - \tau \int_D [\Delta u - (\tau^2\epsilon_0 + \tau\gamma_0)u]\bar{u}dx \\ &= \int_D \frac{1}{\tau\epsilon_c + \gamma_c}|\Delta u - (\tau^2\epsilon_0 + \tau\gamma_0)u|^2 dx + \tau \int_D |\nabla u|^2 dx + \tau^2 \int_D (\tau\epsilon_0 + \gamma_0)|u|^2 dx \\ &\geqslant \int_D \frac{1}{\tau\epsilon_c + \gamma_c}|\Delta u - (\tau^2\epsilon_0 + \tau\gamma_0)u|^2 dx. \end{aligned}$$

Case 1. $\epsilon_c, \gamma_c > 0$. The above inequality implies $\Delta u - (\tau^2\epsilon_0(x) + \tau\gamma_0(x))u = 0$ in D from $\tau > 0$. However, we also have $u = \frac{\partial u}{\partial \nu} = 0$ on Γ_α. Therefore the uniqueness for the Cauchy problem for the elliptic equation yields $u \equiv 0$ in D.

Case 2. $\epsilon_c, \gamma_c < 0$. We apply (2.11), which essentially exchanges the position of (ϵ_0, γ_0) and (ϵ_1, γ_1), to yield

$$0 \geqslant \int_D \frac{-1}{\tau\epsilon_c + \gamma_c}|\Delta u - (\tau^2\epsilon_1 + \tau\gamma_1)u|^2 dx.$$

Then the similar arguments lead to $u \equiv 0$ in D.

Case 3. $\epsilon_c, \gamma_c > 0$ for x in some subset $D_0 \subset D$ and $\epsilon_c, \gamma_c < 0$ for x in $D \backslash D_0$. It is not hard to get that

$$0 \geqslant \int_{D\backslash D_0} \frac{-1}{\tau\epsilon_c + \gamma_c}|\Delta u - (\tau^2\epsilon_1 + \tau\gamma_1)u|^2 dx + \int_{D_0} \frac{1}{\tau\epsilon_c + \gamma_c}|\Delta u - (\tau^2\epsilon_0 + \tau\gamma_0)u|^2 dx.$$

Therefore, there is no eigenvalue $k = \mathrm{i}\tau$ for any $\tau > 0$ for absorbing media provided $\epsilon_c\gamma_c > 0$. For non-absorbing media with $\gamma_1(x) \equiv \gamma_0(x) \equiv 0$, the conclusion follows similarly as above. The proof is complete. □

In general, the existence and distribution of TEVs for absorbing media are still open problems. However, if the absorptions γ_0, γ_1 are assumed to be small enough, we can still apply the perturbation theory in [13] to show the existence of TEVs near the real axis. Before showing the main result for non-absorbing media, we first state the following result given in [22] which is essentially the generalization of the Poincare inequality.

Lemma 3.2. *For any $u \in \tilde{H}_{0\alpha}(D)$, we have the estimate $\|\nabla u\|^2_{L^2(D)} \leqslant \frac{1}{\lambda(D)}\|\Delta u\|^2_{L^2(D)}$, where $\lambda(D)$ is the first eigenvalue of the following buckled plate eigenvalue problem*:

$$\begin{cases} -\Delta^2 v = \lambda\Delta v, & \text{in } D, \\ v = 0, & \text{on } \Gamma = \Gamma_m \cup \Gamma_\alpha, \\ \nu \cdot \nabla v = 0, & \text{on } \Gamma_\alpha, \\ \Delta v = 0, & \text{on } \Gamma_m. \end{cases} \tag{3.1}$$

Now we can clarify the distribution of eigenvalues for inhomogeneous non-absorbing media with mixed boundary condition on Γ.

Theorem 3.3. *For non-absorbing medium D with $\gamma_j(x) \equiv 0$ a.e. in D, assume that*

A1. *$\epsilon_j \in L^\infty(D)$ with the bounds $\epsilon_j^* \geqslant \epsilon_j(x) \geqslant \theta_j > 0$ for $j = 0, 1$*;

A2. *$\epsilon_c(x)$ has no zero points in D satisfying $\frac{1}{|\epsilon_c(x)|} > \alpha > 0$ a.e. in D*;

A3. *the bounds on* A1 *satisfy*

$$0 < \frac{\epsilon_0^*}{\theta_1 - \epsilon_0^*} < 1 \quad \textit{for} \quad \epsilon_c(x) > 0, \quad 0 < \frac{\epsilon_1^*}{\theta_0 - \epsilon_1^*} < 1 \quad \textit{for} \quad \epsilon_c(x) < 0.$$

Then we have

(1) *the set of TEVs is at most discrete and does not accumulate at zero*;

(2) *all the real TEVs (if they exist) are such that*

$$k^2 \geqslant \begin{cases} \lambda(D)\alpha\dfrac{1-\alpha\epsilon_0^*}{1+\alpha\epsilon_0^*}, & \text{if} \quad 0<\frac{\epsilon_0^*}{\theta_1-\epsilon_0^*}<1, \\ \lambda(D)\alpha\dfrac{1-\alpha\epsilon_1^*}{1+\alpha\epsilon_1^*}, & \text{if} \quad 0<\frac{\epsilon_1^*}{\theta_0-\epsilon_1^*}<1, \end{cases}$$

with $\lambda(D)$ *the first eigenvalue of* (3.1).

