SCIENCE CHINA
Mathematics

• ARTICLES •

April 2016 Vol. 59 No. 4: 751–768
doi: 10.1007/s11425-015-5105-2

Rosenthal's inequalities for independent and negatively dependent random variables under sub-linear expectations with applications

ZHANG LiXin

School of Mathematics, Zhejiang University, Hangzhou 310027*, China*
Email: stazlx@zju.edu.cn

Received August 22, 2014; accepted July 14, 2015; published online December 23, 2015

Abstract Classical Kolmogorov's and Rosenthal's inequalities for the maximum partial sums of random variables are basic tools for studying the strong laws of large numbers. In this paper, motived by the notion of independent and identically distributed random variables under the sub-linear expectation initiated by Peng (2008), we introduce the concept of negative dependence of random variables and establish Kolmogorov's and Rosenthal's inequalities for the maximum partial sums of negatively dependent random variables under the sub-linear expectations. As an application, we show that Kolmogorov's strong law of larger numbers holds for independent and identically distributed random variables under a continuous sub-linear expectation if and only if the corresponding Choquet integral is finite.

Keywords sub-linear expectation, capacity, Kolmogorov's inequality, Rosenthal's inequality, negative dependence, strong laws of large numbers

MSC(2010) 60F15, 60F05

Citation: Zhang L X. Rosenthal's inequalities for independent and negatively dependent random variables under sub-linear expectations with applications. Sci China Math, 2016, 59: 751–768, doi: 10.1007/s11425-015-5105-2

1 Introduction and notation

Non-additive probabilities and non-additive expectations are useful tools for studying uncertainties in statistics, measures of risk, superhedging in finance and non-linear stochastic calculus, see [2, 5, 7, 10–13], etc. This paper considers the general sub-linear expectations and related non-additive probabilities generated by them. The notion of independent and identically distributed (i.i.d. for short) random variables under the sub-linear expectations was introduced by Peng [12, 14, 15] and the weak convergences such as central limit theorems and weak laws of large numbers has been studied. Because the proofs of classical Kolmogorov's inequalities and Rosenthal's inequalities for the maximum partial sums of random variables depend basically on the additivity of the probabilities and the expectations, such inequalities have not been established under the sub-linear expectations. As a result, very few results on strong laws of large numbers can be found under the sub-linear expectations. Recently, Chen [1] obtained Kolmogorov's strong law of large numbers for i.i.d. random variables under the condition of finite $(1 + \epsilon)$-moments by establishing an inequality of an exponential moment of partial sums of truncated independent random variables. The moment condition is much stronger than the one for the classical Kolmogorov strong law of large numbers. Also, Gao and Xu [3, 4] studied the large deviations and moderate deviations for quasi-continuous random variables in a complete separable metric space under the Choquet capacity generalized by a regular sub-linear expectation. The main purpose of this paper is to establish basic inequalities for the maximum partial sums of independent random variables in the general sub-linear expectation spaces. These inequalities are basic tools to study the strong limit theorems. They are also essential tools to prove the functional central limit theorem (see [23]). In the remainder of this section, we give some notation under the sub-linear expectations. For explaining our main idea, we prove Kolmogorov's inequality as our first result. Then, we introduce the concept of negative dependence under the sub-linear expectation which is an extension of independence as well as the classical negative dependence. In Section 2, we establish Rosenthal's inequalities for this kind of negatively dependent random variables. In Section 3, as applications of these inequalities, we establish the Kolmogorov type strong laws of large numbers under the weakest moment conditions. In particular, we show that Kolmogorov's type strong law of large numbers holds for independent and identically distributed random variables under a continuous sub-linear expectation if and only if the the corresponding Choquet integral is finite.

We use notation of Peng [14]. Let $(\Omega,\mathcal{F})$ be a given measurable space and let $\mathscr{H}$ be a linear space of real functions defined on $(\Omega,\mathcal{F})$ such that if $X_1,\dots,X_n\in\mathscr{H}$ then $\varphi(X_1,\dots,X_n)\in\mathscr{H}$ for each $\varphi\in C_{l,\mathrm{Lip}}(\mathbb{R}_n)$, where $C_{l,\mathrm{Lip}}(\mathbb{R}_n)$ denotes the linear space of (local Lipschitz) functions φ satisfying

$$\begin{aligned}&|\varphi(\boldsymbol{x})-\varphi(\boldsymbol{y})|\leqslant C(1+|\boldsymbol{x}|^m+|\boldsymbol{y}|^m)|\boldsymbol{x}-\boldsymbol{y}|,\quad\forall\,\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}_n,\\&\text{for some } C>0,\quad m\in\mathbb{N}\ \text{ depending on }\ \varphi.\end{aligned}$$

$\mathscr{H}$ is considered as a space of "random variables". In this case we denote $X\in\mathscr{H}$.

Remark 1.1. It is easily seen that if $\varphi_1,\varphi_2\in C_{l,\mathrm{Lip}}(\mathbb{R}_n)$, then $\varphi_1\vee\varphi_2,\varphi_1\wedge\varphi_2\in C_{l,\mathrm{Lip}}(\mathbb{R}_n)$ because $\varphi_1\vee\varphi_2=\frac{1}{2}(\varphi_1+\varphi_2+|\varphi_1-\varphi_2|)$, $\varphi_1\wedge\varphi_2=\frac{1}{2}(\varphi_1+\varphi_2-|\varphi_1-\varphi_2|)$.

Definition 1.2. A sub-linear expectation $\widehat{\mathbb{E}}$ on $\mathscr{H}$ is a functional $\widehat{\mathbb{E}}:\mathscr{H}\to\overline{\mathbb{R}}:=[-\infty,\infty]$ satisfying the following properties: For all $X,Y\in\mathscr{H}$, we have

(a) Monotonicity: If $X\geqslant Y$ then $\widehat{\mathbb{E}}[X]\geqslant\widehat{\mathbb{E}}[Y]$.

(b) Constant preserving: $\widehat{\mathbb{E}}[c]=c$.

(c) Sub-additivity: $\widehat{\mathbb{E}}[X+Y]\leqslant\widehat{\mathbb{E}}[X]+\widehat{\mathbb{E}}[Y]$ whenever $\widehat{\mathbb{E}}[X]+\widehat{\mathbb{E}}[Y]$ is not of the form $+\infty-\infty$ or $-\infty+\infty$.

(d) Positive homogeneity: $\widehat{\mathbb{E}}[\lambda X]=\lambda\widehat{\mathbb{E}}[X]$, $\lambda\geqslant 0$.

The triple $(\Omega,\mathscr{H},\widehat{\mathbb{E}})$ is called a sub-linear expectation space. Give a sub-linear expectation $\widehat{\mathbb{E}}$, let us denote the conjugate expectation $\widehat{\mathcal{E}}$ of $\widehat{\mathbb{E}}$ by

$$\widehat{\mathcal{E}}[X]:=-\widehat{\mathbb{E}}[-X],\quad\forall\,X\in\mathscr{H}.$$

Obviously, for all $X\in\mathscr{H}$, $\widehat{\mathcal{E}}[X]\leqslant\widehat{\mathbb{E}}[X]$. We also call $\widehat{\mathbb{E}}[X]$ and $\widehat{\mathcal{E}}[X]$ the upper-expectation and lower-expectation of X, respectively.

Definition 1.3 (See [12,14])**.** (i) (Identical distribution) Let $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ be two n-dimensional random vectors defined, respectively in sub-linear expectation spaces $(\Omega_1,\mathscr{H}_1,\widehat{\mathbb{E}}_1)$ and $(\Omega_2,\mathscr{H}_2,\widehat{\mathbb{E}}_2)$. They are called identically distributed, denoted by $\boldsymbol{X}_1\overset{d}{=}\boldsymbol{X}_2$, if

$$\widehat{\mathbb{E}}_1[\varphi(\boldsymbol{X}_1)]=\widehat{\mathbb{E}}_2[\varphi(\boldsymbol{X}_2)],\quad\forall\,\varphi\in C_{l,\mathrm{Lip}}(\mathbb{R}_n),$$

whenever the sub-expectations are finite.

(ii) (Independence) In a sub-linear expectation space $(\Omega,\mathscr{H},\widehat{\mathbb{E}})$, a random vector $\boldsymbol{Y}=(Y_1,\dots,Y_n)$, $Y_i\in\mathscr{H}$ is said to be independent to another random vector $\boldsymbol{X}=(X_1,\dots,X_m)$, $X_i\in\mathscr{H}$ under $\widehat{\mathbb{E}}$ if for each test function $\varphi\in C_{l,\mathrm{Lip}}(\mathbb{R}_m\times\mathbb{R}_n)$ we have $\widehat{\mathbb{E}}[\varphi(\boldsymbol{X},\boldsymbol{Y})]=\widehat{\mathbb{E}}[\widehat{\mathbb{E}}[\varphi(\boldsymbol{x},\boldsymbol{Y})]|_{\boldsymbol{x}=\boldsymbol{X}}]$, whenever $\overline{\varphi}(\boldsymbol{x}):=\widehat{\mathbb{E}}\left[|\varphi(\boldsymbol{x},\boldsymbol{Y})|\right]<\infty$ for all $\boldsymbol{x}$ and $\widehat{\mathbb{E}}\left[|\overline{\varphi}(\boldsymbol{X})|\right]<\infty$.

(iii) (IID random variables) A sequence of random variables $\{X_n;n\geqslant 1\}$ is said to be independent, if X_{i+1} is independent to $(X_1,\dots,X_i)$ for each $i\geqslant 1$. It is said to be identically distributed, if $X_i\overset{d}{=}X_1$ for each $i\geqslant 1$.

As shown by Peng [14], it is important to note that under sub-linear expectations the condition that "$\boldsymbol{Y}$ is independent to $\boldsymbol{X}$" does not imply automatically that "$\boldsymbol{X}$ is independent to $\boldsymbol{Y}$".

From the definition of independence, it is easily seen that, if Y is independent to X and $X \geqslant 0, \widehat{\mathbb{E}}[Y] \geqslant 0$, then

$$\widehat{\mathbb{E}}[XY] = \widehat{\mathbb{E}}[X]\widehat{\mathbb{E}}[Y]. \tag{1.1}$$

Furthermore, if Y is independent to X and $X \geqslant 0, Y \geqslant 0$, then

$$\widehat{\mathbb{E}}[XY] = \widehat{\mathbb{E}}[X]\widehat{\mathbb{E}}[Y], \quad \widehat{\mathcal{E}}[XY] = \widehat{\mathcal{E}}[X]\widehat{\mathcal{E}}[Y]. \tag{1.2}$$

If $\{X_n; n \geqslant 1\}$ is a sequence of independent random variables with both the upper-expectations $\widehat{\mathbb{E}}[X_i]$ and lower-expectations $\widehat{\mathcal{E}}[X_i]$ being zeros, then it is easily checked that

$$\widehat{\mathbb{E}}[S_n^2] = \widehat{\mathbb{E}}\bigg[\sum_{k=1}^n X_k^2 + \sum_{i\neq j} X_i X_j\bigg] = \sum_{k=1}^n \widehat{\mathbb{E}}[X_k^2],$$

because $\widehat{\mathbb{E}}[X_iX_j] = \widehat{\mathcal{E}}[X_iX_j] = 0$ for $i \neq j$ by the definition of the independence, where $S_n = \sum_{k=1}^n X_k$. However, when the popular truncation method is used for studying the limit theorems, the truncated random variables usually no longer have zero sub-linear expectations. It is hard to centralize a random variable such that its upper-expectation and lower-expectation are both zeros. But it is easy to centralize a random variable X such that one of $\widehat{\mathbb{E}}[X]$ and $\widehat{\mathcal{E}}[X]$ is zero. For example, the random variable $X - \widehat{\mathbb{E}}[X]$ has zero upper-expectation. So, the moments of S_n with the condition $\widehat{\mathbb{E}}[X_i] = 0$ $(i = 1, \dots, n)$ are much useful than those with the condition $\widehat{\mathbb{E}}[X_i] = \widehat{\mathcal{E}}[X_i] = 0$ $(i = 1, \dots, n)$. Unfortunately, by noting that the independence of X and Y does not imply $\widehat{\mathbb{E}}[(X - \widehat{\mathbb{E}}[X])(Y - \widehat{\mathbb{E}}[Y])] = 0$ (or $\leqslant 0$), even to get a good estimate of the second order moment $\widehat{\mathbb{E}}[(\sum_{k=1}^n (X_k - \widehat{\mathbb{E}}[X_k]))^2]$ is not a trivial work. As for the probability inequalities or moment inequalities for the maximum partial sums $\max_{k\leqslant n} S_k$, in the classical probability space, the proof depends basically on the additivity of the probabilities and the expectations. For example, the integral on the event $\{\max_{i\leqslant n} S_i \geqslant x\}$ is usually split to integrals on $\{\max_{i\leqslant k} S_i < x, S_k \geqslant x\}$, $k = 1, \dots, n$. The methods based on the additivity cannot be used under the framework of sub-linear expectations. Other popular techniques such as the symmetrization, the martingale method and the stopping time method are also not available under the sub-linear expectations because they are essentially based on the additivity property. The main purpose of this paper is to establish the moment inequalities for $\max_{k\leqslant n} S_k$ which can be applied to truncated random variables freely. To explain our main idea, we first give the following result on Kolmogorov's inequality.

