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# Remainder terms for several inequalities on some groups of Heisenberg-type

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**Abstract** We give estimates of the remainder terms for several conformally-invariant Sobolev-type inequalities on the Heisenberg group. By considering the variations of associated functionals, we give a *stability* for two dual inequalities: The fractional Sobolev (FS) and Hardy-Littlewood-Sobolev (HLS) inequalities, in terms of distance to the submanifold of *extremizers*. Then we compare their remainder terms to improve the inequalities in another way. We also compare, in the limit case, the remainder terms of Beckner-Onofri (BO) inequality and its dual logarithmic Hardy-Littlewood-Sobolev (Log-HLS) inequality. Besides, we also list without proof some results for other groups of Iwasawa-type. Our results generalize earlier works on Euclidean spaces of Chen et al. (2013) and Dolbeault and Jankowiak (2014) onto some groups of Heisenberg-type. We worked for "almost" *all* fractions especially for comparing results, and the stability of HLS is also absolutely new, even for Euclidean case.

**Keywords** remainder terms, stability, Sobolev-type inequalities, Heisenberg groups

**MSC(2010)** 26D10, 35R03, 46E35

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## 1 Introduction

In this paper, we consider on (rank one) Iwasawa-type groups the problem of sharpening the HLS-type inequalities obtained recently in [9, 10, 14]. First, we add a remainder term proportional to the distance square to the submanifold of extremizers, motivated originally from the question asked by Brezis and Lieb [5] for Euclidean spaces. We also want to compare the remainder terms of the dual inequalities similarly to the Euclidean case.

On the Euclidean case $\mathbb{R}^n$, we have classical FS inequality: Given any exponent $0 < s < n, q = \frac{2n}{n-s}$, for any function $f$ in $\frac{s}{2}$-order (homogeneous) Sobolev space $\dot{H}^{s/2}$ endowed with the norm $\|f\|_{\dot{H}^{s/2}} = \|(-\Delta)^{s/4} f\|_{L^2} = (\int f(-\Delta)^{s/2} f)^{1/2}$, we have

$$\|f\|_{\dot{H}^{s/2}} \gtrsim \|f\|_{L^q}, \tag{1.1}$$

and the sharp constant was first computed by several pioneers for some special cases and finally obtained by Lieb [19] (there Lieb considered a dual form, the (diagonal) HLS inequality) for exponent $s$ and dimension $n$ of all range and he also proved that the equality for sharp (1.1) can only be achieved by extremal functions (called *extremizers*) of the form $c(1+|\delta x-x_0|^2)^{-\frac{n-s}{2}}, c \in \mathbb{R}\backslash\{0\}, \delta > 0, x_0 \in \mathbb{R}^n$. Note that, (1.1)

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has an equivalent edition on the $n$-sphere $\mathbb{S}^n$ through the stereographic transformation $\mathcal{S} : x \in \mathbb{R}^n \mapsto \zeta = (\frac{2x}{1+|x|^2}, \frac{1-|x|^2}{1+|x|^2}) \in \mathbb{S}^n$. Both the FS and HLS inequalities are invariant under one kind of "conformal actions", including not only the translations, dilations, but also an inversion, which is given for the fractional inequality by $\sigma_{\text{inv}} : f(x) \mapsto |x|^{s-n} f(x|x|^{-2})$. The sharp inequalities and extremizers were got by exploiting this big conformal symmetry group in several beautiful ways. In [5], Brezis and Lieb proposed a stability problem for the sharp inequality (1.1) in case $s = 2$, asking whether there exists a positive constant $\alpha$, such that the following estimate holds:

$$\|f\|^2_{\dot{H}^{s/2}} - C_{\text{sharp}} \|f\|^2_{L^q} \geqslant \alpha d^2(f, \mathcal{M}), \tag{1.2}$$

where $\mathcal{M}$ is the $(n+2)$-dimensional smooth submanifold of $\dot{H}^{s/2}$, consisting of all real-valued extremizers, and $d(f, \mathcal{M})$ is the usual distance of $f$ to $\mathcal{M}$ under the Sobolev norm. Bianchi and Egnell gave a positive answer in [3] still for $s = 2$, which was later extended to more fractions but not of total range in [1, 22]. Chen et al. [7] extended the method of Bianchi and Egnell to obtain the above remainder term inequality (1.2) for all fractions $0 < s < n$, which contains all old results above. Naturally, we guess the stability like (1.2) should also hold for analogous inequalities (Folland-Stein) on the Heisenberg and some more general groups of Heisenberg type motivated from [9, 10, 14]. In a different way, Dolbeault pointed out in [12] that in $s = 2$ case, the duality of FS and HLS inequalities are greatly related to a fast diffusion equation and used that diffusion flow to compare the remainder terms of the two dual inequalities for $n \geqslant 5$. Later, Jin and Xiong [18], and Dobeault and Jankowiak [13] extended the result respectively to the case $s \in (0, 2), n \geqslant 2, n > 2s$ and the case $s = 2, n \geqslant 3$. Actually, for $s = 2, n \geqslant 3, p = q' = \frac{2n}{n+s}$, there exists a positive constant $\alpha$, such that $\forall\, 0 \leqslant f \in \dot{H}^{s/2}$,

$$\|f\|^{2(q-2)}_{L^q} (\|f\|^2_{\dot{H}^{s/2}} - C_{\text{sharp}} \|f\|^2_{L^q}) \geqslant \alpha (\|f^{q/p}\|^2_{L^p} - C_{\text{sharp}} \|f^{q/p}\|^2_{\dot{H}^{-s/2}}). \tag{1.3}$$

Similar result for limit case—BO and Log-HLS inequalities [2, 6]—was also given in [13] in the case $s = n = 2$. Denote $⨍ = \frac{1}{|\mathbb{S}^n|} \int_{\mathbb{S}^n}$, then on the sphere there exists a positive constant $\alpha$, such that if $⨍ \mathrm{e}^f = 1$, then

$$\frac{1}{2n!} ⨍ f A_n f - \log ⨍ \mathrm{e}^{f - ⨍ f} \geqslant \alpha \left( ⨍ \mathrm{e}^f f - \frac{n!}{2} ⨍ \mathrm{e}^f A_n^{-1} \mathrm{e}^f \right), \tag{1.4}$$

where $A_n$ is a spherical picture of $(-\Delta)^{n/2}$, meaning that $A_n f = (|J_{\mathcal{S}}|^{-1} (-\Delta)^{n/2} (f \circ \mathcal{S})) \circ \mathcal{S}^{-1}$, with the spectrum $j(j+1) \cdots (j+n-1)$ on the spherical harmonic subspace $\mathcal{H}_j$, which is injective when restricted on the image space with a fundamental solution $-\frac{2}{(n-1)!} \log |\zeta - \eta|$ (note the kernel of $A_n$ is $\mathcal{H}_0$ and here $A_n^{-1}$ is interpreted after projection). Dolbeault and Jankowiak [13] also gave some bounds about the sharp proportional constants and we naturally think these sophisticated results about (1.3) and (1.4) should hold for *all* fractions both on Euclidean spaces and Heisenberg groups. Indeed, after the completion of this paper handling on Heisenberg groups, we found corresponding results on $\mathbb{R}^n$ (about (1.3) and (1.4)) were also proved simultaneously and independently by Jankowiak and Nguyen [17] for all fractions $0 < s < n, n \geqslant 2$.

On the Heisenberg group $\mathbb{H}^n$, the sharp HLS inequalities were obtained recently in [14] for all exponents. The extension of HLS and its dual FS to other Iwasawa-type groups were given by Christ et al. [9, 10] for partial range of exponents and fractions. Some limit cases were also given in above papers and [4]. The main purpose of this note is to get, for (almost) all fractions $s$, some analogous results to Euclidean inequalities (1.2)–(1.4) obtained in [7, 13], for conformally-invariant Sobolev-type inequalities on the Heisenberg group (and more general Iwasawa-type groups). For stability, we give in particular the following refined HLS estimate,

$$\|f\|^2_{L^p} - C_{\text{sharp}} \|f\|^2_{H^{-s/2}} \geqslant \alpha d_p^2(f, \mathcal{M}_-). \tag{1.5}$$

As usual, it is natural for us to carry out more conveniently the proofs of the problems on spheres, where constant functions are extremizers for related inequalities. For the stability, we first give a local estimate and then use the recovery of compactness lemma to get a global one by contradiction. For the FS inequality, in the local neighborhood domain of $\mathcal{M}$—the submanifold of extremizers, with an additional condition $d(f,\mathcal{M}) < \|f\|_{s/2}$, we can first use the conformal symmetry to assume the nearest point to be constant function 1 on the sphere and write the function to be $f = 1+\varphi$, with $\varphi$ in the normal subspace, then the Taylor expansion tells us that we only need to estimate the second variation around 1, which is observed to be positive definite on the normal subspace. This normal non-degeneracy property of the extremizering submanifold is crucial. For the HLS inequality, we borrow a fantastic local truncation control, due to *Christ's lemma* for general $(p,q)$-functional with $p < 2 \leqslant q$, to make up for the failure of Taylor expansion on $L^p$-distance and therefore to get a local stability again, which implies the global stability by compactness and contradiction. To compare the two dual remainder terms, we first use the idea of *completion of square* to get a global proportional bound and then consider a local case by variational expansion, which in other words, gives an upper bound of the best constant in the global proportional inequality. To our knowledge, the HLS stability (1.5) is absolutely new that even has not been given on Euclidean spaces before, and we believe our global and local variational analysis working for "almost" all fractions on Heisenberg-type groups is pretty new in some sense, both technically and conceptually. Nevertheless, there's no doubt that we're greatly inspired by [7,13].

Our results sharpen the existing geometric inequalities, and it has been evident that the study of function spaces, boundedness of operators and sharp estimates is one of the central topics in harmonic analysis, referring to [11,15,20,23–26], etc.

## 2 Fractional Sobolev and Hardy-Littlewood-Sobolev inequalities

Here, for simplicity, we consider on the Heisenberg group $\mathbb{H}^n$ the fractional Sobolev (Folland-Stein) inequality and its dual Hardy-Littlewood-Sobolev inequality, both of which contain the intrinsic conformal invariance.

