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Topological complexity, minimality and systems of order two on torus

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Abstract The dynamical system on $\mathbb{T}^2$ which is a group extension over an irrational rotation on $\mathbb{T}^1$ is investigated. The criterion when the extension is minimal, a system of order 2 and when the maximal equicontinuous factor is the irrational rotation is given. The topological complexity of the extension is computed, and a negative answer to the latter part of an open question raised by Host et al. (2014) is obtained.

Keywords topological complexity, minimality, 2-step nilsystem

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1 Introduction

Throughout this paper, by a *topological dynamical system* (t.d.s. for short) we mean a pair (X, T), where X is a compact metric space and $T : X \to X$ is a homeomorphism. In this section, we first discuss the motivations and then state the main results of the article.

The study of the complexity of a dynamical system is one of the main topics in the study of the system. There are several ways to measure the complexity of a t.d.s.. Entropy is a topological invariant and a t.d.s. with positive entropy means that the complexity of the system is "big". We now discuss the so-called *topological complexity*, which was formally introduced in [2] and is suitable to measure systems with 'lower' complexity, especially systems with zero entropy. Let (X, T) be a t.d.s. and $\mathcal{U}$ be an open cover of X. Define the complexity function with respect to $\mathcal{U}$ as $n \mapsto \mathcal{N}(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U})$. We remark that studying the topological complexity for a subshift has a long history, which is the complexity with respect to the open cover consisting of cylinders of length 1, see for example [12].

It was shown in [2] that a t.d.s. is equicontinuous if and only if each nontrivial open cover has a bounded topological complexity. Since an equicontinuous system is distal (which has zero topological entropy) and each minimal distal t.d.s. is the result of a transfinite sequence of equicontinuous extensions, and their limits, starting from a t.d.s. consisting of a singleton, it is natural to ask what the complexity of a minimal distal system could be.

For a special class of minimal distal systems, namely systems of order d which are the inverse limits of minimal d-step nilsystems (see Subsection 2.4 for definitions) it was proved in [3] that the complexity function is bounded above by a polynomial. Host et al. [10] refined the result of [3] by giving the explicit degree of the polynomial and showing that the lower bound and the upper bound have the same degree. To state the result we note that the complexity defined by the open cover can be rephrased in the language of (n,ϵ)-spanning sets, namely one may consider the smallest cardinality $r(n,\epsilon)$ of (n,ϵ)-spanning sets instead of the smallest cardinality of the subcovers. In this language one of the main results in [10] can be stated as follows:

Let $(X=G/\Gamma,T)$ be a minimal s-step nilsystem (see Subsection 2.4 for the definition) for some $s\geqslant 2$ and assume that (X,T) is not an $(s-1)$-step nilsystem. Let d_X be a distance on X defining its topology. Then for every $\epsilon>0$ that is sufficiently small, there exist positive constants $c(\epsilon),c'(\epsilon)$ and $p\geqslant s-1$ such that the topological complexity $r(n,\epsilon)$ of (X,T) for the distance d_X satisfies

$$c(\epsilon)n^p\leqslant r(n,\epsilon)\leqslant c'(\epsilon)n^p \quad \text{for every} \quad n\geqslant 1.$$

Moreover, $c(\epsilon)\to+\infty$ as $\epsilon\to 0$.

An open question asked in [10, Question 1] is what systems have the same topological complexity as nilsystems, namely,

Question. *Characterize the minimal t.d.s.* (X,T) *satisfying the following property* (1.1):

$$\begin{aligned}&\text{For every } \epsilon>0 \text{ small enough, there exist constants } c_1(\epsilon),c_2(\epsilon)>0 \text{ such that}\\ &c_1(\epsilon)n\leqslant r(n,\epsilon)\leqslant c_2(\epsilon)n \text{ for every } n\geqslant 1 \text{ and } c_1(\epsilon)\to\infty \text{ as } \epsilon\to 0.\end{aligned} \tag{1.1}$$

If in addition, we assume that (X,T) *is a distal system, then is it a* 2*-step nilsystem?*

We will give a negative answer to the latter part of this question in this paper. To do this, we consider a t.d.s. on $\mathbb{T}^2$ which is a group extension over an irrational rotation on $\mathbb{T}^1$. The criterion when the extension is minimal, a system of order 2 and when the maximal equicontinuous factor is the rotation on $\mathbb{T}^1$ is given. We note that dynamical systems on $\mathbb{T}^2$ have been studied by many authors, see for example [6–9].

In order to explicitly state our results, we denote $\mathbb{T}^2=\mathbb{T}^1\times\mathbb{T}^1$ with the metric

$$d((x_1,y_1),(x_2,y_2))=\max\{\|x_1-x_2\|,\|y_1-y_2\|\},$$

where $\|r-s\|=\min_{m\in\mathbb{Z}}|r-s+m|$. Let $(\mathbb{T}^2,T)$ be the t.d.s., where T acts on $\mathbb{T}^2$ as follows:

$$T:\mathbb{T}^2\to\mathbb{T}^2,\quad (x,y)\mapsto(x+\alpha,f(x)+y) \tag{1.2}$$

with $f\in\mathcal{F}_l:=\{h:\mathbb{R}\to\mathbb{R}: h \text{ is continuous on } \mathbb{R},\ h(x+1)-h(x)\equiv l \text{ for all } x\in\mathbb{R}\}$, $l\in\mathbb{Z}$ and $\alpha\in\mathbb{R}\setminus\mathbb{Q}$.

Now we state the main results of this paper. In Theorem A, we compute the topological complexity for a class of systems $(\mathbb{T}^2,T)$ when the function f satisfies some mild conditions.

Theorem A. *Let* $(\mathbb{T}^2,T)$ *be a t.d.s. defined in* (1.2) *such that* $f\in\mathcal{F}_l$, $l\neq 0$, $\alpha\in\mathbb{R}\setminus\mathbb{Q}$ *and* f *has a bounded variation on* $[0,1]$*. Then* (1.1) *holds.*

Theorem B gives a characterization of equivalence condition for the system $(\mathbb{T}^2,T)$ to be order 2.

Theorem B. *Let* $(\mathbb{T}^2,T)$ *be a t.d.s. defined in* (1.2) *such that* $f\in\mathcal{F}_l$ *and* $l\neq 0$*. Then the following statements are equivalent*:

(1) $(\mathbb{T}^2,T)$ *is a system of order* 2.

(2) *There exist* $\varphi\in\mathcal{F}_0$ *and* $c\in\mathbb{R}$ *such that* $f(x)=lx+\varphi(x+\alpha)-\varphi(x)+c$ *for any* $x\in\mathbb{R}$.

For an irrational number α we may define a number $\nu(\alpha)$ which measures the approximality of α by rational numbers (see Section 5). We remark that the Lebesgue measure of $\{\alpha\in(0,1):\nu(2\pi\alpha)=0\}$ is 1. In Theorem C, we give a minimal distal system whose topological complexity is low, but it is not a system of order 2. Moreover, by the construction of our example, we know that such systems are numerous.

