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Parameter estimation for generalized diffusion processes with reflected boundary

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Abstract Reflected Ornstein-Uhlenbeck process is a process that returns continuously and immediately to the interior of the state space when it attains a certain boundary. It is an extended model of the traditional Ornstein-Uhlenbeck process being extensively used in finance as a one-factor short-term interest rate model. In this paper, under certain constraints, we are concerned with the problem of estimating the unknown parameter in the reflected Ornstein-Uhlenbeck processes with the general drift coefficient. The methodology of estimation is built upon the maximum likelihood approach and the method of stochastic integration. The strong consistency and asymptotic normality of estimator are derived. As a by-product of the use, we also establish Girsanov's theorem of our model in this paper.

Keywords reflected Ornstein-Uhlenbeck process, maximum likelihood estimation, Girsanov's formula, Skorohod embedding, Dambis, Dubins-Schwartz Brownian motion

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1 Introduction

The Ornstein-Uhlenbeck processes, which are markovian, mean-reverting Gaussian prosseses, well describe various real-life phenomena and so have found widespread use in a broad range of application domains, such as finance, life sciences, and operations research. However, all models involve unknown parameters or functions, which need to be estimated from observations of the process. The estimation of these processes is therefore a crucial step in all applications, in particular in applied finance. In the case of Ornstein-Uhlenbeck processes driven by Wiener processes, the statistical inference for these processes has been studied and a comprehensive survey of various methods was given in Prakasa Rao [26] and Bishwal [8], for example, maximum likelihood estimation, minimum contrast estimation, maximum probability estimation and nonparametric inference, etc. By virtue of the markovian and gaussian properties of Wiener processes, both the maximum likelihood estimator (MLE) and the least squares estimator (LSE) are easy to obtain and exhibit asymptotic unbiasedness, efficiency and normality under the usual regularity conditions. We refer to Bishwal [7, 9] and the references therein.

In many situations, though, the stochastic processes involved are not allowed to cross a certain boundary, or are even supposed to remain within two boundaries. The resulting reflected Ornstein-Uhlenbeck

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processes (denoted in the sequel by ROU) behave like the standard Ornstein-Uhlenbeck processes in the interior of its domain. However, when they reach the boundary, the sample path returns to the interior in a manner that the "pushing" force is minimal. This kind of process, which can be applied into the field of queueing system, financial engineering, and mathematical biology, has attracted the attention of scholars of the world.

Ricciardi and Sacerdote [28] applied the ROU processes into the field of mathematical biology. Krugman [21] limited the currency exchange rate dynamics in a target zone by two reflecting barriers. In [16], the asset pricing models with truncated price distributions have been investigated. Ward and Glynn [29–31] showed that the ROU processes serve as a good approximation for a Markovian queue with reneging when the arrival rate is either close to or exceeds the processing rate and the reneging rate is small and the ROU processes also well approximate queues having renewal arrival and service processes in which customers have deadlines constraining total sojourn time. Customers either renege from the queue when their deadline expires or balk if the conditional expected waiting time given the queue-length exceeds their deadline. Linetsky [23] studied the analytical representation of transition density for reflected diffusions in terms of their Sturm-Liouville spectral expansions. Recently, Bo et al. [11,12] applied the ROU processes to model the dynamics of asset prices in a regulated market, and they calculated the conditional default probability with incomplete (or partial) market information.

Given a filtered probability space $\Lambda := (\Omega, \mathcal{F}, \mathbb{P})$ with the filtration $(\mathcal{F}_t)_{t\geqslant 0}$ satisfying the usual conditions, the diffusion processes $\{X_t, t\geqslant 0\}$ reflected at the boundary $b\in\mathbb{R}^+$ on Λ is defined as follows. Let $\{X_t, t\geqslant 0\}$ be the strong solution whose existence is guaranteed by an extension of the results of Lions and Sznitman [24] to the stochastic differential equation

$$\begin{cases} dX_t = a(t, X_t, \alpha)dt + \sigma dW_t + dL_t, \\ X_t \geqslant b^L, \quad \text{for all} \quad t\geqslant 0, \\ X_0 = x, \end{cases} \tag{1.1}$$

where $b^L\in\mathbb{R}^+$, $x\in[b^L, +\infty)$, $\sigma\in(0,+\infty)$, $\alpha\in\Theta$ (Θ is an open subset of $\mathbb{R}$) and $\{W_t, t\geqslant 0\}$ is a one-dimensional standard Wiener process. Here, the drift coefficient $a(\cdot,\cdot,\cdot)$ is progressively measurable and adapted. $L=(L_t)_{t\geqslant 0}$ is the minimal non-decreasing and non-negative process, which makes the process $X_t\geqslant b^L$ for all $t\geqslant 0$. The process L increases only when X hits the boundary b^L, so that