Theorem 1.4 (Kolmogorov's inequality). *Let $\{X_1, \dots, X_n\}$ be a sequence of random variables in $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$ with $\widehat{\mathbb{E}}[X_k] = 0$, $k = 1, \dots, n$. Suppose that X_k is independent to $(X_{k+1}, \dots, X_n)$ for each $k = 1, \dots, n-1$. Denote $S_k = X_1 + \cdots + X_k$, $S_0 = 0$. Then*

$$\widehat{\mathbb{E}}\Big[\Big(\max_{k\leqslant n} S_k\Big)^2\Big] \leqslant \sum_{k=1}^n \widehat{\mathbb{E}}[X_k^2]. \tag{1.3}$$

In particular, $\widehat{\mathbb{E}}[(S_n^+)^2] \leqslant \sum_{k=1}^n \widehat{\mathbb{E}}[X_k^2]$.

Proof. Set $T_k = \max(X_k, X_k + X_{k+1}, \dots, X_k + \cdots + X_n)$. Then $T_k, T_k^+ \in \mathscr{H}$, and $T_k = X_k + T_{k+1}^+$, $T_k^2 = X_k^2 + 2X_kT_{k+1}^+ + (T_{k+1}^+)^2$. It follows that $\widehat{\mathbb{E}}[T_k^2] \leqslant \widehat{\mathbb{E}}[X_k^2] + 2\widehat{\mathbb{E}}[X_kT_{k+1}^+] + \widehat{\mathbb{E}}[(T_{k+1}^+)^2]$. Note $\widehat{\mathbb{E}}[X_kT_{k+1}^+] = 0$ by (1.1). We conclude that $\widehat{\mathbb{E}}[T_k^2] \leqslant \widehat{\mathbb{E}}[X_k^2] + \widehat{\mathbb{E}}[(T_{k+1}^+)^2] \leqslant \widehat{\mathbb{E}}[X_k^2] + \widehat{\mathbb{E}}[T_{k+1}^2]$. Hence, $\widehat{\mathbb{E}}[T_1^2] \leqslant \sum_{k=1}^n \widehat{\mathbb{E}}[X_k^2]$. The proof is completed. □

In the above proof, the independence is utilized to get $\widehat{\mathbb{E}}[X_kT_{k+1}^+] \leqslant 0$ and so can be weakened. Recall that in the probability $(\Omega, \mathcal{F}, \boldsymbol{P})$, two random vectors $\boldsymbol{Y} = (Y_1, \dots, Y_n)$ and $\boldsymbol{X} = (X_1, \dots, X_m)$ are said to be negatively dependent if for each pair of coordinatewise nondecreasing (resp. non-increasing) functions $\varphi_1(\boldsymbol{x})$ and $\varphi_2(\boldsymbol{y})$ we have $E_{\boldsymbol{P}}[\varphi_1(\boldsymbol{X})\varphi_2(\boldsymbol{Y})] \leqslant E_{\boldsymbol{P}}[\varphi_1(\boldsymbol{X})]E_{\boldsymbol{P}}[\varphi_2(\boldsymbol{Y})]$ whenever the expectations considered exist.

We introduce the concept of negative dependence under the sub-linear expectation.

Definition 1.5 (Negative dependence)**.** In a sub-linear expectation space $(\Omega,\mathscr{H},\widehat{\mathbb{E}})$, a random vector $\boldsymbol{Y}=(Y_1,\ldots,Y_n)$, $Y_i\in\mathscr{H}$ is said to be negatively dependent to another random vector $\boldsymbol{X}=(X_1,\ldots,X_m)$, $X_i\in\mathscr{H}$ under $\widehat{\mathbb{E}}$ if for each pair of test functions $\varphi_1\in C_{l,\mathrm{Lip}}(\mathbb{R}_m)$ and $\varphi_2\in C_{l,\mathrm{Lip}}(\mathbb{R}_n)$ we have $\widehat{\mathbb{E}}[\varphi_1(\boldsymbol{X})\varphi_2(\boldsymbol{Y})]\leqslant\widehat{\mathbb{E}}[\varphi_1(\boldsymbol{X})]\widehat{\mathbb{E}}[\varphi_2(\boldsymbol{Y})]$ whenever $\varphi_1(\boldsymbol{X})\geqslant 0$, $\widehat{\mathbb{E}}[\varphi_2(\boldsymbol{Y})]\geqslant 0$, $\widehat{\mathbb{E}}[|\varphi_1(\boldsymbol{X})\varphi_2(\boldsymbol{Y})|]<\infty$, $\widehat{\mathbb{E}}[|\varphi_1(\boldsymbol{X})|]<\infty$, $\widehat{\mathbb{E}}[|\varphi_2(\boldsymbol{Y})|]<\infty$, and either φ_1 and φ_2 are coordinatewise nondecreasing or φ_1 and φ_2 are coordinatewise non-increasing.

By the definition, it is easily seen that, if $\boldsymbol{Y}=(Y_1,\ldots,Y_n)$ is negatively dependent to $\boldsymbol{X}=(X_1,\ldots,X_m)$, $\varphi_1\in C_{l,\mathrm{Lip}}(\mathbb{R}_m)$ and $\varphi_2\in C_{l,\mathrm{Lip}}(\mathbb{R}_n)$ are coordinatewise nondecreasing (resp. non-increasing) functions, then $\varphi_2(\boldsymbol{Y})$ is negatively dependent to $\varphi_1(\boldsymbol{X})$. Furthermore, if $Y\in\mathscr{H}$ is negatively dependent to $X\in\mathscr{H}$ and $X\geqslant 0$, $\widehat{\mathbb{E}}[X]<\infty$, $\widehat{\mathbb{E}}[|Y|]<\infty$, $\widehat{\mathbb{E}}[Y]\leqslant 0$, then

$$\widehat{\mathbb{E}}[YX]\leqslant\widehat{\mathbb{E}}[(Y-\widehat{\mathbb{E}}[Y])X]+\widehat{\mathbb{E}}[\widehat{\mathbb{E}}[Y]X]\leqslant\widehat{\mathbb{E}}[Y-\widehat{\mathbb{E}}[Y]]\widehat{\mathbb{E}}[X]\leqslant 0.$$

It is obvious that, if $\boldsymbol{Y}$ is independent to $\boldsymbol{X}$, then $\boldsymbol{Y}$ is negatively dependent to $\boldsymbol{X}$. The following is the classical example introduced by Huber and Strassen [6].

Example 1.6. Let $\mathcal{P}$ be a family of probability measures defined on $(\Omega,\mathcal{F})$. For any random variable ξ, we denote the upper expectation by $\widehat{\mathbb{E}}[\xi]=\sup_{Q\in\mathcal{P}}\boldsymbol{E}_Q[\xi]$. Then $\widehat{\mathbb{E}}[\cdot]$ is a sub-linear expectation. Moreover, if $\boldsymbol{X}$ and $\boldsymbol{Y}$ are independent under each $Q\in\mathcal{P}$, then $\boldsymbol{Y}$ is negatively dependent to $\boldsymbol{X}$ under $\widehat{\mathbb{E}}$. In fact,

$$\begin{aligned}\widehat{\mathbb{E}}[\varphi_1(\boldsymbol{X})\varphi_2(\boldsymbol{Y})]&=\sup_{Q\in\mathcal{P}}\boldsymbol{E}_Q[\varphi_1(\boldsymbol{X})\varphi_2(\boldsymbol{Y})]=\sup_{Q\in\mathcal{P}}\boldsymbol{E}_Q[\varphi_1(\boldsymbol{X})]\boldsymbol{E}_Q[\varphi_2(\boldsymbol{Y})]\\&\leqslant\sup_{Q\in\mathcal{P}}\boldsymbol{E}_Q[\varphi_1(\boldsymbol{X})]\sup_{Q\in\mathcal{P}}\boldsymbol{E}_Q[\varphi_2(\boldsymbol{Y})]=\widehat{\mathbb{E}}[\varphi_1(\boldsymbol{X})]\widehat{\mathbb{E}}[\varphi_2(\boldsymbol{Y})]\end{aligned}$$

whenever $\varphi_1(\boldsymbol{X})\geqslant 0$ and $\widehat{\mathbb{E}}[\varphi_2(\boldsymbol{Y})]\geqslant 0$.

However, $\boldsymbol{Y}$ may be not independent to $\boldsymbol{X}$.

With the similar argument, we can show that $\boldsymbol{Y}$ is negatively dependent to $\boldsymbol{X}$ under $\widehat{\mathbb{E}}$ if $\boldsymbol{X}$ and $\boldsymbol{Y}$ are negatively dependent under each $Q\in\mathcal{P}$.

According to its proof, the conclusion of Theorem 1.4 remains true under the concept of negative dependence.

Corollary 1.7. *Let $\{X_1,\ldots,X_n\}$ be a sequence of random variables in $(\Omega,\mathscr{H},\widehat{\mathbb{E}})$ with $\widehat{\mathbb{E}}[X_k]\leqslant 0$, $k=1,\ldots,n$. Suppose that X_k is negatively dependent to $(X_{k+1},\ldots,X_n)$ for each $k=1,\ldots,n-1$. Then* (1.3) *holds.*

Our basic idea for obtaining Theorem 1.4 comes from Newman and Wright [9] and Matula [8] where Kolmogorov's inequality is estiblished for the classical positively and negatively dependent random variables, respectively.

2 Rosenthal's inequalities

In this section, we extend Kolmogorov's inequality to Rosenthal's inequalities. For moment inequalities of partial sums of the classical negatively dependent random variables and related strong limit theorems, one can refer to Shao [17], Su et al. [18], Yuan and An [19], Zhang [20–22], Zhang and Wen [24], etc. Some techniques from these papers will be used in the lines of our proofs. We let $\{X_1,\ldots,X_n\}$ be a sequence of random variables in $(\Omega,\mathscr{H},\widehat{\mathbb{E}})$, and denote $S_k=X_1+\cdots+X_k$, $S_0=0$.

Theorem 2.1 (Rosnethal's inequality)**.** (a) *Suppose that X_k is negatively dependent to $(X_{k+1},\ldots,X_n)$ for each $k=1,\ldots,n-1$, and $\widehat{\mathbb{E}}[X_k]\leqslant 0$, $k=1,\ldots,n$. Then*

$$\widehat{\mathbb{E}}\Big[\Big|\max_{k\leqslant n}S_k\Big|^p\Big]\leqslant 2^{2-p}\sum_{k=1}^n\widehat{\mathbb{E}}[|X_k|^p],\quad \textit{for }\ 1\leqslant p\leqslant 2 \tag{2.1}$$

and

$$\widehat{\mathbb{E}}\Big[\Big|\max_{k\leqslant n} S_k\Big|^p\Big] \leqslant C_p n^{p/2-1}\sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^p], \quad \textit{for } p\geqslant 2. \tag{2.2}$$

(b) *Suppose that X_k is independent to $(X_{k+1},\dots,X_n)$ for each $k=1,\dots,n-1$, and $\widehat{\mathbb{E}}[X_k]\leqslant 0$, $k=1,\dots,n$. Then*

$$\widehat{\mathbb{E}}\Big[\Big|\max_{k\leqslant n} S_k\Big|^p\Big] \leqslant C_p\bigg\{\sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^p] + \bigg(\sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^2]\bigg)^{p/2}\bigg\}, \quad \textit{for } p\geqslant 2. \tag{2.3}$$

(c) *In general, suppose that X_k is negatively dependent to $(X_{k+1},\dots,X_n)$ for each $k=1,\dots,n-1$, or X_{k+1} is negatively dependent to $(X_1,\dots,X_k)$ for each $k=1,\dots,n-1$. Then*

$$\widehat{\mathbb{E}}\Big[\max_{k\leqslant n} |S_k|^p\Big] \leqslant C_p\bigg\{\sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^p] + \bigg(\sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^2]\bigg)^{p/2} + \bigg(\sum_{k=1}^n[(\widehat{\mathcal{E}}[X_k])^- + (\widehat{\mathbb{E}}[X_k])^+]\bigg)^p\bigg\}. \tag{2.4}$$

Here C_p is a positive constant depending only on p.