Proof. **Case 1.** $\epsilon_c > 0$. It follows from (2.10) for $\gamma_0 \equiv \gamma_1 \equiv 0$ a.e. in D that

$$\begin{aligned} 0 &= \int_D \frac{1}{\epsilon_c(x)}[\Delta u + k^2\epsilon_0(x)u][\Delta\bar{\psi} + k^2\epsilon_1(x)\bar{\psi}]dx \\ &= \int_D \frac{1}{\epsilon_c(x)}[\Delta u + k^2\epsilon_0(x)u][\Delta\bar{\psi} + k^2\epsilon_0(x)\bar{\psi}]dx + k^4\int_D \epsilon_0(x)u\bar{\psi}dx - k^2\int_D \nabla u\cdot\nabla\bar{\psi}dx \end{aligned}$$

for $u, \psi \in \tilde{H}_{0\alpha}(D)$. We therefore arrive at the following equivalent form of finding a nonzero function $u \in \tilde{H}_{0\alpha}(D)$ and $k \in \mathbb{C}$ such that

$$\tilde{A}_{k^2}(u,\psi) - k^2\tilde{B}(u,\psi) = 0 \quad \text{for all} \quad \psi \in \tilde{H}_{0\alpha}(D) \tag{3.2}$$

with

$$\begin{aligned} &\tilde{A}_{k^2}(u,\psi) = \left(\frac{1}{\epsilon_c(x)}[\Delta u + k^2\epsilon_0(x)u], \Delta\psi + k^2\epsilon_0(x)\psi\right)_{L^2(D)} + k^4(\epsilon_0(x)u,\psi)_{L^2(D)}, \\ &\tilde{B}(u,\psi) = (\nabla u, \nabla\psi)_{L^2(D)}. \end{aligned}$$

Furthermore, we have from $\epsilon_{c^*} := \inf_D |\epsilon_c(x)| > 0$ that

$$\begin{aligned} |\tilde{A}_{k^2}(u,\psi)| &= \left|\int_D \frac{1}{\epsilon_c}(\Delta u + k^2\epsilon_0 u)(\Delta\bar{\psi} + k^2\epsilon_0\bar{\psi})dx + k^4\int_D \epsilon_0 u\bar{\psi}dx\right| \\ &\leqslant \left|\int_D \left[\frac{1}{\epsilon_c}\Delta u\,\Delta\bar{\psi} + \frac{k^4\epsilon_0^2}{\epsilon_c}u\bar{\psi} + k^4\epsilon_0 u\bar{\psi}\right]\right| + \left|\int_D \left[\frac{k^2\epsilon_0}{\epsilon_c}\bar{\psi}\Delta u + \frac{k^2\epsilon_0}{\epsilon_c}u\Delta\bar{\psi}\right]\right| \\ &\leqslant C\|\Delta u\|\|\Delta\psi\| + C\|u\|\|\psi\| + \left|\int_D \left[\nabla u\cdot\nabla\left(\frac{k^2\epsilon_0}{\epsilon_c}\bar{\psi}\right) + \nabla\bar{\psi}\cdot\nabla\left(\frac{k^2\epsilon_0}{\epsilon_c}u\right)\right]\right| \\ &\leqslant C(\|\Delta u\|\|\Delta\psi\| + \|u\|\|\psi\|) + C(\|\nabla u\|\|\psi\| + \|\nabla u\|\|\nabla\psi\| + \|u\|\|\nabla\psi\|) \\ &\leqslant C(\|\Delta u\|\|\Delta\psi\| + \|u\|\|\psi\| + \|\nabla u\|\|\nabla\psi\|) \end{aligned}$$

owing to the Poincare inequality. Thus we have

$$|\tilde{A}_{k^2}(u,\psi)|^2 \leqslant C^2[\|\Delta u\|^2\|\Delta\psi\|^2 + \|u\|^2\|\psi\|^2 + \|\nabla u\|^2\|\nabla\psi\|^2] \leqslant C^2\|u\|^2_{\tilde{H}(D)}\|\psi\|^2_{\tilde{H}(D)},$$

which leads to

$$|\tilde{A}_{k^2}(u,\psi)| \leqslant C\|u\|_{\tilde{H}(D)}\|\psi\|_{\tilde{H}(D)}, \tag{3.3}$$

with the constant C depending on $(k, \epsilon_{c^*}, \epsilon_0^*, \|\nabla\frac{\epsilon_0}{\epsilon_c}\|_{L^\infty(D)})$. It is easy to see that

$$|\tilde{B}(u,\psi)|^2 = \left|\int_D \nabla u\cdot\nabla\bar{\psi}dx\right|^2 \leqslant \|\nabla u\|^2\|\nabla\psi\|^2 \leqslant \|u\|^2_{\tilde{H}(D)}\|\psi\|^2_{\tilde{H}(D)}. \tag{3.4}$$

From the above two estimates, we know that both $\tilde{A}_{k^2}(\cdot,\cdot)$ and $\tilde{B}(\cdot,\cdot)$ are continuous sesquilinear forms on $\tilde{H}_{0\alpha}(D)\times\tilde{H}_{0\alpha}(D)$. Denote by A_{k^2} and B the bounded linear operators from $\tilde{H}_{0\alpha}(D)$ to $\tilde{H}_{0\alpha}(D)$ defined using Riesz representation theorem by

$$(A_{k^2}u,\psi)_{\tilde{H}_{0\alpha}(D)} = \tilde{A}_{k^2}(u,\psi), \tag{3.5}$$

$$(Bu,\psi)_{\tilde{H}_{0\alpha}(D)}=\tilde{B}(u,\psi) \tag{3.6}$$

for all $\psi\in\tilde{H}_{0\alpha}(D)$.