$$\int_{[0,\infty)} I(X_t > b^L)dL_t = 0, \tag{1.2}$$

where $I(\cdot)$ denotes the indicator function. Sometimes L is called the regulator of the point b^L(see [17]) and by virtue of Ata et al. [3], the paths of the regulator are nondecreasing, right continuous with left limits and possess the support property

$$\int_0^t I(X_s = b^L)dL_s = L_t. \tag{1.3}$$

It can be shown that (see [17]) the process L has an explicit expression as

$$L_t = \max\Big\{0, \sup_{s\in[0,\ t]}(L_s - X_s)\Big\}. \tag{1.4}$$

It is often the case that the reflecting barrier is assumed to be zero in applications to queueing system, storage model, engineering, finance, etc. This is mainly due to the physical restriction of the state processes such as queue-length, inventory level, content process, stock prices and interest rates, which take non-negative values. Of course, considering some other applications of the ROU processes, we still assume that b^L is non-negative except for being mentioned specially.

In contrast to the case for the standard Ornstein-Uhlenbeck processes, the statistical inference for the ROU processes has been in the ascendant. Recently, based on continuous observations, Bo et al. [13] first presented the maximum likelihood estimation for the ROU processes with the ergodic case, i.e.,

$$\begin{cases} dX_t = -\alpha X_t dt + \sigma dW_t + dL_t, \\ X_t \geqslant b^L, \quad \text{for all} \quad t \geqslant 0, \\ X_0 = x, \end{cases} \tag{1.5}$$

where $b^L \in \mathbb{R}^+$, $x \in [b^L, +\infty)$, $\sigma \in (0, +\infty)$, $\alpha \in \mathbb{R}^+$. $\{W_t, t \geqslant 0\}$ and $L = (L_t)_{t \geqslant 0}$ are defined similarly as above.

Our interest lies in the statistical inference for diffusion processes defined in (1.1) and (1.2). More specifically, we would like to estimate the unknown parameter α included in the general drift coefficient based on continuous observations of the state process $\{X_t, t \geqslant 0\}$. Noting that σ is an unknown constant which is independent of the parameter α and the quadratic variation process $[X]_t$ equals to $\sigma^2 t$, $t \geqslant 0$, we assume that σ is known and set it equal to one in the situation of continuous observations.

The rest of this paper is organized as follows. In Section 2, the equivalent martingale measure of our model is formulated. In Section 3, we propose the maximum likelihood estimation for the model, and its statistical properties are established. In Section 4, the performance of our main results is demonstrated through a nonstationary example. In Section 5, the paper is concluded and some opportunities for future research are outlined.

2 Preliminaries

In this section, we review some of the basic concepts concerning our context and build the equivalent martingale measure of the ROU processes with the general drift coefficient, which will be the basis of our further study.

Throughout this paper, we denote P_α^{T}(also $P_\beta^{\mathrm{T}}, \ldots$) for the probability measure generated by the process $\{X_t^\alpha, 0 \leqslant t \leqslant T\}$(also $\{X_t^\beta, 0 \leqslant t \leqslant T\}, \ldots$) on the space $(\mathcal{C}[0,T], \mathcal{B}_T)$, where $\mathcal{C}[0,T]$ denotes the space of continuous functions endowed with the supremum norm and $\mathcal{B}_T$ is the corresponding Borel σ-algebra. Let E_α (also $E_\beta, \ldots$) denote expectation with respect to P_α^{T} (also $P_\beta^{\mathrm{T}}, \ldots$) and P_W^{T} be the probability measure induced by the standard Wiener process. Now we establish the equivalent martingale measure of the ROU processes with the general drift coefficient.

Theorem 2.1 (Girsanov's formula of our model). *Suppose that X^β and X^γ are two ROU processes satisfying the following stochastic differential equations, respectively,*

$$\begin{cases} dX_t^\beta = a(t, X_t^\beta, \beta)dt + dW_t + dL_t^\beta, \\ X_t \geqslant b^L, \quad \textit{for all} \quad t \geqslant 0, \\ X_0 = x, \end{cases} \tag{2.1}$$

and

$$\begin{cases} dX_t^\gamma = a(t, X_t^\gamma, \gamma)dt + dW_t + dL_t^\gamma, \\ X_t \geqslant b^L, \quad \textit{for all} \quad t \geqslant 0, \\ X_0 = x, \end{cases} \tag{2.2}$$

where, $b^L \in \mathbb{R}^+$, $x \in [b^L, +\infty)$, $a(t, X_t^\beta, \cdot)$ is $\mathcal{F}_t^\beta = \sigma\{X_s^\beta, 0 \leqslant s \leqslant t\}$-measurable for almost all t ($0 \leqslant t \leqslant T, T > 0$). L^β and L^γ are the corresponding regulators at b^L. Let the following assumptions[1] *be fulfilled*:

(A1) *Let $\eta_t(\omega) = a(t, X_t^\beta, \beta) - a(t, X_t^\beta, \gamma)$ satisfy*

$$P\bigg(\int_0^t a^2(s, X_s^\beta, \gamma)ds < \infty\bigg) = P\bigg(\int_0^t a^2(s, X_s^\beta, \beta)ds < \infty\bigg) = 1, \tag{2.3}$$

1) Here, our assumptions are inspired by Theorem 7.18 and its Corollary of Liptser and Shiryayev [25]

for all $0 \leqslant t < \infty$; *and*

(A2)

$$E \exp\bigg(- \int_0^T \eta_s(\omega) dW_s - \frac{1}{2} \int_0^T \eta_s^2(\omega) ds \bigg) = 1. \tag{2.4}$$

Then, for each $T > 0$, *we have*

$$\frac{dP_\gamma^T}{dP_\beta^T}\bigg|_{\mathcal{F}_t^\beta} = \exp\bigg(- \int_0^T \eta_s(\omega) dW_s - \frac{1}{2} \int_0^T \eta_s^2(\omega) ds \bigg). \tag{2.5}$$

Proof. Due to the assumption (A1) and Hölder's inequality, it follows that

$$P\bigg(\int_0^t \eta_s^2(\omega) ds < \infty \bigg) = 1,$$

for $0 \leqslant t < \infty$. Then, according to the assertion of Karatzas and Shreve [20], we have $\int_0^T \eta_s(\omega) dW_s$ is a continuous local martingale with quadratic variation process $[\int_0^T \eta_s(\omega) dW_s] = \int_0^T \eta_s^2(\omega) ds$. Furthermore, by virtue of [19, Theorem 15.1], it implies that

$$\exp\bigg(- \int_0^T \eta_s(\omega) dW_s - \frac{1}{2} \int_0^T \eta_s^2(\omega) ds \bigg)$$

is a positive local martingale and hence a supermartingale. Together with the assumption (A2) and the fact that a supermartingale with constant expectation is a martingale, it is verified that

$$Z(T) \triangleq \exp\bigg(- \int_0^T \eta_s(\omega) dW_s - \frac{1}{2} \int_0^T \eta_s^2(\omega) ds \bigg)$$

is a martingale. For a given $T > 0$, we define the equivalent martingale measure $\tilde{\mathbb{P}}$ as follows:

$$d\tilde{\mathbb{P}} = Z(T) d\mathbb{P}.$$

Then, by the usual Girsanov's theorem, we know that the process

$$\tilde{W}_t = W_t + \int_0^t \eta_s(\omega) ds, \quad t \in [0, T]$$

is a standard Brownian motion under $\tilde{\mathbb{P}}$ and

$$\begin{aligned} dX_t^\beta &= a(t, X_t^\beta, \beta) dt + dW_t + dL_t^\beta \\ &= a(t, X_t^\beta, \beta) dt - \eta_t(\omega) dt + d\tilde{W}_t + dL_t^\beta \\ &= a(t, X_t^\beta, \beta) dt - a(t, X_t^\beta, \beta) dt + a(t, X_t^\beta, \gamma) dt + d\tilde{W}_t + dL_t^\beta \\ &= a(t, X_t^\beta, \gamma) dt + d\tilde{W}_t + dL_t^\beta, \end{aligned}$$

for $t \in [0, T]$. Thus X^β under $\tilde{\mathbb{P}}$ has the same distribution as X^γ under $\mathbb{P}$. Therefore, for each measurable subset A of the space $(\mathcal{C}[0, T], \mathcal{B}_T)$, we have

$$\tilde{\mathbb{P}}(X^\beta \in A) = \mathbb{P}(X^\gamma \in A),$$

and then by our preliminaries and the definition of conditional expectation, we have

$$\begin{aligned} P_\gamma^T(A) = \mathbb{P}(X^\gamma \in A) = \tilde{\mathbb{P}}(X^\beta \in A) &= \int_{[X^\beta \in A]} Z(T) d\mathbb{P} \\ &= \int_{[X^\beta \in A]} E[Z(T) \,|\, \mathcal{F}_t^\beta] d\mathbb{P} \\ &= \int_A E[Z(T) \,|\, \mathcal{F}_t^\beta] dP_\beta^T. \end{aligned}$$

Thus the proof is complete by the fact that $Z(T)$ is $\mathcal{F}_t^\beta$-measurable. □

3 Maximum likelihood estimation with respect to our model

In this section, via the maximum likelihood method and assuming that the state process is continuously observable, we shall estimate the unknown parameter included in the drift function concerning our model. It is natural that we will make some appropriate limitations to the drift function, and these limitations will not impact the generality of our main results.