If we consider the sequence $\{X_1,X_2,\dots,X_n\}$ in the reverse order as $\{X_n,X_{n-1},\dots,X_1\}$, by Theorems 2.1(a) and 2.1(b) we have the following corollary.

Corollary 2.2. *Let $\{X_1,\dots,X_n\}$ be a sequence of random variables in $(\Omega,\mathscr{H},\widehat{\mathbb{E}})$ with $\widehat{\mathbb{E}}[X_k]\leqslant 0$, $k=1,\dots,n$.*

(a) *Suppose that X_{k+1} is negatively dependent to $(X_1,\dots,X_k)$ for each $k=1,\dots,n-1$. Then*

$$\widehat{\mathbb{E}}\Big[\Big|\max_{k\leqslant n}(S_n-S_k)\Big|^p\Big] \leqslant 2^{2-p}\sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^p], \quad \textit{for } 1\leqslant p\leqslant 2 \tag{2.5}$$

and

$$\widehat{\mathbb{E}}\Big[\Big|\max_{k\leqslant n}(S_n-S_k)\Big|^p\Big] \leqslant C_p n^{p/2-1}\sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^p], \quad \textit{for } p\geqslant 2. \tag{2.6}$$

In particular,

$$\widehat{\mathbb{E}}[(S_n^+)^p] \leqslant \begin{cases} 2^{2-p}\displaystyle\sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^p], & \textit{for } 1\leqslant p\leqslant 2,\\ C_p n^{p/2-1}\displaystyle\sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^p], & \textit{for } p\geqslant 2.\end{cases} \tag{2.7}$$

(b) *Suppose that X_{k+1} is independent to $(X_1,\dots,X_k)$ for each $k=1,\dots,n-1$. Then*

$$\widehat{\mathbb{E}}\Big[\Big|\max_{k\leqslant n}(S_n-S_k)\Big|^p\Big] \leqslant C_p\bigg\{\sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^p] + \bigg(\sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^2]\bigg)^{p/2}\bigg\}, \quad \textit{for } p\geqslant 2. \tag{2.8}$$

In particular,

$$\widehat{\mathbb{E}}[(S_n^+)^p] \leqslant C_p\bigg\{\sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^p] + \bigg(\sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^2]\bigg)^{p/2}\bigg\}, \quad \textit{for } p\geqslant 2.$$

For the moments under $\widehat{\mathcal{E}}$, we have the following estimates.

Theorem 2.3. *Let $\{X_1,\dots,X_n\}$ be a sequence of random variables in $(\Omega,\mathscr{H},\widehat{\mathbb{E}})$ with $\widehat{\mathcal{E}}[X_k]\leqslant 0$, $k=1,\dots,n$, and $1\leqslant p\leqslant 2$. If X_k is independent to $(X_{k+1},\dots,X_n)$ for each $k=1,\dots,n-1$, then*

$$\widehat{\mathcal{E}}\Big[\Big|\max_{k\leqslant n} S_k\Big|^p\Big] \leqslant 2^{2-p}\sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^p], \quad \textit{for } 1\leqslant p\leqslant 2. \tag{2.9}$$

If X_{k+1} is independent to $(X_1,\dots,X_k)$ for each $k=1,\dots,n-1$, then

$$\widehat{\mathcal{E}}\Big[\Big|\max_{k\leqslant n}(S_n-S_k)\Big|^p\Big] \leqslant 2^{2-p}\sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^p], \quad \textit{for } 1\leqslant p\leqslant 2. \tag{2.10}$$

To prove Theorems 2.1–2.3, we need Hölder's inequality under the sub-linear expectation which can be proved by the same way under the linear expectation due to the properties of the monotonicity and sub-additivity (see [16, Proposition 1.16]).

Lemma 2.4 (Hölder's inequality). *Let $p, q > 1$ be two real numbers satisfying $\frac{1}{p}+\frac{1}{q}=1$. Then for two random variables X, Y in $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$ we have* $\widehat{\mathbb{E}}[|XY|] \leqslant (\widehat{\mathbb{E}}[|X|^p])^{\frac{1}{p}} (\widehat{\mathbb{E}}[|Y|^p])^{\frac{1}{q}}$.

Proof of Theorem 2.1. Let T_k be defined as in the proof of Theorem 1.4.

(a) We first prove (2.1). Substituting $x = X_k$ and $y = T_{k+1}^+$ to the following elementary inequality:

$$|x+y|^p \leqslant 2^{2-p}|x|^p + |y|^p + px|y|^{p-1}\text{sgn}(y), \quad 1 \leqslant p \leqslant 2 \tag{2.11}$$

yields

$$\begin{aligned}\widehat{\mathbb{E}}[|T_k|^p] &\leqslant 2^{2-p}\widehat{\mathbb{E}}[|X_k|^p] + \widehat{\mathbb{E}}[(T_{k+1}^+)^p] + p\widehat{\mathbb{E}}[X_k(T_{k+1}^+)^{p-1}] \\ &\leqslant 2^{2-p}\widehat{\mathbb{E}}[|X_k|^p] + \widehat{\mathbb{E}}[|T_{k+1}|^p]\end{aligned}$$

by the definition of negative dependence and the facts that $\widehat{\mathbb{E}}[X_k] \leqslant 0$, $T_{k+1}^+ \geqslant 0$, and T_{k+1}^+ is a coordinatewise nondecreasing function of $X_{k+1}, \ldots, X_n$. Hence, $\widehat{\mathbb{E}}[|T_1|^p] \leqslant 2^{2-p}\sum_{k=1}^{n-1}\widehat{\mathbb{E}}[|X_k|^p] + \widehat{\mathbb{E}}[|X_n|^p]$. So, (2.1) is proved.

For (2.2), by the following elementary inequality:

$$|x+y|^p \leqslant 2^p p^2 |x|^p + |y|^p + px|y|^{p-1}\text{sgn}(y) + 2^p p^2 x^2 |y|^{p-2}, \quad p \geqslant 2,$$

we have

$$|T_k|^p \leqslant 2^p p^2 |X_k|^p + |T_{k+1}|^p + pX_k(T_{k+1}^+)^{p-1} + 2^p p^2 X_k^2 (T_{k+1}^+)^{p-2}.$$

It follows that

$$|T_i|^p \leqslant 2^p p^2 \sum_{k=i}^{n} |X_k|^p + p\sum_{k=i}^{n-1} X_k(T_{k+1}^+)^{p-1} + 2^p p^2 \sum_{k=i}^{n-1} X_k^2 (T_{k+1}^+)^{p-2}. \tag{2.12}$$

Hence by the definition of the negative dependence and Hölder's inequality,

$$\begin{aligned}\widehat{\mathbb{E}}[|T_i|^p] &\leqslant 2^p p^2 \widehat{\mathbb{E}}\bigg[\sum_{k=i}^{n}|X_k|^p\bigg] + p\sum_{k=i}^{n-1}\widehat{\mathbb{E}}[X_k(T_{k+1}^+)^{p-1}] + 2^p p^2\sum_{k=i}^{n-1}\widehat{\mathbb{E}}[X_k^2(T_{k+1}^+)^{p-2}] \\ &\leqslant 2^p p^2 \widehat{\mathbb{E}}\bigg[\sum_{k=1}^{n}|X_k|^p\bigg] + 2^p p^2 \sum_{k=1}^{n-1}(\widehat{\mathbb{E}}[|X_k|^p])^{\frac{2}{p}}(\widehat{\mathbb{E}}[|T_{k+1}|^p])^{1-\frac{2}{p}}.\end{aligned}$$

Let $A_n = \max_{k\leqslant n}\widehat{\mathbb{E}}[|T_k|^p]$. Then $A_n \leqslant 2^p p^2 \sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^p] + 2^p p^2 \sum_{k=1}^{n-1}(\widehat{\mathbb{E}}[|X_k|^p])^{\frac{2}{p}} A_n^{1-\frac{2}{p}}$. From the above inequalities, it can be shown that

$$A_n \leqslant C_p\bigg\{\sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^p] + \bigg(\sum_{k=1}^{n-1}(\widehat{\mathbb{E}}[|X_k|^p])^{\frac{2}{p}}\bigg)^{\frac{p}{2}}\bigg\} \leqslant C_p n^{p/2-1}\sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^p].$$

(2.2) is proved.

(b) Note the independence. From (2.12) it follows that

$$\begin{aligned}\widehat{\mathbb{E}}[|T_i|^p] &\leqslant 2^p p^2 \widehat{\mathbb{E}}\bigg[\sum_{k=i}^{n}|X_k|^p\bigg] + p\sum_{k=i}^{n-1}\widehat{\mathbb{E}}[X_k(T_{k+1}^+)^{p-1}] + 2^p p^2\sum_{k=i}^{n-1}\widehat{\mathbb{E}}[X_k^2(T_{k+1}^+)^{p-2}] \\ &= 2^p p^2 \widehat{\mathbb{E}}\bigg[\sum_{k=i}^{n}|X_k|^p\bigg] + p\sum_{k=i}^{n-1}\widehat{\mathbb{E}}[X_k]\widehat{\mathbb{E}}[(T_{k+1}^+)^{p-1}] + 2^p p^2\sum_{k=i}^{n-1}\widehat{\mathbb{E}}[X_k^2]\widehat{\mathbb{E}}[(T_{k+1}^+)^{p-2}] \\ &\leqslant 2^p p^2 \widehat{\mathbb{E}}\bigg[\sum_{k=1}^{n}|X_k|^p\bigg] + 2^p p^2 \sum_{k=1}^{n-1}\widehat{\mathbb{E}}[X_k^2](\widehat{\mathbb{E}}[|T_{k+1}|^p])^{1-\frac{2}{p}}.\end{aligned}$$

Let $A_n = \max_{k\leqslant n} \widehat{\mathbb{E}}[|T_k|^p]$. Then

$$A_n \leqslant 2^p p^2 \widehat{\mathbb{E}}\bigg[\sum_{k=1}^n |X_k|^p\bigg] + 2^p p^2 \sum_{k=1}^{n-1} \widehat{\mathbb{E}}[X_k^2] A_n^{1-\frac{2}{p}}.$$

From the above inequality, it can be shown that

$$A_n \leqslant C_p \bigg\{ \sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^p] + \bigg(\sum_{k=1}^n \widehat{\mathbb{E}}[X_k^2]\bigg)^{\frac{p}{2}} \bigg\}.$$

(2.2) is proved. □

(c) We first show the Marcinkiewicz-Zygmund inequality:

$$\widehat{\mathbb{E}}\Big[\max_{k\leqslant n} |S_k|^p\Big] \leqslant C_p \bigg\{ \bigg(\sum_{k=1}^n ((\widehat{\mathbb{E}}[X_k])^+ + (\widehat{\mathcal{E}}[X_k])^-)\bigg)^p + \widehat{\mathbb{E}}\bigg(\sum_{k=1}^n X_k^2\bigg)^{\frac{p}{2}} \bigg\}. \tag{2.13}$$

Without loss of generality, we assume that X_k is negatively dependent to $(X_{k+1}, \dots, X_n)$ for all $k = 1, 2, \dots, n-1$. If X_{k+1} is negatively dependent to $(X_1, \dots, X_k)$ for all $k = 1, 2, \dots, n-1$, then (2.13) will hold with $\max_{k\leqslant n} |S_k|$ being replaced by $\max_{0\leqslant k\leqslant n} |S_n - S_k|$. By noting the fact $\max_{k\leqslant n} |S_k| \leqslant \max_{0\leqslant k\leqslant n} |S_n - S_k| + |S_n| \leqslant 2\max_{0\leqslant k\leqslant n} |S_n - S_k|$, (2.13) also is true.

