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Strong laws of large numbers for sub-linear expectations

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Abstract We investigate three kinds of strong laws of large numbers for capacities with a new notion of independently and identically distributed (IID) random variables for sub-linear expectations initiated by Peng. It turns out that these theorems are natural and fairly neat extensions of the classical Kolmogorov's strong law of large numbers to the case where probability measures are no longer additive. An important feature of these strong laws of large numbers is to provide a frequentist perspective on capacities.

Keywords capacity, strong law of large numbers, independently and identically distributed, nonlinear expectation

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1 Introduction

The classical strong laws of large numbers (strong LLN) as fundamental limit theorems in probability theory play a fruitful role in the development of probability theory and its applications. The key in the proofs of these limit theorems is the additivity of probability measures and mathematical expectations. However, such an additivity assumption is not feasible in many areas of applications because many uncertain phenomena cannot be well modelled using additive probabilities or additive expectations. More specifically, motivated by some problems in mathematical economics, statistics, quantum mechanics and finance, a number of papers have used non-additive probabilities (called capacities) and nonlinear expectations (for example Choquet integral/expectation, g-expectation) to describe and interpret the phenomena which are generally nonadditive (see [1, 5–7, 10, 11, 19, 21]). A natural question is what is the law of large numbers under nonadditive probabilities or nonlinear expectations? Recently, motivated by the risk measures, super-hedge pricing and model uncertainty in finance, Peng [12–17] initiated the notion of independently and identically distributed (IID) random variables under sub-linear expectations. Under this framework, he proved a weak law of large numbers (LLN) and a central limit theorem (CLT). In this paper, we investigate three strong laws of large numbers for capacities in Peng's framework. All of them are natural and fairly neat extensions of the classical Kolmogorov's strong law of large numbers, but the proofs here are different from the original proofs of the classical strong law of large numbers.

Now we describe the problem in more details. For a given set $\mathcal{P}$ of multiple prior probability measures on $(\Omega, \mathcal{F})$, we define a pair $(\mathbb{V}, v)$ of capacities by

$$\mathbb{V}(A) := \sup_{P\in\mathcal{P}} P(A), \quad v(A) := \inf_{P\in\mathcal{P}} P(A), \quad \forall\, A \in \mathcal{F}.$$

The corresponding Choquet integrals/expectations $(C_{\mathbb{V}}, C_v)$ are defined by

$$C_V[X] := \int_0^{\infty} V(X \geqslant t)dt + \int_{-\infty}^0 [V(X \geqslant t) - 1]dt,$$

where V is replaced by $\mathbb{V}$ and v, respectively.

The pair of so-called maximum-minimum expectations $(\mathbb{E}, \mathcal{E})$ is defined by

$$\mathbb{E}[\xi] := \sup_{P \in \mathcal{P}} E_P[\xi], \quad \mathcal{E}[\xi] := \inf_{P \in \mathcal{P}} E_P[\xi].$$

Here and in the sequel, E_P denotes the classical expectation under probability P.

In general, the relation between Choquet integral and maximum-minimum expectations is as follows: For any random variable X,

$$\mathbb{E}[X] \leqslant C_{\mathbb{V}}[X], \quad C_v[X] \leqslant \mathcal{E}[X].$$

Note that under some very special assumptions on $\mathcal{P}$ and $\mathbb{V}$, both inequalities could become equalities (see [6, 7, 18]).

Given a sequence $\{X_i\}_{i=1}^{\infty}$ of IID random variables for capacities, the earlier papers related to strong laws of large numbers for capacities can be found in [3, 20]. However, the more general results for strong laws of large numbers for capacities were given by Maccheroni and Marinacci [8], Marinacci [9] and Epstein and Schneider [4]. They show that, on full set, any cluster point of empirical averages lies between the lower Choquet integral $C_v[X_1]$ and the upper Choquet integral $C_{\mathbb{V}}[X_1]$ with probability one under capacity v, i.e.,

$$v\left(\omega \in \Omega : C_v[X_1] \leqslant \liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^n X_i(\omega) \leqslant \limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^n X_i(\omega) \leqslant C_{\mathbb{V}}[X_1]\right) = 1.$$

Marinacci [9] obtained his result under the assumptions that $\mathbb{V}$ is a totally monotone capacity on a Polish space Ω random variables $\{X_i\}_{i=1}^{\infty}$ are bounded or continuous. Epstein and Schneider [4] also showed the same result under the assumptions that $\mathbb{V}$ is rectangular and the set $\mathcal{P}$ is finite.