Theorem C. *For a given* $l\neq 0$ *and an irrational number* α *with* $v(2\pi\alpha)=0$*, there exists a function* $f\in\mathcal{F}_l$ *such that the t.d.s.* $(\mathbb{T}^2,T)$ *defined in* (1.2) *by* f *is a minimal distal system but not a system of order* 2*, and* (1.1) *holds.*

For readers interested in zero entropy diffeomorphisms on manifolds (particular $\mathbb{T}^2$), it is worth mentioning that, Frączek [6, 7] concentrated on ergodic diffeomorphisms of $\mathbb{T}^2$ with polynomial (or linear) growth of the derivative and obtained that they are (in some sense) “conjugate” to (1.2) with $l\neq 0$.

The paper is organized as follows. In Section 2, some definitions and related results are introduced. In Section 3, we prove that if f satisfies some mild conditions, then its topological complexity of $(\mathbb{T}^2, T)$ is low. In Section 4, we show that some system $(\mathbb{T}^2, T)$ is minimal distal and its maximal equicontinuous factor is the irrational rotation. In Section 5, we give the main result in this paper, i.e., there exists a minimal distal system $(\mathbb{T}^2, T)$ such that its topological complexity is low and it is not a system of order 2.

2 Preliminaries

2.1 Topological dynamical systems

A *topological dynamical system* (t.d.s. for short) is a pair (X, T), where X is a compact metric space and $T : X \to X$ is a homeomorphism from X to itself. We use d to denote the metric on X.

A t.d.s. (X, T) is *transitive* if for any non-empty open sets U and V in X, there exists $n \in \mathbb{Z}$ such that $U \cap T^n V \neq \emptyset$. We say $x \in X$ is a *transitive point* if its orbit $\mathrm{orb}(x, T) = \{x, Tx, T^2x, \dots\}$ is dense in X. A t.d.s. (X, T) is *minimal* if the orbit of any point is dense in X. We say $x \in X$ is a *minimal point* if $(\overline{\mathrm{orb}(x, T)}, T)$ is a minimal subsystem of (X, T).

Let (X, T) be a t.d.s. and $(x, y) \in X \times X$. We say (x, y) is a *proximal pair* if $\inf_{n\in\mathbb{Z}} d(T^n x, T^n y) = 0$, and it is a *distal pair* if it is not proximal. A t.d.s. (X, T) is called *distal* if (x, y) is distal whenever $x, y \in X$ are distinct. The following result is classical.

Lemma 2.1 (See [1]). *Suppose (X, T) is a distal t.d.s., then for any point $x \in X$, x is a minimal point. In particular, if (X, T) is distal, then (X, T) is minimal if and only if (X, T) has a transitive point.*

A *homomorphism* $\pi : X \to Y$ between topological dynamical systems (X, T) and (Y, S) is a continuous onto map such that $\pi \circ T = S \circ \pi$; one says that (Y, S) is a *factor* of (X, T) and that (X, T) is an *extension* of (Y, S), and one also refers to π as a *factor map* or an *extension*. The systems are said to be *conjugate* if π is bijective.

Given a t.d.s. (X, T), define the *regionally proximal relation*:

$$Q(X, T) = \bigcap_{k=1}^{+\infty} \overline{\bigcup_{n=-\infty}^{+\infty} (T \times T)^{-n} \Delta_{\frac{1}{k}}},$$

where $\Delta_{\frac{1}{k}} := \{(x, y) \in X \times X : d(x, y) < 1/k\}$. It is clear that $(x, y) \in Q(X, T)$ if and only if for any $\epsilon > 0$, any neighbourhoods U and V of x and y, respectively, there exist $x' \in U$, $y' \in V$ and $n \in \mathbb{Z}$ such that $d(T^n x', T^n y') < \epsilon$.

A t.d.s. (X, T) is said to be *equicontinuous* if the family of $\{T^n : n \in \mathbb{Z}\}$ is equicontinuous, i.e., for any $\epsilon > 0$, there exists $\delta > 0$ such that if $d(x_1, x_2) < \delta$, then $d(T^n x_1, T^n x_2) < \epsilon$ for any $n \in \mathbb{Z}$. The following result is well known.

Lemma 2.2 (See [1, Chapter 5]). *Suppose $\pi : (X, T) \to (Y, S)$ is an extension between two t.d.s.. Then the following are equivalent*:

(1) (Y, S) *is equicontinuous.*

(2) $Q(X, T) \subset R_\pi$, *where* $R_\pi = \{(x, y) \in X \times X : \pi(x) = \pi(y)\}$.

In particular, the maximal equicontinuous factor of (X, T) is induced by the smallest closed invariant equivalence relation containing $Q(X, T)$.

2.2 Topological complexity

Let (X, T) be a t.d.s. and denote by d the metric on X. For any $n \in \mathbb{N}$ and $\epsilon > 0$, a subset F of X is said to be an (n, ϵ)-*spanning set of* X with respect to T if for any $x \in X$, there exists $y \in F$ with $d_n(x, y) \leqslant \epsilon$, where

$$d_n(x, y) = \max_{0\leqslant i\leqslant n-1} d(T^i(x), T^i(y)).$$

Let $r(n,\epsilon)$ denote the smallest cardinality of all (n,ϵ)-spanning subsets of X with respect to T, we call $r(n,\epsilon)$ the *topological complexity* of the system (X,T). We write $r(n,\epsilon,T)$ to emphasise T if we need to. We can also define topological complexity in terms of (n,ϵ)-separated set. A subset E of X is said to be an (n,ϵ)-*separated* set of X with respect to T if $x,y\in E, x\neq y$, implies $d_n(x,y)>\epsilon$, where $d_n(x,y)$ is defined as mentioned above. Let $s(n,\epsilon)$ denote the largest cardinality of all (n,ϵ) separated subsets of X with respect to T. We write $s(n,\epsilon,T)$ to emphasise T if we need to.

We have $r(n,\epsilon)\leqslant s(n,\epsilon)\leqslant r(n,\epsilon/2)$ for any $\epsilon>0$ and $n\in\mathbb{N}$ (see [13, p. 169] for details).

2.3 Unique ergodicity

Suppose $(X,\mathcal{B}(X),\mu)$ is a probability space, where X is a compact metrizable space and $\mathcal{B}(X)$ is the smallest σ-algebra generated by all open subsets of X.

A continuous transformation $T: X\to X$ is called *uniquely ergodic* if there is only one T-invariant Borel probability measure μ on X, i.e., $\mu(T^{-1}(B))=\mu(B)$ for all $B\in\mathcal{B}(X)$.