To establish the invertibility of operator A_{k^2}, we apply the Lax-Milgram theorem. Indeed, following the scheme in [7], we obtain

$$\begin{aligned}
\tilde{A}_{k^2}(u,u)&=\left(\frac{1}{\epsilon_c(x)}[\Delta u+k^2\epsilon_0(x)u],\Delta u+k^2\epsilon_0(x)u\right)_{L^2(D)}+k^4(\epsilon_0(x)u,u)_{L^2(D)}\\
&\geqslant \alpha\|\Delta u+k^2\epsilon_0u\|^2_{L^2(D)}+\frac{1}{\epsilon_0^*}k^4\|\epsilon_0u\|^2_{L^2(D)}\\
&\geqslant \alpha(\|\Delta u\|^2+k^4\|\epsilon_0u\|^2-2k^2\|\Delta u\|\|\epsilon_0u\|)+\frac{1}{\epsilon_0^*}k^4\|\epsilon_0u\|^2\\
&\geqslant \alpha\left[X^2+Y^2-\left(\frac{X^2}{\epsilon}+\epsilon Y^2\right)\right]+\frac{1}{\epsilon_0^*}Y^2\\
&=\alpha\left(1-\frac{1}{\epsilon}\right)X^2+\left(\alpha+\frac{1}{\epsilon_0^*}-\alpha\epsilon\right)Y^2
\end{aligned} \tag{3.7}$$

for any $\epsilon>0$, where $X=\|\Delta u\|_{L^2(D)}$ and $Y=k^2\|\epsilon_0u\|_{L^2(D)}$.

Under the condition A3 that

$$0<\frac{\epsilon_0^*}{\theta_1-\epsilon_0^*}<1, \tag{3.8}$$

it automatically yields $\epsilon_c:=\epsilon_1(x)-\epsilon_0(x)>\theta_1-\epsilon_0^*>0$ and

$$\alpha\epsilon_0^*<\frac{\epsilon_0^*}{\epsilon_1(x)-\epsilon_0(x)}<\frac{\epsilon_0^*}{\theta_1-\epsilon_0^*}<1.$$

Now we take $\epsilon=\frac{1}{2}(1+\frac{1}{\alpha\epsilon_0^*})>1$, and (3.7) becomes

$$\tilde{A}_{k^2}(u,u)\geqslant\alpha\frac{1-\alpha\epsilon_0^*}{1+\alpha\epsilon_0^*}X^2+\frac{1}{2}\left(\alpha+\frac{1}{\epsilon_0^*}\right)Y^2. \tag{3.9}$$

Using Lemma 3.2, we have

$$\begin{aligned}
\|u\|^2_{\tilde{H}(D)}&=(u,u)_{L^2(D)}+(\nabla u,\nabla u)_{L^2(D)}+(\Delta u,\Delta u)_{L^2(D)}\\
&\leqslant\frac{1}{k^4\theta_0^2}Y^2+\|\nabla u\|^2_{L^2(D)}+X^2\\
&\leqslant\frac{1}{k^4\theta_0^2}Y^2+\frac{1}{\lambda(D)}\|\Delta u\|^2_{L^2(D)}+X^2=\left(1+\frac{1}{\lambda(D)}\right)X^2+\frac{1}{k^4\theta_0^2}Y^2.
\end{aligned} \tag{3.10}$$

Then (3.9)–(3.10) show that there exist a positive constant c_k independent of u such that $\tilde{A}_{k^2}(u,u)\geqslant c_k\|u\|^2_{\tilde{H}(D)}$. So $A_{k^2}:\tilde{H}_{0\alpha}(D)\to\tilde{H}_{0\alpha}(D)$ is bijection for any fixed $k\in\mathbb{R}$ from the Lax-Milgram theorem, since the sesquilinear form $\tilde{A}_{k^2}(\cdot,\cdot)$ is coercive in $\tilde{H}_{0\alpha}(D)\times\tilde{H}_{0\alpha}(D)$ for any fixed $k\in\mathbb{R}$.

In order to use the analytic Fredholm theory, we first have the following observations.

(I) The sesquilinear form $\tilde{A}_{k^2}(\cdot,\cdot)$ is analytic in $k\in\mathbb{C}$.

(II) Denote by $\mathbb{L}(\cdot,\cdot)$ the set of all bound linear operators from one Banach space to another. We define the operator-valued function $f:k\in\mathbb{C}\to A_{k^2}\in\mathbb{L}(\tilde{H}_{0\alpha}(D),\tilde{H}_{0\alpha}(D))$ such that for each $u\in\tilde{H}_{0\alpha}(D)$, the function $f_u:k\in\mathbb{C}\to A_{k^2}u\in\tilde{H}_{0\alpha}(D)$ is weakly analytic. This is true since for each $l\in[\tilde{H}_{0\alpha}(D)]^*=\mathbb{L}(\tilde{H}_{0\alpha}(D),\mathbb{C})$ where $*$ represents the dual space, we get that

$$l(f_u(k))=l(A_{k^2}u)=(A_{k^2}u,\psi)_{\tilde{H}_{0\alpha}(D)}=\tilde{A}_{k^2}(u,\psi)\in\mathbb{C}\quad\text{for some}\quad\psi\in\tilde{H}_{0\alpha}(D)$$

is analytic in $k\in\mathbb{C}$. Thus, f is analytic in $\mathbb{C}$.

(III) By the Lax-Milgram theorem, we firstly know that there exists a bounded linear inverse operator $A_{k^2}^{-1}$ of A_{k^2} for $k\in\mathbb{R}$. Therefore A_{k^2} is also invertible in a neighborhood of the positive real axis from the analytic property of A_{k^2}. Moreover, $A_{k^2}^{-1}$ is also analytic in this neighborhood.