Since the gap between the Choquet integrals $C_{\mathbb{V}}[X]$ and $C_v[X]$ is bigger than that of the maximum-minimum expectations $\mathbb{E}[X]$ and $\mathcal{E}[X]$ for all X, it is of interest to ask whether we can obtain a more precise result if the Choquet integrals/expectations in the above equality are replaced by maximum-minimum expectations, i.e.,

$$v\left(\omega \in \Omega : \mathcal{E}[X_1] \leqslant \liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^n X_i(\omega) \leqslant \limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^n X_i(\omega) \leqslant \mathbb{E}[X_1]\right) = 1.$$

The first result in this paper is to show that the above equality is still true in Peng's framework. Furthermore, motivated by this result, we establish two new laws of large numbers. The first is to show that there exist two cluster points of empirical averages which reach the minimum expectation $\mathcal{E}[X_1]$ and the maximum expectation $\mathbb{E}[X_1]$, respectively under capacity $\mathbb{V}$, i.e.,

$$\mathbb{V}\left(\omega \in \Omega : \limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^n X_i(\omega) = \mathbb{E}[X_1]\right) = 1,$$

$$\mathbb{V}\left(\omega \in \Omega : \liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^n X_i(\omega) = \mathcal{E}[X_1]\right) = 1.$$

The second is to prove that the cluster set of empirical averages coincides with the interval between minimum expectation $\mathcal{E}[X_1]$ and maximum expectation $\mathbb{E}[X_1]$, i.e., let $C(\{x_n\})$ be the cluster set of $\{x_n\}$, then, for any $b \in [\mathcal{E}[X_1], \mathbb{E}[X_1]]$,

$$\mathbb{V}\left(\omega \in \Omega : b \in C\left(\left\{\frac{1}{n}\sum_{i=1}^n X_i(\omega)\right\}\right)\right) = 1.$$

Obviously, if either $\mathbb{V}$ or v in the above results is a probability measure, all of our main results are natural and fairly neat extensions of the classical Kolmogorov's strong law of large numbers. Moreover, an important feature of our strong laws of large numbers is to provide a frequentist perspective on capacities.

2 Notation and lemmas

In order to prove our results in Peng's framework, we shall recall briefly the notions of both IID random variables and sub-linear expectations initiated by Peng [14].

Let $(\Omega, \mathcal{F})$ be a measurable space, and $\mathcal{L}$ be a subset of all random variables on $(\Omega, \mathcal{F})$ such that for any $A \in \mathcal{F}$, $I_A \in \mathcal{L}$, where I_A is the indicator function of event A.

Definition 2.1. A functional $\mathbb{E}$ on $\mathcal{L} \mapsto (-\infty, +\infty)$ is called a sub-linear expectation, if it satisfies the following properties: For all $X, Y \in \mathcal{L}$,

(a) Monotonicity: $X \geqslant Y$ implies $\mathbb{E}[X] \geqslant \mathbb{E}[Y]$.

(b) Constant preserving: $\mathbb{E}[c] = c, \forall\, c \in \mathbb{R}$.

(c) Sub-additivity: $\mathbb{E}[X+Y] \leqslant \mathbb{E}[X] + \mathbb{E}[Y]$.

(d) Positive homogeneity: $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X], \forall\, \lambda \geqslant 0$.

Given a sub-linear expectation $\mathbb{E}$, let us denote the conjugate expectation $\mathcal{E}$ of sub-linear $\mathbb{E}$ by

$$\mathcal{E}[X] := -\mathbb{E}[-X], \quad \forall\, X \in \mathcal{L}.$$

Obviously, for all $X \in \mathcal{L}$, $\mathcal{E}[X] \leqslant \mathbb{E}[X]$. By the sub-additivity of $\mathbb{E}$, we have the following lemma.

Lemma 2.2. *If $X, Y \in \mathcal{L}$, then*

$$\mathcal{E}[X] \leqslant \mathbb{E}[X+Y] - \mathbb{E}[Y].$$

Given a sub-linear expectation, we can define a pair of capacities $(\mathbb{V}, v)$ as follows:

Definition 2.3. A pair $(\mathbb{V}, v)$ of capacities is said to be generated by a sub-linear expectation $\mathbb{E}$, if

$$\mathbb{V}(A) := \mathbb{E}[I_A], \quad v(A) := \mathcal{E}[I_A], \quad \forall\, A \in \mathcal{F}.$$

It is easy to check that such capacities have the following properties:

Lemma 2.4. (1) $\mathbb{V}(\varnothing) = v(\varnothing) = 0, \mathbb{V}(\Omega) = v(\Omega) = 1$.

(2) $\mathbb{V}(A) \leqslant \mathbb{V}(B)$, $v(A) \leqslant v(B)$, *whenever $A \subset B$ and $A, B \in \mathcal{F}$.*

(3) $\mathbb{V}(A \cup B) \leqslant \mathbb{V}(A) + \mathbb{V}(B)$, $A, B \in \mathcal{F}$.

(4) $\mathbb{V}(A) + v(A^c) = 1$, $\forall\, A \in \mathcal{F}$, *where A^c is the complement set of A.*

Motivated by the notion of IID random variables under sub-linear expectations initiated by Peng [14], we adopt the following notion of IID random variables under sub-linear expectations to study strong law of large numbers for non-additive probabilities.