Write $\widetilde{T}_1 = \max_{k\leqslant n} |S_k|$. It is easily seen that $S_k + T_{k+1}^+ = \max(S_k, S_{k+1}, \dots, S_n) \leqslant T_1$. So, $T_{k+1}^+ \leqslant 2\widetilde{T}_1$. Note (2.12). By the the definition of the negative dependence,

$$\begin{aligned}
\widehat{\mathbb{E}}[X_k (T_{k+1}^+)^{p-1}] &\leqslant \begin{cases} \widehat{\mathbb{E}}[X_k]\widehat{\mathbb{E}}[(T_{k+1}^+)^{p-1}], & \text{if } \ \widehat{\mathbb{E}}[X_k] \geqslant 0 \\ 0, & \text{if } \ \widehat{\mathbb{E}}[X_k] < 0 \end{cases} \\
&\leqslant 2^{p-1}(\widehat{\mathbb{E}}[X_k])^+ \widehat{\mathbb{E}}[\widetilde{T}_1^{p-1}] \leqslant 2^{p-1}(\widehat{\mathbb{E}}[X_k])^+ (\widehat{\mathbb{E}}[\widetilde{T}_1^p])^{1-\frac{1}{p}}
\end{aligned}$$

by Hölder's inequality. By (2.12) and Hölder's inequality again, it follows that

$$\begin{aligned}
\widehat{\mathbb{E}}[|T_1|^p] &\leqslant 2^p p^2 \widehat{\mathbb{E}}\bigg[\sum_{k=1}^n |X_k|^p\bigg] + p\sum_{k=1}^{n-1} \widehat{\mathbb{E}}[X_k (T_{k+1}^+)^{p-1}] + 2^p p^2 \widehat{\mathbb{E}}\bigg[\sum_{k=1}^{n-1} X_k^2 (T_{k+1}^+)^{p-2}\bigg] \\
&\leqslant 2^p p^2 \widehat{\mathbb{E}}\bigg[\sum_{k=1}^n |X_k|^p\bigg] + 2^{p-1} p \sum_{k=1}^{n-1} (\widehat{\mathbb{E}}[X_k])^+ (\widehat{\mathbb{E}}[\widetilde{T}_1^p])^{1-\frac{1}{p}} \\
&\quad + 2^p p^2 2^{p-2} \widehat{\mathbb{E}}\bigg[\sum_{k=1}^{n-1} X_k^2 \widetilde{T}_1^{p-2}\bigg] \\
&\leqslant 2^p p^2 \widehat{\mathbb{E}}\bigg[\sum_{k=1}^n |X_k|^p\bigg] + 2^{p-1} p \bigg(\sum_{k=1}^{n-1} (\widehat{\mathbb{E}}[X_k])^+\bigg) (\widehat{\mathbb{E}}[\widetilde{T}_1^p])^{1-\frac{1}{p}} \\
&\quad + 2^{2p-2} p^2 \bigg[\widehat{\mathbb{E}}\bigg(\sum_{k=1}^{n-1} X_k^2\bigg)^{\frac{p}{2}}\bigg]^{\frac{2}{p}} (\widehat{\mathbb{E}}[\widetilde{T}_1^p])^{1-\frac{2}{p}}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\widehat{\mathbb{E}}\bigg[\Big|\max_{k\leqslant n}(-S_k)\Big|^p\bigg] &\leqslant 2^p p^2 \widehat{\mathbb{E}}\bigg[\sum_{k=1}^n |X_k|^p\bigg] + 2^{p-1} p \bigg(\sum_{k=1}^{n-1} (\widehat{\mathbb{E}}[-X_k])^+\bigg) (\widehat{\mathbb{E}}[\widetilde{T}_1^p])^{1-\frac{1}{p}} \\
&\quad + 2^{2p-2} p^2 \bigg[\widehat{\mathbb{E}}\bigg(\sum_{k=1}^{n-1} X_k^2\bigg)^{\frac{p}{2}}\bigg]^{\frac{2}{p}} (\widehat{\mathbb{E}}[\widetilde{T}_1^p])^{1-\frac{2}{p}}.
\end{aligned}$$

Hence,

$$\widehat{\mathbb{E}}[\widetilde{T}_1^p] \leqslant 2^{p+1} p^2 \widehat{\mathbb{E}}\bigg[\sum_{k=1}^n |X_k|^p\bigg] + 2^{p-1} p \bigg(\sum_{k=1}^n [(\widehat{\mathbb{E}}[X_k])^+ + (\widehat{\mathcal{E}}[X_k])^-]\bigg) (\widehat{\mathbb{E}}[\widetilde{T}_1^p])^{1-\frac{1}{p}}$$

$$+ 2^{2p-1}p^2\bigg[\widehat{\mathbb{E}}\bigg(\sum_{k=1}^{n-1}X_k^2\bigg)^{\frac{p}{2}}\bigg]^{\frac{2}{p}}(\widehat{\mathbb{E}}[\widetilde{T}_1^p])^{1-\frac{2}{p}},$$

which implies

$$\widehat{\mathbb{E}}[\widetilde{T}_1^p] \leqslant C_p\bigg\{\widehat{\mathbb{E}}\bigg[\sum_{k=1}^{n}|X_k|^p\bigg] + \bigg(\sum_{k=1}^{n}((\widehat{\mathbb{E}}[X_k])^+ + (\widehat{\mathcal{E}}[X_k])^-)\bigg)^p + \widehat{\mathbb{E}}\bigg(\sum_{k=1}^{n}X_k^2\bigg)^{\frac{p}{2}}\bigg\}.$$

Note

$$\bigg(\sum_{k=1}^{n}|x_k|^p\bigg)^{\frac{2}{p}} = \bigg(\sum_{k=1}^{n}|x_k^2|^{\frac{p}{2}}\bigg)^{\frac{2}{p}} \leqslant \sum_{k=1}^{n}x_k^2 \quad \text{for} \quad \frac{2}{p} \leqslant 1.$$

So

$$\sum_{k=1}^{n}|X_k|^p \leqslant \bigg(\sum_{k=1}^{n}X_k^2\bigg)^{\frac{p}{2}}.$$

The Marcinkiewicz-Zygmund inequality (2.13) is proved.

Now, for $2 \leqslant p \leqslant 4$, applying (2.1) to the sequences $\{(X_1^+)^2, \ldots, (X_n^+)^2\}$ yields

$$\widehat{\mathbb{E}}\bigg[\bigg(\bigg\{\sum_{k=1}^{n}[(X_k^+)^2 - \widehat{\mathbb{E}}[(X_k^+)^2]]\bigg\}^+\bigg)^{\frac{p}{2}}\bigg]$$
$$\leqslant 2^{2-\frac{p}{2}}\sum_{k=1}^{n}\widehat{\mathbb{E}}[|(X_k^+)^2 - \widehat{\mathbb{E}}[(X_k^+)^2]|^{\frac{p}{2}}] \leqslant C_p\sum_{k=1}^{n}\widehat{\mathbb{E}}[|X_k|^p].$$

It follows that

$$\widehat{\mathbb{E}}\bigg(\sum_{k=1}^{n}(X_k^+)^2\bigg)^{\frac{p}{2}} \leqslant C_p\bigg\{\bigg(\sum_{k=1}^{n}\widehat{\mathbb{E}}[(X_k^+)^2]\bigg)^{\frac{p}{2}} + \sum_{k=1}^{n}\widehat{\mathbb{E}}[|X_k|^p]\bigg\}.$$

Similarly

$$\widehat{\mathbb{E}}\bigg(\sum_{k=1}^{n}(X_k^-)^2\bigg)^{\frac{p}{2}} \leqslant C_p\bigg\{\bigg(\sum_{k=1}^{n}\widehat{\mathbb{E}}[(X_k^-)^2]\bigg)^{\frac{p}{2}} + \sum_{k=1}^{n}\widehat{\mathbb{E}}[|X_k|^p]\bigg\}.$$

Hence,

$$\widehat{\mathbb{E}}\bigg(\sum_{k=1}^{n}X_k^2\bigg)^{\frac{p}{2}} \leqslant C_p\bigg\{\bigg(\sum_{k=1}^{n}\widehat{\mathbb{E}}[X_k^2]\bigg)^{\frac{p}{2}} + \sum_{k=1}^{n}\widehat{\mathbb{E}}[|X_k|^p]\bigg\}.$$

Substituting the above estimate to (2.13) yield (2.3).

Suppose (2.3) is proved for $2^l < p \leqslant 2^{l+1}$. Then applying it to the sequences $\{(X_1^+)^2, \ldots, (X_n^+)^2\}$ and $\{(X_1^-)^2, \ldots, (X_n^-)^2\}$, respectively with $2^l < p/2 \leqslant 2^{l+1}$ yields

$$\widehat{\mathbb{E}}\bigg[\bigg(\sum_{k=1}^{n}(X_k^+)^2\bigg)^{\frac{p}{2}}\bigg] \leqslant C_p\bigg\{\sum_{k=1}^{n}\widehat{\mathbb{E}}[|(X_k^+)^2|^{\frac{p}{2}}] + \bigg(\sum_{k=1}^{n}(\widehat{\mathbb{E}}[(X_k^+)^2])^+\bigg)^{\frac{p}{2}} + \bigg(\sum_{k=1}^{n}\widehat{\mathbb{E}}[[(X_k^+)^2]^2]\bigg)^{\frac{p}{4}}\bigg\}$$
$$\leqslant C_p\bigg\{\sum_{k=1}^{n}\widehat{\mathbb{E}}[|X_k|^p] + \bigg(\sum_{k=1}^{n}\widehat{\mathbb{E}}[X_k^2]\bigg)^{\frac{p}{2}} + \bigg(\sum_{k=1}^{n}\widehat{\mathbb{E}}[X_k^4]\bigg)^{\frac{p}{4}}\bigg\}$$

and

$$\widehat{\mathbb{E}}\bigg[\bigg(\sum_{k=1}^{n}(X_k^-)^2\bigg)^{\frac{p}{2}}\bigg] \leqslant C_p\bigg\{\sum_{k=1}^{n}\widehat{\mathbb{E}}[|X_k|^p] + \bigg(\sum_{k=1}^{n}\widehat{\mathbb{E}}[X_k^2]\bigg)^{\frac{p}{2}} + \bigg(\sum_{k=1}^{n}\widehat{\mathbb{E}}[X_k^4]\bigg)^{\frac{p}{4}}\bigg\}.$$

Hence,

$$\widehat{\mathbb{E}}\bigg[\bigg(\sum_{k=1}^n X_k^2\bigg)^{\frac{p}{2}}\bigg] \leqslant C_p\bigg\{\sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^p] + \bigg(\sum_{k=1}^n \widehat{\mathbb{E}}[X_k^2]\bigg)^{\frac{p}{2}} + \bigg(\sum_{k=1}^n \widehat{\mathbb{E}}[X_k^4]\bigg)^{\frac{p}{4}}\bigg\}. \tag{2.14}$$

By applying Hölder's inequality, it follows that

$$\widehat{\mathbb{E}}[X_k^4] = \widehat{\mathbb{E}}[(X_k^2)^{\frac{p-4}{p-2}}(|X_k|^p)^{\frac{2}{p-2}}] \leqslant (\widehat{\mathbb{E}}[X_k^2])^{\frac{p-4}{p-2}}(\widehat{\mathbb{E}}[|X_k|^p])^{\frac{2}{p-2}},$$

which implies

$$\bigg(\sum_{k=1}^n \widehat{\mathbb{E}}[X_k^4]\bigg)^{p/4} \leqslant C_p\bigg\{\sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^p] + \bigg(\sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^2]\bigg)^{p/2}\bigg\} \tag{2.15}$$

by some elementary calculation. Substituting (2.14) and (2.15) to (2.13), we conclude that (2.3) is also valid for $2^{l+1} < p \leqslant 2^{l+2}$. By the induction, (2.3) proved. □

Proof of Theorem 2.3. Suppose that X_k is independent to $(X_{k+1},\dots,X_n)$ for each $k=1,\dots,n-1$. Due to (2.11), we have $|T_k|^p \leqslant 2^{2-p}|X_k|^p + (T_{k+1}^+)^p + pX_k(T_{k+1}^+)^{p-1}$. By the independence and the fact that $\widehat{\mathcal{E}}[X+Y] \leqslant \widehat{\mathcal{E}}[X] + \widehat{\mathbb{E}}[Y]$, it follows that

$$\begin{aligned}&\widehat{\mathcal{E}}[2^{2-p}|X_k|^p + (T_{k+1}^+)^p + pX_k(T_{k+1}^+)^{p-1} \mid T_{k+1}^+]\\ &\quad\leqslant 2^{2-p}\widehat{\mathbb{E}}[|X_k|^p] + (T_{k+1}^+)^p + p\widehat{\mathcal{E}}[X_k](T_{k+1}^+)^{p-1} \leqslant 2^{2-p}\widehat{\mathbb{E}}[|X_k|^p] + (T_{k+1}^+)^p.\end{aligned}$$

So

$$\widehat{\mathcal{E}}[|T_k|^p] \leqslant \widehat{\mathcal{E}}[2^{2-p}|X_k|^p + (T_{k+1}^+)^p + pX_k(T_{k+1}^+)^{p-1}] \leqslant 2^{2-p}\widehat{\mathbb{E}}[|X_k|^p] + \widehat{\mathcal{E}}[|T_{k+1}|^p].$$

It follows that $\widehat{\mathcal{E}}[|T_1|^p] \leqslant 2^{2-p}\sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^p]$. Now, (2.9) is proved. (2.10) follows from (2.9) by considering the sequence $\{X_1, X_2, \dots, X_n\}$ in the reverse order as $\{X_n, X_{n-1}, \dots, X_1\}$. □

3 Strong laws of large numbers under capacities

Let $\mathcal{G} \subset \mathcal{F}$. A function $V : \mathcal{G} \to [0,1]$ is called a capacity if

$$V(\emptyset) = 0, \quad V(\Omega) = 1 \quad \text{and} \quad V(A) \leqslant V(B), \quad \forall\, A \subset B, \quad A, B \in \mathcal{G}.$$

It is called to be sub-additive if $V(A\cup B) \leqslant V(A) + V(B)$ for all $A, B \in \mathcal{G}$ with $A \cup B \in \mathcal{G}$.