It is well known that if $T(x)=ax$ is a rotation on the compact metrizable group G, then T is uniquely ergodic iff T is minimal. The Haar measure is the only T-invariant measure (see for example [13, p. 162]).

Lemma 2.3 (See [11])**.** *Let $T: X\to X$ be a continuous transformation of a compact metrizable space X. Then the following are equivalent*:

(1) *T is uniquely ergodic.*

(2) *There exists a T-invariant Borel probability measure μ such that for all $f\in C(X)$ and all $x\in X$, $\frac{1}{n}\sum_{i=0}^{n-1}f(T^ix)\to\int_X fd\mu$ as $n\to+\infty$.*

2.4 Nilpotent groups and systems of order 2

Let G be a group. For $g,h\in G$, we write $[g,h]=ghg^{-1}h^{-1}$ for the commutator of g and h and for $A,B\subset G$, we write $[A,B]$ for the subgroup spanned by $\{[a,b]: a\in A, b\in B\}$. The commutator subgroups $G_j, j\geqslant 1$, are defined inductively by setting $G_1=G$ and $G_{j+1}=[G_j,G]$. Let $d\geqslant 1$ be an integer, we say that G is *d-step nilpotent* if G_{d+1} is the trivial subgroup.

Let G be a d-step nilpotent Lie group and Γ a discrete cocompact subgroup of G. The compact manifold $X=G/\Gamma$ is called a *d-step nilmanifold*. The group G acts on X by left translations and we write this action as $(g,x)\mapsto gx$. Let $\tau\in G$ and T be the transformation $x\mapsto\tau x$, then (X,T) is called a *d-step nilsystem*.

The enveloping semigroup (or Ellis semigroup) $E(X)$ of a topological dynamical system (X,T) is defined as the closure in X^X of the set $\{T^n: n\in\mathbb{Z}\}$ endowed with the product topology.

Let (Y,S) be a t.d.s., K a compact group, and $\phi: Y\to K$ a continuous mapping. Form $X=Y\times K$ and define $T: X\to X$ by $T(y,k)=(Sy,\phi(y)k)$. The resulting system (X,T) is called a *group extension* of (Y,S). It is obvious that the system $(\mathbb{T}^2,T)$ defined in (1.2) is a group extension of an irrational rotation on $\mathbb{T}^1$ by taking $\phi=f$.

The following theorem relates the notion of system of order 2 and nilpotent group which will be used in this paper. We recall that a minimal topological dynamical system is *a system of order d* if it is an inverse limit of d-step nilsystems. In particular, a 2-step nilsystem is a system of order 2.

Theorem 2.1 (See [4, Theorem 1.2])**.** *Let (X,T) be a minimal t.d.s.. Then the following are equivalent*:

(1) *(X,T) is a system of order 2.*

(2) *$E(X)$ is a 2-step nilpotent group and (X,T) is a group extension of an equicontinuous system.*

The following theorem gives a more explicit characterization for the enveloping semigroup $E(\mathbb{T}^2)$ to become 2-step nilpotent.

Theorem 2.2 (See [9, Theorem 2.3])**.** *Suppose $(\mathbb{T}^2,T)$ is minimal, which is the t.d.s. defined in (1.2) such that $f\in\mathcal{F}_l$, $l\neq 0$, $\alpha\in\mathbb{R}\setminus\mathbb{Q}$ and that the projection $(\mathbb{T}^2,T)\xrightarrow{\pi}(\mathbb{T}^1,\tau)$ onto the first coordinate is the maximal equicontinuous factor, where $\tau:\mathbb{T}^1\to\mathbb{T}^1, x\mapsto x+\alpha$. Then the following are equivalent*:

(1) *There exist $\varphi\in\mathcal{F}_0$ and $c\in\mathbb{R}$ such that $f(x)=\varphi(x+\alpha)-\varphi(x)+lx+c$.*

(2) *The system $(\mathbb{T}^2,T)$ satisfies that $E(\mathbb{T}^2)$ (as an abstract group) is 2-step nilpotent.*

3 Proof of Theorem A

In this section, the topological complexity of the dynamical system on $\mathbb{T}^2$ is computed. We will show that their topological complexity is low in some cases. Firstly, we introduce some notation. Let f be a real-valued function on $[a,b]$, $\Delta : a = x_0 < x_1 < x_2 < \cdots < x_n = b$ be a partition,

$$v_\Delta = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|, \quad \text{and} \quad \bigvee_a^b (f) = \sup\{v_\Delta : \Delta \text{ is a partition over } [a,b]\}.$$

We say that f is a function with bounded variation if $\bigvee_a^b(f) < +\infty$.

It is well known that f has a bounded variation on $[a,b]$ if and only if $f(x) = g(x) - h(x)$ for all $x \in [a,b]$, where $g(x) = \frac{1}{2}(\bigvee_a^x(f) + f(x))$ and $h(x) = \frac{1}{2}(\bigvee_a^x(f) - f(x))$ are increasing functions on $[a,b]$.

Lemma 3.1. *Let $(\mathbb{T}^2, T)$ be a t.d.s. defined in* (1.2) *such that $f \in \mathcal{F}_l, l \neq 0$ and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. If f has a bounded variation on $[0,1]$, then*

$$s(n,\epsilon) \leqslant 20\bigg(\bigvee_0^1 (f) + 1\bigg) n/\epsilon^2 \tag{3.1}$$

for any $n \in \mathbb{N}$ and $\epsilon > 0$ small enough.

Proof. Clearly, $T^n(x,y) = (x + n\alpha, f_n(x) + y)$, where $f_n(x) = \sum_{i=0}^{n-1} f(x + i\alpha)$.

Let $g(x) = \frac{1}{2}(\bigvee_0^x(f) + f(x))$ and $h(x) = \frac{1}{2}(\bigvee_0^x(f) - f(x))$. Then g and h are increasing functions on $[0,1]$ satisfying

$$f(x) = g(x) - h(x), \quad g(x+1) - g(x) = M \quad \text{and} \quad h(x+1) - h(x) = M - l$$

for any $x \in \mathbb{R}$. Take $M = \frac{1}{2}(\bigvee_0^1(f) + l)$, then $M \geqslant 0$ and $M - l \geqslant 0$.

For $n \in \mathbb{N}$, let

$$g_n(x) = \sum_{i=0}^{n-1} g(x + i\alpha) \quad \text{and} \quad h_n(x) = \sum_{i=0}^{n-1} h(x + i\alpha).$$

We can choose a partition

$$\Delta_1 : 0 = s_0 < s_1 < s_2 < \cdots < s_m = 1$$

such that $s_{i+1} - s_i \leqslant \epsilon/2$ for $0 \leqslant i \leqslant m-1$ with $m = [2/\epsilon] + 1$, and a partition

$$\Delta_2 : 0 = t_0 < t_1 < t_2 < \cdots < t_r = 1$$

such that

$$g_n(t_{j+1}) - g_n(t_j) \leqslant \epsilon/4 \quad \text{and} \quad h_n(t_{j+1}) - h_n(t_j) \leqslant \epsilon/4$$

for any $0 \leqslant j \leqslant r-1$ with $r \leqslant [4Mn/\epsilon] + [4(M-l)n/\epsilon] + 1$.