**The Heisenberg group.** We identify the Heisenberg group $\mathbb{H}^n$ with its Lie algebra $\mathbb{C}^n\times\mathbb{R}$ endowed with group law $uu' = (z,t)(z',t') = (z+z', t+t'+2\,\mathrm{Im}\, z\cdot\overline{z'})$ for group elements $u=(z,t), u'=(z',t')$, where $z,z'\in\mathbb{C}^n, t,t'\in\mathbb{R}$ and $z\cdot\overline{z'} = \sum_{j=1}^n z_j\overline{z'_j}$. The dilation for $\delta>0$ is $\delta: u=(z,t)\mapsto\delta(u) = (\delta z,\delta^2 t)$, and we denote the related homogeneous dimension by $Q=2n+2$, the homogeneous norm by $|u| = (|z|^4+|t|^2)^{1/4}$. The left invariant vector fields which coincide respectively with $\frac{\partial}{\partial x_j},\frac{\partial}{\partial y_j},\frac{\partial}{\partial t}$ at the origin are given by

$$X_j = \frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t},\quad Y_j = \frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t},\quad T = \frac{\partial}{\partial t},\quad 1\leqslant j\leqslant n.$$

The *sub-Laplacian* is a second order left invariant differential operator given by $\mathcal{L} = -\frac{1}{4}(X_j^2+Y_j^2)$, which is hypoelliptic from a celebrated Hörmander's theorem, and we recall that this essentially self-adjoint positive operator does not depend on the choice of an orthonormal basis. An explicit computation gives the following formula,

$$\mathcal{L} = -\frac{1}{4}\Delta_z - |z|^2\frac{\partial^2}{\partial t^2} + \frac{1}{2}N\frac{\partial}{\partial t},$$

where

$$\Delta_z = \sum_{j=1}^n\left(\frac{\partial^2}{\partial x_j^2}+\frac{\partial^2}{\partial y_j^2}\right),\quad N = \sum_{j=1}^n\left(x_j\frac{\partial}{\partial y_j} - y_j\frac{\partial}{\partial x_j}\right)$$

are respectively the standard Laplacian and corresponding rotation operator on $\mathbb{C}^n$. Using the (boundary) Cayley transform, a generalization of stereographic transform, $\mathcal{C}:\mathbb{H}^n\to\mathbb{S}^{2n+1}\setminus\{o\}$ with $o$ being the south pole $(0,\dots,0,-1)$, defined by

$$u=(z,t)\mapsto\zeta = (\zeta',\zeta_{n+1}) = \left(\frac{2z}{1+|z|^2-\mathrm{i}t},\frac{1-|z|^2+\mathrm{i}t}{1+|z|^2-\mathrm{i}t}\right),$$

we can identify the Heisenberg group with the complex sphere. The Jacobian of the Cayley transform is

$$|J_{\mathcal{C}}| = 2^{Q-1}((1+|z|^2)^2+|t|^2)^{-Q/2} = 2^{-1}|1+\zeta_{n+1}|^Q.$$

A similar sub-Laplacian on $\mathbb{S}^{2n+1}$ can be defined from $\mathcal{L}$ by the Cayley transform and explicitly to be

$$\mathcal{L}' = -\sum_{j=1}^{n+1} \frac{\partial^2}{\partial\zeta_j \partial\overline{\zeta_j}} + \sum_{j,k=1}^{n+1} \zeta_j\overline{\zeta_k} \frac{\partial^2}{\partial\zeta_j\partial\overline{\zeta_k}} + \frac{n}{2}\sum_{j=1}^{n+1}\left(\zeta_j\frac{\partial}{\partial\zeta_j} + \overline{\zeta_j}\frac{\partial}{\partial\overline{\zeta_j}}\right),$$

and the Geller-type *conformal sub-Laplacian* is defined to be

$$\mathcal{D} = \mathcal{L}' + \frac{n^2}{4}.$$

The "sub-Laplacians" above play as important counterparts of positive Laplacian $-\Delta$ on Euclidean space $\mathbb{R}^n$ and there is an important relation between $\mathcal{L}$ and $\mathcal{D}$,

$$\mathcal{L}((2|J_{\mathcal{C}}|)^{\frac{Q-2}{2Q}} f\circ\mathcal{C}) = (2|J_{\mathcal{C}}|)^{\frac{Q+2}{2Q}} (\mathcal{D}f)\circ\mathcal{C}.$$

It is well known that the fundamental solutions of $\mathcal{L}$ and $\mathcal{D}$ are multiples of $(2-Q)$-power of distance functions:

$$\mathcal{L}^{-1} = \frac{2^{n-2}\Gamma^2(\frac{n}{2})}{\pi^{n+1}}|u^{-1}v|^{2-Q}, \quad \mathcal{D}^{-1} = \frac{\Gamma^2(\frac{n}{2})}{2\pi^{n+1}}|1-\zeta\cdot\bar{\eta}|^{\frac{2-Q}{2}}.$$

**Fractional Sobolev inequality.** The sharp FS inequality characterises the embedding $H^{s/2}\hookrightarrow L^q$, which generalizes a Jerison-Lee inequality (for $s=2$): $\forall\ 0<s<Q, q=\frac{2Q}{Q-s}$ and $f\in H^{s/2}$, we have sharp inequality

$$\|f\|_{s/2}^2 \geqslant C|f|_q^2, \tag{2.1}$$

where

$$C = \left(\frac{4\pi^{Q/2}}{n!}\right)^{s/Q}\frac{\Gamma^2(\frac{Q+s}{4})}{\Gamma^2(\frac{Q-s}{4})} \tag{2.2}$$

is the best constant and will be fixed in this note. Here, for simplicity, we use $|\cdot|_q$ to denote the $L^q$ norm, and $H^{s/2}$ is the fractional Sobolev (Folland-Stein) space, which is the completion of $C_0^\infty$ w.r.t. the Sobolev norm $\|f\|_{s/2}^2 = \langle f,f\rangle_{s/2} = \langle f,\mathcal{L}_s f\rangle = \int f\mathcal{L}_s f$, with $\mathcal{L}_s$ being an *intertwining operator* for complementary series representations of $\mathrm{SU}(n+1,1)$. We have the following characterization of $\mathcal{L}_s$:

$$\mathcal{L}_s = |2T|^{s/2}\frac{\Gamma(\mathcal{L}|2T|^{-1}+\frac{2+s}{4})}{\Gamma(\mathcal{L}|2T|^{-1}+\frac{2-s}{4})}, \quad \mathcal{L}_s^{-1} = \frac{2^{n-1-s/2}\Gamma^2(\frac{Q-s}{4})}{\pi^{n+1}\Gamma(\frac{s}{2})}|u|^{s-Q}.$$

Note that $\|\cdot\|_{s/2}$ is also equivalent to $|(I+\mathcal{L})^{s/2}|_2$, and $\mathcal{L}_2=\mathcal{L}$ is nothing but the sub-Laplacian. The FS inequality (2.1) is invariant under the conformal action $\sigma: f\mapsto\sigma(f) = f\circ\sigma|J_\sigma|^{1/q}$, where $\sigma$ is any conformal transformation on $\mathbb{H}^n$ with Jacobian determinant $|J_\sigma|$. Actually, $|\cdot|_q$ is obviously invariant under the conformal actions and so is the inner product $\langle\cdot,\cdot\rangle_{s/2}$, i.e., $\langle\sigma(f),\sigma(g)\rangle_{s/2} = \langle f,g\rangle_{s/2}$, considering the intertwining property of $\mathcal{L}_s$:

$$|J_\sigma|^{\frac{Q+s}{2Q}}(\mathcal{L}_s f)\circ\sigma = \mathcal{L}_s(|J_\sigma|^{\frac{Q-s}{2Q}} f\circ\sigma).$$

Using the Cayley transform $\mathcal{C}:\mathbb{H}^n\to\mathbb{S}^{2n+1}\setminus\{o\}$, we can move the inequality onto the complex sphere $\mathbb{S}^{2n+1}\subset\mathbb{C}^{n+1}$:

$$\|f\|_*^2 \geqslant C|f|_q^2, \tag{2.3}$$

where $\|\cdot\|_*$ is the norm induced by the inner product $\langle f,g\rangle_* = \langle\mathcal{C}^{-1}(f),\mathcal{C}^{-1}(g)\rangle_{s/2}$ and the two FS inequalities (2.1) and (2.3) are equivalent under the conformal correspondence of functions on the group and complex sphere $\mathcal{C}: f\mapsto\mathcal{C}(f) = f\circ\mathcal{C}|J_{\mathcal{C}}|^{1/q}$. This inner product $\langle\cdot,\cdot\rangle_*$ is a quadratic form involving an intertwining operator $\mathcal{A}_s$ on $\mathbb{S}^{2n+1}$, a spherical picture of $\mathcal{L}_s$. Actually, $\mathcal{A}_s$ is uniquely given by the relation

$$\mathcal{L}_s(|J_{\mathcal{C}}|^{\frac{Q-s}{2Q}} f\circ\mathcal{C}) = |J_{\mathcal{C}}|^{\frac{Q+s}{2Q}}(\mathcal{A}_s f)\circ\mathcal{C},$$

and more precisely characterized by the bispherical harmonic decomposition or its fundamental solution as:

$$\lambda_{j,k} = \mathcal{A}_s|_{\mathcal{H}_{j,k}} = 2^{s/Q}\frac{\Gamma(j+\frac{Q+s}{4})\Gamma(k+\frac{Q+s}{4})}{\Gamma(j+\frac{Q-s}{4})\Gamma(k+\frac{Q-s}{4})}, \quad \mathcal{A}_s^{-1} = \frac{2^{-1-s/Q}\Gamma^2(\frac{Q-s}{4})}{\pi^{n+1}\Gamma(\frac{s}{2})}|1-\zeta\cdot\bar{\eta}|^{\frac{s-Q}{2}}, \tag{2.4}$$