Here we only consider the capacities generated by a sub-linear expectation. Let $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$ be a sub-linear space, and $\widehat{\mathcal{E}}$ be the conjugate expectation of $\widehat{\mathbb{E}}$. Furthermore, let us denote a pair $(\mathbb{V}, \mathcal{V})$ of capacities by

$$\mathbb{V}(A) := \inf\{\widehat{\mathbb{E}}[\xi] : I_A \leqslant \xi, \xi \in \mathscr{H}\}, \quad \mathcal{V}(A) := 1 - \mathbb{V}(A^c), \quad \forall\, A \in \mathcal{F},$$

where A^c is the complement set of A. Then

$$\begin{aligned}&\mathbb{V}(A) := \widehat{\mathbb{E}}[I_A], \quad \mathcal{V}(A) := \widehat{\mathcal{E}}[I_A], \quad \text{if} \quad I_A \in \mathscr{H},\\ &\widehat{\mathbb{E}}[f] \leqslant \mathbb{V}(A) \leqslant \widehat{\mathbb{E}}[g], \quad \widehat{\mathcal{E}}[f] \leqslant \mathcal{V}(A) \leqslant \widehat{\mathcal{E}}[g], \quad \text{if} \quad f \leqslant I_A \leqslant g, \quad f, g \in \mathscr{H}.\end{aligned} \tag{3.1}$$

The corresponding Choquet integrals/expecations $(C_{\mathbb{V}}, C_{\mathcal{V}})$ are defined by

$$C_V[X] = \int_0^\infty V(X \geqslant t)dt + \int_{-\infty}^0 [V(X \geqslant t) - 1]\, dt$$

with V being replaced by $\mathbb{V}$ and $\mathcal{V}$, respectively.

Definition 3.1. (I) A sub-linear expectation $\widehat{\mathbb{E}}:\mathscr{H}\to\mathbb{R}$ is called to be countably sub-additive if it satisfies

(e) Countable sub-additivity: $\widehat{\mathbb{E}}[X]\leqslant\sum_{n=1}^{\infty}\widehat{\mathbb{E}}[X_n]$, whenever $X\leqslant\sum_{n=1}^{\infty}X_n$, $X,X_n\in\mathscr{H}$ and $X\geqslant 0$, $X_n\geqslant 0$, $n=1,2,\dots$

It is called continuous if it satisfies

(f) Continuity from below: $\widehat{\mathbb{E}}[X_n]\uparrow\widehat{\mathbb{E}}[X]$ if $0\leqslant X_n\uparrow X$, where $X_n,X\in\mathscr{H}$.

(g) Continuity from above: $\widehat{\mathbb{E}}[X_n]\downarrow\widehat{\mathbb{E}}[X]$ if $0\leqslant X_n\downarrow X$, where $X_n,X\in\mathscr{H}$.

(II) A function $V:\mathcal{F}\to[0,1]$ is called to be countably sub-additive if

$$V\bigg(\bigcup_{n=1}^{\infty}A_n\bigg)\leqslant\sum_{n=1}^{\infty}V(A_n),\quad\forall\, A_n\in\mathcal{F}.$$

(III) A capacity $V:\mathcal{F}\to[0,1]$ is called a continuous capacity if it satisfies:

(III1) Continuity from below: $V(A_n)\uparrow V(A)$ if $A_n\uparrow A$, where $A_n,A\in\mathcal{F}$.

(III2) Continuity from above: $V(A_n)\downarrow V(A)$ if $A_n\downarrow A$, where $A_n,A\in\mathcal{F}$.

Example 1.6 (Continued)**.** The sub-linear expectation $\widehat{\mathbb{E}}$ defined in Example 1.6 is continuous from below, and so is countably sub-additive. If $\mathscr{H}$ is the set of all random variables and $\mathcal{P}$ is a weakly compact set of probability measures defined on $(\Omega,\mathcal{F})$, then $(\mathbb{V},\mathcal{V})$ is a pair of continuous capacities.

Definition 3.2. Let $\{X_n;n\geqslant 1\}$ be a sequence of random variables in the sub-linear expectation space $(\Omega,\mathscr{H},\widehat{\mathbb{E}})$. $X_1,X_2,\dots$ are said to be independent if X_{i+1} is independent to $(X_1,\dots,X_i)$ for each $i\geqslant 1$, they are said to be negatively dependent if X_{i+1} is negatively dependent to $(X_1,\dots,X_i)$ for each $i\geqslant 1$, and they are said to be identically distributed if $X_i\overset{d}{=}X_1$ for each $i\geqslant 1$.

It is obvious that, if $\{X_n;n\geqslant 1\}$ is a sequence of independent random variables and $f_1(x),f_2(x),\dots\in C_{l,\mathrm{Lip}}(\mathbb{R})$, then $\{f_n(X_n);n\geqslant 1\}$ is also a sequence of independent random variables; if $\{X_n;n\geqslant 1\}$ is a sequence of negatively dependent random variables and $f_1(x),f_2(x),\dots\in C_{l,\mathrm{Lip}}(\mathbb{R})$ are non-decreasing (resp. non-increasing) functions, then $\{f_n(X_n);n\geqslant 1\}$ is also a sequence of negatively dependent random variables.

For a sequence $\{X_n;n\geqslant 1\}$ of random variables in the sub-linear expectation space $(\Omega,\mathscr{H},\widehat{\mathbb{E}})$, we denote $S_n=\sum_{k=1}^{n}X_k$, $S_0=0$. The main purpose of this section is to establish the following Kolmogorov type strong laws of large numbers.

Theorem 3.3. (a) *Let $\{X_n;n\geqslant 1\}$ be a sequence of negatively dependent and identically distributed random variables. Suppose that $\mathbb{V}$ is countably sub-additive, $C_{\mathbb{V}}[|X_1|]<\infty$ and $\lim_{c\to\infty}\widehat{\mathbb{E}}[(|X_1|-c)^+]=0$. Then*

$$\mathbb{V}\bigg(\bigg\{\liminf_{n\to\infty}\frac{S_n}{n}<\widehat{\mathcal{E}}[X_1]\bigg\}\cup\bigg\{\limsup_{n\to\infty}\frac{S_n}{n}>\widehat{\mathbb{E}}[X_1]\bigg\}\bigg)=0. \tag{3.2}$$

(b) *Suppose that $\{X_n;n\geqslant 1\}$ is a sequence of independent and identically distributed random variables, and $\mathbb{V}$ is continuous. If*

$$\mathbb{V}\bigg(\limsup_{n\to\infty}\frac{|S_n|}{n}=+\infty\bigg)<1, \tag{3.3}$$

then $C_{\mathbb{V}}[|X_1|]<\infty$.

(c) *Suppose that $\{X_n;n\geqslant 1\}$ is a sequence of independent and identically distributed random variables with $C_{\mathbb{V}}[|X_1|]<\infty$ and $\lim_{c\to\infty}\widehat{\mathbb{E}}[(|X_1|-c)^+]=0$. If $\mathbb{V}$ is continuous, then*

$$\mathbb{V}\bigg(\liminf_{n\to\infty}\frac{S_n}{n}=\widehat{\mathcal{E}}[X_1]\quad\textit{and}\quad\limsup_{n\to\infty}\frac{S_n}{n}=\widehat{\mathbb{E}}[X_1]\bigg)=1 \tag{3.4}$$

and

$$\mathbb{V}\bigg(C\bigg\{\frac{S_n}{n}\bigg\}=[\widehat{\mathcal{E}}[X_1],\widehat{\mathbb{E}}[X_1]]\bigg)=1, \tag{3.5}$$

where $C(\{x_n\})$ denotes the cluster set of a sequence of $\{x_n\}$ in $\mathbb{R}$.

The following corollary follows from Theorem 3.3 immediately.

Corollary 3.4. *Suppose that $\mathscr{H}$ is a monotone class in the sense that $X \in \mathscr{H}$ whenever $\mathscr{H} \ni X_n \downarrow X \geqslant 0$. Assume that $\widehat{\mathbb{E}}$ is continuous. Let $\{X_n; n \geqslant 1\}$ be a sequence of independent and identically distributed random variables in $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$. Then* (3.3) $\Rightarrow C_{\mathbb{V}}[|X_1|] < \infty \Rightarrow$ (3.2).

Because $\mathbb{V}$ may be not countably sub-additive in general, we define an outer capacity $\mathbb{V}^*$ by

$$\mathbb{V}^*(A) = \inf\left\{\sum_{n=1}^{\infty} \mathbb{V}(A_n) : A \subset \bigcup_{n=1}^{\infty} A_n\right\}, \quad \mathcal{V}^*(A) = 1 - \mathbb{V}^*(A^c), \quad A \in \mathcal{F}.$$

Then it can be shown that $\mathbb{V}^*(A)$ is a countably sub-additive capacity with $\mathbb{V}^*(A) \leqslant \mathbb{V}(A)$ and the following properties:

(a*) If $\mathbb{V}$ is countably sub-additive, then $\mathbb{V}^* \equiv \mathbb{V}$.

(b*) If $I_A \leqslant g$, $g \in \mathscr{H}$, then $\mathbb{V}^*(A) \leqslant \widehat{\mathbb{E}}[g]$. Furthermore, if $\widehat{\mathbb{E}}$ is countably sub-additive, then

$$\widehat{\mathbb{E}}[f] \leqslant \mathbb{V}^*(A) \leqslant \mathbb{V}(A) \leqslant \widehat{\mathbb{E}}[g], \quad \forall\, f \leqslant I_A \leqslant g, \quad f, g \in \mathscr{H}. \tag{3.6}$$

(c*) $\mathbb{V}^*$ is the largest countably sub-additive capacity satisfying the property that $\mathbb{V}^*(A) \leqslant \widehat{\mathbb{E}}[g]$ whenever $I_A \leqslant g \in \mathscr{H}$, i.e., if V is also a countably sub-additive capacity satisfying $V(A) \leqslant \widehat{\mathbb{E}}[g]$ whenever $I_A \leqslant g \in \mathscr{H}$, then $V(A) \leqslant \mathbb{V}^*(A)$.

In fact, it is obvious that (c*) implies (a*). For (b*) and (c*), suppose $A \subset \bigcup_{n=1}^{\infty} A_n$, $\sum_{n=1}^{\infty} \mathbb{V}(A_n) \leqslant \mathbb{V}^*(A) + \epsilon/2$ with $I_{A_n} \leqslant f_n \in \mathscr{H}$ and $\widehat{\mathbb{E}}[f_n] \leqslant \mathbb{V}(A_n) + \epsilon/2^{n+2}$. If $\mathscr{H} \ni f \leqslant I_A$, then $f \leqslant \sum_{n=1}^{\infty} I_{A_n} \leqslant \sum_{n=1}^{\infty} f_n$, which implies

$$\widehat{\mathbb{E}}[f] \leqslant \sum_{n=1}^{\infty} \widehat{\mathbb{E}}[f_n] \leqslant \sum_{n=1}^{\infty} \mathbb{V}(A_n) + \sum_{n=1}^{\infty} \epsilon/2^{n+2} \leqslant \mathbb{V}^*(A) + \epsilon$$

by the countable sub-additivity of $\widehat{\mathbb{E}}$. While, if V is countably sub-additive, then

$$V(A) \leqslant \sum_{n=1}^{\infty} V(A_n) \leqslant \sum_{n=1}^{\infty} \widehat{\mathbb{E}}[f_n] \leqslant \sum_{n=1}^{\infty} \mathbb{V}(A_n) + \sum_{n=1}^{\infty} \epsilon/2^{n+2} \leqslant \mathbb{V}^*(A) + \epsilon.$$

Theorem 3.5. *Let $\{X_n; n \geqslant 1\}$ be a sequence identically distributed random variables in $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$.*

(a) *Suppose that $X_1, X_2, \ldots$ are negatively dependent with $C_{\mathbb{V}}[|X_1|] < \infty$ and $\lim_{c\to\infty} \widehat{\mathbb{E}}[(|X_1| - c)^+] = 0$. Then*

$$\mathbb{V}^*\left(\left\{\liminf_{n\to\infty} \frac{S_n}{n} < \widehat{\mathcal{E}}[X_1]\right\} \cup \left\{\limsup_{n\to\infty} \frac{S_n}{n} > \widehat{\mathbb{E}}[X_1]\right\}\right) = 0. \tag{3.7}$$

(b) *Suppose that $X_1, X_2, \ldots$ are independent, $\mathbb{V}^*$ is continuous and $\widehat{\mathbb{E}}$ is countably sub-additive. If*

$$\mathbb{V}^*\left(\limsup_{n\to\infty} \frac{|S_n|}{n} = +\infty\right) < 1, \tag{3.8}$$

then $C_{\mathbb{V}}[|X_1|] < \infty$.