For ease of the following exposition, we introduce some technical assumptions used in the paper.

Assumption 1. For the equation (1.1), let

$$P\left(\int_0^t a^2(s, X_s, \alpha)ds < \infty\right) = 1,$$

for all $0 \leqslant t < \infty$, and

$$P\left(\int_0^\infty [a'(s, X_s, \alpha)]^2 ds = \infty\right) = 1.$$

Assumption 2. Suppose that the following function $l_T(\alpha)$ is twice continuously differentiable in a neighborhood V_α of α for every $\alpha \in \Theta$, and that

$$E_\alpha\left(\int_0^T (a'(t, X_t, \alpha))^2 dt\right) < \infty$$

and

$$E_\alpha\left(\int_0^T (a''(t, X_t, \alpha))^2 dt\right) < \infty.$$

Assumption 3. For any $\alpha \in \Theta$, there exists a neighborhood V'_α of $\alpha \in \Theta$ such that

$$P\left(\int_0^T [a(t, X_t, \alpha_0) - a(t, X_t, \alpha)]^2 dt < \infty\right) = 1, \quad 0 < T < \infty$$

and

$$P\left(\int_0^\infty [a(t, X_t, \alpha_0) - a(t, X_t, \alpha)]^2 dt = \infty\right) = 1$$

for all $\alpha_0 \in \Theta - \{\alpha\}$.

The first part of Assumption 1 conventionally proposed in the study of the problem in this paper (see [25]), together with Theorem 2.1, ensures that $P_\alpha^{\mathrm{T}} \ll dP_W^{\mathrm{T}}$ and we can define the log-likelihood function as follows:

$$\begin{aligned} l_T(\alpha) &= \log \frac{dP_\alpha^{\mathrm{T}}}{dP_W^{\mathrm{T}}} \\ &= \int_0^T a(t, X_t, \alpha)dX_t - \frac{1}{2}\int_0^T a^2(t, X_t, \alpha)dt - \int_0^T a(t, X_t, \alpha)dL_t. \end{aligned}$$

The maximum likelihood estimator (MLE) is naturally defined as

$$\hat{\alpha}_T := \arg\sup_{\alpha \in \Theta} l_T(\alpha).$$

Then, we can derive the maximum likelihood estimator (MLE) of the parameter α by solving the likelihood equation $l'_T(\alpha) = 0$.

By imposing suitable regularity conditions on the drift function $a(\cdot, \cdot, \cdot)$, we will establish the following lemma which will play a key role in our main proofs.

Lemma 3.1. *For the log-likelihood function $l_T(\alpha)$, we have*

$$l'_T(\alpha) = \int_0^T a'(t, X_t, \alpha)dW_t \tag{3.1}$$

and

$$l''_T(\alpha) = \int_0^T a''(t, X_t, \alpha)dW_t - \int_0^T [a'(t, X_t, \alpha)]^2 dt, \tag{3.2}$$

where, a' and a'' denote the first and second order derivatives with respect to α, respectively.

Proof. For any function $g(t, x, \alpha)$, let g' and g'' denote the first and second order derivatives with respect to α, respectively. Suppose that $a(t, x, \alpha)$ is continuously differentiable in x. Let

$$F(t, x, \alpha) = \int_0^x a(t, y, \alpha)dy. \tag{3.3}$$

Also suppose F is jointly continuous in $(t, x) \in [0, \infty) \times R$ with the partial derivatives F_x, F_{xx} and F_t. Apply Itô's formula to $F(T, X_T, \alpha)$, due to the quadratic variation $[X]_t = t$, it follows that

$$\begin{aligned} F(T, X_T, \alpha) &= F(0, X_0, \alpha) + \int_0^T F_t(s, X_s, \alpha)ds + \int_0^T F_x(s, X_s, \alpha)dX_s \\ &\quad + \frac{1}{2}\int_0^T F_{xx}(s, X_s, \alpha)d[X]_s \\ &= F(0, X_0, \alpha) + \int_0^T F_t(s, X_s, \alpha)ds + \int_0^T a(s, X_s, \alpha)dX_s \\ &\quad + \frac{1}{2}\int_0^T a_x(s, X_s, \alpha)d[X]_s \\ &= \int_0^T F_t(s, X_s, \alpha)ds + \frac{1}{2}\int_0^T a_x(s, X_s, \alpha)ds + \int_0^T a(s, X_s, \alpha)dX_s. \end{aligned}$$

Let

$$f(t, x, \alpha) = F_t(t, x, \alpha) + \frac{1}{2}a_x(t, x, \alpha). \tag{3.4}$$

This relation, together with (3.5), leads to

$$\int_0^T a(s, X_s, \alpha)dX_s = F(T, X_T, \alpha) - \int_0^T f(s, X_s, \alpha)ds. \tag{3.5}$$