where $\mathcal{H}_{j,k}$ is the bispherical harmonic subspace spanned by harmonic polynomials of degree $j,k$, respectively on $\zeta$ and $\bar{\zeta}$ and we have the irreducible decomposition $L^2(\mathbb{S}^{2n+1}) = \bigoplus_{j,k=1}^{\infty}\mathcal{H}_{j,k}$ (also $= \bigoplus_{j=1}^{\infty}\mathcal{H}_j$, where $\mathcal{H}_j$ is the classical real spherical harmonic subspace spanned by harmonic polynomials on real variables of degree $j$). There is also an intertwining relation for $\mathcal{A}_s$, just like that for $\mathcal{L}_s$, which, modulo a constant, uniquely determines the operator. Note here $\mathcal{A}_2 = 2^{2/Q}\mathcal{D}$, where $\mathcal{D}$ is the Geller-type conformal sub-Laplacian and $\mathcal{D}|_{\mathcal{H}_{j,k}} = (j+\frac{n}{2})(k+\frac{n}{2})$. We will use $H^*$ for the $\frac{s}{2}$-order Sobolev space on $\mathbb{S}^{2n+1}$, endowed with the norm $\|\cdot\|_*$. The conformal symmetry group of the complex spherical picture FS inequality (2.3) consists of

$$\tau : f \to \tau(f) = f\circ\tau|J_\tau|^{1/q} \tag{2.5}$$

for any conformal transformation $\tau = \mathcal{C}\circ\sigma\circ\mathcal{C}^{-1}$ on $S^{2n+1}$, induced by the Cayley transform from any conformal transformation $\sigma$ on $\mathbb{H}^n$. We also have the invariance of the inner product $\langle\cdot,\cdot\rangle_*$ under the conformal actions.

It was proved in [14] (in a dual reformulation) that all real-valued extremizers of the sharp FS inequality (2.1) are right all functions of the form $cH(\delta(u\cdot))$, where $c\in\mathbb{R}\setminus\{0\}, \delta>0, u\in\mathbb{H}^n$, and

$$H = ((1+|z|^2)^2+|t|^2)^{-\frac{Q-s}{4}} = 2^{-\frac{(Q-1)(Q-s)}{2Q}}|J_{\mathcal{C}}|^{1/q},$$

i.e., (2.1) reaches equality only on $cH$, up to the conformal invariant actions. Similarly, all real-valued extremizers of (2.3) are right given by $c|1-\xi\cdot\bar{\zeta}|^{-\frac{Q-s}{2}}$, where $c\in\mathbb{R}\setminus\{0\}, \xi\in\mathbb{C}^{n+1}$ and $|\xi|<1$, i.e., (2.3) reaches equality only on constant functions, up to the conformal symmetry group. Take $\mathcal{M},\mathcal{M}_*$ respectively to be the $(Q+1)$-dimensional smooth submanifolds of all real-valued extremizers for (2.1) and (2.3) and define naturally the distances

$$d(f,\mathcal{M}) := \inf\{\|f-g\|_{s/2} : g\in\mathcal{M}\}, \quad d(f,\mathcal{M}_*) := \inf\{\|f-g\|_* : g\in\mathcal{M}_*\}.$$

Then we have the following theorem characterising the stability of extremizers.

**Theorem 2.1** (Stability for FS). *Let $0<s<Q=2n+2, q=\frac{2Q}{Q-s}$. Then there exists $\alpha>0$ only depending on the dimension $Q$ and fraction $s$, such that*

$$d^2(f,\mathcal{M}) \geqslant \|f\|_{s/2}^2 - C|f|_q^2 \geqslant \alpha\, d^2(f,\mathcal{M}), \quad \forall f\in H^{s/2}, \tag{2.6}$$

$$d^2(f,\mathcal{M}_*) \geqslant \|f\|_*^2 - C|f|_q^2 \geqslant \alpha\, d^2(f,\mathcal{M}_*), \quad \forall f\in H^*, \tag{2.7}$$

*and if $d(f,\mathcal{M})$ or $d(f,\mathcal{M}_*)>0$, the left inequality is strict.*

**Hardy-Littlewood-Sobolev inequality.** A duality argument writes the FS inequality (2.1) into the (diagonal) HLS inequality, which gives the $(p,p')$-boundness of the fractional integral operator $|u|^{-\lambda}*$: $\forall\, 0<\lambda<Q$, $p=\frac{2Q}{2Q-\lambda}$ and $f,g\in L^p$, we have

$$||u|^{-\lambda}*f|_{p'} \lesssim |f|_p, \quad \text{or equivalently} \quad \left|\iint_{\mathbb{H}^n\times\mathbb{H}^n}\frac{f(u)g(v)}{|u^{-1}v|^\lambda}dudv\right| \lesssim |f|_p|g|_p,$$

and by the boundary Cayley transform, there is an equivalent HLS inequality on the complex sphere $\mathbb{S}^{2n+1}$,

$$\left|\iint_{\mathbb{S}^{2n+1}\times\mathbb{S}^{2n+1}}\frac{f(\zeta)g(\eta)}{|1-\zeta\cdot\bar{\eta}|^{\lambda/2}}d\zeta d\eta\right| \lesssim |f|_p|g|_p.$$

Modulo a constant multiple (see from the fundamental solution formulas of $\mathcal{L}_s$ and $\mathcal{A}_s$), we write the above HLS inequalities in the following sharp form: $\forall\, 0<s<Q,\ p=\frac{2Q}{Q+s},\ f\in L^p$ (on $\mathbb{H}^n$ or $\mathbb{S}^{2n+1}$ depending on the corresponding context),

$$\|f\|_{-s/2}^2 \leqslant C^{-1}|f|_p^2, \quad \|f\|_{-*}^2 \leqslant C^{-1}|f|_p^2, \tag{2.8}$$

where $\|\cdot\|_{-s/2}$ is the negative fractional Sobolev norm on $\mathbb{H}^{\mathrm{n}}$ under the meaning of $\|\cdot\|_{-s/2}^2 = \langle\cdot, \mathcal{L}_s^{-1}\cdot\rangle$ and $\|\cdot\|_{-*}$ is the correspondence on $\mathbb{S}^{2\mathrm{n}+1}$ induced from the inner product $\langle\cdot,\cdot\rangle_{-*} = \langle\cdot, \mathcal{A}_s^{-1}\cdot\rangle$. The two HLS inequalities (2.8) are respectively invariant under the conformal action $\sigma : f \mapsto f\circ\sigma|J_\sigma|^{1/p}$ and $\tau : f \mapsto f\circ\tau|J_\tau|^{1/p}$, where $\sigma$ and $\tau$ are respectively the conformal transformation on the group and sphere. This can be seen from the conformal invariance of the FS inequalities (2.1) and (2.3) by dual argument or derived directly from the following formulae:

$$|\sigma(u)^{-1}\sigma(v)| = |u^{-1}v||J_\sigma(u)J_\sigma(v)|^{\frac{1}{2Q}}, \quad |1-\tau(\zeta)\cdot\overline{\tau(\eta)}|^{1/2} = |1-\zeta\cdot\bar{\eta}|^{1/2}|J_\tau(\zeta)J_\tau(\eta)|^{\frac{1}{2Q}}.$$

Take the submanifold of real-valued extremizers for HLS inequalities on the group and sphere, respectively to be

$$\begin{aligned}\mathcal{M}_- &= \{c(|J_{\mathcal{C}}|^{1/p})(\delta(u\cdot)) : c\in\mathbb{R}\setminus\{0\}, \delta>0, u\in\mathbb{H}^n\},\\ \mathcal{M}_{-*} &= \{c|J_\tau|^{1/p} = c|1-\xi\cdot\bar{\zeta}|^{-\frac{Q+s}{2}} : c\in\mathbb{R}\setminus\{0\}, \xi\in\mathbb{C}^{n+1}, |\xi|<1,\\ &\qquad \tau \text{ is a conformal transformation on } \mathbb{S}^{2\mathrm{n}+1}\}.\end{aligned}$$

Define the $L^p$ distances to the extremizering submanifolds to be

$$d_p(f,\mathcal{M}_-) = \inf\{|f-g|_p : g\in\mathcal{M}_-\}, \quad d_p(f,\mathcal{M}_{-*}) = \inf\{|f-g|_p : g\in\mathcal{M}_{-*}\},$$

and denote $H^{-s/2}, H^{-*}$ to be the associated negative fractional Sobolev spaces on the group and sphere. We then have a similar stability result for the HLS inequalities (2.8).

**Theorem 2.2** (Stability for HLS). *Let $0<s<Q=2n+2$, $p=\frac{2Q}{Q+s}$. Then there exist $\alpha_0,\alpha_1>0$ only depending on the dimension $Q$ and fraction $s$, such that $\forall\ 0\not\equiv f\in L^p$ (on the group or sphere depending on the context)*

$$\alpha_1\frac{d_p(f,\mathcal{M}_-)}{|f|_p} \geqslant C^{-1} - \frac{\|f\|_{-s/2}^2}{|f|_p^2} \geqslant \alpha_0\frac{d_p^2(f,\mathcal{M}_-)}{|f|_p^2}, \tag{2.9}$$

$$\alpha_1\frac{d_p(f,\mathcal{M}_{-*})}{|f|_p} \geqslant C^{-1} - \frac{\|f\|_{-*}^2}{|f|_p^2} \geqslant \alpha_0\frac{d_p^2(f,\mathcal{M}_{-*})}{|f|_p^2}. \tag{2.10}$$

We remark a difference from the problem for the FS inequality. For the upper bound, we can only do for the first power as we do not have any orthogonality, and still do not know about the square distance. This global bound, a stability, is still got from some local estimate. For the lower local bound, the Taylor expansion fails as the exponent $p$ is strictly less than 2 and however, we can borrow one useful lemma of Christ's, who first noticed the failure of $C^2(L^p)$ of the functional as $p<2$ and used very sophisticated estimates to get a bound for the functional using second variation of suitably chosen small truncation. See details in Subsection 4.2.

**Relation between the remainder terms of FS and HLS inequalities.** Now we want to compare the dual remainder terms of FS and HLS inequalities by a constant multiple, both globally and locally. The similar problem on Euclidean spaces has been studied in [13,18] and is now extended to the Heisenberg group for total range of $s$.