By joining Δ_1 with Δ_2, we get a new partition

$$0 = x_0 < x_1 < x_2 < \cdots < x_p = 1$$

for $\epsilon > 0$ small enough with $p \leqslant m + r \leqslant 5(\bigvee_0^1(f) + 1)n/\epsilon$.

We can also take a partition

$$0 = y_0 < y_1 < y_2 < \cdots < y_q = 1$$

such that $y_{i+1} - y_i < \epsilon/3$ for any $0 \leqslant i \leqslant q-1$ and $\epsilon > 0$ small enough with $q = [3/\epsilon] + 1 \leqslant 4/\epsilon$.

Now we show that $s(n,\epsilon) \leqslant pq \leqslant 20(\bigvee_0^1(f) + 1)n/\epsilon^2$ for $\epsilon > 0$ small enough. Suppose this result dose not hold, then there exists an (n,ϵ)-separated set E of $(\mathbb{T}^2, T)$ such that $|E| > 20(\bigvee_0^1(f) + 1)n/\epsilon^2$. Since $|E| > pq$, there exist $0 \leqslant i \leqslant p-1$ and $0 \leqslant j \leqslant q-1$ such that there are at least two points

$$(u_1, v_1), (u_2, v_2) \in E \cap ([x_i, x_{i+1}] \times [y_j, y_{j+1}])$$

with $(u_1,v_1)\neq(u_2,v_2)$. Thus,

$$\begin{aligned}d_n((u_1,v_1),(u_2,v_2)) &= \max_{0\leqslant i\leqslant n-1} d(T^i(u_1,v_1),T^i(u_2,v_2))\\ &= \max_{0\leqslant i\leqslant n-1}\max\{\|u_1-u_2\|,\|g_i(u_1)-g_i(u_2)-h_i(u_1)+h_i(u_2)+v_1-v_2\|\}\\ &\leqslant \max_{0\leqslant i\leqslant n-1}\max\{\|u_1-u_2\|,\|g_i(u_1)-g_i(u_2)\|+\|h_i(u_1)-h_i(u_2)\|+\|v_1-v_2\|\}\\ &\leqslant \max_{0\leqslant i\leqslant n-1}\{|u_1-u_2|,|g_i(u_1)-g_i(u_2)|+|h_i(u_1)-h_i(u_2)|+|v_1-v_2|\}\\ &\leqslant \max\{|u_1-u_2|,|g_n(u_1)-g_n(u_2)|+|h_n(u_1)-h_n(u_2)|+|v_1-v_2|\}\\ &<\epsilon.\end{aligned}$$

This is a contradiction with the definition of the (n,ϵ)-separated set E. This shows that (3.1) holds. □

Lemma 3.2. *Let $(\mathbb{T}^2,T)$ be a t.d.s. defined in (1.2) such that $f\in\mathcal{F}_l$ and $l\neq 0$. Then*

$$s(n,\epsilon)\geqslant n|l|/3(\epsilon+\eta(\epsilon)) \tag{3.2}$$

for any $n\in\mathbb{N}$ and $\epsilon>0$ small enough, where $\eta(\epsilon)=\sup_{|x-y|\leqslant\epsilon}|f(x)-f(y)|$.

Proof. Clearly, $T^n(x,y)=(x+n\alpha,f_n(x)+y)$, where $f_n(x)=\sum_{i=0}^{n-1}f(x+i\alpha)$. Since $f_n(1)-f_n(0)=nl$, one has either $f_n(1/2)-f_n(0)\geqslant nl/2$ or $f_n(1)-f_n(1/2)\geqslant nl/2$. Without loss of generality, we suppose $f_n(1/2)-f_n(0)\geqslant nl/2$. For the other case, the argument is similar. Since f is continuous and $f(x+1)-f(x)=l$, it is not hard to check that $\epsilon\searrow 0$ implies $\eta(\epsilon)\searrow 0$. Take $\epsilon_0>0$ such that $\epsilon_0+\eta(\epsilon_0)<1/3$. We can find a sequence $0=x_0<x_1<x_2\cdots<x_k\leqslant 1/2$ such that

$$f_n(x_{i+1})-f_n(x_i)=\eta(\epsilon)+\epsilon$$

for any $0\leqslant i\leqslant k-1$ with $k>n|l|/3(\eta(\epsilon)+\epsilon)$ and $\epsilon_0\geqslant\epsilon>0$.

Fix $\epsilon\in(0,\epsilon_0]$. To show (3.2) it suffices to show that $\{(x_i,0)\mid 1\leqslant i\leqslant k\}$ is an (n,ϵ)-separated set of $(\mathbb{T}^2,T)$. In fact, for any $1\leqslant i<j\leqslant k$, if $x_j-x_i>\epsilon$, then $\|x_i-x_j\|>\epsilon$ which implies that $d_n((x_i,0),(x_j,0))>\epsilon$. Otherwise, by the definition of $\eta(\epsilon)$, we have

$$-\eta(\epsilon)\leqslant f_1(x_j)-f_1(x_i)\leqslant\eta(\epsilon),$$

and

$$f_m(x_j)-f_m(x_i)\in[f_{m-1}(x_j)-f_{m-1}(x_i)-\eta(\epsilon),f_{m-1}(x_j)-f_{m-1}(x_i)+\eta(\epsilon)]$$

for any $2\leqslant m\leqslant n$.

Since $f_n(x_j)-f_n(x_i)\geqslant\eta(\epsilon)+\epsilon$, we can define $l:=\min\{1\leqslant k\leqslant n: f_k(x_j)-f_k(x_i)>\epsilon\}$. This means

$$\epsilon<f_l(x_j)-f_l(x_i)\leqslant\eta(\epsilon)+\epsilon.$$

Thus,

$$d_n((x_i,0),(x_j,0))\geqslant d_l((x_i,0),(x_j,0))\geqslant\|f_l(x_i)-f_l(x_j)\|>\epsilon.$$

Summarizing up, we always have $d_n((x_i,0),(x_j,0))>\epsilon$ for any $1\leqslant i<j\leqslant k$. This completes the proof. □

Now we turn to prove Theorem A.