Definition 2.5. **Independence.** Suppose that $Y_1, Y_2, \ldots, Y_n$ is a sequence of random variables such that $Y_i \in \mathcal{L}$. Random variable Y_n is said to be independent of $X := (Y_1, \ldots, Y_{n-1})$ under $\mathbb{E}$, if for each Borel-measurable function φ on $\mathbb{R}^n$ with $\varphi(X, Y_n) \in \mathcal{L}$ and $\varphi(x, Y_n) \in \mathcal{L}$ for each $x \in \mathbb{R}^{n-1}$, we have

$$\mathbb{E}[\varphi(X, Y_n)] = \mathbb{E}[\overline{\varphi}(X)],$$

where $\overline{\varphi}(x) := \mathbb{E}[\varphi(x, Y_n)]$ and $\overline{\varphi}(X) \in \mathcal{L}$.

Identical distribution. Random variables X and Y are said to be identically distributed, denoted by $X \overset{d}{=} Y$, if for each Borel-measurable function φ such that $\varphi(X),\ \varphi(Y) \in \mathcal{L}$,

$$\mathbb{E}[\varphi(X)] = \mathbb{E}[\varphi(Y)].$$

IID random variables. A sequence of random variables $\{X_i\}_{i=1}^{\infty}$ is said to be IID, if $X_i \overset{d}{=} X_1$ and X_{i+1} is independent of $Y := (X_1, \ldots, X_i)$ for each $i \geqslant 1$.

The following lemma shows the relation between our independence and pairwise independence in [9].

Lemma 2.6. *Suppose that $X, Y \in \mathcal{L}$ are two random variables. $\mathbb{E}$ is a sub-linear expectation and $(\mathbb{V}, v)$ is the pair of capacities generated by $\mathbb{E}$. If random variable X is independent of Y under $\mathbb{E}$, then X is also independent of Y under capacities $\mathbb{V}$ and v, i.e., for all subsets D and $G \in \mathcal{B}(\mathbb{R})$,*

$$V(X \in D, Y \in G) = V(X \in D)V(Y \in G)$$

holds for both capacities $\mathbb{V}$ and v.

Proof. If we choose $\varphi(x, y) = xy$, by the definition of independence in Definition 2.5, it is easy to obtain the independence for events,

$$\mathbb{V}(X \in D, Y \in G) = \mathbb{E}[I_{\{X\in D\}} I_{\{Y \in G\}}] = \mathbb{E}[\varphi(I_{\{X\in D\}}, I_{\{Y\in G\}})] = \mathbb{V}(X \in D)\mathbb{V}(Y \in G).$$

Similarly, we can prove that X is independent of Y under capacity v by choosing $\varphi(x, y) = -xy$. □

Chen et al. [2] proved that Borel-Cantelli lemma is still true for capacity under some assumptions.

Lemma 2.7 (See [2, Lemma 2.2])**.** *Let $\{A_n, n \geqslant 1\}$ be a sequence of events in $\mathcal{F}$ and $(\mathbb{V}, v)$ be a pair of capacities generated by sub-linear expectation $\mathbb{E}$.*

(1) If $\sum_{n=1}^{\infty} \mathbb{V}(A_n) < \infty$, then $\mathbb{V}(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i) = 0$.

(2) If further $\mathbb{V}$ is upper continuous and $\{A_n^c\}_{n=1}^{\infty}$ are mutually independent with respect to v, i.e., for any $n \in \mathbb{N}$,

$$v\bigg(\bigcap_{i=n}^{\infty} A_i^c\bigg) = \prod_{i=n}^{\infty} v(A_i^c).$$

If $\sum_{n=1}^{\infty} \mathbb{V}(A_n) = \infty$, then

$$\mathbb{V}\bigg(\bigcap_{n=1}^{\infty}\bigcup_{i=n}^{\infty} A_i\bigg) = 1.$$

Suppose that $C_b(\mathbb{R})$ is the set of all continuous and bounded functions on $\mathbb{R}$ and $C_b^2(\mathbb{R})$ is the set of all continuous and bounded functions on $\mathbb{R}$ whose second derivatives exist in $C_b(\mathbb{R})$.

With the notion of IID under sub-linear expectation, we can obtain the following lemma.

Lemma 2.8. *Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of IID random variables with finite means $\overline{\mu} := \mathbb{E}[X_1]$, $\underline{\mu} := \mathcal{E}[X_1]$, and $S_n := \sum_{i=1}^{n} X_i$ with $S_0 := 0$. Suppose $\mathbb{E}[|X_1|^{1+\alpha}] < \infty$ for some $\alpha > 0$. Then for any positive function $\varphi \in C_b(\mathbb{R})$,*

$$\liminf_{n\to\infty} \mathbb{E}\bigg[\varphi\bigg(\frac{S_n}{n}\bigg)\bigg] \geqslant \sup_{\underline{\mu}\leqslant x\leqslant\overline{\mu}} \varphi(x).$$

Proof. We turn the proof into three steps. Let x^* be the maximal point of φ over $[\underline{\mu}, \overline{\mu}]$.