**Theorem 2.3** (Dual remainder terms inequality). *About the remainder terms of the sharp FS and HLS inequalities on $\mathbb{S}^{2\mathrm{n}+1}$, we have the following estimates (an equivalence on $\mathbb{H}^{\mathrm{n}}$ exists):*

(1) (*Global estimate*)

$$|f|_q^{2(q-2)}(\|f\|_*^2 - C|f|_q^2) \geqslant C(|f^{q/p}|_p^2 - C\|f^{q/p}\|_{-*}^2), \quad \forall\, 0\leqslant f\in H^*. \tag{2.11}$$

(2) (*Local estimate*)

$$\liminf_{0<d(f,\mathcal{M}_*)\to 0,\ 0\leqslant f\in H^*,\ d(f,\mathcal{M}_*)<\|f\|_*} \frac{|f|_q^{2(q-2)}(\|f\|_*^2 - C|f|_q^2)}{|f^{q/p}|_p^2 - C\|f^{q/p}\|_{-*}^2} = \frac{Q+4+s}{Q+4-s}C. \tag{2.12}$$

The above results can be extended to all (rank one) Iwasawa-type groups. We list on the sphere the results without proofs as they are absolutely similar to the Heisenberg group case. First, we introduce something about groups of Iwasawa-type, which can be seen as the nilpotent part of the Iwasawa decomposition of a semisimple Lie group of rank one. It is one kind of groups of Heisenberg-type, satisfying additional $J_2$-condition. A two-step nilpotent Lie algebra $\mathfrak{g}$ with center $\mathfrak{z}$, endowed with an inner product $\langle\cdot,\cdot\rangle$, for which $\mathfrak{g}=\mathfrak{z}^{\perp}\oplus\mathfrak{z}$, is said to be of Heisenberg-type, if the Lie structure satisfies the following condition: For any element $|t|=1$ in the center $\mathfrak{z}$, an endomorphism $J_t$ defined on $\mathfrak{z}^{\perp}$ by $\langle J_t z, z'\rangle=\langle t,[z,z']\rangle$ for any $z,z'\in\mathfrak{z}^{\perp}$, is an orthogonal map. A simply connected Lie group is said to be of Heisenberg type if its Lie algebra is of Heisenberg type. The $J_2$-condition is defined to be: For any $t,t'\in\mathfrak{z}$ satisfying $\langle t,t'\rangle=0$, there exists a $t''\in\mathfrak{z}$, such that $J_{t''}=J_tJ_t'$. It was well known that any group of Heisenberg-type is of Iwasawa-type if and only if the Lie structure satisfies the $J_2$-condition. Besides, from a geometrical point of view, the Iwasawa groups can be seen as the nilpotent part in the Iwasawa decomposition of the isometry group of the associated non-compact Riemannian symmetric spaces of rank one and finally Iwasawa-type groups are identified to four cases having center of dimension 0,1,3,7. We write isometrically the group in a unifying form to be $\mathbb{K}^n\times\operatorname{Im}\mathbb{K}$ for appropriate $n$, where $\mathbb{K}$ is one of the four real division algebra $\mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O}$. For the above results and FS, HLS and related inequalities in the later two cases, see [9, 10] and references there. If we still use the same notation as those on the Heisenberg group and denote $m$ to be the dimension of center, we can state the following result.

**Theorem 2.4** (For general Iwasawa-type group). *We have analogous sharp FS and HLS inequalities on the Iwasawa-type group for fractions of partial range $s<Q-4[\frac{m}{2}]$ and $\lambda>4[\frac{m}{2}]$, where $m$ is the dimension of center and $Q$ is the homogeneous dimension. Moreover, Theorems* 2.1–2.3 *still hold similarly on the Iwasawa-type group for the corresponding inequality with exponent of partial range. More precisely, we have the following properties for the remainder terms: Let $p=q'=\frac{2Q}{Q+s}$.*

(1) (*Stability for FS and HLS*) *There exist two positive constants $\alpha_0,\alpha_1$ only depending on $Q$ and $s$, such that*

$$d^2(f,\mathcal{M}_*)\geqslant\|f\|_*^2-C|f|_q^2\geqslant\alpha_0\ d^2(f,\mathcal{M}_*),\quad\forall\, f\in H^*,$$
$$\alpha_1|f|_p d_p(f,\mathcal{M}_{-*})\geqslant|f|_p^2-C\|f\|_{-*}^2\geqslant\alpha_0 d_p^2(f,\mathcal{M}_{-*}),\quad\forall\, f\in L^p,$$

*where $C,\mathcal{M},\mathcal{M}_*$ are respectively the sharp constant and extremizering submanifolds, computed in* [9, 10, 14, 19], *and $d(f,\mathcal{M}_*)$, $d_p(f,\mathcal{M}_{-*})$ are respectively the Sobolev and Lebesgue distance to the submanifold.*

(2) (*Dual remainder terms inequality*) *About the remainder terms of the sharp FS and HLS inequalities on the sphere, we have*

(a) (*Global estimate*)

$$|f|_q^{2(q-2)}(\|f\|_*^2-C|f|_q^2)\geqslant C(|f^{q/p}|_p^2-C\|f^{q/p}\|_{-*}^2),\quad\forall\, 0\leqslant f\in H^*.$$

(b) (*Local estimate*)

$$\liminf_{0<d(f,\mathcal{M}_*)\to 0,\ 0\leqslant f\in H^*,\ d(f,\mathcal{M}_*)<\|f\|_*}\frac{|f|_q^{2(q-2)}(\|f\|_*^2-C|f|_q^2)}{|f^{q/p}|_p^2-C\|f^{q/p}\|_{-*}^2}=\frac{Q+2+2\operatorname{sign}m+s}{Q+2+2\operatorname{sign}m-s}\,C.$$

We add a note here about the sharp proportional constant between the two dual remainder terms, which is still open even for the Euclidean case. The global estimate tells that the sharp constant, denoted by $C_{\mathrm{sharp}}$, is bigger than $C$, i.e., $C_{\mathrm{sharp}}\geqslant C$, while the local estimate tells $C_{\mathrm{sharp}}\leqslant\frac{Q+4+s}{Q+4-s}\,C$. However, we expect a fast diffusion method like in [13] may improve a bit the lower bound strictly to be $C_{\mathrm{sharp}}>C$.

## 3 Limit case—Beckner-Onofri and logarithmic Hardy-Littlewood-Sobolev inequalities

Now, as in [4, 14], we can consider on the Heisenberg group the endpoint limit case ($s=Q,\lambda=0$), using the functional differential argument. We use $⨍$ to denote the mean integral on the sphere, i.e., $⨍ f=|\mathbb{S}^{2n+1}|^{-1}\int_{\mathbb{S}^{2n+1}}f$.

**Beckner-Onofri inequality.** For the limit case $s = Q$, the generalization of sharp Beckner-Onofri (BO) inequality on the Heisenberg group and complex sphere was first given by [4]. For simplicity, we only deal on the sphere, which involves the *conditional intertwining operator* $\mathcal{A}'_Q$, a kind of differentiation of $\mathcal{A}_s(0 < s < Q)$ at $s = Q$. The operator, defined on $H^* \cap \mathcal{P}$, where $H^*$, in abuse of notation, is the $\frac{Q}{2}$-order Sobolev space and $\mathcal{P} = \bigoplus_{jk=0} \mathcal{H}_{j,k}$ ($\mathbb{R}\mathcal{P}$) is the space of $L^2$ (real-valued) CR-pluriharmonic functions, is given by:

$$\lambda_j = \mathcal{A}'_Q|_{\mathcal{H}_{j,0}} = \mathcal{A}'_Q|_{\mathcal{H}_{0,j}} = \frac{\Gamma(j+n+1)}{\Gamma(j)} = j(j+1)\cdots(j+n). \tag{3.1}$$

If in this subsection, we denote (a constant difference from (2.4) in the last section)

$$\lambda_{j,k} = \mathcal{A}_s|_{\mathcal{H}_{j,k}} = \frac{\Gamma(j+\frac{Q+s}{4})\Gamma(k+\frac{Q+s}{4})}{\Gamma(j+\frac{Q-s}{4})\Gamma(k+\frac{Q-s}{4})}, \tag{3.2}$$

then

$$\mathcal{A}'_Q = -\frac{4}{n!}\frac{d}{ds}\bigg|_{s=Q}\mathcal{A}_s = \lim_{s\to Q}\frac{4}{n!}\frac{1}{Q-s}\mathcal{A}_s. \tag{3.3}$$

The sharp BO inequality states that: $\forall\, f \in H^* \cap \mathbb{R}\mathcal{P}$ (with zero mean), we have

$$\frac{1}{2(n+1)!}⨍ f\mathcal{A}'_Q f \geqslant \log ⨍ \mathrm{e}^{(f-⨍ f)} \tag{3.4}$$

(without the mean term on the right-hand side). Obviously, the BO inequality (3.4) is invariant up to constant translations, and besides, there is also a big conformal invariance action group for this inequality,

$$\tau : f \mapsto \tau(f) = f\circ\tau + \log|J_\tau|. \tag{3.5}$$

We can easily see the exponential integral $⨍ \mathrm{e}^f$ on the right-hand side is invariant under the conformal action, while we need to use the limit differentiation argument and conformal invariance property of FS inequality to get the whole invariance for the BO inequality (3.4): From (2.2), we define the FS functional of (2.3) modified by (3.2) to be

$$I(g) = ⨍ g\mathcal{A}_s g - \frac{\Gamma^2(\frac{Q+s}{4})}{\Gamma^2(\frac{Q-s}{4})}\left(⨍ |g|^q\right)^{2/q},$$

then it satisfies the following formulae: Take $\lambda = Q - s \to 0$, $\forall\, f \in H^* \cap \mathbb{R}\mathcal{P}$,

(1) if $⨍ f = 0$, we have

$$\begin{aligned} I(1+\lambda f) &= \langle 1, \mathcal{A}_s 1\rangle + \lambda^2\langle f, \mathcal{A}_s f\rangle - \left(\left(\frac{\lambda}{4n!}\right)^2 + o(1)\right)\left(⨍ \mathrm{e}^{2Qf} + o(1)\right)^{\lambda/Q} \\ &= \frac{n!}{4}\langle f, \mathcal{A}'_Q f\rangle\lambda^3 - \left(\left(\frac{n!}{4}\right)^2\lambda^2 + o(1)\right)\left(\left(⨍ \mathrm{e}^{2Qf} + o(1)\right)^{\lambda/Q} - 1\right) \\ &= \frac{n!}{4}\langle f, \mathcal{A}'_Q f\rangle\lambda^3 - \left(\left(\frac{n!}{4}\right)^2\lambda^2 + o(1)\right)\left(\frac{1}{Q}\log⨍ \mathrm{e}^{2Qf}\lambda + o(\lambda)\right) \\ &= \frac{1}{Q}\left(\frac{n!}{4}\right)^2\left(\frac{4Q}{n!}⨍ f\mathcal{A}'_Q f - \log⨍ \mathrm{e}^{2Qf}\right)\lambda^3 + o(\lambda^3), \end{aligned}$$

where we use the notation $\langle f, g\rangle = ⨍ fg$.