Proof of Theorem A. Since $r(n,\epsilon)\leqslant s(n,\epsilon)\leqslant r(n,\epsilon/2)$ for any $\epsilon>0$ and $n\in\mathbb{N}$, we have

$$n|l|/3(2\epsilon+\eta(2\epsilon))\leqslant r(n,\epsilon)\leqslant 20\left(\bigvee_0^1(f)+1\right)n/\epsilon^2$$

for $\epsilon>0$ small enough, where $\eta(\epsilon)$ comes from Lemma 3.2. Take

$$c_1(\epsilon)=|l|/3(2\epsilon+\eta(2\epsilon))\quad\text{and}\quad c_2(\epsilon)=20\left(\bigvee_0^1(f)+1\right)/\epsilon^2,$$

then we get the result. □

We now translate Theorem A into the language of topological complexity using open covers. Let $\mathcal{U}$ be an open cover of X, and for every integer $n \in \mathbb{N}$, $\mathcal{N}(\mathcal{U}, n)$ be the minimal cardinality of a subcover of $\bigvee_{j=0}^{n-1} T^{-j}\mathcal{U}$. The complexity function of $\mathcal{U}$ is the map $n \mapsto \mathcal{N}(\mathcal{U}, n)$.

We know that $r(n,\epsilon) \leqslant c(\epsilon)n$ for every $\epsilon > 0$ is equivalent to $\mathcal{N}(\mathcal{U}, n) \leqslant C(\mathcal{U})n$ for every open cover $\mathcal{U}$ of X; $r(n,\epsilon) \geqslant c(\epsilon)n$ for every $\epsilon > 0$ is equivalent to $\mathcal{N}(\mathcal{U}, n) \geqslant C(\mathcal{U})n$ for every open cover $\mathcal{U}$ of X (see [10] for details).

4 Minimality and the maximal equicontinuous factor

Auslander [1, Chapter 5] presented a criterion for the minimality of a class of group extensions of minimal systems, and applied the criterion to show the minimality of a class of skew products on $\mathbb{T}^{k+1}$, the $(k+1)$ torus, namely

$$T(z, w_1, w_2, \ldots, w_k) = (\alpha z, \varphi(z)w_1, \varphi(\beta z)w_2, \ldots, \varphi(\beta^{k-1}z)w_k),$$

where $\alpha, \beta \in \mathbb{T}^1$ and φ is chosen appropriately. In this section, we will give another way to prove that when $f \in \mathcal{F}_l (l \neq 0)$, the system $(\mathbb{T}^2, T)$ defined as before is minimal and the regionally proximal relation

$$Q(\mathbb{T}^2, T) = \{((x, y_1), (x, y_2)) : x, y_1, y_2 \in \mathbb{T}^1\}.$$

For this purpose, the following result is needed and very useful.

Lemma 4.1. *Let $f \in \mathcal{F}_0$. Then there exist $x_1, x_2 \in \mathbb{T}^1$ such that*

$$\sup_{n\geqslant 1}\left(f_n(x_1) - n\int_0^1 f(x)dx\right) \leqslant 2 \tag{4.1}$$

and

$$\inf_{n\geqslant 1}\left(f_n(x_2) - n\int_0^1 f(x)dx\right) \geqslant -2, \tag{4.2}$$

where $f_n(x) = \sum_{i=0}^{n-1} f(x + i\alpha)$. Moreover, the sets

$$A = \left\{x \in \mathbb{T}^1 : \textit{there esixts } M_1(x) \in \mathbb{R} \textit{ such that } \sup_{n\geqslant 1}\left(g_n(x_1) - n\int_0^1 g(x)dx\right) \leqslant M_1(x)\right\}$$

and

$$B = \left\{x \in \mathbb{T}^1 : \textit{there exists } M_2(x) \in \mathbb{R} \textit{ such that } \inf_{n\geqslant 1}\left(g_n(x_2) - n\int_0^1 g(x)dx\right) \geqslant M_2(x)\right\}$$

are dense in $\mathbb{T}^1$.

Proof. Firstly, we prove (4.1). Suppose (4.1) dose not hold, then for any $y \in \mathbb{T}^1$, there exists $n_y \geqslant 1$ such that $f_{n_y}(y) - n_y\int_0^1 f(x)dx > 2$. By the continuity of f, there exists an open neighborhood U_y of y such that for any $y' \in U_y$, $f_{n_y}(y') - n_y\int_0^1 f(x)dx > 2$. Since $\{U_y : y \in \mathbb{T}^1\}$ is an open cover of $\mathbb{T}^1$, there exists $\{y_1, y_2, \ldots, y_l\} \subset \mathbb{T}^1$ such that $\bigcup_{i=1}^{l} U_{y_i} = \mathbb{T}^1$. For $y_0 \in \mathbb{T}^1$, we define $\{s_i\}_{i\in\mathbb{N}} \subset \mathbb{N}$ and $\{k_i\}_{i\in\mathbb{N}} \subset \{1, 2, \ldots, l\}$ by induction that $s_0 = 0$ and $s_{i+1} = s_i + n_{y_{k_i}}$, where k_i is taken such that $y_0 + s_i\alpha \in U_{y_{k_i}}$. We claim that

$$f_{s_i}(y_0) - s_i\int_0^1 f(x)dx \geqslant 2i \quad \text{for any} \quad i \geqslant 1. \tag{4.3}$$

In fact, for $i = 1$, it is clear, and if we assume it is true for $i = p$, then it also holds for $i = p+1$ because

$$f_{s_{p+1}}(y_0) - s_{p+1}\int_0^1 f(x)dx = f_{s_p}(y_0) + f_{n_{y_{k_p}}}(y_0 + s_p\alpha) - (s_p + n_{y_{k_p}})\int_0^1 f(x)dx$$

$$
\begin{aligned}
&= \left(f_{n_{y_{k_p}}}(y_0 + s_p\alpha) - n_{y_{k_p}} \int_0^1 f(x)dx\right) + \left(f_{s_p}(y_0) - s_p \int_0^1 f(x)dx\right) \\
&\geqslant 2 + \left(f_{s_p}(y_0) - s_p \int_0^1 f(x)dx\right) \text{ (by the definition of } \{s_i\}) \\
&\geqslant 2(p+1) \text{ (by the induction assumption).}
\end{aligned}
$$

Thus, by induction, (4.3) holds.

Let $M = \max\{n_{y_1}, n_{y_2}, \ldots, n_{y_l}\}$, then $i \leqslant s_i \leqslant Mi$ for any $i \geqslant 1$. On one hand, we have

$$
\limsup_{i\to+\infty} \frac{f_{s_i}(y_0)}{s_i} \geqslant \limsup_{i\to+\infty} \frac{s_i \int_0^1 f(x)dx + 2i}{s_i} \geqslant \int_0^1 f(x)dx + \frac{2}{M}. \tag{4.4}
$$

On the other hand, since $\tau : \mathbb{T}^1 \to \mathbb{T}^1, x \mapsto x + \alpha$ is uniquely ergodic, by Lemma 2.3, we have

$$
\lim_{n\to+\infty} \frac{f_n(y_0)}{n} = \int_0^1 f(x)dx,
$$

a contradiction with (4.4). This implies that (4.1) holds.