Then, the log-likelihood function $l_T(\alpha)$ can be written in the form[2)]

$$\begin{aligned} l_T(\alpha) = \log \frac{dP_\alpha^{\mathrm{T}}}{dP_W^{\mathrm{T}}} &= \int_0^T a(t, X_t, \alpha)dX_t - \frac{1}{2}\int_0^T a^2(t, X_t, \alpha)dt - \int_0^T a(t, X_t, \alpha)dL_t \\ &= F(T, X_T, \alpha) - \int_0^T \left[f(t, X_t, \alpha) + \frac{1}{2}a^2(t, X_t, \alpha)\right]dt - \int_0^T a(t, X_t, \alpha)dL_t. \end{aligned}$$

Now, we suppose that the integral defined by (3.4) and the integral on the right-hand side of (3.8) can be differentiated under the integral sign with respect to α. Further assume that F' and F'' have the same properties as F. Then, $l_T(\alpha)$ is twice differentiable with respect to α and a further application of Itô's formula to F', repeating the same procedures, it follows that

$$\int_0^T a'(s, X_s, \alpha)dX_s = F'(T, X_T, \alpha) - \int_0^T f'(s, X_s, \alpha)ds. \tag{3.6}$$

Furthermore,

$$l'_T(\alpha) = F'(T, X_T, \alpha) - \int_0^T [f'(t, X_t, \alpha) + a(t, X_t, \alpha)a'(t, X_t, \alpha)]dt$$

2) It is easy to see that an advantage of this form of the log-likelihood function is that it does not involve a stochastic integral except for the last term on the right-hand side of (3.8) and is amenable to deal with. While the integrator of the last term is of finite variation on any finite interval, the integral is defined as "pathwise" (i.e., for each $\omega \in \Omega$, separately) and then it can be coped with as usual Lebesgue-Stieltjes integral. Then, the following assumption concerning being differentiated under the integral sign is natural and weak.

$$
\begin{aligned}
&-\int_0^T a'(t,X_t,\alpha)dL_t \\
&=\int_0^T a'(t,X_t,\alpha)dX_t-\int_0^T a(t,X_t,\alpha)a'(t,X_t,\alpha)dt-\int_0^T a'(t,X_t,\alpha)dL_t \\
&=\int_0^T a'(t,X_t,\alpha)(a(t,X_t,\alpha)dt+dW_t+dL_t)-\int_0^T a(t,X_t,\alpha)a'(t,X_t,\alpha)dt \\
&\quad-\int_0^T a'(t,X_t,\alpha)dL_t \\
&=\int_0^T a'(t,X_t,\alpha)dW_t.
\end{aligned}
$$

Apply the same procedure on the function F''. Then

$$
\int_0^T a''(s,X_s,\alpha)dX_s=F''(T,X_T,\alpha)-\int_0^T f''(s,X_s,\alpha)ds. \tag{3.7}
$$

Hence, it follows that

$$
\begin{aligned}
l''_T(\alpha)&=F''(T,X_T,\alpha)-\int_0^T [f''(t,X_t,\alpha)+[a'(t,X_t,\alpha)]^2 \\
&\quad+a(t,X_t,\alpha)a''(t,X_t,\alpha)]dt-\int_0^T a''(t,X_t,\alpha)dL_t \\
&=\int_0^T a''(t,X_t,\alpha)dX_t-\int_0^T [[a'(t,X_t,\alpha)]^2+a(t,X_t,\alpha)a''(t,X_t,\alpha)]dt \\
&\quad-\int_0^T a''(t,X_t,\alpha)dL_t \\
&=\int_0^T a''(t,X_t,\alpha)(a(t,X_t,\alpha)dt+dW_t+dL_t) \\
&\quad-\int_0^T [[a'(t,X_t,\alpha)]^2+a(t,X_t,\alpha)a''(t,X_t,\alpha)]dt-\int_0^T a''(t,X_t,\alpha)dL_t \\
&=\int_0^T a''(t,X_t,\alpha)dW_t-\int_0^T [a'(t,X_t,\alpha)]^2dt.
\end{aligned}
$$

Thus, the proof of Lemma 3.1 is completed. □

Now, we state our main results concerning the strong consistency and asymptotic normality of the maximum likelihood estimation.