(2) For general $f$, we have

$$I\left(1+\frac{\lambda}{2Q}f\right) = \frac{1}{Q}\left(\frac{n!}{4}\right)^2\left(\frac{1}{2(n+1)!}⨍ f\mathcal{A}'_Q f - \log⨍ \mathrm{e}^{f-⨍ f}\right)\lambda^3 + o(\lambda^3).$$

(3) From the invariance of the FS inequality functional under the corresponding conformal action (2.5), we have

$$\begin{aligned} I\left(1+\frac{\lambda}{2Q}f\right) &= I\left(\left(1+\frac{\lambda}{2Q}f\right)\circ\tau\ |J_\tau|^{1/q}\right) \\ &= I\left(1+\frac{\lambda}{2Q}(f\circ\tau+\log|J_\tau|)+o(\lambda)\right) \\ &= I\left(1+\frac{\lambda}{2Q}(f\circ\tau+\log|J_\tau|)\right)+o(\lambda^3). \end{aligned}$$

Then we see the invariance of (3.4) under (3.5) from

$$\frac{1}{2(n+1)!}\fint f\mathcal{A}'_Q f-\log\fint \mathrm{e}^{f-\fint f}=Q\left(\frac{4}{n!}\right)^2\frac{I(1+\frac{\lambda}{2Q}f)}{\lambda^3}+o(1).$$

We know extremizers for BO inequality (3.4) are right all the functions of the following form $\log(c|J_\tau|)=\log c+Q\log|1-\xi\cdot\bar{\zeta}|^{-1}$, where $c>0,\xi\in\mathbb{C}^{n+1},|\xi|<1$ and $\tau$ is any conformal transformation on $\mathbb{S}^{2\mathrm{n}+1}$ ($c=1$ when adding zero mean).

For stability, it seems that we cannot find a good distance norm, which is invariant under the conformal action, to characterize the relation between the remainder terms and distance to the $(Q+1)$-dimensional smooth submanifold of all real-valued extremizers. (The norm $\fint f\mathcal{A}'_Q f$, and also the subspace of zero-mean functions in $\mathcal{P}$, are not invariant under the conformal action, while the functional $\frac{1}{2(n+1)!}\fint f\mathcal{A}'_Q f$ $-\fint f$ is unable to characterize the distance.)

**Logarithmic Hardy-Littlewood-Sobolev inequality.** The generalized sharp logarithmic Hardy-Littlewood-Sobolev (Log-HLS) inequality on the Heisenberg group and complex sphere was first obtained in [4] and recovered in [14], which states that: For any non-negative normalized $f\in L\log L,\fint f=1$, we have

$$(n+1)\fint\fint\log\frac{1}{|1-\zeta\cdot\bar{\eta}|}f(\zeta)f(\eta)d\zeta d\eta\leqslant\fint f\log f.$$

Because the fundamental solution of operator $\mathcal{A}'_Q$ restricted on the image subspace is given by

$$\mathcal{A}'^{-1}_Q|_{f\in\mathcal{P},\fint f=0}=\frac{1}{\pi^{n+1}}\log\frac{1}{|1-\zeta\cdot\bar{\eta}|},$$

the above Log-HLS inequality is equivalent to the following dual reformulation of the BO inequality (3.4):

$$\frac{(n+1)!}{2}\fint(f-1)\mathcal{A}'^{-1}_Q(P(f-1))\leqslant\fint f\log f, \tag{3.6}$$

where $P$ denotes the projection operator from $L^2$ onto $\mathcal{P}$ and for simplicity we often leave $P$ out and use directly $\mathcal{A}'^{-1}_Q$ for $\mathcal{A}'^{-1}_Q\circ P$. The Log-HLS inequality is invariant under the conformal action $f\mapsto f\circ\tau|J_\tau|$.

**Relations between the remainder terms of BO and Log-HLS inequality.** Just as estimates for the FS and HLS remainder terms, we can get both a global and a local bound for the BO and Log-HLS remainder terms. Differentiation is used to get the global one while the local one is got directly from Taylor expansion and the property of variations.

**Theorem 3.1** (Dual inequality for BO and Log-HLS)**.** *About the remainder terms of the sharp BO and Log-HLS inequalities on* $\mathbb{S}^{2\mathrm{n}+1}$*, we have the following estimates* (*an equivalence on* $\mathbb{H}^{\mathrm{n}}$ *exists*)*:*

(1) (*Global estimate*)

$$\begin{aligned} \frac{1}{2(n+1)!}\fint f\mathcal{A}'_Q f-\log\fint \mathrm{e}^{f-\fint f}\geqslant\frac{1}{\fint \mathrm{e}^f}\fint \mathrm{e}^f f-\frac{(n+1)!}{2(\fint \mathrm{e}^f)^2}\fint\left(\mathrm{e}^f-\fint \mathrm{e}^f\right)\mathcal{A}'^{-1}_Q\left(\mathrm{e}^f-\fint \mathrm{e}^f\right) \\ -\log\fint \mathrm{e}^f,\quad\forall\, f\in H^*\cap\mathbb{R}\mathcal{P}. \end{aligned}\tag{3.7}$$

*Note that both the left- and right-hand sides are invariant under the constant translation $f \mapsto f - c$, then for any $f$, we can choose $c = \log \fint \mathrm{e}^f$, such that after a constant translation, new $f$ satisfies $\fint \mathrm{e}^f = 1$ and the inequality becomes*

$$\frac{1}{2(n+1)!}\fint f\mathcal{A}'_Q f - \log \fint \mathrm{e}^{f-\fint f} \geqslant \fint \mathrm{e}^f f - \frac{(n+1)!}{2}\fint (\mathrm{e}^f - 1)\mathcal{A}'^{-1}_Q(\mathrm{e}^f - 1).$$

(2) (*Local estimate*)

$$\liminf_{0<\fint f\mathcal{A}'_Q f \to 0, f\in H^*\cap\mathcal{P}, f\perp\mathcal{H}_1} \frac{\frac{1}{2(n+1)!}\fint f\mathcal{A}'_Q f - \log\fint \mathrm{e}^{f-\fint f}}{\frac{1}{\fint \mathrm{e}^f}\fint \mathrm{e}^f f - \frac{(n+1)!}{2(\fint \mathrm{e}^f)^2}\fint(\mathrm{e}^f - \fint \mathrm{e}^f)\mathcal{A}'^{-1}_Q(\mathrm{e}^f - \fint \mathrm{e}^f) - \log\fint \mathrm{e}^f} \geqslant n+2. \tag{3.8}$$

On the other two Iwasawa-type groups, there is difficulty in getting the sharp FS and HLS inequality for big $s$ and small $\lambda$, as the eigenvalue can be negative. Therefore, the endpoint case—some putative sharp BO and Log-HLS inequalities are also still unknown.

## 4 Proofs of main results

We only prove the second formula in Theorems 2.1 and 2.2, recalling the equivalence of those two formulae on the group and sphere.

### 4.1 Proof of FS stability

For Theorem 2.1, we follow closely Bianchi-Egnell argument [3, 7]. First we need a local result.

**Proposition 4.1** (Local stability for FS). *We have the following local remainder term estimate: For all $f \in H^*$, satisfying $d(f, \mathcal{M}_*) < \|f\|_*$,*

$$d^2(f,\mathcal{M}_*) \geqslant \|f\|_*^2 - C|f|_q^2 \geqslant \frac{2s}{Q+4+s} d^2(f,\mathcal{M}_*) + o(d^2(f,\mathcal{M}_*)), \tag{4.1}$$

*and if $d(f, \mathcal{M}_*) > 0$, the left inequality is strict.*

*Proof.* Consider any $f \in H^*$. From definition, there exists a function $g \in \overline{\mathcal{M}_*}$, such that $\|f - g\|_* = d(f, \mathcal{M}_*)$. From the condition $d(f, \mathcal{M}_*) < \|f\|_*$, we know $g \not\equiv 0$, and there exist $c \in \mathbb{R}\setminus\{0\}, \xi \in \mathbb{C}^{n+1}, |\xi| < 1$, and a conformal transformation $\tau$, such that $g = c\tau(1) = c|J_\tau|^{1/q} = c|1 - \xi\cdot\bar{\zeta}|^{-\frac{Q-s}{2}}$. So, from the invariance of $\mathcal{M}_*$ and above inequality (4.1) under the conformal action, we may assume $f = 1 + \varphi$ with $\varphi \perp T_1\mathcal{M}_*$ (normal to the tangent space at point—function 1) under the inner product $\langle\cdot,\cdot\rangle_*$. We take the difference functional $I(f) = \|f\|_*^2 - C|f|_q^2$, which is in $C^2(H^*)$, then for the local lower bound, by Taylor expansion and the critical property of extremizer 1, we have

$$I(f) = \frac{1}{2}I''(1,\varphi) + o(\|\varphi\|_*^2), \tag{4.2}$$

where

$$I'(1,\varphi) = \frac{d}{dt}\Big|_{t=0} I(1+t\varphi) = 2\langle 1,\varphi\rangle_* - 2C|1|_q^{2-q}\int \varphi \equiv 0,$$
$$I''(1,\varphi) = \frac{d^2}{dt^2}\Big|_{t=0} I(1+t\varphi) = 2\|\varphi\|_*^2 - 2(q-1)C|1|_q^{2-q}|\varphi|_2^2.$$