Now we show that the set A is dense in $\mathbb{T}^1$. For any $m \in \mathbb{N}$,

$$
\begin{aligned}
&\sup_{n\geqslant1} \left[g_n(x_1 + (m+1)\alpha) - n \int_0^1 g(x)dx\right] \\
&\quad = \sup_{n\geqslant1} \left[g_{n+1}(x_1 + m\alpha) - g(x_1 + m\alpha) - n \int_0^1 g(x)dx\right] \\
&\quad = \sup_{n\geqslant1} \left[g_{n+1}(x_1 + m\alpha) - (n+1) \int_0^1 g(x)dx\right] + \left[\int_0^1 g(x)dx - g(x_1 + m\alpha)\right].
\end{aligned}
$$

So we have $\{x_1 + m\alpha : m \in \mathbb{N}\} \subset A$, hence A is dense in $\mathbb{T}^1$. By a similar argument, we obtain that (4.2) holds and the set B is dense in $\mathbb{T}^1$. □

Lemma 4.2. *Let $(\mathbb{T}^2, T)$ be a t.d.s. defined in (1.2) such that $f \in \mathcal{F}_l$, $l \neq 0$ and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then $(\mathbb{T}^2, T)$ is minimal.*

Proof. It is clear that $(\mathbb{T}^2, T)$ is distal. To show $(\mathbb{T}^2, T)$ is minimal, by Lemma 2.1, it suffices to show that $(\mathbb{T}^2, T)$ is transitive. Consider non-empty open sets $U_1 \times V_1$ and $U_2 \times V_2$ of $\mathbb{T}^2$. There exist $x_1, x_2, y_1, y_2 \in (0, 1)$ and $\delta > 0$ such that

$$
(x_1 - \delta, x_1 + \delta) \times (y_1 - \delta, y_1 + \delta) \subset U_1 \times V_1
$$

and

$$
(x_2 - \delta, x_2 + \delta) \times (y_2 - \delta, y_2 + \delta) \subset U_2 \times V_2.
$$

Let $f(x) = g(x) + lx$. Since $f \in \mathcal{F}_l$, we have $g \in \mathcal{F}_0$.

In the following, we divide the proof into two cases.

Case 1. $l > 0$. By Lemma 4.1, there exist $x' \in (x_1 + \delta/4, x_1 + \delta/2)$, $x'' \in (x_1 - \delta/2, x_1 - \delta/4)$, and M_1, $M_2 \in \mathbb{R}$ such that

$$
\inf_{n\geqslant1} \left(g_n(x') - n \int_0^1 g(x)dx\right) \geqslant M_1 \quad \text{and} \quad \sup_{n\geqslant1} \left(g_n(x'') - n \int_0^1 g(x)dx\right) \leqslant M_2.
$$

Thus,

$$
f_n(x') - f_n(x'') = g_n(x') - g_n(x'') + nl(x' - x'') \geqslant M_1 - M_2 + nl\delta/2 \to +\infty \tag{4.5}
$$

as $n \to +\infty$. Since $\{n\alpha \mid n \in \mathbb{Z}^+\}$ is dense in $\mathbb{T}^1$, there are infinitely many $n_i \in \mathbb{N}$ such that

$$
(n_i\alpha + x_1 - \delta/2, n_i\alpha + x_1 + \delta/2) \subset (x_2 - \delta, x_2 + \delta).
$$

Therefore, for any $x \in (x_1 - \delta/2, x_1 + \delta/2)$, we have $x + n_i\alpha \in (x_2 - \delta, x_2 + \delta)$. By (4.5), there exists $N_1 \in \{n_i\}$ such that $f_{N_1}(x') - f_{N_1}(x'') > 1$. By the continuity of f, there exists $x_0 \in (x'', x') \subset (x_1 - \delta/2, x_1 + \delta/2)$ such that

$$f_{N_1}(x_0) + y_1 \in (y_2 - \delta, y_2 + \delta) \quad \text{and} \quad x_0 + N_1\alpha \in (x_2 - \delta, x_2 + \delta),$$

i.e.,

$$(x_0, y_1) \in (x_1 - \delta, x_1 + \delta) \times (y_1 - \delta, y_1 + \delta) \subset U_1 \times V_1$$

and

$$T^{N_1}(x_0, y_1) \in (x_2 - \delta, x_2 + \delta) \times (y_2 - \delta, y_2 + \delta) \subset U_2 \times V_2.$$

Therefore, $(x_0, y_1) \in (U_1 \times V_1) \cap T^{-N_1}(U_2 \times V_2)$. Since U_1, U_2, V_1 and V_2 are arbitrary, $(\mathbb{T}^2, T)$ is transitive.

Case 2. $l < 0$. By Lemma 4.1, there exist $z' \in (x_1 + \delta/4, x_1 + \delta/2)$, $z'' \in (x_1 - \delta/2, x_1 - \delta/4)$ and $K_1, K_2 \in \mathbb{R}$ such that

$$\sup_{n \geqslant 1} \left(g_n(z') - n \int_0^1 g(x)dx \right) \leqslant K_1 \quad \text{and} \quad \inf_{n \geqslant 1} \left(g_n(z'') - n \int_0^1 g(x)dx \right) \geqslant K_2.$$

Thus,

$$f_n(z') - f_n(z'') = g_n(z') - g_n(z'') + nl(z' - z'') \leqslant K_1 - K_2 + nl\delta \to -\infty \tag{4.6}$$

as $n \to +\infty$. Since $\{n\alpha \mid n \in \mathbb{Z}^+\}$ is dense in $\mathbb{T}^1$, there are infinitely many $m_i \in \mathbb{N}$ such that

$$(m_i\alpha + x_1 - \delta/2, m_i\alpha + x_1 + \delta/2) \subset (x_2 - \delta, x_2 + \delta).$$

Therefore, for any $x \in (x_1 - \delta/2, x_1 + \delta/2)$, we have $x + m_i\alpha \in (x_2 - \delta, x_2 + \delta)$. By (4.6), there exists $N_2 \in \{m_i\}$ such that $f_{N_2}(x') - f_{N_2}(x'') < -1$. By the continuity of f, there exists $z_0 \in (z'', z') \subset (x_1 - \delta/2, x_1 + \delta/2)$ such that

$$f_{N_2}(z_0) + y_1 \in (y_2 - \delta, y_2 + \delta) \quad \text{and} \quad z_0 + N_2\alpha \in (x_2 - \delta, x_2 + \delta),$$

i.e.,

$$(z_0, y_1) \in (x_1 - \delta, x_1 + \delta) \times (y_1 - \delta, y_1 + \delta) \subset U_1 \times V_1$$

and

$$T^{N_2}(z_0, y_1) \in (x_2 - \delta, x_2 + \delta) \times (y_2 - \delta, y_2 + \delta) \subset U_2 \times V_2.$$