Theorem 3.2. *Under Assumptions* 1–3 *and the regularity conditions involved in the above proof, there exists a root of the likelihood equation $l'_T(\alpha)=0$, which is strongly consistent for α as $T\to\infty$.*

Proof. For any $\delta>0$ such that $\alpha\pm\delta\in\Theta$, it follows from (3.1) that

$$
\begin{aligned}
l_T(\alpha\pm\delta)-l_T(\alpha)&=\log\frac{dP^{\mathrm{T}}_{\alpha\pm\delta}}{dP^{\mathrm{T}}_{\alpha}} \\
&=\int_0^T [a(t,X_t,\alpha\pm\delta)-a(t,X_t,\alpha)]d(X_t-L_t) \\
&\quad-\frac{1}{2}\int_0^T [a^2(t,X_t,\alpha\pm\delta)-a^2(t,X_t,\alpha)]dt \\
&=\int_0^T [a(t,X_t,\alpha\pm\delta)-a(t,X_t,\alpha)]d[a(t,X_t,\alpha)dt+dW_t] \\
&\quad-\frac{1}{2}\int_0^T [a^2(t,X_t,\alpha\pm\delta)-a^2(t,X_t,\alpha)]dt
\end{aligned}
$$

$$= \int_0^T [a(t, X_t, \alpha \pm \delta) - a(t, X_t, \alpha)] dW_t - \frac{1}{2} \int_0^T [a(t, X_t, \alpha \pm \delta) - a(t, X_t, \alpha)]^2 dt.$$

Let

$$A_t^\alpha \triangleq a(t, X_t, \alpha \pm \delta) - a(t, X_t, \alpha) \tag{3.8}$$

and

$$K_T \triangleq \int_0^T (A_t^\alpha)^2 dt. \tag{3.9}$$

Then, it follows that

$$\frac{l_T(\alpha \pm \delta) - l_T(\alpha)}{K_T} = \frac{\int_0^T A_t^\alpha dW_t}{\int_0^T (A_t^\alpha)^2 dt} - \frac{1}{2}. \tag{3.10}$$

Under Assumption 3, $\int_0^T A_t^\alpha dW_t$ is a continuous local martingale, using the Skorohod embedding of the continuous local martingale $\int_0^T A_t^\alpha dW_t$, we have

$$\frac{\int_0^T A_t^\alpha dW_t}{\int_0^T (A_t^\alpha)^2 dt} = \frac{\tilde{B}_{\int_0^T (A_t^\alpha)^2 dt}}{\int_0^T (A_t^\alpha)^2 dt}, \tag{3.11}$$

where $\tilde{B}$ is another Brownian motion with respect to the enlarged filtration $(\mathcal{F}_{\tau_t})_{t \geqslant 0}$ with $\tau_t = \inf\{s : \int_0^s (A_r^\alpha)^2 dr > t\}$ and $\tilde{B}$ independent of $\int_0^T (A_t^\alpha)^2 dt$. By Assumption 3 and the fact that as $T \to \infty$,

$$\lim_{T \to \infty} \frac{\tilde{B}_T}{T} = 0 \quad \text{a.s.},$$

then as $T \to \infty$,

$$\frac{\int_0^T A_t^\alpha dW_t}{\int_0^T (A_t^\alpha)^2 dt} \to 0 \quad \text{a.s.} \tag{3.12}$$

This leads to, as $T \to \infty$,

$$\frac{l_T(\alpha \pm \delta) - l_T(\alpha)}{K_T} \to -\frac{1}{2} \quad \text{a.s.} \tag{3.13}$$

Note that also from Assumption 3, $K_T > 0$ a.s. for large enough T. Hence, there exists T_0 such that $T > T_0$, we have

$$l_T(\alpha \pm \delta) < l_T(\alpha) \quad \text{a.s.} \tag{3.14}$$

Since $l_T(\alpha)$ is continuous on the closed interval $[\alpha - \delta, \alpha + \delta]$, by virtue of (3.20), it attains the maximum in the interior of the interval, i.e., there exists $\hat{\alpha}_T \in (\alpha - \delta, \alpha + \delta)$ such that

$$l_T(\hat{\alpha}_T) = \sup_{\alpha \in (\alpha - \delta, \alpha + \delta)} l_T(\alpha), \tag{3.15}$$

which leads to $l_T'(\hat{\alpha}_T) = 0$. Thus, The proof of the desired result is complete. □

Theorem 3.3. *One has*

$$(\hat{\alpha}_T - \alpha) \sqrt{\int_0^T [a'(t, X_t, \alpha)]^2 dt} \xrightarrow{\mathcal{D}} \mathcal{N}, \tag{3.16}$$

as $T \to \infty$, *where* $\xrightarrow{\mathcal{D}}$ *denotes the convergence in distribution and* $\mathcal{N}$ *is the standard normal random variable.*