More precisely, for the Taylor expansion (4.2), we only need to prove

$$|1+\varphi|_q^2 = |1|_q^2 + (q-1)|1|_q^{2-q}|\varphi|_2^2 + o(|\varphi|_q^2), \quad \forall\, q > 2, \quad \varphi \in L^q, \quad \int \varphi = 0. \tag{4.3}$$

We note that the quantity $|1+\varphi|^q$ cannot be expanded by Taylor series around 1 unless $|\varphi|$ is very small relative to 1, which gives the difficulty for the Taylor expansion of the difference (or $L^q$) functional. To prove (4.3), we use a truncation considered for $|\varphi| \leqslant \delta$ and $|\varphi| > \delta$, respectively, setting $\delta = |\varphi|_q^{1-\epsilon}$ and $|\varphi|_q \to 0$,

$$\forall\, |\varphi| \leqslant \delta, \quad |1+\varphi|^q = 1 + q\varphi + \frac{q(q-1)}{2}\varphi^2 + \frac{q(q-1)(q-2)}{3!}|1+\theta\varphi|^{q-3}\varphi^3,$$
$$\forall\, |\varphi| > \delta, \quad \frac{||1+\varphi|^q - (1+q\varphi)|}{|\varphi|^q} \leqslant \left(\frac{q(q-1)}{2} + \epsilon\right)\delta^{2-q}, \quad |\varphi|^2 < \delta^{2-q}|\varphi|^q.$$

Then we have a uniform estimate for $q > 2$,

$$\left| |1+\varphi|^q - \left(1 + q\varphi + \frac{q(q-1)}{2}\varphi^2\right)\right| \lesssim \delta\varphi^2 + \delta^{2-q}|\varphi|^q,$$

and by integration, we have

$$\left| |1+\varphi|_q^q - \left(|1|_q^q + \frac{q(q-1)}{2}|\varphi|_2^2\right)\right| \lesssim \delta|1|_q^{q-2}|\varphi|_q^2 + \delta^{2-q}|\varphi|_q^q = o(|\varphi|_q^2),$$

which gives the assertion (4.3) and therefore (4.2) after raised to the power $2/q$ as

$$|1+\varphi|_q^2 = \left(|1|_q^q + \frac{q(q-1)}{2}|\varphi|_2^2 + o(|\varphi|_q^2)\right)^{\frac{2}{q}} = |1|_q^2 + \frac{2}{q}\frac{q(q-1)}{2}|1|_q^{2-q}|\varphi|_2^2 + o(|\varphi|_q^2).$$

Now we return to the Taylor expansion of the difference functional and to prove the local lower bound. Note that $T_1\mathcal{M}_* = \mathrm{Span}\{1, \mathrm{Re}\,\zeta_j, \mathrm{Im}\,\zeta_j\}_{j=1}^{2n+2} = \mathcal{H}_0 \oplus \mathcal{H}_1 = \mathcal{H}_{0,0} \oplus \mathcal{H}_{1,0} \oplus \mathcal{H}_{0,1}$ and from bispherical harmonics expansion $\varphi = \sum_{j,k}\varphi_{j,k}$, where $\varphi_{j,k} \in \mathcal{H}_{j,k}$, we see

$$\frac{|\varphi|_2^2}{\|\varphi\|_*^2} = \frac{\sum_{j+k\geqslant 2}|\varphi_{j,k}|_2^2}{\sum_{j+k\geqslant 2}\lambda_{j,k}|\varphi_{j,k}|_2^2},$$

and

$$\begin{aligned} I(f) &= \|\varphi\|_*^2 - (q-1)C|\mathbb{S}^{2n+1}|^{\frac{2-q}{q}}|\varphi|_2^2 + o(\|\varphi\|_*^2) \\ &= \|\varphi\|_*^2\left(1 - (q-1)C|\mathbb{S}^{2n+1}|^{\frac{2-q}{q}}\frac{|\varphi|_2^2}{\|\varphi\|_*^2} + o(1)\right). \end{aligned}$$

From the increasing of $\lambda_{j,k}$ on $j,k$ (see (2.4) and note $\lambda_{1,1} \geqslant \lambda_{2,0}$), we have $\frac{|\varphi|_2^2}{\|\varphi\|_*^2} \leqslant \lambda_{2,0}^{-1}$ and recalling the fact $\frac{\lambda_{1,0}}{C} = (q-1)|\mathbb{S}^{2n+1}|^{\frac{2-q}{q}}$ (see (2.2) and (2.4)), we finally get

$$\begin{aligned} I(f) &\geqslant \|\varphi\|_*^2(1 - (q-1)C|\mathbb{S}^{2n+1}|^{\frac{2-q}{q}}\lambda_{2,0}^{-1} + o(1)) \\ &= \|\varphi\|_*^2\left(\frac{2s}{Q+4+s} + o(1)\right). \end{aligned}$$

So, for all $s \in (0, Q)$, we give a positive local lower bound. For the upper bound, we use $\langle\varphi, 1\rangle_* = 0$ and get $I(f) = \|\varphi\|_*^2 + \|1\|_*^2 - C|1+\varphi|_q^2 \leqslant \|\varphi\|_*^2$, recalling $|1+\varphi|_q^2 \geqslant |\mathbb{S}^{2n+1}|^{2/q-1}|1+\varphi|_2^2 \geqslant |\mathbb{S}^{2n+1}|^{2/q} = |1|_q^2$, which reaches equality if and only if $\varphi \equiv 0$, or equivalently $d(f, \mathcal{M}_*) = 0$. The inequality (4.1) and therefore Proposition 4.1 are proved. □

For global estimate of the stability, we also need the following lemma asserting the possibility of recovering compactness from the conformal symmetries. The lemma is proved in a technical $TT^*$ way for dual HLS inequality in [14] and can also be derived from suitably adapted *concentration compactness* argument, originated from Lions's work for HLS inequality on $\mathbb{R}^n$ [21]. It is also natural to apply directly the adapted profile decomposition from Gérard's work [16] on Euclidean spaces.

**Lemma 4.2** (Recovery of compactness for FS). *Let* $(f_j)$ *be a* (*non-vanishing*) *extremizing sequence of inequality* (2.3) (*or its functional*), *i.e.*, $\frac{\|f_j\|_*^2}{|f_j|_q^2} \xrightarrow{j\to\infty} C$, *then* $\frac{d(f_j, \mathcal{M}_*)}{\|f_j\|_*} \xrightarrow{j\to\infty} 0$.

We can now use the local stability—Proposition 4.1 and above recovery of compactness property—Lemma 4.2 to prove the global stability for FS inequality.

*Proof of Theorem* 2.1. The core is the lower bound for (2.7). For the upper bound, as $0 \in \overline{\mathcal{M}_*}$, we have $d(f, \mathcal{M}_*) \leqslant \|f\|_*$, then it is trivial from the upper bound of (4.1) in Proposition 4.1. For the lower bound of (2.7), we argue by contradiction (this works only for existence of a positive constant). If the constant $\alpha > 0$ does not exist, then we can get a sequence $(f_j)$ satisfying $\frac{\|f_j\|_*^2 - C|f_j|_q^2}{d^2(f_j,\mathcal{M}_*)} \xrightarrow{j\to\infty} 0$ with $d(f_j, \mathcal{M}_*) > 0$. From $\frac{d^2(f,\mathcal{M}_*)}{\|f\|_*^2} \leqslant 1$, we have $\frac{\|f_j\|_*^2}{|f_j|_q^2} \xrightarrow{j\to\infty} C$, and then from Lemma 4.2, we have $\frac{d(f_j,\mathcal{M}_*)}{\|f_j\|_*} \xrightarrow{j\to\infty} 0$, which tells that $(f_j)$ is not only an extremizing sequence but also a local one to $\mathcal{M}_*$, which implies from the local lower bound of (4.1) in Proposition 4.1 that, $\liminf_{j\to\infty} \frac{\|f_j\|_*^2 - C|f_j|_q^2}{d^2(f_j,\mathcal{M}_*)} \geqslant \frac{2s}{Q+4+s} > 0$. So we have got a contradiction and therefore proven the global theorem totally. □

## 4.2 Proof of HLS stability

We now still use the contradiction idea to prove Theorem 2.2, the global stability of the HLS inequality. For (2.10), first we need to consider locally, and recalling the failure of Taylor expansion of associated functional (see the second variation in *a priori* Taylor expansion $|\varphi|_2^2 - \lambda_{1,0}\|\varphi\|_{-*}$, which tells the falseness of the expansion when $\varphi$ is not in $L^2 \subset L^p$). We borrow a lemma due to Christ [8] to get the lower bound, which Christ used to consider the stability for the Hausdorff-Young inequality functional.