On the other hand, it follows from (3.9) and Lemma 3.2 that

$$\tilde{A}_{k^2}(u,u) - k^2\tilde{B}(u,u) \geqslant \alpha\frac{1-\alpha\epsilon_0^*}{1+\alpha\epsilon_0^*}X^2 + \frac{1}{2}\left(\alpha+\frac{1}{\epsilon_0^*}\right)Y^2 - k^2\|\nabla u\|_{L^2(D)}^2$$
$$\geqslant \left(\alpha\frac{1-\alpha\epsilon_0^*}{1+\alpha\epsilon_0^*} - k^2\frac{1}{\lambda(D)}\right)X^2 + \frac{1}{2}\left(\alpha+\frac{1}{\epsilon_0^*}\right)Y^2. \tag{3.11}$$

Therefore, if $k^2 < \lambda(D)\alpha\frac{1-\alpha\epsilon_0^*}{1+\alpha\epsilon_0^*}$, then $A_{k^2} - k^2B$ is invertible. Using the observation (III), this statement yields that $I - k^2A_{k^2}^{-1}B$ is also invertible for real small k from the decomposition

$$A_{k^2} - k^2B \equiv A_{k^2}(I - k^2A_{k^2}^{-1}B).$$

However, $I - k^2A_{k^2}^{-1}B$ is also analytic from observation (III). Combining these two conclusions together, the analytic Fredholm theory (see [12, Theorem 8.26]) yields that $(I - k^2A_{k^2}^{-1}B)^{-1}$ does not exist at most in a discrete set of $\mathbb{C}$, i.e., the set of TEV is at most discrete set.

Moreover, if real wave number k satisfies $k^2 < \lambda(D)\alpha\frac{1-\alpha\epsilon_0^*}{1+\alpha\epsilon_0^*}$, then k is not a TEV, since $A_{k^2} - k^2B$ is invertible. Therefore all real TEVs satisfy $k^2 \geqslant \lambda(D)\alpha\frac{1-\alpha\epsilon_0^*}{1+\alpha\epsilon_0^*}$, if (3.8) holds.

Case 2. $\epsilon_c < 0$. By using (2.11) (i.e., we exchange the position $\epsilon_0(x)$ and $\epsilon_1(x)$) and taking the same arguments as those for Case 1, we conclude that, under the condition $0 < \frac{\epsilon_1^*}{\theta_0-\epsilon_1^*} < 1$, the first conclusion of the theorem holds and all real TEVs satisfy $k^2 \geqslant \lambda(D)\alpha\frac{1-\alpha\epsilon_1^*}{1+\alpha\epsilon_1^*}$.

The proof is complete. □

Remark 3.4. The second conclusion in this theorem is in fact the so-called Faber-Krahn inequality, which gives the lower bounds on the transmission eigenvalues.

Remark 3.5. By checking the proof procedure carefully, we see that (3.11) is the key estimate for giving the lower bounds for real eigenvalues. An important observation is that the second term $-k^2\tilde{B}(u,u)$ generates the estimate on k^2, which enters only the coefficient of X^2. Therefore, we can improve the lower bound on the real eigenvalues, if we can enlarge the coefficient of X^2 in (3.7) and ensure $1-\frac{1}{\epsilon} > 0, \alpha+\frac{1}{\epsilon_0^*}-\alpha\epsilon > 0$ by choosing $\epsilon > 0$ suitably. Using this procedure, it is possible to improve our second result of the theorem.

Now we will consider the existence of TEV for non-absorbing media and extend the results to the inhomogeneous background media with our mixed boundary conditions. We firstly state the following general result given in [6] as the basis of the existence result.

Lemma 3.6. *Let $\tau \to \mathbb{A}_\tau$ be a continuous mapping from $[0,\infty)$ to the set of self-adjoint and positive definite bounded linear operators on U and $\mathbb{B}$ be a self-adjoint and non-negative compact bounded linear operator on U. We assume that there exist two positive constants $\tau_1 > \tau_0 > 0$ such that*

(I) $\mathbb{A}_{\tau_0} - \tau_0\mathbb{B}$ *is positive on U;*

(II) $\mathbb{A}_{\tau_1} - \tau_1\mathbb{B}$ *is non-positive on an m dimensional subspace of U.*

Then each of the equations $\lambda_j(\tau) = \tau$ for $j = 1,\dots,m$ has at least one solution in $[\tau_0,\tau_1]$ where $\lambda_j(\tau)$ is the j-th eigenvalue (counting multiplicity) of $\mathbb{A}_\tau$ with respect to $\mathbb{B}$, i.e., $\ker(\mathbb{A}_\tau - \lambda_j(\tau)\mathbb{B}) \neq \{0\}$.

Theorem 3.7. *Under the assumptions* A1 *and* A2 *stated in Theorem* 3.3 *and the assumption*

B3. *the bounds on* A1 *satisfy*

either

$$\epsilon_c(x) > 0, \quad 0 < \frac{\epsilon_0^*}{\theta_1 - \epsilon_0^*} < \frac{1}{8} \tag{3.12}$$

or

$$\epsilon_c(x) < 0, \quad 0 < \frac{\epsilon_1^*}{\theta_0 - \epsilon_1^*} < \frac{1}{8}, \tag{3.13}$$

there exists an infinite number of eigenvalues with $+\infty$ as the only possible accumulation point for (2.1)–(2.5) *with $\gamma_0 = \gamma_1 = 0$.*

Proof. The discreteness of the TEV has been proved in Theorem 3.3. For existence, we adopt the proof scheme of [6, Theorem 2.5] with essential modifications when constructing the m-dimensional subspace required in Lemma 3.6.

We only consider the first situation (3.12). The situation (3.13) can be treated analogously by exchanging the position of $\epsilon_0(x)$ and $\epsilon_1(x)$. Set

$$M := \max \frac{1}{\epsilon_1(x) - \epsilon_0(x)} \leqslant \frac{1}{\theta_1 - \epsilon_0^*}.$$

Take $U = \tilde{H}_{0\alpha}(D)$. It follows from (3.11) that for any fixed $\tau_0 := k^2 < \lambda(D)\alpha\frac{1-\alpha\epsilon_0^*}{1+\alpha\epsilon_0^*}$, the operator $A_{k^2} - k^2 B$ is positive and therefore the condition (I) in Lemma 3.6 holds.