Proof. Applying Taylor's expansion to $l'_T(\alpha)$ around $\hat{\alpha}_T$, it follows that

$$\begin{aligned} l'_T(\alpha) &= l'_T(\hat{\alpha}_T) + (\alpha - \hat{\alpha}_T) l''_T(\hat{\alpha}_T + \beta_T(\alpha - \hat{\alpha}_T)) \\ &= (\alpha - \hat{\alpha}_T) l''_T(\hat{\alpha}_T + \beta_T(\alpha - \hat{\alpha}_T)), \end{aligned}$$

where from Theorem 3.2 $|\beta_T| \leqslant 1$ for large enough T. On the other hand, under Assumption 2, it is obvious to see that $l'_T(\alpha) = \int_0^T a'(t, X_t, \alpha) dW_t$ is a zero mean square integrable martingale with quadratic variation process

$$\int_0^T [a'(t, X_t, \alpha)]^2 dt,$$

i.e., for $M_t^\alpha := l'_t(\alpha)$, we have $[M^\alpha]_t = \int_0^t [a'(s, X_s, \alpha)]^2 ds$. Setting

$$L_t^\alpha = \inf\{s : [M^\alpha]_s > t\}, \tag{3.17}$$

then, $\tilde{\tilde{B}}_t = M^\alpha_{L_t^\alpha}$ is a Brownian motion with respect to the enlarged filtration $\mathcal{F}_{L_t^\alpha}$ and $M_t^\alpha = \tilde{\tilde{B}}_{[M^\alpha]_t}$[3]. In fact, $\tilde{\tilde{B}}_t$ is the so-called *Dambis, Dubins-Schwarz* Brownian motion of M_t^α (see [27, Theorem 1.6, Chapter 5, p. 173]). Furthermore, note that $\int_0^t [a'(s, X_s, \alpha)]^2 ds > 0$, a.s. for t sufficiently large. The scaling property of Brownian motion implies that

$$\frac{M_t^\alpha}{\sqrt{[M^\alpha]_t}} = \frac{1}{\sqrt{[M^\alpha]_t}} \tilde{\tilde{B}}_{[M^\alpha]_t} \overset{\mathcal{D}}{=} \mathcal{N}.$$

Hence, we have

$$\frac{l'_T(\alpha)}{\sqrt{\int_0^T [a'(t, X_t, \alpha)]^2 dt}} \overset{\mathcal{D}}{=} \mathcal{N}. \tag{3.18}$$

On the other hand, in view of Theorem 3.2 and the continuity of $l''_T(\alpha)$ with respect to α, it is easy to see that

$$l''_T(\hat{\alpha}_T + \beta_T(\alpha - \hat{\alpha}_T)) - l''_T(\alpha) \to 0$$

in probability as $T \to \infty$. Furthermore, together with Theorem 3.2 and the second part of Assumption 1, it holds

$$(\alpha - \hat{\alpha}_T) \frac{l''_T(\alpha)}{\sqrt{\int_0^T [a'(t, X_t, \alpha)]^2 dt}} \overset{\mathcal{D}}{\to} \mathcal{N}. \tag{3.19}$$

Under Assumption 2, $\int_0^T a''(t, X_t, \alpha) dW_t$ is a zero mean square integrable martingale. This assertion, coupled with the second part of Assumption 1, ensures that [15, Theorem 2.1] holds, i.e., as $T \to \infty$,

$$\frac{\int_0^T a''(t, X_t, \alpha) dW_t}{\int_0^T [a'(t, X_t, \alpha)]^2 dt} \to 0. \tag{3.20}$$

Therefore, one has

$$\frac{l''_T(\alpha)}{\int_0^T [a'(t, X_t, \alpha)]^2 dt} \to -1$$

in probability as $T \to \infty$ from (3.12). This, together with (3.26), yields the desired result. □

[3] See [20, Remark 4.1] for the exact construction of the extended probability space. It is important to note that the extension does not change the law of the local martingale. In this paper, we study properties of sequences of such laws and therefore we may assume that each continuous local martingale in question is embedded in a Brownian motion in the sense of the above theorem.

4 A nonstationary case

In this section, we propose to a nonstationary case concerning our main results. Consider the following stochastic differential equation[4)]:

$$\begin{cases} dX_t = \alpha t X_t dt + dW_t + dL_t, \\ X_t \geqslant 0, \quad \text{for all} \quad t \geqslant 0, \\ X_0 = 0, \quad \alpha > 0. \end{cases} \tag{4.1}$$

Here, $L = (L_t)_{t \geqslant 0}$ is the minimal non-decreasing and non-negative process, which makes the process $X_t \geqslant 0$ for all $t \geqslant 0$. The process L increases only when X hits the boundary zero, so that

$$\int_{[0,\infty)} I(X_t > 0) dL_t = 0. \tag{4.2}$$

Then, $\{X_t\}_{t \geqslant 0}$ is a non-stationary and nonhomogeneous process. It is easy to see that, to check the above assumptions, we only need to verify that

$$\int_0^T t^2 X_t^2 dt \to \infty \quad \text{a.s.} \quad \text{as} \quad T \to \infty. \tag{4.3}$$