For proving the theorems, we need some properties of the sub-linear expectations and capacities. We define an extension of $\widehat{\mathbb{E}}$ on the space of all random variables by $\mathbb{E}^*[X] = \inf\{\widehat{\mathbb{E}}[Y] : X \leqslant Y, Y \in \mathscr{H}\}$. Then $\mathbb{E}^*$ is a sub-linear expectation on the space of all random variables, and

$$\mathbb{E}^*[X] = \widehat{\mathbb{E}}[X], \quad \forall\, X \in \mathscr{H}, \quad \mathbb{V}(A) = \mathbb{E}^*[I_A], \quad \forall\, A \in \mathcal{F}.$$

We have the following properties.

Lemma 3.6. (P1) *If $\widehat{\mathbb{E}}$ is continuous from below, then it is countably sub-additive. Similarly, if $\mathbb{V}$ is continuous from below, then it is countably sub-additive.*

(P2) *If $\mathbb{V}$ is continuous from above, then $\mathbb{V}$ and $\mathcal{V}$ are continuous.*

(P3) *If $\widehat{\mathbb{E}}$ is continuous from above, then $\widehat{\mathbb{E}}$ is continuous from below controlled, i.e., $\widehat{\mathbb{E}}[X_n] \uparrow \widehat{\mathbb{E}}[X]$ if $0 \leqslant X_n \uparrow X$, where $X_n, X \in \mathscr{H}$ and $\widehat{\mathbb{E}}X < \infty$.*

(P4) *Suppose that* $\widehat{\mathbb{E}}$ *is countably sub-additive. If* $X \leqslant \sum_{n=1}^{\infty} X_n$, $X, X_n \geqslant 0$ *and* $X \in \mathscr{H}$, *then* $\widehat{\mathbb{E}}[X] \leqslant \sum_{n=1}^{\infty} \mathbb{E}^*[X_n]$.

(P5) *Set* $\mathcal{H} = \{A : I_A \in \mathscr{H}\}$, *then* $\mathbb{V}$ *is a countably sub-additive capacity in* $\mathcal{H}$ *if* $\widehat{\mathbb{E}}$ *is countably sub-additive in* $\mathscr{H}$, *and* $(\mathbb{V}, \mathcal{V})$ *is a pair of continuous capacities in* $\mathcal{H}$ *if* $\widehat{\mathbb{E}}$ *is continuous in* $\mathscr{H}$.

Proof. For (P1), if $0 \leqslant X \leqslant \sum_{k=1}^{\infty} X_n$, $0 \leqslant X, X_n \in \mathscr{H}$, then

$$\widehat{\mathbb{E}}[X] = \widehat{\mathbb{E}}\bigg[\bigg(\sum_{k=1}^{\infty} X_k\bigg) \wedge X\bigg] = \lim_{n\to\infty} \widehat{\mathbb{E}}\bigg[\bigg(\sum_{k=1}^{n} X_k\bigg) \wedge X\bigg]$$
$$\leqslant \lim_{n\to\infty} \widehat{\mathbb{E}}\bigg[\sum_{k=1}^{n} X_k\bigg] \leqslant \lim_{n\to\infty} \sum_{k=1}^{n} \widehat{\mathbb{E}}[X_k] \leqslant \sum_{k=1}^{\infty} \widehat{\mathbb{E}}[X_k].$$

(P1) is proved.

For (P2), it is sufficient to note that, if $A_n \uparrow A$, then $A \backslash A_n \downarrow \emptyset$ and $0 \leqslant \mathbb{V}(A) - \mathbb{V}(A_n) \leqslant \mathbb{V}(A \backslash A_n)$. Similarly, for (P3), it is sufficient to note that $X - X_n \downarrow 0$ and $0 \leqslant \widehat{\mathbb{E}}[X] - \widehat{\mathbb{E}}[X_n] \leqslant \widehat{\mathbb{E}}[X - X_n]$.

For (P4), choose $0 \leqslant Y_n \in \mathscr{H}$ such that $Y_n \geqslant X_n$, $\widehat{\mathbb{E}}[Y_n] \leqslant \mathbb{E}^*[X_n] + \frac{\epsilon}{2^{n+1}}$. Then $X \leqslant \sum_{n=1}^{\infty} Y_n$. By the countable sub-additivity of $\widehat{\mathbb{E}}$,

$$\widehat{\mathbb{E}}[X] \leqslant \sum_{n=1}^{\infty} \widehat{\mathbb{E}}[Y_n] \leqslant \sum_{n=1}^{\infty} \bigg(\mathbb{E}^*[X_n] + \frac{\epsilon}{2^{n+1}}\bigg) \leqslant \sum_{n=1}^{\infty} \mathbb{E}^*[X_n] + \epsilon.$$

(P4) is proved. (P5) is obvious. □

The following is the "the convergence part" of the Borel-Cantelli lemma for a countably sub-additive capacity.

Lemma 3.7 (Borel-Cantelli's lemma). *Let* $\{A_n, n \geqslant 1\}$ *be a sequence of events in* $\mathcal{F}$. *Suppose that* V *is a countably sub-additive capacity. If* $\sum_{n=1}^{\infty} V(A_n) < \infty$, *then*

$$V(A_n \ i.o.) = 0, \quad where \quad \{A_n \ i.o.\} = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i.$$

Proof. By the monotonicity and countable sub-additivity, it follows that

$$0 \leqslant V\bigg(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i\bigg) \leqslant V\bigg(\bigcup_{i=n}^{\infty} A_i\bigg) \leqslant \sum_{i=n}^{\infty} V(A_i) \to 0 \quad \text{as} \quad n \to \infty.$$ □

Remark 3.8. It is important to note that the condition that "X is independent to Y under $\widehat{\mathbb{E}}$" does not imply that "X is independent to Y under $\mathbb{V}$" because the indicator functions $I\{X \in A\}$ and $I\{X \in A\}$ are not in $C_{l,\mathrm{Lip}}(\mathbb{R})$, and also, "$X$ is independent to Y under $\mathbb{V}$" does not imply that "X is independent to Y under $\widehat{\mathbb{E}}$" because $\widehat{\mathbb{E}}$ is not an integral with respect to $\mathbb{V}$. So, we do not have "the divergence part" of the Borel-Cantelli lemma.

Similarly, the conditions that "X and Y are identically distributed under $\widehat{\mathbb{E}}$" and that "X and Y are identically distributed under $\mathbb{V}$" do not imply each other.

Lemma 3.9. *Suppose that* $X \in \mathscr{H}$ *and* $C_{\mathbb{V}}(|X|) < \infty$.

(a) *Then*

$$\sum_{j=1}^{\infty} \frac{\widehat{\mathbb{E}}[(|X| \wedge j)^2]}{j^2} < \infty. \tag{3.9}$$

(b) *Furthermore, if* $\lim_{c\to\infty} \widehat{\mathbb{E}}[|X| \wedge c] = \widehat{\mathbb{E}}[|X|]$, *then*

$$\widehat{\mathbb{E}}[|X|] \leqslant C_{\mathbb{V}}(|X|). \tag{3.10}$$

(c) *If* $\widehat{\mathbb{E}}$ *is countably sub-additive, then*

$$\widehat{\mathbb{E}}[|Y|] \leqslant C_{\mathbb{V}}(|Y|), \quad \forall Y \in \mathscr{H} \tag{3.11}$$

and

$$\lim_{c\to\infty}\widehat{\mathbb{E}}[(|X|-c)^+]=0,\quad \lim_{c\to\infty}\widehat{\mathbb{E}}[|X|\wedge c]=\widehat{\mathbb{E}}[|X|] \tag{3.12}$$

whenever $C_{\mathbb{V}}(|X|)<\infty$.

Proof. (a) Note

$$\begin{aligned}(|X|\wedge j)^2&=\sum_{i=1}^j|X|^2I\{i-1<|X|\leqslant i\}+jI\{|X|>j\}\\&\leqslant\sum_{i=1}^j i^2I\{i-1<|X|\leqslant i\}+jI\{|X|>j\}\\&=\sum_{i=0}^{j-1}(i+1)^2I\{|X|>i\}-\sum_{i=1}^j i^2I\{|X|>i\}+jI\{|X|>j\}\\&\leqslant 1+\sum_{i=1}^{j-1}(2i+1)I\{|X|>i\}+jI\{|X|>j\}\\&\leqslant 1+3\sum_{i=1}^j iI\{|X|>i\}.\end{aligned}$$

So, $\widehat{\mathbb{E}}[(|X|\wedge j)^2]=\mathbb{E}^*[(|X|\wedge j)^2]\leqslant 1+3\sum_{i=1}^j i\mathbb{V}(|X|>i)$, by the (finite) sub-additivity of $\mathbb{E}^*$. It follows that

$$\begin{aligned}\sum_{j=1}^\infty\frac{\widehat{\mathbb{E}}[(|X|\wedge j)^2]}{j^2}&\leqslant\sum_{j=1}^\infty\frac{1+3\sum_{i=1}^j i\mathbb{V}(|X|>i)}{j^2}\\&\leqslant 2+3\sum_{i=1}^\infty i\mathbb{V}(|X|>i)\sum_{j=i+1}^\infty\frac{1}{j^2}\leqslant 2+3\sum_{i=1}^\infty\mathbb{V}(|X|>i)\leqslant 2+3C_{\mathbb{V}}(|X|).\end{aligned}$$

(3.9) is proved.

(b) For $n>2$, note

$$\begin{aligned}|X|\wedge n&=\sum_{i=1}^n|X|I\{i-1<|X|\leqslant i\}+nI\{|X|>n\}\\&\leqslant\sum_{i=1}^n i(I\{|X|>i-1\}-I\{|X|>i\})+nI\{|X|>n\}\leqslant 1+\sum_{i=1}^n I\{|X|>i\}.\end{aligned}$$

It follows that $\widehat{\mathbb{E}}[|X|\wedge n]=\mathbb{E}^*[|X|\wedge n]\leqslant 1+\sum_{i=1}^n\mathbb{V}(|X|\geqslant i)\leqslant 1+\int_0^n\mathbb{V}(|X|\geqslant x)dx$. Taking $n\to\infty$ yields $\widehat{\mathbb{E}}[|X|]=\lim_{n\to\infty}\widehat{\mathbb{E}}[|X|\wedge n]\leqslant 1+C_{\mathbb{V}}(|X|)$. By considering $|X|/\epsilon$ instead of $|X|$, we have

$$\widehat{\mathbb{E}}\left[\frac{|X|}{\epsilon}\right]\leqslant 1+C_{\mathbb{V}}\left(\frac{|X|}{\epsilon}\right)=1+\frac{1}{\epsilon}C_{\mathbb{V}}(|X|),$$

i.e., $\widehat{\mathbb{E}}[|X|]\leqslant\epsilon+C_{\mathbb{V}}(|X|)$. Taking $\epsilon\to 0$ yields (3.10).

(c) Now, from the fact that $|Y|\leqslant 1+\sum_{i=1}^\infty I\{|Y|\geqslant i\}$, by the countable sub-additivity of $\widehat{\mathbb{E}}$ and Property (P4) in Lemma 3.6, it follows that

$$\widehat{\mathbb{E}}[|Y|]\leqslant 1+\sum_{i=1}^\infty\mathbb{E}^*[I\{|Y|\geqslant i\}]=1+\sum_{i=1}^\infty\mathbb{V}(|Y|\geqslant i)\leqslant 1+C_{\mathbb{V}}(|Y|).$$

Then (3.11) is proved by the same argument in (b) above.

Letting $Y=(|X|-c)^+$ in (3.11) yields

$$\widehat{\mathbb{E}}[(|X|-c)^+]\leqslant C_{\mathbb{V}}((|X|-c)^+)=\int_c^\infty\mathbb{V}(|X|\geqslant x)dx\to 0\quad\text{as}\quad c\to\infty.$$

So

$$0 \leqslant \widehat{\mathbb{E}}[|X|] - \widehat{\mathbb{E}}[|X| \wedge c] \leqslant \widehat{\mathbb{E}}[(|X| - c)^+] \to 0 \quad \text{as} \quad c \to \infty.$$

(3.12) is proved. □

Proof of Theorems 3.3 *and* 3.5. We first prove Theorem 3.5(a). Theorem 3.3(a) follows from Theorem 3.5(a) because $\mathbb{V}^* = \mathbb{V}$ when $\mathbb{V}$ is countably sub-additive.