To verify that the condition (II) in Lemma 3.6 is also satisfied, we introduce the Dirichlet eigenfunction $u^{S_\epsilon^j}(x)$ of $-\Delta$ in S_ϵ^j corresponding to the first eigenvalue $\lambda_1(S_\epsilon^j)$, where S_ϵ^j is a disk with center $x_j \in D$ and radius $\epsilon > 0$, i.e.,

$$\begin{cases} -\Delta u^{S_\epsilon^j} = \lambda_1(S_\epsilon^j) u^{S_\epsilon^j}, & \text{in } S_\epsilon^j, \\ u^{S_\epsilon^j} = 0, & \text{on } \partial S_\epsilon^j. \end{cases} \tag{3.14}$$

Take $\epsilon > 0$ small enough and choose the center $x_j \in D$ suitably such that D contains $m := m(\epsilon) \geqslant 1$ disjoint discs $S_\epsilon^1, S_\epsilon^2, \ldots, S_\epsilon^m$. Without loss of generality, we assume that $\|u^{S_\epsilon^j}\|_{L^2(S_\epsilon^j)} = 1$. Then we extend each $u^{S_\epsilon^j}(x)$ from S_ϵ^j to $\overline{D}$ by defining $u^{S_\epsilon^j}(x) \equiv 0$ outside of S_ϵ^j. Denote by $u_j(x)$ the function from this extension. Obviously any two functions of $u_1(x), \ldots, u_m(x)$ are L^2-orthogonal each other and $\|u_j\|_{L^2(D)} = 1$. Now we take the m-dimensional subspace as $span\{u_1(x), \ldots, u_m(x)\}$ in $\tilde{H}_{0\alpha}(D)$.

For any $u = \sum_{j=1}^m c_j u_j \in \operatorname{span}\{u_1(x), \ldots, u_m(x)\}$ and any $\tau > 0$, we have

$$\begin{aligned} &((A_\tau - \tau B)u, u)_{\tilde{H}_{0\alpha}(D)} \\ &\quad = \tilde{A}_\tau(u, u) - \tau \tilde{B}(u, u) \\ &\quad = \int_D \frac{|\Delta u + \tau\epsilon_0(x)u|^2}{\epsilon_1(x) - \epsilon_0(x)} dx + \tau^2 \int_D \epsilon_0(x)|u|^2 dx - \tau \int_D |\nabla u|^2 dx \\ &\quad = \sum_{j=1}^m |c_j|^2 \bigg[\int_{S_\epsilon^j} \frac{|\Delta u^{S_\epsilon^j} + \tau\epsilon_0(x)u^{S_\epsilon^j}|^2}{\epsilon_1(x) - \epsilon_0(x)} dx + \tau^2 \int_{S_\epsilon^j} \epsilon_0(x)|u^{S_\epsilon^j}|^2 dx - \tau \int_{S_\epsilon^j} |\nabla u^{S_\epsilon^j}|^2 dx \bigg] \end{aligned} \tag{3.15}$$

from the property of $u^{S_\epsilon^j}(x)$. On the other hand, it follows from (3.14) that $\|\nabla u^{S_\epsilon^j}\|^2 = \lambda_1(S_\epsilon^j)$ and $\|\Delta u^{S_\epsilon^j}\|^2 = \lambda_1^2(S_\epsilon^j)$ from straightforward computations. Therefore we have

$$\begin{aligned} &\int_{S_\epsilon^j} \frac{|\Delta u^{S_\epsilon^j} + \tau\epsilon_0(x)u^{S_\epsilon^j}|^2}{\epsilon_1(x) - \epsilon_0(x)} dx + \tau^2 \int_{S_\epsilon^j} \epsilon_0(x)|u^{S_\epsilon^j}|^2 dx - \tau \int_{S_\epsilon^j} |\nabla u^{S_\epsilon^j}|^2 dx \\ &\quad = \int_{S_\epsilon^j} \frac{|\Delta u^{S_\epsilon^j}|^2}{\epsilon_1(x) - \epsilon_0(x)} dx + \tau^2 \int_{S_\epsilon^j} \frac{\epsilon_0^2(x)|u^{S_\epsilon^j}|^2}{\epsilon_1(x) - \epsilon_0(x)} dx + \tau \int_{S_\epsilon^j} \frac{\epsilon_0(x)(\bar{u}^{S_\epsilon^j}\Delta u^{S_\epsilon^j} + u^{S_\epsilon^j}\Delta\bar{u}^{S_\epsilon^j})}{\epsilon_1(x) - \epsilon_0(x)} dx \\ &\qquad + \tau^2 \int_{S_\epsilon^j} \epsilon_0(x)|u^{S_\epsilon^j}|^2 dx - \tau \int_{S_\epsilon^j} |\nabla u^{S_\epsilon^j}|^2 dx \\ &\quad \leqslant M\|\Delta u^{S_\epsilon^j}\|^2 + \tau^2[\epsilon_0^* + (\epsilon_0^*)^2 M] + 2\tau\epsilon_0^* M\|\Delta u^{S_\epsilon^j}\| - \tau\|\nabla u^{S_\epsilon^j}\|^2 \\ &\quad = M\lambda_1^2(S_\epsilon^j) + \tau^2[\epsilon_0^* + (\epsilon_0^*)^2 M] + 2\tau\epsilon_0^* M\lambda_1(S_\epsilon^j) - \tau\lambda_1(S_\epsilon^j). \end{aligned} \tag{3.16}$$