Step 1. We first prove that if $\{X_i\}_{i=1}^{\infty}$ is an IID sequence, then

$$\mathbb{E}\bigg[\varphi\bigg(\frac{1}{n}\sum_{i=1}^{n} X_i\bigg)\bigg] - \varphi(x^*) \geqslant n \inf_{x\in\mathbb{R}} \bigg\{\mathbb{E}\bigg[\varphi\bigg(x + \frac{X_{n-m} - x^*}{n}\bigg)\bigg] - \varphi(x)\bigg\}.$$

In fact, set $T_k := \frac{1}{n}\sum_{i=1}^{k} X_i$ with $T_0 = 0, k = 1, 2, \ldots, n$, and $y := \frac{x^*}{n}$.

$$\begin{aligned}\mathbb{E}[\varphi(T_n)] - \varphi(x^*) &= \mathbb{E}[\varphi(T_n)] - \mathbb{E}[\varphi(T_{n-1} + y)]\\ &\quad + \mathbb{E}[\varphi(T_{n-1} + y)] - \mathbb{E}[\varphi(T_{n-2} + 2y)] + \cdots\\ &\quad + \mathbb{E}[\varphi(T_{n-m} + my)] - \mathbb{E}[\varphi(T_{n-(m+1)} + (m+1)y)] + \cdots\\ &\quad + \mathbb{E}[\varphi(T_1 + (n-1)y)] - \mathbb{E}[\varphi(ny)]\\ &= \sum_{m=0}^{n-1}\{\mathbb{E}[\varphi(T_{n-m} + my)] - \mathbb{E}[\varphi(T_{n-(m+1)} + (m+1)y)]\}. \qquad (2.1)\end{aligned}$$

We now evaluate each term inside the summation. Let

$$h(x) := \mathbb{E}\left[\varphi\left(x + \frac{X_{n-m}}{n}\right)\right].$$

Then because of independence of $\{X_i\}_{i=1}^n$,

$$\begin{aligned}\mathbb{E}[\varphi(T_{n-m} + my)] &= \mathbb{E}\left[\mathbb{E}\left[\varphi\left(x + \frac{X_{n-m}}{n}\right)\right]\Big|_{x=T_{n-(m+1)}+my}\right]\\ &= \mathbb{E}[h(T_{n-(m+1)} + my)].\end{aligned}$$

Then by the sub-linearity of $\mathbb{E}$ in Lemma 2.2, we have

$$\begin{aligned}&\mathbb{E}[\varphi(T_{n-m} + my)] - \mathbb{E}[\varphi(T_{n-(m+1)} + (m+1)y)]\\ &\quad= \mathbb{E}[h(T_{n-(m+1)} + my)] - \mathbb{E}[\varphi(T_{n-(m+1)} + my + y)]\\ &\quad\geqslant \mathcal{E}[h(T_{n-(m+1)} + my) - \varphi(T_{n-(m+1)} + my + y)]\\ &\quad\geqslant \inf_{x\in\mathbb{R}}(h(x) - \varphi(x + y))\\ &\quad= \inf_{x\in\mathbb{R}}\left\{\mathbb{E}\left[\varphi\left(x + \frac{X_{n-m}}{n}\right)\right] - \varphi\left(x + \frac{x^*}{n}\right)\right\}\\ &\quad= \inf_{x\in\mathbb{R}}\left\{\mathbb{E}\left[\varphi\left(x + \frac{X_{n-m} - x^*}{n}\right)\right] - \varphi(x)\right\}.\end{aligned}$$

It then follows that $\{X_i\}_{i=1}^\infty$ is identical. The proof of Step 1 is complete.

Step 2. For $\varphi \in C_b^2(\mathbb{R})$, we shall prove that

$$\liminf_{n\to\infty} n \inf_{x\in\mathbb{R}}\left\{\mathbb{E}\left[\varphi\left(x + \frac{X_{n-m} - x^*}{n}\right)\right] - \varphi(x)\right\} \geqslant 0.$$

The Taylor expansion of function φ implies that for some random variables $\{\theta_i\}_{i=1}^n$ valued in $[0, 1]$,

$$\varphi\left(x + \frac{X_i - x^*}{n}\right) - \varphi(x) = \varphi'(x)\frac{X_i - x^*}{n} + J_n(x, X_i, x^*), \tag{2.2}$$

where

$$J_n(x, X_i, x^*) := \left[\varphi'\left(x + \theta_i\frac{X_i - x^*}{n}\right) - \varphi'(x)\right]\frac{X_i - x^*}{n}, \quad 1 \leqslant i \leqslant n.$$

Taking sub-linear expectation $\mathbb{E}$ on both sides of (2.2), and applying the sub-linearity of $\mathbb{E}$, we have

$$-\mathbb{E}[|J_n(x, X_i, x^*)|] + \mathbb{E}\left[\varphi'(x)\frac{X_i - x^*}{n}\right] \leqslant \mathbb{E}\left[\varphi\left(x + \frac{X_i - x^*}{n}\right) - \varphi(x)\right].$$