**Lemma 4.3** (Christ's lemma). *For any exponents $p < 2 \leqslant q$, there exist positive constants, $c_0, c_1, \eta_0$, $\gamma$ $(> 1)$, such that given any small $\eta, \delta$ satisfying $0 < \eta \leqslant \eta_0$, $0 < \delta < \eta^\gamma$, any $(L^p, L^q)$-bounded linear operator for an arbitrary pair of measure spaces with norm $\|T\|$ and some non-vanishing extremizer $F$ (i.e., $|TF|_q = \|T\||F|_p$), and any $f \in L^p$ satisfying $|f|_p \leqslant \delta|F|_p$, $\operatorname{Re}\int f\bar{F}|F|^{p-2} = 0$. If we decompose $f = f_1 + f_2$, where $f_1 = f\chi_{|f|\leqslant\eta}$, then we have the following estimate of the functional*:

$$\frac{|T(F+f)|_q}{\|T\||F+f|_p} \leqslant 1 + \phi(f_1) + c_1\eta|f_1|_p^2|F|_p^{-2} - c_0\eta^{2-p}|f_2|_p^p|F|_p^{-p}, \tag{4.4}$$

*where*

$$\begin{aligned}\phi(f) = \frac{q-1}{2}|TF|_q^{-q}\int\left(\operatorname{Re}\frac{Tf}{TF}\right)^2|TF|^q + \frac{1}{2}|TF|_q^{-q}\int\left(\operatorname{Im}\frac{Tf}{TF}\right)^2|TF|^q \\ - \frac{p-1}{2}|F|_p^{-p}\int\left(\operatorname{Re}\frac{f}{F}\right)^2|F|^p - \frac{1}{2}|F|_p^{-p}\int\left(\operatorname{Im}\frac{f}{F}\right)^2|F|^p.\end{aligned} \tag{4.5}$$

The proof of this general lemma is to use the expansion property of function $|1+z|^r$, $r = p, q$, similar to that in last subsection, but here things are much more complicated, and we refer to [8] for details. Then we have the following local estimate:

**Proposition 4.4** (Local stability for HLS). *Let $Q, s, p$ be the same as those in Theorem 2.2. Then there exist positive constants $\alpha_0, \alpha_1$ depending only on dimension $Q$ and $s$, such that $\forall\, 0 \not\equiv f \in L^p$,*

$$o\left(\frac{d_p(f,\mathcal{M}_{-*})}{|f|_p}\right) + \alpha_1\,\frac{d_p(f,\mathcal{M}_{-*})}{|f|_p} \geqslant C^{-1} - \frac{\|f\|_{-*}^2}{|f|_p^2} \geqslant \alpha_0\,\frac{d_p^2(f,\mathcal{M}_{-*})}{|f|_p^2} + o\left(\frac{d_p^2(f,\mathcal{M}_{-*})}{|f|_p^2}\right). \tag{4.6}$$

*Proof.* We denote $f_p$ to be the $L^p$-nearest point, and locally, we may assume $f_p = c|J_\tau|^{1/p}$ for some $c \neq 0$ and conformal transformation $\tau$, noting $\frac{c}{|f|_p} \to 1$ as $\frac{d_p(f,\mathcal{M}_{-*})}{|f|_p} \to 0$. Denote $g = c^{-1} f\circ\tau^{-1}|J_{\tau^{-1}}|^{1/p}$, then $g_p = 1$ and we write $g = 1 + \varphi$ with $|\varphi|_p \to 0$ and $\int\varphi = 0$ (this condition needs to be satisfied in Christ's lemma, and if $\int\varphi \neq 0$, then update $\varphi$ by $\frac{\varphi - \fint\varphi}{1+\fint\varphi}$ and note that $|\varphi - \fint\varphi|_p = d_p(g, 1+\fint\varphi) \geqslant d_p(g,\mathcal{M}_{-*}) = |\varphi|_p$). Then it suffices to prove (4.6) for $g$ as the inequality is invariant under the above transformation. For the lower bound, we use Lemma 4.3 (taking there $T = \mathcal{A}_s^{-1/2}, q = 2, p = \frac{2Q}{Q+s}, F = 1, \|T\| = C^{-1/2}, f = \varphi, \delta = |\varphi|_p/|1|_p, \eta = (1+\epsilon)\delta^{1/\gamma}$), then from (4.4) and (4.5) we have

$$\frac{C\|g\|_{-*}^2}{|g|_p^2} \leqslant \left(1 + \frac{1}{2}\|1\|_{-*}^{-2}\|\varphi_1\|_{-*}^2 - \frac{p-1}{2}|1|_p^{-p}|\varphi_1|_2^2 + c_1\eta|\varphi_1|_p^2|1|_p^{-2} - c_0\eta^{2-p}|\varphi_2|_p^p|1|_p^{-p}\right)^2.$$

We easily see $\|\varphi_1\|_{-*}^2 \lesssim |\varphi_1|_p^2 = o(|\varphi|_p^2), |\varphi_1|_2^2 \leqslant \eta^{2-p}|\varphi|_p^p$ and it follows from $\eta \gtrsim |\varphi|_p$ that

$$1-\frac{C\|g\|_{-*}^2}{|g|_p^2} \gtrsim \eta^{2-p}|\varphi|_p^p + o(|\varphi|_p^2) \gtrsim |\varphi|_p^2 + o(|\varphi|_p^2),$$

which gives the right-hand side of (4.6). For the upper bound, we denote the $H^{-*}$-nearest point $g_* = c'|J'_\tau|^{1/p}$, where $c' \neq 0$ as $g_* \xrightarrow{\|\cdot\|_{-*}} 1$. Then from the conformal invariance and extremizer property, we have

$$|g|_p^2 - C\|g\|_{-*}^2 \leqslant |g|_p^2 - C\|g_*\|_{-*}^2 = |g|_p^2 - |g_*|_p^2 = |g|_p^2 - |1|_p^2 + (1-|c'|^2)|1|_p^2.$$

Note that, $|g|_p^2 - |1|_p^2 \leqslant |\varphi|_p(|g|_p + |1|_p) = 2|1|_p|\varphi|_p + o(|\varphi|_p)$ and $|c'| \to 1$. Actually, $|1-|c'|| \lesssim |1-g_*|_{-*} \lesssim |\varphi|_p + |g-g_*|_{-*} \lesssim |\varphi|_p$. Above all, we get

$$|g|_p^2 - C\|g\|_{-*}^2 \lesssim |\varphi|_p + o(|\varphi|_p),$$

then we get a local upper bound for $f$ via the relation between $f$ and $g$,

$$1-\frac{C\|f\|_{-*}^2}{|f|_p^2} \lesssim \frac{c}{|f|_p}\bigg(\frac{d_p(f,\mathcal{M}_{-*})}{|f|_p} + o\bigg(\frac{d_p(f,\mathcal{M}_{-*})}{|f|_p}\bigg)\bigg).$$

Finally from $\frac{|c|}{|f|_p} \lesssim 1 + |1-\frac{|c|}{|f|_p}|1|_p| \lesssim 1 + |\frac{f}{|f|_p} - \frac{f_p}{|f|_p}|_p = 1 + \frac{d_p(f,\mathcal{M}_{-*})}{|f|_p}$, we get the left-hand side of (4.6) (we can also see directly from the conformal invariance). Proposition 4.4 is then proved. □

For the global stability, we only need the recovery of compactness (dual of Lemma 4.2, see a different formulation in [14, Lemma 4.6] and we can also use suitably adapted concentration compactness argument) and then apply the contradiction argument.

**Lemma 4.5** (Recovery of compactness for HLS). *Let* $(f_j)$ *be a* (*non-vanishing*) *extremizing sequence of* (2.8) (*or its functional*) *on the sphere, i.e.,* $\frac{\|f_j\|_{-*}^2}{|f_j|_p^2} \xrightarrow{j\to\infty} C^{-1}$, then $\frac{d_p(f_j,\mathcal{M}_{-*})}{|f_j|_p} \xrightarrow{j\to\infty} 0$.

*Proof of Theorem* 2.2. We assume the statement for (2.10) in Theorem 2.2 is wrong, then there exist two non-degenerate sequences of functions $(f_j)$ and $(g_j)$ satisfying

$$\lim_{j\to\infty} \frac{C^{-1} - \frac{\|f_j\|_{-*}^2}{|f_j|_p^2}}{\frac{d_p^2(f_j,\mathcal{M}_{-*})}{|f_j|_p^2}} = 0 \quad \text{and} \quad \lim_{j\to\infty} \frac{C^{-1} - \frac{\|g_j\|_{-*}^2}{|g_j|_p^2}}{\frac{d_p(g_j,\mathcal{M}_{-*})}{|g_j|_p}} = \infty. \tag{4.7}$$

Then $\lim_{j\to\infty} \frac{\|f_j\|_{-*}^2}{|f_j|_p^2} = C^{-1}$, recalling $\frac{d_p(f,\mathcal{M}_{-*})}{|f|_p} \leqslant 1$, which gives $\lim_{j\to\infty} \frac{d_p(f_j,\mathcal{M}_{-*})}{|f_j|_p} = 0$ from Lemma 4.5. Simultaneously, we have $\lim_{j\to\infty} \frac{d_p(g_j,\mathcal{M}_{-*})}{|g_j|_p} = 0$. Actually, we see this from Lemma 4.5 if $\lim_{j\to\infty} \frac{\|g_j\|_{-*}^2}{|g_j|_p^2} = C^{-1}$, and otherwise after moving to a subsequence as $\liminf_{j\to\infty} \frac{\|g_j\|_{-*}^2}{|g_j|_p^2} < C^{-1}$. So, the formulae in (4.7) are local limitations, which contradicts the local bound (4.6) in Proposition 4.4. □

### 4.3 Proof of proportional inequalities between dual remainder terms

Now we are going to prove the relations between the remainder terms of FS and HLS inequalities and those of BO and Log-HLS inequalities. First, for the FS and HLS.