This estimate holds for all $\tau > 0$. Consider the right-hand side as a polynomial of degree 2 with respect to real variable $\tau \in \mathbb{R}^+$. Especially, at

$$\tau_1 := \frac{(1 - 2M\epsilon_0^*)\lambda_1(S_\epsilon^j)}{2(\epsilon_0^* + (\epsilon_0^*)^2 M)} > 0$$

for θ_1, ϵ_0^* satisfying (3.12), which is the minimizer of the right-hand side, we have

$$\int_{S_\epsilon^j} \frac{|\Delta u^{S_\epsilon^j} + \tau_1\epsilon_0(x)u^{S_\epsilon^j}|^2}{\epsilon_1(x) - \epsilon_0(x)} dx + \tau_1^2 \int_{S_\epsilon^j} \epsilon_0(x)|u^{S_\epsilon^j}|^2 dx - \tau_1 \int_{S_\epsilon^j} |\nabla u^{S_\epsilon^j}|^2 dx$$

$$
\begin{aligned}
&\leqslant M\lambda_1^2(S_\epsilon^j)+\tau_1^2[\epsilon_0^*+(\epsilon_0^*)^2M]+2\tau_1\epsilon_0^*M\lambda_1(S_\epsilon^j)-\tau_1\lambda_1(S_\epsilon^j)\\
&=\frac{-1+8\epsilon_0^*M}{4(\epsilon_0^*+(\epsilon_0^*)^2M)}\lambda_1^2(S_\epsilon^j).
\end{aligned}\tag{3.17}
$$

Noticing that $\lambda_1(S_\epsilon^j)$ is in fact independent of the point x_j and therefore denoted by $\lambda_{1,\epsilon}$, by combining the above estimates together, we have in the m-dimensional space $\operatorname{span}\{u_1(x),\dots,u_m(x)\}$ that

$$
((A_{\tau_1}-\tau_1B)u,u)_{\tilde H_{0\alpha}(D)}\leqslant\frac{-1+8\epsilon_0^*M}{4(\epsilon_0^*+(\epsilon_0^*)^2M)}\lambda_{1,\epsilon}^2\sum_{j=1}^m|c_j|^2=\frac{-1+8\epsilon_0^*M}{4(\epsilon_0^*+(\epsilon_0^*)^2M)}\lambda_{1,\epsilon}^2\|u\|^2<0
$$

for constants θ_1,ϵ_0^* satisfying (3.12). Therefore we have verified the condition (II) in Lemma 3.6 for $\tau_1>0$ defined above.

Hence we know that there are $m(\epsilon)$ TEVs (counting multiplicity) inside $[\tau_0,\tau_1]$. Letting $\epsilon\to0$ which means we can take $m:=m(\epsilon)\to\infty$. Therefore there exists an infinite countable set of TEVs with $+\infty$ as the only possible accumulation point. The proof is complete. □

We already establish the distribution property of the transmission eigenvalues for inhomogeneous non-absorbing media with mixed boundary condition. The next issue is to extend the result to the absorbing media. Since the mixed boundary condition is incorporated into the functional space $\tilde H_{0\alpha}(D)$, which has the same inner product as that of $H_0^2(D)$, the mixed boundary condition has no essential influence on the extension as already established in [5] for absorbing media with Dirichlet boundary condition in ∂D. Since the conclusion and the proof are completely the same as those for [5, Theorem 2.5], we just state the results for our setting and give the outline for the proof.

Lemma 3.8. *Assume that k^* is a real transmission eigenvalue for non-absorbing media corresponding to inhomogeneous media $(\epsilon_0(x),\epsilon_1(x))$. Then there exists a constant $\eta(k^*)>0$ such that, if the absorption $\gamma(x):=(\gamma_0(x),\gamma_1(x))$ satisfies*

$$
0\leqslant\sup_D\gamma_0(x)+\sup_D\gamma_1(x)\leqslant\eta(k^*),\tag{3.18}
$$

then there exists at least one TEV in the complex plane near k^*.

Proof. Define the linear operators E,F_γ,G_ϵ: $L^2(D)\times L^2(D)\to L^2(D)\times L^2(D)$ by

$$
E=\begin{pmatrix}\Delta_{00}&0\\0&\Delta\end{pmatrix},\quad F_\gamma=\begin{pmatrix}\mathrm{i}\gamma_1&\mathrm{i}\gamma_c\\0&\mathrm{i}\gamma_0\end{pmatrix},\quad G_\epsilon=\begin{pmatrix}\epsilon_1&\epsilon_c\\0&\epsilon_0\end{pmatrix},
$$

where Δ_{00} indicates that the Laplacian acts on a function in $\tilde H_{0\alpha}(D)$. Then the TEV problem can be reformulated as the following quadratic eigenvalue problem,

$$
Ep+kF_\gamma p+k^2G_\epsilon p=0,\quad p\in L^2(D)\times L^2(D).\tag{3.19}
$$

By introducing $U:=(p,kG_\epsilon p)^{\mathrm T}$, the eigenvalue problem (3.19) is transformed into

$$
(\mathbb K-k\mathbb I_{\epsilon,\gamma})U=0,\quad U\in(L^2(D)\times L^2(D))^2,\tag{3.20}
$$

where the 4×4 matrix operators $\mathbb K$ and $\mathbb I_{\epsilon,\gamma}$ are given by

$$
\mathbb K=\begin{pmatrix}E&0\\0&\mathbb I\end{pmatrix},\quad \mathbb I_{\epsilon,\gamma}=\begin{pmatrix}-F_\gamma&-\mathbb I\\G_\epsilon&0\end{pmatrix},
$$

and $\mathbb I$ is the identity operator in $L^2(D)\times L^2(D)$. Define the unbounded operator $\mathbb T_{\epsilon,\gamma}:=\mathbb I_{\epsilon,\gamma}^{-1}\mathbb K$. The proof is complete by considering $\mathbb T_{\epsilon,\gamma}$ as a small perturbation for $\mathbb T_{\epsilon,\gamma\equiv0}$ from the estimate

$$
\hat\delta(\mathbb T_{\epsilon,\gamma},\mathbb T_{\epsilon,\gamma=0})\leqslant\|\mathbb P\|\leqslant\|F_\gamma G_\epsilon^{-1}\|\leqslant\frac{4(\sup_D\epsilon_0+\sup_D\epsilon_1)(\sup_D\gamma_0+\sup_D\gamma_1)}{(\inf_D\epsilon_0)(\inf_D\epsilon_1)}\tag{3.21}
$$

and (3.18) with $\hat\delta(\cdot,\cdot)$ being the distance between two operators, noticing we already prove that there exists a real transmission eigenvalue k^* for $\mathbb T_{\epsilon,\gamma\equiv0}$ in Theorem 3.7 under some assumptions on $\epsilon_i(x)$ for $i=1,2$. □

Using this result, we have the following theorem.

