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A hybridized weak Galerkin finite element scheme for the Stokes equations

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Abstract In this paper a hybridized weak Galerkin (HWG) finite element method for solving the Stokes equations in the primary velocity-pressure formulation is introduced. The WG method uses weak functions and their weak derivatives which are defined as distributions. Weak functions and weak derivatives can be approximated by piecewise polynomials with various degrees. Different combination of polynomial spaces leads to different WG finite element methods, which makes WG methods highly flexible and efficient in practical computation. A Lagrange multiplier is introduced to provide a numerical approximation for certain derivatives of the exact solution. With this new feature, the HWG method can be used to deal with jumps of the functions and their flux easily. Optimal order error estimates are established for the corresponding HWG finite element approximations for both primal variables and the Lagrange multiplier. A Schur complement formulation of the HWG method is derived for implementation purpose. The validity of the theoretical results is demonstrated in numerical tests.

Keywords hybridized weak Galerkin finite element methods, weak gradient, weak divergence, Stokes equation

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1 Introduction

Weak Galerkin (WG) refers to a general finite element technique for partial differential equations (PDEs) in which differential operators are approximated by their weak forms as distributions. Since their introduction, WG finite element methods have been applied successfully to the discretization of several classes of partial differential equations, e.g., second order elliptic equations [6, 14, 15, 18, 22, 24], the biharmonic equations [13, 17, 19, 25], the Stokes equations [23], and the Brinkman equations [16]. WG methods, by design, make use of discontinuous piecewise polynomials on finite element partitions with arbitrary shape of polygons and polyhedrons. Weak functions and weak derivatives can be approximated by piecewise polynomials with various degrees. The flexibility of WG method on these aspects of approximating polynomials makes it an excellent candidate for the numerical solution of incompressible flow problems.

Hybridization of finite element methods is a technique where Lagrange multipliers are introduced to relax certain constraints such as some continuity requirements. The main feature of the HWG method is that their approximate solutions can be expressed in an element-by-element fashion. Hybridization [1]

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can be employed to obtain an efficient implementation for solving PDEs. The generalization of this idea to mixed finite elements has been investigated in [2–5, 7, 21]. The idea of hybridization was also used in discontinuous Galerkin methods [11, 12, 20] to derive hybridizable discontinuous Galerkin (HDG) [8–10].

In this paper, the WG finite element formulation developed in [23] is hybridized to obtain our new hybridized weak Galerkin finite element method for solving Stokes equations. This HWG formulation can be modified easily to solve interface problems by adding two functionals arising from the jump condition to the right-hand side. A Schur complement formulation of the HWG method is derived for implementation purpose. By eliminating the interior unknowns and the Lagrange multipliers, the Schur complement formulation yields a system with much smaller size. We shall show that hybridization is a natural approach for the weak Galerkin finite element method of [23]. We shall also establish a theoretical foundation to address critical issues such as stability and convergence for the HWG finite element method.

The paper is organized as follows. In Section 2, we briefly discuss the continuous Stokes problem and recall some basic results for later reference. After presenting some standard notations in Sobolev spaces in Section 3, we introduce two weakly-defined differential operators: weak gradient and weak divergence. The HWG finite element scheme for the Stokes problem is developed in Section 4. In Section 5, we shall study the stability and solvability of the HWG scheme. In particular, the usual inf-sup condition is established for the HWG scheme. In Section 6, we shall derive an error equation for the HWG approximations. Optimal-order error estimates for the WG finite element approximations are also derived in this Section. The equivalence of HWG formulation and its Schur complement formulation is proved in Section 7. Finally in Section 8, numerical experiments are conducted.

2 The model problem

Let $\Omega \subset \mathbb{R}^d$ be a polygonal or polyhedral domain in $\mathbb{R}^d$ for $d = 2, 3$ respectively. As a model for the flow of an incompressible viscous fluid confined in Ω, we consider the stationary Stokes problem with nonhomogeneous Dirichlet boudary conditions, given by

$$-\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f}, \quad \text{in } \Omega, \tag{2.1}$$

$$\nabla \cdot \boldsymbol{u} = 0, \quad \text{in } \Omega, \tag{2.2}$$

$$\boldsymbol{u} = \boldsymbol{g}, \quad \text{on } \partial\Omega. \tag{2.3}$$

Throughout the presentation, we assume that the unit external volumetric force acts on the fluid $\boldsymbol{f} \in [L^2(\Omega)]^d$.