Since $\mathbb{E}[X_i] = \overline{\mu}$, $\mathcal{E}[X_i] = \underline{\mu}$ and $x^* \in [\underline{\mu}, \overline{\mu}]$,

$$\mathbb{E}\left[\varphi'(x)\frac{X_i - x^*}{n}\right] = (\varphi'(x))^+\,\frac{\overline{\mu} - x^*}{n} + (\varphi'(x))^-\,\frac{x^* - \underline{\mu}}{n} \geqslant 0.$$

Therefore, we only need to prove that

$$\sum_{i=1}^n \sup_{x\in\mathbb{R}} \mathbb{E}[|J_n(x, X_i, x^*)|] \to 0, \quad n \to \infty. \tag{2.3}$$

In fact, for any $\epsilon > 0$, using Hölder's and Chebyshev's inequalities and the fact that $\{X_i\}$ is identical, we get

$$\sum_{i=1}^n \sup_{x\in\mathbb{R}} \mathbb{E}[|J_n(x, X_i, x^*)|]$$

$$
\begin{aligned}
&\leqslant \sum_{i=1}^{n}\left\{\sup_{x\in\mathbb{R}}\mathbb{E}[|J_n(x,X_i,x^*)|I_{\{|\frac{X_i-x^*}{n}|>\epsilon\}}]+\sup_{x\in\mathbb{R}}\mathbb{E}[|J_n(x,X_i,x^*)|I_{\{|\frac{X_i-x^*}{n}|\leqslant\varepsilon\}}]\right\}\\
&\leqslant \sum_{i=1}^{n}\left\{\mathbb{E}\left[\left(\sup_{x\in\mathbb{R}}\left|\varphi'\left(x+\theta_i\frac{X_i-x^*}{n}\right)\right|+\sup_{x\in\mathbb{R}}|\varphi'(x)|\right)\frac{|X_i-x^*|}{n}I_{\{|X_i-x^*|>n\epsilon\}}\right]\right.\\
&\quad\left.+\,\mathbb{E}\left[\sup_{x\in\mathbb{R}}\left|\varphi''\left(x+\theta_i\bar{\theta}_i\frac{X_i-x^*}{n}\right)\right|\frac{(X_i-x^*)^2}{n^2}I_{\{|X_i-x^*|\leqslant n\epsilon\}}\right]\right\}\\
&\leqslant n\left\{\frac{2\|\varphi'\|}{n}(\mathbb{E}[|X_1-x^*|^{1+\alpha}])^{\frac{1}{1+\alpha}}(\mathbb{E}[I_{\{|X_1-x^*|>n\epsilon\}}])^{\frac{\alpha}{1+\alpha}}+\frac{\epsilon}{n}\|\varphi''\|\mathbb{E}[|X_1-x^*|]\right\}\\
&\leqslant n\left\{\frac{2\|\varphi'\|}{n}(\mathbb{E}[|X_1-x^*|^{1+\alpha}])^{\frac{1}{1+\alpha}}\left(\frac{\mathbb{E}[|X_1-x^*|^{1+\alpha}]}{(n\epsilon)^{1+\alpha}}\right)^{\frac{\alpha}{1+\alpha}}+\frac{\epsilon}{n}\|\varphi''\|\mathbb{E}[|X_1-x^*|]\right\}\\
&\leqslant n\left\{\frac{2}{n^{1+\alpha}\epsilon^{\alpha}}\|\varphi'\|\mathbb{E}[|X_1-x^*|^{1+\alpha}]+\frac{\epsilon}{n}\|\varphi''\|\mathbb{E}[|X_1-x^*|]\right\}\\
&=\frac{2}{(n\epsilon)^{\alpha}}\|\varphi'\|\mathbb{E}[|X_1-x^*|^{1+\alpha}]+\epsilon\|\varphi''\|\mathbb{E}[|X_1-x^*|]\\
&\to \epsilon\|\varphi''\|\mathbb{E}[|X_1-x^*|],\quad \text{as}\quad n\to\infty,
\end{aligned}
$$

where $\{\bar{\theta}_i\}_{i=1}^{\infty}$ are random variables valued in $[0,1]$. For arbitrariness of ϵ, we obtain the conclusion (2.3).

Hence, Lemma 2.8 holds for $\varphi\in C_b^2(\mathbb{R})$.

Step 3. If $\varphi\in C_b(\mathbb{R})$, then for any $\epsilon>0$ there exists $\overline{\varphi}\in C_b^2(\mathbb{R})$ such that

$$\sup_{x\in\mathbb{R}}|\varphi(x)-\overline{\varphi}(x)|\leqslant\epsilon.$$

Apply Step 2 for function $\overline{\varphi}(x)$ and the fact that

$$
\begin{aligned}
&\liminf_{n\to\infty}\mathbb{E}\left[\varphi\left(\frac{S_n}{n}\right)\right]-\sup_{\underline{\mu}\leqslant x\leqslant\overline{\mu}}\varphi(x)\\
&\quad=\liminf_{n\to\infty}\mathbb{E}\left[\varphi\left(\frac{S_n}{n}\right)-\overline{\varphi}\left(\frac{S_n}{n}\right)+\overline{\varphi}\left(\frac{S_n}{n}\right)\right]-\sup_{\underline{\mu}\leqslant x\leqslant\overline{\mu}}[\varphi(x)-\overline{\varphi}(x)+\overline{\varphi}(x)]\\
&\quad\geqslant\liminf_{n\to\infty}\mathbb{E}\left[\overline{\varphi}\left(\frac{S_n}{n}\right)\right]-\sup_{\underline{\mu}\leqslant x\leqslant\overline{\mu}}\overline{\varphi}(x)-2\epsilon\\
&\quad\geqslant -2\epsilon.
\end{aligned}
$$

For arbitrariness of ϵ, the proof of this lemma is complete. □

3 Main result

The following theorem is our main result.