Therefore, $(z_0, y_1) \in (U_1 \times V_1) \cap T^{-N_2}(U_2 \times V_2)$. Since U_1, U_2, V_1 and V_2 are arbitrary, $(\mathbb{T}^2, T)$ is transitive. Summarizing up, we complete the proof. □

Lemma 4.3. *Let $(\mathbb{T}^2, T)$ be a t.d.s. defined in* (1.2) *such that $f \in \mathcal{F}_l$, $l \neq 0$ and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then*

$$Q(\mathbb{T}^2, T) = \{((x, y_1), (x, y_2)) : x, y_1, y_2 \in \mathbb{T}^1\}.$$

Proof. Firstly, we show that for any $(x_1, y_1), (x_2, y_2) \in \mathbb{T}^2$ with $x_1 \neq x_2$, we have $((x, y_1), (x, y_2)) \notin Q(\mathbb{T}^2, T)$. Fix $x_1,\ x_2,\ y_1\ , y_2 \in \mathbb{T}^1$ and let $\epsilon_0 = \|x_1 - x_2\|/4$. Consider $(x_1 - \epsilon_0, x_1 + \epsilon_0) \times V_1$ and $(x_2 - \epsilon_0, x_2 + \epsilon_0) \times V_2$, where V_1, V_2 are non-empty open neighborhoods of y_1 and y_2 respectively. For any $(x', y') \in (x_1 - \epsilon_0, x_1 + \epsilon_0) \times V_1$ and $(x'', y'') \in (x_2 - \epsilon_0, x_2 + \epsilon_0) \times V_2$, we have

$$d(T^n(x', y'), T^n(x'', y'')) \geqslant \|(x' + n\alpha) - (x'' + n\alpha)\| = \|x' - x''\| \geqslant 2\epsilon_0$$

for each $n \in \mathbb{Z}$. So $((x_1, y_1), (x_2, y_2)) \notin Q(\mathbb{T}^2, T)$ whenever $x_1 \neq x_2$.

It remains to show that $((x,\ y_1), (x,\ y_2)) \in Q(\mathbb{T}^2, T)$ for any $x,\ y_1,\ y_2 \in \mathbb{T}^1$. Fix $x,\ y_1\ y_2 \in \mathbb{T}^1$. For any $\epsilon > 0$, suppose $U_1 \times V_1$ and $U_2 \times V_2$ are non-empty open neighborhoods of (x, y_1) and (x, y_2) respectively, then there exists $\delta > 0$ $(\delta < \epsilon)$ such that

$$(x - \delta, x + \delta) \subset U_1 \cap U_2, \quad (y_1 - \delta, y_1 + \delta) \subset V_1 \quad \text{and} \quad (y_2 - \delta, y_2 + \delta) \subset V_2.$$

In the following, we divide the proof into two cases.

Case 1. $l > 0$. By Lemma 4.1, there exist $x' \in (x_1 + \delta/4, x_1 + \delta/2), x'' \in (x_1 - \delta/2, x_1 - \delta/4)$, and $M_1,\ M_2 \in \mathbb{R}$ such that

$$\inf_{n\geqslant 1}\left(g_n(x') - n\int_0^1 g(x)dx\right) \geqslant M_1 \quad \text{and} \quad \sup_{n\geqslant 1}\left(g_n(x'') - n\int_0^1 g(x)dx\right) \leqslant M_2.$$

Thus,

$$f_n(x') - f_n(x'') = g_n(x') - g_n(x'') + nl(x' - x'') \geqslant M_1 - M_2 + nl\delta/2 \to +\infty$$

as $n \to +\infty$. Therefore, there exists $N_1 \in \mathbb{N}$ such that $f_{N_1}(x') - f_{N_1}(x'') > 1$. By the continuity of f, there exists $x_0 \in (x'', x') \subset (x_1 - \delta/2, x_1 + \delta/2)$ such that $\|f_{N_1}(x_0) + y_1 - f_{N_1}(x') - y_2\| < \epsilon$, i.e., there exist $(x_0, y_1) \in U_1 \times V_1$, $(x', y_2) \in U_2 \times V_2$ and $N_1 \in \mathbb{N}$ such that $d(T^{N_1}(x_0, y_1), T^{N_1}(x', y_2)) < \epsilon$.

Case 2. $l < 0$. By Lemma 4.1, there exist $z' \in (x_1 + \delta/4, x_1 + \delta/2), z'' \in (x_1 - \delta/2, x_1 - \delta/4)$ and $K_1, K_2 \in \mathbb{R}$ such that

$$\sup_{n\geqslant 1}\left(g_n(z') - n\int_0^1 g(x)dx\right) \leqslant K_1 \quad \text{and} \quad \inf_{n\geqslant 1}\left(g_n(z'') - n\int_0^1 g(x)dx\right) \geqslant K_2.$$

Thus,

$$f_n(z') - f_n(z'') = g_n(z') - g_n(z'') + nl(z' - z'') \leqslant K_1 - K_2 + nl\delta \to -\infty \tag{4.7}$$

as $n \to +\infty$. Therefore, there exists $N_2 \in \mathbb{N}$ such that $f_{N_2}(z') - f_{N_2}(z'') < -1$. By the continuity of f, there exists $z_0 \in (z'', z') \subset (x_1 - \frac{\delta}{2}, x_1 + \frac{\delta}{2})$ such that $\|f_{N_2}(z_0) + y_1 - f_{N_2}(z') - y_2\| < \epsilon$, i.e., there exist $(z_0, y_1) \in U_1 \times V_1$, $(z', y_2) \in U_2 \times V_2$ and $N_2 \in \mathbb{N}$ such that $d(T^{N_2}(z_0, y_1), T^{N_2}(z', y_2)) < \epsilon$. Summarizing up, we finish the proof. □

The following result follows from Lemmas 2.2 and 4.3.

Lemma 4.4. *Let $(\mathbb{T}^2, T)$ be a t.d.s. defined in (1.2) such that $f \in \mathcal{F}_l$, $l \neq 0$ and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Suppose*

$$\pi : (\mathbb{T}^2, T) \to (\mathbb{T}^1, \tau), \quad (x, y) \mapsto x,$$

where $\tau : \mathbb{T}^1 \to \mathbb{T}^1, x \mapsto x + \alpha$. Then $(\mathbb{T}^1, \tau)$ is the maximal equicontinuous factor of $(\mathbb{T}^2, T)$.

Now we prove Theorem B.