The weak form in the primary velocity-pressure formulation for the Stokes problem (2.1)–(2.3) seeks $\boldsymbol{u} \in [H^1(\Omega)]^d$ and $p \in L_0^2(\Omega)$ satisfying $\boldsymbol{u} = \boldsymbol{g}$ on $\partial\Omega$ and

$$(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) - (\nabla \cdot \boldsymbol{v}, p) = (\boldsymbol{f}, \boldsymbol{v}), \tag{2.4}$$

$$(\nabla \cdot \boldsymbol{u}, q) = 0, \tag{2.5}$$

for all $\boldsymbol{v} \in [H_0^1(\Omega)]^d$ and $q \in L_0^2(\Omega)$.

Recently a weak Galerkin finite element method has been developed for solving the Stokes equations in [23]. The main idea of weak Galerkin finite element methods is the introduction of weak functions and their corresponding weak derivatives in the algorithm design. With well-defined weak functions and weak derivatives, a weak Galerkin finite element formulation for the Stokes equations is derived from the variational form of the PDE (2.4)–(2.5) by replacing regular derivatives with weak derivatives and possibly adding a parameter independent stabilizer: find $\boldsymbol{u}_h$ and p_h from properly-defined finite element spaces satisfying

$$(\nabla_w \boldsymbol{u}_h, \nabla_w \boldsymbol{v}) - (\nabla_w \cdot \boldsymbol{v}, p_h) + s(\boldsymbol{u}_h, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}), \tag{2.6}$$

$$(\nabla_w \cdot \boldsymbol{u}_h, q) = 0, \tag{2.7}$$

for all test functions $\boldsymbol{v}$ and q in test spaces. In this paper, the WG finite element formulation developed in [23] is hybridized to obtain our new hybridized weak Galerkin finite element method for solving Stokes equation (2.1)–(2.3).

3 Weak differential operators and discrete weak gradient

Let D be any open bounded domain with Lipschitz continuous boundary in $\mathbb{R}^d, d = 2, 3$. We use the standard definition for the Sobolev space $H^s(D)$ and its associated inner products $(\cdot,\cdot)_{s,D}$, norms $\|\cdot\|_{s,D}$, and seminorms $|\cdot|_{s,D}$ for any $s \geqslant 0$. For example, for any integer $s \geqslant 0$, the seminorm $|\cdot|_{s,D}$ is given by

$$|v|_{s,D} = \left(\sum_{|\alpha|=s} \int_D |\partial^\alpha v|^2 \mathrm{d}D \right)^{\frac{1}{2}}$$

with the usual notation

$$\alpha = (\alpha_1, \ldots, \alpha_d), \quad |\alpha| = \alpha_1 + \cdots + \alpha_d, \quad \partial^\alpha = \prod_{j=1}^d \partial_{x_j}^{\alpha_j}.$$

The Sobolev norm $\|\cdot\|_{m,D}$ is given by

$$\|v\|_{m,D} = \left(\sum_{j=0}^m |v|_{j,D}^2 \right)^{\frac{1}{2}}.$$

The space $H^0(D)$ coincides with $L^2(D)$, for which the norm and the inner product are denoted by $\|\cdot\|_D$ and $(\cdot,\cdot)_D$, respectively. When $D = \Omega$, we shall drop the subscript D in the norm and in the inner product notation.

The space $H(\mathrm{div}; D)$ is defined as the set of vector-valued functions on D which, together with their divergence, are square integrable, i.e.,

$$H(\mathrm{div}; D) = \{\boldsymbol{v} : \boldsymbol{v} \in [L^2(D)]^d, \nabla \cdot \boldsymbol{v} \in L^2(D)\}.$$

Let T be a polygonal or polyhedral domain with boundary ∂T. A weak vector-valued function on the region T refers to a vector-valued function $\boldsymbol{v} = \{\boldsymbol{v}_0, \boldsymbol{v}_b\}$ such that $\boldsymbol{v}_0 \in [L^2(T)]^d$ and $\boldsymbol{v}_b \in [H^{\frac{1}{2}}(\partial T)]^d$. Let

$$\mathcal{V}(T) = \{\boldsymbol{v} = \{\boldsymbol{v}_0, \boldsymbol{v}_b\} : \boldsymbol{v}_0 \in [L^2(T)]^d, \boldsymbol{v}_b \in [H^{\frac{1}{2}}(\partial T)]^d\}. \tag{3.1}$$