Theorem 3.9. *Assume that $\epsilon_i(x) \in L^\infty(D)$ such that $\epsilon_i(x) \geqslant \theta_i > 0$ for $i = 0, 1$ and $\frac{1}{\epsilon_c(x)} \geqslant \alpha > 0$ a.e. in D. Let $k_i > 0$, $i = 0, 1, \dots, l$ be the first $l + 1$ real TEVs (multiple eigenvalues are counted once) corresponding to (2.1)–(2.5) for non-absorbing media ($\gamma_0 = \gamma_1 = 0$ a.e. in D). Then for every $\sigma > 0$, there exists $\bar{\eta} > 0$ depending on σ such that if the absorption in the media is such that $\sup_D \gamma_0 + \sup_D \gamma_1 < \bar{\eta}$, then there exist at least $l + 1$ TEVs against (2.1)–(2.5) each in a σ-neighborhood of k_i, $i = 0, 1, 2, \dots, l$.*

Proof. It is clear that from (3.21) we can select $\bar{\eta} = \min\{\bar{\eta}_{k_0}, \bar{\eta}_{k_1}, \dots, \bar{\eta}_{k_l}\}$, where

$$\bar{\eta}_{k_i} < \frac{\eta_{k_i}(\inf_D \epsilon_0)(\inf_D \epsilon_1)}{4(\sup_D \epsilon_0 + \sup_D \epsilon_1)}$$

and η_{k_i} is the size of the perturbation corresponding to real transmission eigenvalue k_i, $i = 0, 1, \dots, l$ for non-absorbing media. Now we can apply Lemma 3.8 by choosing $k^* = k_i$ for $i = 0, 1, \dots, l$. The proof is complete. □

4 Transmission eigenvalue-free zones

The goal of this section is to give some components of eigenvalue-free zones in the complex plane, of particular interest from a practical point of view are the estimates for real TEVs (if they exist) since they can be measured from the scattering data [4]. We restrict ourselves to homogeneous absorbing media. For general inhomogeneous media, the description of TEV-free zone is very difficult.

Let $k = a + b\mathrm{i}$ be a complex wave number. From (2.10) for $u \in \tilde{H}_{0\alpha}(D)$, we have

$$\begin{aligned} 0 &= \int_D [\Delta u + (k^2\epsilon_0 + k\mathrm{i}\gamma_0)u][\Delta \bar{u} + (k^2\epsilon_1 + k\mathrm{i}\gamma_1)\bar{u}]dx \\ &= \int_D |\Delta u + (k^2\epsilon_0 + k\mathrm{i}\gamma_0)u|^2 dx - (k^2\epsilon_1 + k\mathrm{i}\gamma_1 - \overline{k^2}\epsilon_0 + \mathrm{i}\bar{k}\gamma_0)\int_D |\nabla u|^2 dx \\ &\quad + (k^2\epsilon_1 + k\mathrm{i}\gamma_1 - \overline{k^2}\epsilon_0 + \mathrm{i}\bar{k}\gamma_0)(k^2\epsilon_0 + k\mathrm{i}\gamma_0)\int_D |u|^2 dx. \end{aligned} \tag{4.1}$$

Substituting $k = a + b\mathrm{i}$ into (4.1), we obtain that

$$\begin{aligned} 0 = &\int_D |\Delta u + [(a^2 - b^2 - 2ab\mathrm{i})\epsilon_0 + ak\mathrm{i}\gamma_0 - bk\gamma_0]u|^2 dx - (\dot{M}_0 + \dot{N}_0\mathrm{i})\int_D |\nabla u|^2 dx \\ &+ (\dot{P}_0 + \dot{Q}_0\mathrm{i})\int_D |u|^2 dx \end{aligned} \tag{4.2}$$

with

$$\begin{aligned} \dot{M}_0 &= (a^2 - b^2)\epsilon_c - b\gamma_c, \\ \dot{N}_0 &= 2ab(\epsilon_0 + \epsilon_1) + a(\gamma_0 + \gamma_1), \\ \dot{P}_0 &= \dot{M}_0[(a^2 - b^2)\epsilon_0 - b\gamma_0] - \dot{N}_0(2ab\epsilon_0 + a\gamma_0), \\ \dot{Q}_0 &= \dot{N}_0[(a^2 - b^2)\epsilon_0 - b\gamma_0] + \dot{M}_0(2ab\epsilon_0 + a\gamma_0). \end{aligned}$$

For given constants (ϵ_i, γ_i) for $i = 0, 1$, if there exists a nonzero solution $u \in L^2(D)$ to (4.2), then $k = a + b\mathrm{i}$ is the TEVs. In other words, if the point (a, b) is such that (4.2) does not hold for any nonzero u in $L^2(D)$, then (a, b) is in the TEVs-free zone. Now we derive the sufficient condition for these statements.