Proof of Theorem B. By Lemma 4.2, $(\mathbb{T}^2, T)$ is minimal. By Lemma 4.4, we know that $(\mathbb{T}^1, \tau)$ is the maximal equicontinuous factor of $(\mathbb{T}^2, T)$, where

$$\tau : \mathbb{T}^1 \to \mathbb{T}^1, \quad x \mapsto x + \alpha.$$

It is clear that $(\mathbb{T}^2, T)$ is an isometric extension of $(\mathbb{T}^1, \tau)$. We can easily get Theorem B by applying Theorems 2.1 and 2.2. □

5 An example

In this section, we will give a negative answer to the latter part of the open question raised by Host-Kra-Maass mentioned in the introduction, i.e., we will construct a system whose topological complexity is low but it is not a system of order 2. Precisely, we will find a bounded variation function f which belongs to $\mathcal{F}_l$ with $l \neq 0$, and at the same time, we define $(\mathbb{T}^2, T)$ in (1.2) such that f also satisfies that for any $\varphi \in \mathcal{F}_0$ and $c \in \mathbb{R}$, the equation $f(x) = \varphi(x + \alpha) - \varphi(x) + lx + c$ does not hold. To do this we start with continued fractions and some related results.

5.1 Continued fractions

A (*simple*) *continued fraction* is a formal expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \ddots}}}},$$

which we will also denote by $[a_0; a_1, a_2, , a_3, \ldots]$ with $a_n \in \mathbb{N}$ for $n \geqslant 1$ and $a_0 \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$. The numbers a_n are *the partial quotients* of the continued fraction. We also write $[a_0; a_1, a_2, \ldots, a_n]$ for the finite fraction

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}.$$

We state some basic properties about continued fractions for convenience (see for example [5] for details):

(1) The infinite continued fraction converges to a real number, namely, there exists a real number α such that

$$\alpha = [a_0; a_1, a_2, \ldots] = \lim_{n\to\infty} [a_0; a_1, a_2, \ldots, a_n].$$

We say that $[a_0; a_1, a_2, \ldots]$ is the continued fraction expansion for α.

(2) Let $a_n \in \mathbb{N}$ for all $n \geqslant 0$. Then $[a_0; a_1, a_2, \ldots]$ is irrational.

(3) The map that sends the sequence $(a_0, a_1, a_2, \ldots) \in \mathbb{N}_0 \times \mathbb{N}^{\mathbb{N}}$ to $[a_0; a_1, a_2, \ldots]$ is injective.

(4) For any irrational number $\alpha \in (0, 1)$, there exists a continued fraction expansion for α.

A real number $\alpha = [a_0; a_1, a_2, \ldots] \in (0, 1)$ is called *badly approximable* if there is some M such that $a_n \leqslant M$ for all $n \geqslant 1$. The following result is well known (see for example [5, p. 87]).

Lemma 5.1. *A real number $\alpha \in (0, 1)$ is badly approximable if and only if there exists some constant $c = c(\alpha) > 0$ such that*

$$\left|\alpha - \frac{p}{q}\right| > \frac{c}{q^2}$$

for every rational number $\frac{p}{q}$.

We define $v(\alpha) = \liminf_{n\to+\infty} n\|n\alpha\|$. It is clear that $v(\alpha) > 0$ if and only if α is badly approximable. It is well known that the set of all badly approximable numbers in $(0, 1)$ is a null set with respect to Lebesgue measure (see, e.g., [5, p. 87]). Hence, the Lebesgue measure of the set $\{\alpha \in (0, 1) : v(\alpha) = 0\}$ is one.

Now we prove Theorem C.

Proof of Theorem C. By the definition of $v(2\pi\alpha)$, we know that

$$\liminf_{n\to+\infty} n|\mathrm{e}^{2\pi\mathrm{i}n\alpha} - 1| = 0.$$

So there exists an increasing sequence $\{n_k\}_{k=1}^{+\infty}$ of positive integers such that $n_k|\mathrm{e}^{2\pi\mathrm{i}n_k\alpha} - 1| < 1/k^2$ for every $k \in \mathbb{N}$. Take a function

$$f(x) = lx + \sum_{n=-\infty}^{+\infty} a_n \mathrm{e}^{2\pi\mathrm{i}nx},$$

where

$$a_n = \begin{cases} \mathrm{e}^{2\pi\mathrm{i}n_k\alpha} - 1, & \text{if} \quad n = \pm n_k, \quad k \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Since $f(x) = \overline{f}(x)$, f is a real valued function. Since

$$f'(x) = l + \sum_{n=-\infty}^{+\infty} 2\pi i n a_n \mathrm{e}^{2\pi i n x} \quad \text{and} \quad |f'(x)| \leqslant 4\pi \sum_{k=1}^{+\infty} 1/k^2 + |l|,$$

we know that f is a continuous function with a bounded variation. By Theorem A, we know that (1.1) holds.

By the construction of f and Lemma 4.2, we know that $(\mathbb{T}^2, T)$ is minimal. It is clear that $(\mathbb{T}^2, T)$ is distal.

Next, we show that for the function f defined above, the system $(\mathbb{T}^2, T)$ is not a system of order 2. Suppose $(\mathbb{T}^2, T)$ is a system of order 2, by Theorem B, we can assume that there exist $\varphi \in \mathcal{F}_0$ and $c \in \mathbb{R}$ such that $f(x) = \varphi(x+\alpha) - \varphi(x) + lx + c$ for any $x \in \mathbb{R}$.

Let $\varphi(x) = \sum_{n=-\infty}^{+\infty} b_n \mathrm{e}^{2\pi i n x}$ be the Fourier series of periodic function φ. Comparing the Fourier coefficients of the equation $f(x) - lx = \varphi(x+\alpha) - \varphi(x) + c$, we have

$$a_n = \begin{cases} b_n(\mathrm{e}^{2\pi i n \alpha} - 1), & n \neq 0, \\ c, & n = 0. \end{cases}$$

This implies that $\sum_{n=-\infty}^{+\infty} |b_n|^2 = +\infty$, a contradiction with $\sum_{n=-\infty}^{+\infty} |b_n|^2 = \int_0^1 |\varphi(x)|^2 dx < +\infty$. Thus, by Theorem B, we conclude that $(\mathbb{T}^2, T)$ is not a system of order 2. □

Remark 5.1. Let m be the Lebesgue measure on $\mathbb{R}$,

$$A = \{\alpha \in \mathbb{R} \setminus \mathbb{Q} : v(\alpha) = 0\} \quad \text{and} \quad B = \{\alpha \in \mathbb{R} \setminus \mathbb{Q} : v(2\pi\alpha) = 0\}.$$

Since $m(\{\alpha \in (0,1) : v(\alpha) = 0\}) = 1$, we have $m(A \cap (0, 2\pi)) = 2\pi$, which implies that

$$m\left(\frac{A}{2\pi} \cap (0,1)\right) = 1,$$

i.e., $m(B \cap (0,1)) = 1$. Therefore, for almost all $\alpha \in (0,1)$ in the sense of Lebesgue measure, there exists $f \in \mathcal{F}_l$ such that Theorem C holds for the system $(\mathbb{T}^2, T)$.

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