Theorem 3.1. *Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of IID random variables for sublinear expectation $\mathbb{E}$. Suppose $\mathbb{E}[|X_1|^{1+\alpha}]<\infty$ for some $\alpha\in(0,1]$. Set $\overline{\mu}:=\mathbb{E}[X_1]$, $\underline{\mu}=\mathcal{E}[X_1]$ and $S_n:=\sum_{i=1}^{n}X_i$. Then*

(I)

$$\mathbb{V}\left(\left\{\liminf_{n\to\infty}S_n/n<\underline{\mu}\right\}\cup\left\{\limsup_{n\to\infty}S_n/n>\overline{\mu}\right\}\right)=0. \tag{3.1}$$

Also

$$v\left(\underline{\mu}\leqslant\liminf_{n\to\infty}S_n/n\leqslant\limsup_{n\to\infty}S_n/n\leqslant\overline{\mu}\right)=1. \tag{3.2}$$

If furthermore $\mathbb{V}$ is upper continuous, then

(II)

$$\mathbb{V}\left(\limsup_{n\to\infty}S_n/n=\overline{\mu}\right)=1,\quad \mathbb{V}\left(\liminf_{n\to\infty}S_n/n=\underline{\mu}\right)=1.$$

(III) *Suppose that* $C(\{x_n\})$ *is the cluster set of a sequence of* $\{x_n\}$ *in* $\mathbb{R}$, *then, for any* $b \in [\underline{\mu}, \overline{\mu}]$,

$$\mathbb{V}(b \in C(\{S_n/n\})) = 1.$$

Proof. (I) can be deduced from [2, Theorem 3.1] directly, so we omit the details.

We now prove (II). If $\overline{\mu} = \underline{\mu}$, it is trivial. Suppose $\overline{\mu} > \underline{\mu}$, then we only need to prove that there exists an increasing subsequence $\{n_k\}$ of $\mathbb{N}$ such that for any $0 < \epsilon < \overline{\mu} - \underline{\mu}$,

$$\mathbb{V}\Big(\bigcap_{m=1}^{\infty}\bigcup_{k=m}^{\infty}\{S_{n_k}/n_k \geqslant \overline{\mu} - \epsilon\}\Big) = 1. \tag{3.3}$$

Since $\mathbb{E}$ is upper continuous, we have

$$\mathbb{V}\Big(\limsup_{k\to\infty} S_{n_k}/n_k \geqslant \overline{\mu}\Big) = 1.$$

This together with (I) suffices to yield the desired result (II).

Indeed, choose $n_k = k^k$ for $k \geqslant 1$. Set $\overline{S}_n := \sum_{i=1}^{n}(X_i - \overline{\mu})$, then

$$\begin{aligned}\mathbb{V}\bigg(\frac{S_{n_k} - S_{n_{k-1}}}{n_k - n_{k-1}} \geqslant \overline{\mu} - \epsilon\bigg) &= \mathbb{V}\bigg(\frac{S_{n_k - n_{k-1}}}{n_k - n_{k-1}} \geqslant \overline{\mu} - \epsilon\bigg)\\ &= \mathbb{V}\bigg(\frac{S_{n_k - n_{k-1}} - (n_k - n_{k-1})\overline{\mu}}{n_k - n_{k-1}} \geqslant -\epsilon\bigg)\\ &= \mathbb{V}\bigg(\frac{\overline{S}_{n_k - n_{k-1}}}{n_k - n_{k-1}} \geqslant -\epsilon\bigg)\\ &\geqslant \mathbb{E}\bigg[\phi\bigg(\frac{\overline{S}_{n_k - n_{k-1}}}{n_k - n_{k-1}}\bigg)\bigg],\end{aligned}$$

where $\phi(x)$ is defined by

$$\phi(x) = \begin{cases} 1 - \mathrm{e}^{-(x+\epsilon)}, & x \geqslant -\epsilon,\\ 0, & x < -\epsilon.\end{cases}$$

Consider the sequence of IID random variables $\{X_i - \overline{\mu}\}_{i=1}^{\infty}$. Obviously,

$$\mathbb{E}[X_i - \overline{\mu}] = 0, \quad \mathcal{E}[X_i - \overline{\mu}] = -(\overline{\mu} - \underline{\mu}).$$