Taking real part of (4.2), we have

$$0 = \int_D |\Delta u + [(a^2 - b^2 - 2ab\mathrm{i})\epsilon_0 + ak\mathrm{i}\gamma_0 - bk\gamma_0]u|^2 dx - \dot{M}_0\int_D |\nabla u|^2 dx + \dot{P}_0\int_D |u|^2 dx. \tag{4.3}$$

Denote by $\mu(D)$ the first (smallest) Dirichlet eigenvalue of $-\Delta$. Then we have

$$\mu(D)=\min_{u\in H_0^1(D),u\neq 0}\frac{\|\nabla u\|^2}{\|u\|^2}$$

from the Rayleigh-Ritz characterization of the first Dirichlet eigenvalue, which generates $\|\nabla u\|^2 \geqslant \mu(D)\|u\|^2$ for all $u\in H_0^1(D)$.

If $\dot{M}_0\leqslant 0$, we have

$$-\dot{M}_0\int_D|\nabla u|^2dx+\dot{P}_0\int_D|u|^2dx\geqslant[-\dot{M}_0\mu(D)+\dot{P}_0]\|u\|_{L^2}^2.$$

So the real part of (4.3) cannot be zero for any nonzero $u\in L^2(D)$ if $-\dot{M}_0\mu(D)+\dot{P}_0>0$, noticing that the first term in the right-hand side of (4.3) is positive. Hence we know that

$$D_1^0:=\{(a,b):\dot{M}_0\leqslant 0,-\dot{M}_0\mu(D)+\dot{P}_0>0\}$$

is one of the components of TEVs-free zone.

Taking imaginary part of (4.2), we have $0=\dot{Q}_0\int_D|u|^2dx-\dot{N}_0\int_D|\nabla u|^2dx\geqslant[\dot{Q}_0-\dot{N}_0\mu(D)]\|u\|_{L^2}^2$ for $\dot{N}_0\leqslant 0$, i.e., $D_2^0:=\{(a,b):\dot{N}_0\leqslant 0,\dot{Q}_0-\dot{N}_0\mu(D)>0\}$ is also the component of TEVs-free zone. Moreover, we have

$$0=\dot{Q}_0\int_D|u|^2dx-\dot{N}_0\int_D|\nabla u|^2dx\leqslant\left(\frac{\dot{Q}_0}{\mu(D)}-\dot{N}_0\right)\|\nabla u\|^2$$

for $\dot{Q}_0\geqslant 0$, i.e., $D_3^0:=\{(a,b):\dot{Q}_0\geqslant 0,\dot{Q}_0-\dot{N}_0\mu(D)<0\}$ is also the component of TEVs-free zone.

In a similar way, we can deal with (2.11). Defining

$$\begin{aligned}
&\dot{M}_1=-\dot{M}_0,\\
&\dot{N}_1=\dot{N}_0,\\
&\dot{P}_1=\dot{M}_1[(a^2-b^2)\epsilon_1-b\gamma_1]-\dot{N}_1(2ab\epsilon_1+a\gamma_1),\\
&\dot{Q}_1=\dot{N}_1[(a^2-b^2)\epsilon_1-b\gamma_1]+\dot{M}_1(2ab\epsilon_1+a\gamma_1),
\end{aligned}$$

we know that

$$\begin{aligned}
&D_1^1:=\{(a,b):\dot{M}_1\leqslant 0,-\dot{M}_1\mu(D)+\dot{P}_1>0\},\\
&D_2^1:=\{(a,b):\dot{N}_1\leqslant 0,\dot{Q}_1-\dot{N}_1\mu(D)>0\},\\
&D_3^1:=\{(a,b):\dot{Q}_1\geqslant 0,\dot{Q}_1-\dot{N}_1\mu(D)<0\}
\end{aligned}$$

are the other three components of eigenvalue-free zone.

So we can assert that the TEVs-free zone is at least $F:=D_1^0\cup D_2^0\cup D_3^0\cup D_1^1\cup D_2^1\cup D_3^1$.

Remark 4.1. We extract six components of the eigenvalue-free zone based on (4.1) and its version by exchanging (ϵ_1,γ_1) and (ϵ_0,γ_0), which are special equalities coming from the variational form of our transmission problem with mixed boundary condition. However, if the mixed boundary condition is replaced by the Cauchy boundary condition $u=\frac{\partial u}{\partial\nu}\equiv 0$ in the whole boundary ∂D as discussed in [5], i.e., $\Gamma_m=\emptyset$ and therefore $\Gamma=\Gamma_\alpha$ in (2.9), the equality (4.1) as well as its version by exchanging (ϵ_1,γ_1) and (ϵ_0,γ_0) are also true, which means that we will get the same eigenvalue-free components for the media with specified constants ϵ_i,γ_i for $i=0,1$ but different kinds of boundary conditions. In other words, the scheme proposed in this section to give the information about the components of eigenvalue-free zone has nothing to to with the boundary state. This phenomenon is natural, since we only give part of the eigenvalue-free zone.

5 Conclusions and future work

We consider the transmission eigenvalue problem motivated from the wave scattering by a dielectric inhomogeneous absorbing obstacle lying on a perfect conducting surface. Different from the well-known spectral theory for elliptic or self adjoint operators, the spectrum distribution analysis and the computations for such kinds of problems need to develop some new techniques. We prove that the transmission eigenvalues exist and form a discrete set, and there is no eigenvalues with real parts vanishing. Moreover, we derive the Faber-Krahn type inequalities for real transmission eigenvalues in terms of the media parameters. We prove that the transmission eigenvalues also exist and form a discrete set for inhomogeneous media with small absorption. Finally, we present the possible eigenvalue-free zones for homogeneous media quantitatively.

Numerical computations of transmission eigenvalues are challenging due to the existence of zero eigenvalue and complex ones. In the forthcoming works, the efficient algorithms to compute the real transmission eigenvalues and the applications of these eigenvalues to inverse scattering problems are important research topics.

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