Without loss of generality, we assume $\widehat{\mathbb{E}}[X_1] = 0$. Define

$$f_c(x) = (-c) \vee (x \wedge c), \quad \widehat{f}_c(x) = x - f_c(x) \tag{3.13}$$

and

$$\overline{X}_j = f_j(X_j) - \widehat{\mathbb{E}}[f_j(X_j)], \quad \overline{S}_j = \sum_{i=1}^{j} \overline{X}_i, \quad j = 1, 2, \ldots$$

Then $f_c(\cdot), \widehat{f}_c(\cdot) \in C_{l,\mathrm{Lip}}(\mathbb{R})$, and $\overline{X}_j$, $j = 1, 2, \ldots$ are negatively dependent. Let $\theta > 1$, $n_k = [\theta^k]$. For $n_k < n \leqslant n_{k+1}$, we have

$$\begin{aligned}
\frac{S_n}{n} &= \frac{1}{n}\left\{\overline{S}_{n_{k+1}} + \sum_{j=1}^{n_{k+1}} \widehat{\mathbb{E}}[f_j(X_j)] + \sum_{j=1}^{n} \widehat{f}_j(X_j) - \sum_{j=n+1}^{n_{k+1}} f_j(X_j)\right] \\
&\leqslant \frac{\overline{S}^{+}_{n_{k+1}}}{n_k} + \frac{\sum_{j=1}^{n_{k+1}} |\widehat{\mathbb{E}}[f_j(X_1)]|}{n_k} + \frac{\sum_{j=1}^{n_{k+1}} |\widehat{f}_j(X_j)|}{n_k} \\
&\quad + \frac{\sum_{j=n_k+1}^{n_{k+1}} \{f_j^+(X_j) - \widehat{\mathbb{E}}[f_j^+(X_j)]\}}{n_k} + \frac{\sum_{j=n_k+1}^{n_{k+1}} \{f_j^-(X_j) - \widehat{\mathbb{E}}[f_j^-(X_j)]\}}{n_k} \\
&\quad + \frac{(n_{k+1} - n_k)\widehat{\mathbb{E}}|X_1|}{n_k} \\
&=: (I)_k + (II)_k + (III)_k + (IV)_k + (V)_k + (VI)_k.
\end{aligned}$$

It is obvious that $\lim_{k\to\infty}(VI)_k = (\theta - 1)\widehat{\mathbb{E}}[|X_1|] \leqslant (\theta - 1)C_{\mathbb{V}}(|X_1|)$ by Lemma 3.9(b).

For $(I)_k$, applying (2.7) yields

$$\begin{aligned}
\mathbb{V}(\overline{S}_{n_{k+1}} \geqslant \epsilon n_k) &\leqslant \frac{\sum_{j=1}^{n_{k+1}} \widehat{\mathbb{E}}[\overline{X}_j^2]}{\epsilon^2 n_k^2} \leqslant \frac{4\sum_{j=1}^{n_{k+1}} \widehat{\mathbb{E}}[f_j^2(X_1)]}{\epsilon^2 n_k^2} \\
&\leqslant \frac{4n_{k+1}}{\epsilon^2 n_k^2} + \frac{4\sum_{j=1}^{n_{k+1}} \widehat{\mathbb{E}}[(|X_1| \wedge j)^2]}{\epsilon^2 n_k^2}.
\end{aligned}$$

It is obvious that $\sum_k \frac{n_{k+1}}{n_k^2} < \infty$. Also,

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{\sum_{j=1}^{n_{k+1}} \widehat{\mathbb{E}}[(|X_1| \wedge j)^2]}{n_k^2} &\leqslant \sum_{j=1}^{\infty} \widehat{\mathbb{E}}[(|X_1| \wedge j)^2] \sum_{k: n_{k+1} \geqslant j} \frac{1}{n_k^2} \\
&\leqslant C\sum_{j=1}^{\infty} \widehat{\mathbb{E}}[(|X_1| \wedge j)^2]\frac{1}{j} < \infty
\end{aligned}$$

by Lemma 3.9(a). Hence,

$$\sum_{k=1}^{\infty} \mathbb{V}^*((I)_k \geqslant \epsilon) \leqslant \sum_{k=1}^{\infty} \mathbb{V}((I)_k \geqslant \epsilon) < \infty.$$

By the Borel-Cantelli's lemma and the countable sub-additivity of $\mathbb{V}^*$, it follows that

$$\mathbb{V}^*\left(\limsup_{k\to\infty}(I)_k > \epsilon\right) = 0, \quad \forall\, \epsilon > 0.$$

Similarly,

$$\mathbb{V}^*\Big(\limsup_{k\to\infty}(IV)_k > \epsilon\Big) = 0, \quad \mathbb{V}^*\Big(\limsup_{k\to\infty}(V)_k > \epsilon\Big) = 0, \quad \forall\, \epsilon > 0.$$

For $(II)_k$, note that by the (finite) sub-additivity,

$$|\widehat{\mathbb{E}}[f_j(X_1)]| = |\widehat{\mathbb{E}}[f_j(X_1)] - \widehat{\mathbb{E}}X_1| \leqslant \widehat{\mathbb{E}}[|\widehat{f}_j(X_1)|] = \widehat{\mathbb{E}}[(|X_1| - j)^+] \to 0.$$

It follows that

$$(II)_k = \frac{n_{k+1}}{n_k} \frac{\sum_{j=1}^{n_{k+1}} |\widehat{\mathbb{E}}[f_j(X_1)]|}{n_{k+1}} \to 0.$$

At last, we consider $(III)_k$. By the Borel-Cantelli's lemma, we will have

$$\mathbb{V}^*\Big(\limsup_{k\to\infty}(III)_k > 0\Big) \leqslant \mathbb{V}^*(\{|X_j| > j\} \text{ i.o.}) = 0$$

if we have shown that

$$\sum_{j=1}^{\infty} \mathbb{V}^*(|X_j| > j) \leqslant \sum_{j=1}^{\infty} \mathbb{V}(|X_j| > j) < \infty. \tag{3.14}$$

Let g_ϵ be a function satisfying that its derivatives of each order are bounded, $g_\epsilon(x) = 1$ if $x \geqslant 1$, $g_\epsilon(x) = 0$ if $x \leqslant 1 - \epsilon$, and $0 \leqslant g_\epsilon(x) \leqslant 1$ for all x, where $0 < \epsilon < 1$. Then

$$g_\epsilon(\cdot) \in C_{l,\mathrm{Lip}}(\mathbb{R}) \quad \text{and} \quad I\{x \geqslant 1\} \leqslant g_\epsilon(x) \leqslant I\{x > 1 - \epsilon\}.$$

Hence, by (3.1),

$$\begin{aligned}\sum_{j=1}^{\infty} \mathbb{V}(|X_j| > j) &\leqslant \sum_{j=1}^{\infty} \widehat{\mathbb{E}}[g_{1/2}(|X_j|/j)] = \sum_{j=1}^{\infty} \widehat{\mathbb{E}}[g_{1/2}(|X_1|/j)] \quad (\text{since } X_j \overset{d}{=} X_1) \\ &\leqslant \sum_{j=1}^{\infty} \mathbb{V}(|X_1| > j/2) \leqslant 1 + C_{\mathbb{V}}(2|X_1|) < \infty.\end{aligned}$$

(3.14) is proved. So, we conclude that

$$\mathbb{V}^*\left(\limsup_{n\to\infty} \frac{S_n}{n} > \epsilon\right) = 0, \quad \forall\, \epsilon > 0,$$

by the arbitrariness of $\theta > 1$. Hence

$$\mathbb{V}^*\left(\limsup_{n\to\infty} \frac{S_n}{n} > 0\right) = \mathbb{V}^*\left(\bigcup_{k=1}^{\infty}\left\{\limsup_{n\to\infty} \frac{S_n}{n} > \frac{1}{k}\right\}\right) \leqslant \sum_{k=1}^{\infty} \mathbb{V}^*\left(\limsup_{n\to\infty} \frac{S_n}{n} > \frac{1}{k}\right) = 0.$$

Finally,

$$\mathbb{V}^*\left(\liminf_{n\to\infty} \frac{S_n}{n} < \widehat{\mathcal{E}}[X_1]\right) = \mathbb{V}^*\left(\limsup_{n\to\infty} \frac{\sum_{k=1}^n (-X_k - \widehat{\mathbb{E}}[-X_k])}{n} > 0\right) = 0.$$

The proof of (3.2) is now completed.

For (b) of Theorems 3.3 and 3.5, suppose $C_{\mathbb{V}}(|X_1|) = \infty$. Then, by (3.1),

$$\begin{aligned}\sum_{j=1}^{\infty} \widehat{\mathbb{E}}\left[g_{1/2}\left(\frac{|X_j|}{Mj}\right)\right] &= \sum_{j=1}^{\infty} \widehat{\mathbb{E}}\left[g_{1/2}\left(\frac{|X_1|}{Mj}\right)\right] \quad (\text{since } X_j \overset{d}{=} X_1) \\ &\geqslant \sum_{j=1}^{\infty} \mathbb{V}(|X_1| > Mj) = \infty, \quad \forall\, M > 0.\end{aligned} \tag{3.15}$$

For any $l \geqslant 1$,

$$\mathcal{V}\left(\sum_{j=1}^{n} g_{1/2}\left(\frac{|X_j|}{Mj}\right) < l\right) = \mathcal{V}\left(\exp\left\{-\frac{1}{2}\sum_{j=1}^{n} g_{1/2}\left(\frac{|X_j|}{Mj}\right)\right\} > \mathrm{e}^{-l/2}\right)$$

$$\leqslant \mathrm{e}^{l/2}\widehat{\mathcal{E}}\bigg[\exp\bigg\{-\sum_{j=1}^n g_{1/2}\bigg(\frac{|X_j|}{Mj}\bigg)\bigg\}\bigg]$$
$$= \mathrm{e}^{l/2}\prod_{j=1}^n \widehat{\mathcal{E}}\bigg[\exp\bigg\{-\frac{1}{2}g_{1/2}\bigg(\frac{|X_j|}{Mj}\bigg)\bigg\}\bigg]$$

by (3.1) again and the independence because $0 \leqslant \exp\{-\frac{1}{2}g_{1/2}(\frac{|x_j|}{Mj})\} \in C_{l,\mathrm{Lip}}(\mathbb{R})$. Applying the elementary inequality

$$\mathrm{e}^{-x} \leqslant 1 - \frac{1}{2}x \leqslant \mathrm{e}^{-x/2}, \quad \forall\, 0 \leqslant x \leqslant \frac{1}{2}$$

yields

$$\widehat{\mathcal{E}}\bigg[\exp\bigg\{-\frac{1}{2}g_{1/2}\bigg(\frac{|X_j|}{Mj}\bigg)\bigg\}\bigg] \leqslant 1 - \frac{1}{4}\widehat{\mathbb{E}}\bigg[g_{1/2}\bigg(\frac{|X_j|}{Mj}\bigg)\bigg] \leqslant \exp\bigg\{-\frac{1}{4}\widehat{\mathbb{E}}\bigg[g_{1/2}\bigg(\frac{|X_j|}{Mj}\bigg)\bigg]\bigg\}.$$

It follows that

$$\mathcal{V}\bigg(\sum_{j=1}^n g_{1/2}\bigg(\frac{|X_j|}{Mj}\bigg) < l\bigg) \leqslant \mathrm{e}^{l/2}\exp\bigg\{-\frac{1}{4}\sum_{j=1}^n \widehat{\mathbb{E}}\bigg[g_{1/2}\bigg(\frac{|X_j|}{Mj}\bigg)\bigg]\bigg\} \to 0 \quad \text{as} \quad n \to \infty,$$

by (3.15). So

$$\mathbb{V}\bigg(\sum_{j=1}^n g_{1/2}\bigg(\frac{|X_j|}{Mj}\bigg) > l\bigg) \to 1 \quad \text{as} \quad n \to \infty.$$