Applying Lemma 2.8, we have $n_k - n_{k-1} \to \infty$ as $k \to \infty$ and

$$\liminf_{n\to\infty} \mathbb{E}\bigg[\phi\bigg(\frac{\overline{S}_{n_k - n_{k-1}}}{n_k - n_{k-1}}\bigg)\bigg] \geqslant \sup_{-(\overline{\mu}-\underline{\mu})\leqslant y\leqslant 0} \phi(y) = \phi(0) = 1 - \mathrm{e}^{-\epsilon} > 0.$$

Thus

$$\sum_{k=1}^{\infty}\mathbb{V}\bigg(\frac{S_{n_k} - S_{n_{k-1}}}{n_k - n_{k-1}} \geqslant \overline{\mu} - \epsilon\bigg) \geqslant \sum_{k=1}^{\infty}\mathbb{E}\bigg[\phi\bigg(\frac{\overline{S}_{n_k - n_{k-1}}}{n_k - n_{k-1}}\bigg)\bigg] = \infty.$$

Note the fact that $\{S_{n_k} - S_{n_{k-1}}\}_{k\geqslant 1}$ is a sequence of independent random variables for $k \geqslant 1$. Using the second Borel-Cantelli lemma, we have

$$\limsup_{k\to\infty} \frac{S_{n_k} - S_{n_{k-1}}}{n_k - n_{k-1}} \geqslant \overline{\mu} - \epsilon, \quad \text{a.s. } \mathbb{V}.$$

But

$$\frac{S_{n_k}}{n_k} \geqslant \frac{S_{n_k} - S_{n_{k-1}}}{n_k - n_{k-1}} \cdot \frac{n_k - n_{k-1}}{n_k} - \frac{|S_{n_{k-1}}|}{n_{k-1}} \cdot \frac{n_{k-1}}{n_k}.$$

From the fact that

$$\frac{n_k - n_{k-1}}{n_k} \to 1, \quad \frac{n_{k-1}}{n_k} \to 0, \quad \text{as } \ k \to \infty,$$

and

$$\limsup_{n\to\infty} S_n/n \leqslant \overline{\mu}, \quad \limsup_{n\to\infty}(-S_n)/n \leqslant -\underline{\mu}, \quad \text{a.s. } v,$$

we have

$$\limsup_{n\to\infty} |S_n|/n \leqslant \max\{|\overline{\mu}|, |\underline{\mu}|\}, \quad \text{a.s. } v.$$

Hence,

$$\limsup_{k\to\infty} \frac{S_{n_k}}{n_k} \geqslant \limsup_{k\to\infty} \frac{S_{n_k}-S_{n_{k-1}}}{n_k-n_{k-1}} \lim_{k\to\infty} \frac{n_k-n_{k-1}}{n_k} - \limsup_{k\to\infty} \frac{|S_{n_{k-1}}|}{n_{k-1}} \lim_{k\to\infty} \frac{n_{k-1}}{n_k}.$$

We conclude that

$$\limsup_{k\to\infty} \frac{S_{n_k}}{n_k} \geqslant \overline{\mu} - \epsilon, \quad \text{a.s. } \mathbb{V}.$$

Since ϵ is arbitrary and $\mathbb{V}$ is upper continuous, we have

$$\mathbb{V}\Big(\limsup_{k\to\infty} S_{n_k}/n_k \geqslant \overline{\mu}\Big) = 1.$$

By (I), we know $\mathbb{V}(\limsup_{n\to\infty} S_n/n > \overline{\mu}) = 0$, thus

$$\begin{aligned}\mathbb{V}\Big(\limsup_{n\to\infty} S_n/n = \overline{\mu}\Big) &= \mathbb{V}\Big(\limsup_{n\to\infty} S_n/n = \overline{\mu}\Big) + \mathbb{V}\Big(\limsup_{n\to\infty} S_n/n > \overline{\mu}\Big) \\ &\geqslant \mathbb{V}\Big(\limsup_{n\to\infty} S_n/n \geqslant \overline{\mu}\Big) = 1.\end{aligned}$$

Considering the sequence of $\{-X_n\}_{n=1}^{\infty}$, we have

$$\mathbb{V}\Big(\limsup_{n\to\infty}(-S_n)/n = \mathbb{E}[-X_1]\Big) = 1.$$

Therefore,

$$\mathbb{V}\Big(\liminf_{n\to\infty} S_n/n = -\mathbb{E}[-X_1]\Big) = 1.$$

But $\underline{\mu} = -\mathbb{E}[-X_1]$, thus

$$\mathbb{V}\Big(\liminf_{n\to\infty} S_n/n = \underline{\mu}\Big) = 1.$$

The proof of (II) is complete.

To prove (III), we only need to prove that for $b \in (\underline{\mu}, \overline{\mu})$,

$$\mathbb{V}\Big(\liminf_{n\to\infty} |S_n/n - b| = 0\Big) = 1.$$

To do so, we only need to prove that for any $\epsilon > 0$ there exists an increasing subsequence $\{n_k\}$ of $\mathbb{N}$ such that for any $b \in (\underline{\mu}, \overline{\mu})$,

$$\mathbb{V}\Big(\bigcap_{m=1}^{\infty}\bigcup_{k=m}^{\infty}\{|S_{n_k}/n_k - b| \leqslant \epsilon\}\Big) = 1. \tag{3.4}$$

Indeed, for any $0 < \epsilon \leqslant \min\{\overline{\mu} - b, b - \underline{\mu}\}$, let us choose $n_k = k^k$ for $k \geqslant 1$.