Some other conditions are naturally satisfied. Apply Itô's formula to the function $x\mathrm{e}^{-\frac{\alpha t^2}{2}}$, it follows that

$$X_t = \mathrm{e}^{\frac{\alpha t^2}{2}} \int_0^t \mathrm{e}^{-\frac{\alpha s^2}{2}} dW_s + \mathrm{e}^{\frac{\alpha t^2}{2}} \int_0^t \mathrm{e}^{-\frac{\alpha s^2}{2}} dL_s.$$

In comparison to the model (4.1), we introduce the non-stationary process $\{Y_t, t \geqslant 0\}$ satisfying the following stochastic differential equation

$$\begin{cases} dY_t = \alpha t Y_t dt + dW_t, \\ Y_0 = 0, \quad \alpha > 0. \end{cases} \tag{4.4}$$

Thus, we have

$$Y_t = \mathrm{e}^{\frac{\alpha t^2}{2}} \int_0^t \mathrm{e}^{-\frac{\alpha s^2}{2}} dW_s \tag{4.5}$$

and

$$X_t - Y_t = \mathrm{e}^{\frac{\alpha t^2}{2}} \int_0^t \mathrm{e}^{-\frac{\alpha s^2}{2}} dL_s \geqslant 0, \tag{4.6}$$

for all $t \geqslant 0$. On the other hand, let $\xi_t = \int_0^t \mathrm{e}^{-\frac{\alpha s^2}{2}} dW_s$, since the quadratic variation process

$$[\xi]_\infty = \sqrt{\frac{\pi}{\alpha}} < \infty.$$

Thus, $\{\xi_t\}_{t \geqslant 0}$ is a square-integrable $\mathcal{F}_t$-martingale. Moreover, ξ_t is a normal random variable with distribution $\mathcal{N}(0, \int_0^t \mathrm{e}^{-\alpha s^2} ds)$. Therefore, by martingale convergence theorem, it follows that

$$\lim_{t \to \infty} \xi_t = \int_0^\infty \mathrm{e}^{-\frac{\alpha s^2}{2}} dW_s.$$

Then, by L'Hospital rule as $T \to \infty$,

$$\frac{\int_0^T t^2 Y_t^2 dt}{\int_0^T t^2 \mathrm{e}^{\alpha t^2} dt} \to \left(\int_0^\infty \mathrm{e}^{-\frac{\alpha t^2}{2}} dW_t \right)^2 \quad \text{a.s.,}$$

4) In most practical applications, the reflecting barrier is usually taken as $b = 0$ (see [1, 2, 5, 6, 32]). Thus, it does not impact on the application of our results, if we set b equal to zero here.

which is a chi-square random variable. Hence, $\int_0^T t^2 Y_t^2 dt \to \infty$ a.s. as $T \to \infty$, and then (4.3) holds through (4.6).

Note that

$$\hat{\alpha}_T - \alpha = \frac{\int_0^T tX_t dX_t}{\int_0^T t^2 X_t^2 dt} - \alpha = \frac{\int_0^T tX_t dW_t}{\int_0^T t^2 X_t^2 dt}$$

by the fact that the process L increases only when X hits the boundary zero, and then together with (4.3),

$$\hat{\alpha}_T - \alpha = \frac{W_{\int_0^T t^2 X_t^2 dt}}{\int_0^T t^2 X_t^2 dt} \to 0 \quad \text{a.s.}$$

On the other hand, we have

$$E\bigg[\int_0^T tX_t dW_t\bigg] = 0$$

and

$$E\bigg[\int_0^T tX_t dW_t\bigg]^2 = E\bigg[\int_0^T t^2 X_t^2 dt\bigg].$$

By virtue of central limit theorem (see [26]), we have

$$(\hat{\alpha}_T - \alpha)\sqrt{\int_0^T t^2 X_t^2 dt} \xrightarrow{\mathcal{D}} \mathcal{N}.$$

5 Conclusions

Based on the continuous observations, this paper contributes a method for maximum-likelihood estimation (MLE) of the reflected Ornstein-Uhlenbeck (ROU) processes with the general drift coefficient. Under some sufficient (but not necessary) technical conditions, the strong consistency and asymptotic normality of the maximum likelihood estimation are theoretically justified. Owing to the limited space of this paper which focuses on maximum-likelihood estimation and its statistical properties, investigations on more asymptotic properties related to the MLE can be regarded as a future research topic, for example, a similar estimation for our model based on discrete observations, as well as its consistency and asymptotic distribution (see [18]).

On the other hand, some future work may investigate the other estimators for the other reflected diffusions. For example, Lee et al. [22] proposed a sequential maximum likelihood estimation (SMLE) of the unknown drift of the ROU process without jumps; the reflected jump-diffusion or Lévy processes have been extensively investigated in the literature (see [1, 2, 4–6, 10, 12, 14, 32]).

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