If $\mathbb{V}$ is continuous as assumed in Theorem 3.3, then $\mathbb{V} \equiv \mathbb{V}^*$. If $\widehat{\mathbb{E}}$ is countably sub-additive as assumed in Theorem 3.5, then

$$\mathbb{V}^*(|X| \geqslant c) \leqslant \mathbb{V}(|X| \geqslant c) \leqslant \widehat{\mathbb{E}}[g_\epsilon(|X|/c)] \leqslant \mathbb{V}^*(|X| \geqslant c(1-\epsilon)),$$

by (3.1) and (3.6). In either case, we have

$$\mathbb{V}^*\bigg(\sum_{j=1}^n g_{1/2}\bigg(\frac{|X_j|}{Mj}\bigg) > \frac{l}{2}\bigg) \geqslant \mathbb{V}\bigg(\sum_{j=1}^n g_{1/2}\bigg(\frac{|X_j|}{Mj}\bigg) > l\bigg) \to 1 \quad \text{as} \quad n \to \infty.$$

Now, by the continuity of $\mathbb{V}^*$,

$$\mathbb{V}^*\bigg(\limsup_{n\to\infty}\frac{|X_n|}{n} > \frac{M}{2}\bigg) = \mathbb{V}^*\bigg(\bigg\{\frac{|X_j|}{Mj} > \frac{1}{2}\bigg\}\ \text{i.o.}\bigg) \geqslant \mathbb{V}^*\bigg(\sum_{j=1}^\infty g_{1/2}\bigg(\frac{|X_j|}{Mj}\bigg) = \infty\bigg)$$
$$= \lim_{l\to\infty}\mathbb{V}^*\bigg(\sum_{j=1}^\infty g_{1/2}\bigg(\frac{|X_j|}{Mj}\bigg) > \frac{l}{2}\bigg)$$
$$= \lim_{l\to\infty}\lim_{n\to\infty}\mathbb{V}^*\bigg(\sum_{j=1}^n g_{1/2}\bigg(\frac{|X_j|}{Mj}\bigg) > \frac{l}{2}\bigg) = 1.$$

On the other hand,

$$\limsup_{n\to\infty}\frac{|X_n|}{n} \leqslant \limsup_{n\to\infty}\bigg(\frac{|S_n|}{n} + \frac{|S_{n-1}|}{n}\bigg) \leqslant 2\limsup_{n\to\infty}\frac{|S_n|}{n}.$$

It follows that

$$\mathbb{V}^*\bigg(\limsup_{n\to\infty}\frac{|S_n|}{n} > m\bigg) = 1, \quad \forall\, m > 0.$$

Hence,

$$\mathbb{V}^*\bigg(\limsup_{n\to\infty}\frac{|S_n|}{n} = +\infty\bigg) = \lim_{m\to\infty}\mathbb{V}^*\bigg(\limsup_{n\to\infty}\frac{|S_n|}{n} > m\bigg) = 1,$$

which contradicts (3.3) and (3.8). So, $C_{\mathbb{V}}(|X_1|) < \infty$.

Finally, we consider Theorem 3.3(c). For (3.4), we first show that

$$\mathbb{V}\left(\frac{S_n}{n} > \widehat{\mathbb{E}}[X_1] - \epsilon\right) \to 1, \quad \forall\, \epsilon > 0. \tag{3.16}$$

Let $f_c(x)$ and $\widehat{f}_c(x)$ be defined as in (3.13). Then

$$\mathbb{V}\left(\frac{|\sum_{k=1}^n \widehat{f}_c(X_k)|}{n} > \epsilon\right) \leqslant \frac{\sum_{k=1}^n \widehat{\mathbb{E}}[|\widehat{f}_c(X_k)|]}{\epsilon n} \leqslant \frac{\widehat{\mathbb{E}}[(|X_1| - c)^+]}{\epsilon} \to 0 \quad \text{as} \quad c \to \infty,$$

$\widehat{\mathbb{E}}[X_1] - \widehat{\mathbb{E}}[f_c(X_1)] \to 0$ as $c \to \infty$, and by Theorem 2.3,

$$\begin{aligned}\mathcal{V}\left(\frac{\sum_{k=1}^n f_c(X_k)}{n} \leqslant \widehat{\mathbb{E}}[f_c(X_1)] - \epsilon\right) &= \mathcal{V}\left(\sum_{k=1}^n (-f_c(X_k) - \widehat{\mathcal{E}}[-f_c(X_k)]) \geqslant n\epsilon\right)\\ &\leqslant \frac{\widehat{\mathcal{E}}[|(\sum_{k=1}^n(-f_c(X_k) - \widehat{\mathcal{E}}[-f_c(X_k)]))^+|^2]}{n^2\epsilon^2}\\ &\leqslant 2\frac{\widehat{\mathbb{E}}[(-f_c(X_1) - \widehat{\mathcal{E}}[-f_c(X_1)])^2]}{n\epsilon^2}\\ &\leqslant \frac{2(2c)^2}{n\epsilon^2} \to 0 \quad \text{as} \quad n \to \infty.\end{aligned}$$

Then (3.16) is proved. By considering $\{-X_n; n \geqslant 1\}$ instead, from (3.16) we have

$$\mathbb{V}\left(\frac{S_n}{n} \leqslant \widehat{\mathcal{E}}[X_1] + \epsilon\right) \to 1, \quad \forall\, \epsilon > 0. \tag{3.17}$$

Note the independence. We conclude that

$$\begin{aligned}&\mathbb{V}\left(\frac{S_n}{n} < \widehat{\mathcal{E}}[X_1] + \epsilon \text{ and } \frac{S_{n^2} - S_n}{n^2 - n} > \widehat{\mathbb{E}}[X_1] - \epsilon\right)\\ &\quad\geqslant \widehat{\mathbb{E}}\left[\phi\left(\frac{S_n}{n} - \widehat{\mathcal{E}}[X_1]\right)\phi\left(\widehat{\mathbb{E}}[X_1] - \frac{S_{n^2} - S_n}{n^2 - n}\right)\right]\\ &\quad\geqslant \widehat{\mathbb{E}}\left[\phi\left(\frac{S_n}{n} - \widehat{\mathcal{E}}[X_1]\right)\right] \cdot \widehat{\mathbb{E}}\left[\phi\left(\widehat{\mathbb{E}}[X_1] - \frac{S_{n^2} - S_n}{n^2 - n}\right)\right]\\ &\quad\geqslant \mathbb{V}\left(\frac{S_n}{n} < \widehat{\mathcal{E}}[X_1] + \frac{\epsilon}{2}\right) \cdot \mathbb{V}\left(\frac{S_{n^2} - S_n}{n^2 - n} > \widehat{\mathbb{E}}[X_1] - \frac{\epsilon}{2}\right) \to 1, \quad \forall\, \epsilon > 0,\end{aligned}$$

where $\phi(x) \in C_{l,\mathrm{Lip}}(\mathbb{R})$ is a function such that $I\{x \leqslant \epsilon\} \geqslant \phi(x) \geqslant I\{x \leqslant \epsilon/2\}$. Now, by (3.2) and the continuity of $\mathbb{V}$,

$$\begin{aligned}&\mathbb{V}\left(\liminf_{n\to\infty} \frac{S_n}{n} \leqslant \widehat{\mathcal{E}}[X_1] + \epsilon \text{ and } \limsup_{n\to\infty} \frac{S_n}{n} \geqslant \widehat{\mathbb{E}}[X_1] - \epsilon\right)\\ &\quad\geqslant \mathbb{V}\left(\liminf_{n\to\infty} \frac{S_n}{n} \leqslant \widehat{\mathcal{E}}[X_1] + \epsilon \text{ and } \limsup_{n\to\infty} \frac{S_{n^2} - S_n}{n^2 - n} \geqslant \widehat{\mathbb{E}}[X_1] - \epsilon\right)\\ &\quad\geqslant \mathbb{V}\left(\frac{S_n}{n} < \widehat{\mathcal{E}}[X_1] + \epsilon \text{ and } \frac{S_{n^2} - S_n}{n^2 - n} > \widehat{\mathbb{E}}[X_1] - \epsilon \text{ i.o.}\right)\\ &\quad\geqslant \limsup_{n\to\infty} \mathbb{V}\left(\frac{S_n}{n} < \widehat{\mathcal{E}}[X_1] + \epsilon \text{ and } \frac{S_{n^2} - S_n}{n^2 - n} > \widehat{\mathbb{E}}[X_1] - \epsilon\right) = 1, \quad \forall\, \epsilon > 0.\end{aligned}$$

By the continuity of $\mathbb{V}$ again,

$$\mathbb{V}\left(\liminf_{n\to\infty} \frac{S_n}{n} \leqslant \widehat{\mathcal{E}}[X_1] \text{ and } \limsup_{n\to\infty} \frac{S_n}{n} \geqslant \widehat{\mathbb{E}}[X_1]\right) = 1,$$

which, together with (3.2) implies (3.4).

Finally, note

$$\frac{S_n}{n} - \frac{S_{n-1}}{n-1} = \frac{X_n}{n} - \frac{S_{n-1}}{n-1}\frac{1}{n} \to 0 \quad \text{a.s. } \mathbb{V}.$$

It can be verified that (3.4) implies (3.5). □

Proof of Corollary 3.4. It is sufficient to note the facts that $\mathbb{V}(A) = \widehat{\mathbb{E}}[I_A]$ is continuous in $\mathcal{H} = \{A, I_A \in \mathscr{H}\}$ and all events we consider are in $\mathcal{H}$ because $\mathscr{H}$ is monotone and $I\{x \geqslant 1\} = \lim_{\epsilon\to 0} g_\epsilon(x)$. □

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant No. 11225104) and the Fundamental Research Funds for the Central Universities.

References

1 Chen Z J. Strong laws of large numbers for capacities. ArXiv:1006.0749, 2010
2 Denis L, Martini C. A theoretical framework for the pricing of contingent claims in the presence of model uncertainty. Ann Appl Probab, 2006, 16: 827–852
3 Gao F Q, Xu M Z. Large deviations and moderate deviations for independent random variables under sublinear expectations (in Chinese). Sci Sin Math, 2011, 41: 337–352
4 Gao F Q, Xu M Z. Relative entropy and large deviations under sublinear expectations. Acta Math Sci Ser B, 2012, 32: 1826–1834
5 Gilboa I. Expected utility theory with purely subjective non-additive probabilities. J Math Econom, 1987, 16: 65–68
6 Huber P, Strassen V. Minimax tests and the Neyman-Pearson lemma for capacity. Ann Statist, 1973, 1: 252–263
7 Marinacci M. Limit laws for non-additive probabilities and their frequentist interpretation. J Econom Theory, 1999, 84: 145–195
8 Matula P. A note on the almost sure convergence of sums of negatively dependent random variables. Statist Probab Lett, 1992, 15: 209–213
9 Newman C M, Wright A L. An invariance principle for certain dependent sequence. Ann Probab, 1981, 9: 671–675
10 Peng S. BSDE and related g-expectation. Pitman Res Notes Math Ser, 1997, 364: 141–159
11 Peng S. Monotonic limit theorem of BSDE and nonlinear decomposition theorem of Doob-Meyer type. Probab Theory Related Fields, 1999, 113: 473–499
12 Peng S. G-expectation, G-Brownian motion and related stochastic calculus of Ito type. In: Proceedings of the 2005 Abel Symposium. Berlin-Heidelberg: Springer, 2008, 541–567
13 Peng S. Multi-dimensional G-Brownian motion and related stochastic calculus under G-expectation. Stochastic Process Appl, 2008, 118: 2223–2253
14 Peng S. A new central limit theorem under sublinear expectations. ArXiv:0803.2656v1, 2008
15 Peng S. Survey on normal distributions, central limit theorem, Brownian motion and the related stochastic calculus under sublinear expectations. Sci China Ser A, 2009, 52: 1391–1411
16 Peng S. Nonlinear expectations and stochastic calculus under uncertainty. ArXiv:1002.4546, 2010
17 Shao Q M. A Comparison theorem on moment inequalities between negatively associated and independent random variables. J Theort Probab, 2000, 13: 343–356
18 Su C, Zhao L C, Wang Y B. Moment inequalities and weak convergence for negatively associated sequences. Sci China Ser A, 1997, 40: 172–182
19 Yuan D M, An J. Rosenthal type inequalities for asymptotically almost negatively associated random variables and applications. Sci China Ser A, 2009, 52: 1887–1904
20 Zhang L X. A functional central limit theorem for asymptotically negatively dependent random fields. Acta Math Hungar, 2000, 83: 237–259
21 Zhang L X. A Strassen's law of the iterated logarithm for negatively associated random vectors. Stoch Process Appl, 2001, 95: 311–328
22 Zhang L X. The weak convergence for functions of negatively associated random variables. J Multiv Anal, 2001, 78: 272–298
23 Zhang L X. Donsker's invariance principle under the sub-linear expectation with an application to Chung's law of the iterated logarithm. Commun Math Stat, 2015, 3: 187–214
24 Zhang L X, Wen J W. A weak convergence for negatively associated fields. Statist Probab Lett, 2001, 53: 259–267