Set $\overline{S}_n := \sum_{i=1}^n (X_i - b)$, then

$$\begin{aligned}\mathbb{V}\bigg(\bigg|\frac{S_{n_k}-S_{n_{k-1}}}{n_k-n_{k-1}} - b\bigg| \leqslant \epsilon\bigg) &= \mathbb{V}\bigg(\bigg|\frac{S_{n_k-n_{k-1}}}{n_k-n_{k-1}} - b\bigg| \leqslant \epsilon\bigg) \\ &= \mathbb{V}\bigg(\bigg|\frac{S_{n_k-n_{k-1}} - (n_k-n_{k-1})b}{n_k-n_{k-1}}\bigg| \leqslant \epsilon\bigg) \\ &= \mathbb{V}\bigg(\bigg|\frac{\overline{S}_{n_k-n_{k-1}}}{n_k-n_{k-1}}\bigg| \leqslant \epsilon\bigg) \\ &\geqslant \mathbb{E}\bigg[\phi\bigg(\frac{\overline{S}_{n_k-n_{k-1}}}{n_k-n_{k-1}}\bigg)\bigg],\end{aligned}$$

where $\phi(x)$ is defined by

$$\phi(x)=\begin{cases}1-\mathrm{e}^{|x|-\epsilon}, & |x|\leqslant\epsilon,\\ 0, & |x|>\epsilon.\end{cases}$$

Consider the sequence of IID random variables $\{X_i-b\}_{i=1}^{\infty}$. Obviously,

$$\mathbb{E}[X_i-b]=\overline{\mu}-b>0,\quad \mathcal{E}[X_i-b]=\underline{\mu}-b<0.$$

Applying Lemma 2.8, we have

$$\liminf_{k\to\infty}\mathbb{E}\left[\phi\left(\frac{\overline{S}_{n_k-n_{k-1}}}{n_k-n_{k-1}}\right)\right]\geqslant\sup_{\underline{\mu}-b\leqslant y\leqslant\overline{\mu}-b}\phi(y)=\phi(0)=1-\mathrm{e}^{-\epsilon}>0.$$

Thus

$$\sum_{k=1}^{\infty}\mathbb{V}\left(\left|\frac{S_{n_k}-S_{n_{k-1}}}{n_k-n_{k-1}}-b\right|\leqslant\epsilon\right)\geqslant\sum_{k=1}^{\infty}\mathbb{E}\left[\phi\left(\frac{\overline{S}_{n_k-n_{k-1}}}{n_k-n_{k-1}}\right)\right]=\infty.$$

Note the fact that the sequence of $\{S_{n_k}-S_{n_{k-1}}\}_{k\geqslant 1}$ are independent random variables. Using the second Borel-Cantelli lemma, we have

$$\liminf_{k\to\infty}\left|\frac{S_{n_k}-S_{n_{k-1}}}{n_k-n_{k-1}}-b\right|\leqslant\epsilon,\quad \text{a.s. } \mathbb{V}.$$

But

$$\left|\frac{S_{n_k}}{n_k}-b\right|\leqslant\left|\frac{S_{n_k}-S_{n_{k-1}}}{n_k-n_{k-1}}-b\right|\cdot\frac{n_k-n_{k-1}}{n_k}+\left[\frac{|S_{n_{k-1}}|}{n_{k-1}}+|b|\right]\frac{n_{k-1}}{n_k}. \tag{3.5}$$

Note that

$$\frac{n_k-n_{k-1}}{n_k}\to 1,\quad \frac{n_{k-1}}{n_k}\to 0,\quad \text{as}\ \ k\to\infty$$

and

$$\limsup_{n\to\infty}S_n/n\leqslant\overline{\mu},\quad \limsup_{n\to\infty}(-S_n)/n\leqslant-\underline{\mu},\quad \text{a.s.}\ \ v,$$

which implies

$$\limsup_{n\to\infty}|S_n|/n\leqslant\max\{|\overline{\mu}|,|\underline{\mu}|\}<\infty\quad \text{a.s.}\ \ v.$$

Hence, from inequality (3.5), for any $\epsilon>0$,

$$\liminf_{k\to\infty}\left|\frac{S_{n_k}}{n_k}-b\right|\leqslant\epsilon,\quad \text{a.s.}\ \ \mathbb{V},$$

i.e.,

$$\mathbb{V}\left(\liminf_{n\to\infty}|S_n/n-b|\leqslant\epsilon\right)=1.$$

Since ϵ is arbitrary, we have

$$\mathbb{V}\left(\liminf_{n\to\infty}|S_n/n-b|=0\right)=1.$$

The proof of (III) is complete. □

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