*Proof of Theorem* 2.3. For the local bound (2.12), we expand the two remainder terms by Taylor expansion. Take $f_*$ to be the $H^*$-nearest point, and as before we may assume $f_* \not\equiv 0$ (we can add other condition like $\|f\|_* = 1$ locally), and further $f_* = 1, f = 1+\varphi$, where $\varphi \perp T_1\,\mathcal{M}_*$ under $\langle\cdot,\cdot\rangle_*$. We denote the two remainder functionals respectively by

$$I_1(f) = \|f\|_*^2 - C|f|_q^2, \quad I_2(f) = |f|_p^2 - C\|f\|_{-*}^2.$$

Then we expand and estimate the two terms

$$\begin{aligned}I_1(f) = I_1(1+\varphi) &= I_1(1) + \frac{d}{dt}\bigg|_{t=0} I_1(1+t\varphi) + \frac{1}{2}\frac{d^2}{dt^2}\bigg|_{t=0} I_1(1+t\varphi) + o(\|\varphi\|_*^2)\\ &= \|\varphi\|_*^2 - (q-1)C|\,\mathbb{S}^{2n+1}\,|^{2/q-1}|\varphi|_2^2 + o(\|\varphi\|_*^2),\\ I_2(f^{q/p}) &= I_1((1+\varphi)^{q/p})\\ &= I_2(1) + \frac{d}{dt}\bigg|_{t=0} I_2((1+t\varphi)^{q/p}) + \frac{1}{2}\frac{d^2}{dt^2}\bigg|_{t=0} I_2((1+t\varphi)^{q/p}) + o(\|\varphi\|_*^2)\\ &= (q-1)^2|\,\mathbb{S}^{2n+1}\,|^{2/p-1}|\varphi|_2^2 - \frac{q}{p}\left(\frac{q}{p}-1\right)C\langle 1,\varphi^2\rangle_{-*} - \left(\frac{q}{p}\right)^2 C\|\varphi\|_{-*}^2 + o(\|\varphi\|_*^2)\\ &= \left((q-1)^2|\,\mathbb{S}^{2n+1}\,|^{2/p-1} - \frac{q}{p}\left(\frac{q}{p}-1\right)C\lambda_{0,0}^{-1}\right)|\varphi|_2^2 - \left(\frac{q}{p}\right)^2 C\|\varphi\|_{-*}^2 + o(\|\varphi\|_*^2)\\ &= (q-1)|\,\mathbb{S}^{2n+1}\,|^{2/p-1}|\varphi|_2^2 - \left(\frac{q}{p}\right)^2 C\|\varphi\|_{-*}^2 + o(\|\varphi\|_*^2),\end{aligned}$$

where we have used $C\lambda_{0,0}^{-1} = |\,\mathbb{S}^{2n+1}\,|^{2/p-1}$ and the $C^2(H^*)$ property of $I_1(f)$ and $I_2(f^{q/p})$ (this can be checked easily as in the proof of (4.2), Proposition 4.1). Taking bispherical harmonics decomposition on $\varphi$ and recalling $\varphi \perp \mathcal{H}_0 \oplus \mathcal{H}_1$, we can set $\varphi = \sum_{j+k\geqslant 2}\varphi_{j,k}$, then we want to study the following quotient:

$$\frac{I_2(f^{q/p})}{I_1(f)} = (q-1)|\,\mathbb{S}^{2n+1}\,|^{2/p-1}\frac{\sum_{j+k\geqslant 2}(1-\frac{q}{p}C|\,\mathbb{S}^{2n+1}\,|^{1-2/p}\lambda_{j,k}^{-1})|\varphi_{j,k}|_2^2}{\sum_{j+k\geqslant 2}(\lambda_{j,k}-(q-1)C|\,\mathbb{S}^{2n+1}\,|^{2/q-1})|\varphi_{j,k}|_2^2} + o(1).$$

Note that

$$\frac{1-\frac{q}{p}C|\,\mathbb{S}^{2n+1}\,|^{1-2/p}\lambda_{j,k}^{-1}}{\lambda_{j,k}-(q-1)C|\,\mathbb{S}^{2n+1}\,|^{2/q-1}} = \frac{1}{\lambda_{j,k}},$$

we then have

$$\limsup_{0<d(f,\mathcal{M}_*)\to 0,\, d(f,\mathcal{M}_*)<\|f\|_*} \frac{I_2(f^{q/p})}{|f|_q^{2(q-2)}I_1(f)} = \frac{(q-1)|\,\mathbb{S}^{2n+1}\,|^{1-2/p}}{\lambda_{2,0}} = \frac{\lambda_{1,0}}{C\lambda_{2,0}} = \frac{1}{C}\frac{Q+4-s}{Q+4+s}.$$

For the global bound (2.11), we use the square idea as in [13]. For $r>0$ to be fixed later,

$$\begin{aligned}0 &\leqslant ||f|_q^r\mathcal{A}_s^{1/2}f - C\mathcal{A}_s^{-1/2}f^{q/p}|_2^2\\ &= |f|_q^{2r}|\mathcal{A}_s^{1/2}f|_2^2 + C^2|\mathcal{A}_s^{-1/2}f^{q/p}|_2^2 - 2C|f|_q^r\langle\mathcal{A}_s^{1/2}f,\mathcal{A}_s^{-1/2}f^{q/p}\rangle\\ &= |f|_q^{2r}(\|f\|_*^2 - C|f|_q^{-r+q}) - C(|f^{q/p}|_p^{p(r+q)/q} - C\|f^{q/p}\|_{-*}^2).\end{aligned}$$

Take $r = q-2 = \frac{2s}{Q-s}$, then $-r+q = p(r+q)/q = 2$, and

$$|f|_q^{2r}(\|f\|_*^2 - C|f|_q^2) \geqslant C(|f^{q/p}|_p^2 - C\|f^{q/p}\|_{-*}^2).$$

So Theorem 2.3 is proved. □

Now we are going to prove the endpoint case—relation between the remainder terms of BO and Log-HLS inequalities—using the result about FS and HLS by differentiation argument globally and still the Taylor expansion for the local estimate.

*Proof of Theorem* 3.1. For the global bound (3.7), we can see from the remainder terms control between FS and HLS inequalities (2.11) in Theorem 2.3 by the differential argument. Actually we can write the inequality (2.11) to the form $I_1(f) \geqslant I_2(f)$, where the associated two functionals are

$$I_1(f) = \left(⨍|f|^q\right)^{2s/Q}\left(⨍ f\mathcal{A}_s f - \frac{\Gamma(\frac{Q+s}{4})}{\Gamma(\frac{Q-s}{4})}\left(⨍|f|^q\right)^{2/q}\right),$$

$$I_2(f)=\frac{\Gamma^2(\frac{Q+s}{4})}{\Gamma^2(\frac{Q-s}{4})}\bigg(\bigg(\fint|f|^q\bigg)^{2/p}-\frac{\Gamma^2(\frac{Q+s}{4})}{\Gamma^2(\frac{Q-s}{4})}\fint f^{q/p}\mathcal{A}_s^{-1}f^{q/p}\bigg),$$

then from (3.3), $\mathcal{A}_s=\frac{\Gamma(\frac{Q+s}{4})}{\Gamma(\frac{Q-s}{4})}\mathcal{A}'_Q+o(\lambda)$ and by Taylor expansion on $\lambda$, we get

$$\begin{aligned}I_1\bigg(1+\frac{\lambda}{2Q}f\bigg)&=\bigg(\bigg(\fint \mathrm{e}^f\bigg)^2+o(1)\bigg)\bigg(\fint 1\mathcal{A}_s1+\bigg(\frac{\lambda}{2Q}\bigg)^2\fint\bigg(f-\fint f\bigg)\mathcal{A}_s\bigg(f-\fint f\bigg)\\&\quad-\frac{\Gamma^2(\frac{Q+s}{4})}{\Gamma^2(\frac{Q-s}{4})}\bigg(1+\frac{\lambda}{Q}\log\fint \mathrm{e}^{f-\fint f}+o(\lambda)\bigg)\bigg)\\&=\frac{1}{Q}\bigg(\frac{n!}{4}\bigg)^2\bigg(\fint\mathrm{e}^f\bigg)^2\bigg(\frac{1}{2(n+1)!}\fint f\mathcal{A}'_Qf-\log\fint \mathrm{e}^{f-\fint f}\bigg)\lambda^3+o(\lambda^3),\end{aligned}$$

and

$$\begin{aligned}I_2\bigg(1+\frac{\lambda}{2Q}f\bigg)&=\bigg(\frac{n!\lambda}{4}\bigg)^2\bigg(\bigg(\fint\mathrm{e}^f\bigg)^2\bigg(1-\bigg(\frac{\log\fint\mathrm{e}^f}{Q}+\frac{1}{2Q}\frac{\fint\mathrm{e}^ff^2}{\fint\mathrm{e}^f}\bigg)\lambda+o(\lambda)\bigg)\\&\quad-\bigg(\fint\bigg(1+\frac{\lambda}{2Q}f\bigg)^{\frac{2Q-\lambda}{\lambda}}\bigg)^2-\frac{\Gamma(\frac{Q+s}{4})}{\Gamma(\frac{Q-s}{4})}\fint\bigg(\mathrm{e}^f-\fint\mathrm{e}^f\bigg)\mathcal{A}_Q'^{-1}\bigg(\mathrm{e}^f-\fint\mathrm{e}^f\bigg)\bigg)\\&=\frac{1}{Q}\bigg(\frac{n!}{4}\bigg)^2\bigg(\fint\mathrm{e}^f\fint\mathrm{e}^ff-\bigg(\fint\mathrm{e}^f\bigg)^2\log\fint\mathrm{e}^f-\frac{(n+1)!}{2}\\&\quad\times\fint\bigg(\mathrm{e}^f-\fint\mathrm{e}^f\bigg)\mathcal{A}_Q'^{-1}\bigg(\mathrm{e}^f-\fint\mathrm{e}^f\bigg)\bigg)\lambda^3+o(\lambda^3).\end{aligned}$$

Then (3.7) is proved by comparing the dominating terms: Dividing the two formulas by $\lambda^3$ and taking $\lambda=Q-s\to 0$ .

For local estimate (3.8), by constant translation invariance of the two terms in the quotient formula, we can assume $f\perp\mathcal{H}_0\oplus\mathcal{H}_1$. Now, we consider Taylor expansion of the quotient, which we denote by $I(f)$,

$$\begin{aligned}I(f)&=\frac{\frac{1}{2(n+1)!}(\fint f\mathcal{A}'_Qf-(n+1)!\fint f^2)+o(\fint f\mathcal{A}'_Qf)}{\frac12(\fint f^2-(n+1)!\fint f\mathcal{A}_Q'^{-1}f)+o(\fint f\mathcal{A}'_Qf)}\\&=\frac{1}{(n+1)!}\frac{\sum_{j\geqslant2}(\frac{\Gamma(j+n+1)}{\Gamma(j)}-(n+1)!)(|f_{j,0}|_2^2+|f_{0,j}|_2^2)}{\sum_{j\geqslant2}(1-\frac{\Gamma(j)}{\Gamma(j+n+1)}(n+1)!)(|f_{j,0}|_2^2+|f_{0,j}|_2^2)}\\&\geqslant n+2.\end{aligned}$$

This completes the proof of Theorem 3.1. □

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