Recall that, for any $\boldsymbol{v} \in \mathcal{V}(T)$, the weak gradient of $\boldsymbol{v}$ is defined as a linear functional $\nabla_w \boldsymbol{v}$ in the dual space of $[H(\mathrm{div}, T)]^d$ whose action on each $q \in [H(\mathrm{div}, T)]^d$ is given by

$$(\nabla_w \boldsymbol{v}, q)_T = -(\boldsymbol{v}_0, \nabla \cdot q)_T + \langle \boldsymbol{v}_b, q \cdot \boldsymbol{n} \rangle_{\partial T}, \tag{3.2}$$

where $\boldsymbol{n}$ is the outer unit normal vector along ∂T, $(\cdot,\cdot)_T$ stands for the L^2-inner product in $[L^2(T)]^d$ and $\langle \cdot,\cdot \rangle_{\partial T}$ is the inner product in $[H^{\frac{1}{2}}(\partial T)]^d$.

A discrete version of the weak gradient operator ∇_w is an approximation, denoted by $\nabla_{w,r,T}$ in the space of polynomials of degree r such that

$$(\nabla_{w,r,T} \boldsymbol{v}, q)_T = -(\boldsymbol{v}_0, \nabla \cdot q)_T + \langle \boldsymbol{v}_b, q \cdot \boldsymbol{n} \rangle_{\partial T}, \quad \forall\, q \in [P_r(T)]^{d \times d}. \tag{3.3}$$

From the integration by parts, we have

$$(\boldsymbol{v}_0, \nabla q)_T = -(\nabla \boldsymbol{v}_0, q)_T + \langle \boldsymbol{v}_0, q \cdot \boldsymbol{n} \rangle_{\partial T}.$$

Substituting the above identity into (3.3) yields

$$(\nabla_{w,r,T} \boldsymbol{v}, q)_T - (\nabla \boldsymbol{v}_0, q)_T = \langle \boldsymbol{v}_b - \boldsymbol{v}_0, q \cdot \boldsymbol{n} \rangle_{\partial T}, \tag{3.4}$$

for all $q \in [P_r(T)]^{d\times d}$.

To define a weak divergence, we require weak function $\boldsymbol{v} = \{\boldsymbol{v}_0, \boldsymbol{v}_b\}$ to be such that $\boldsymbol{v}_0 \in [L^2(T)]^d$ and $\boldsymbol{v}_b \cdot \boldsymbol{n} \in H^{-\frac{1}{2}}(\partial T)$. Denote

$$V(T) = \{\boldsymbol{v} = \{\boldsymbol{v}_0, \boldsymbol{v}_b\} : \boldsymbol{v}_0 \in [L^2(T)]^d, \boldsymbol{v}_b \cdot \boldsymbol{n} \in H^{-\frac{1}{2}}(\partial T)\}. \tag{3.5}$$

Recall that, for any $\boldsymbol{v} \in \mathcal{V}(T)$, the weak divergence of $\boldsymbol{v}$ is defined as a linear functional $\nabla_w \cdot \boldsymbol{v}$ in the dual space of $H^1(T)$ whose action on each $\varphi \in H^1(T)$ is given by

$$(\nabla_w \cdot \boldsymbol{v}, \varphi)_T = -(\boldsymbol{v}_0, \nabla\varphi)_T + \langle \boldsymbol{v}_b \cdot \boldsymbol{n}, \varphi\rangle_{\partial T}, \tag{3.6}$$

where $\boldsymbol{n}$ is the outer unit normal vector along ∂T, $(\cdot,\cdot)_T$ stands for the L^2-inner product in $L^2(T)$ and $\langle\cdot,\cdot\rangle_{\partial T}$ is the inner product in $H^{\frac{1}{2}}(\partial T)$.

A discrete version of the weak divergence operator $\nabla_w\cdot$ is an approximation, denoted by $(\nabla_{w,r,T}\cdot)$ in the space of polynomials of degree r such that

$$(\nabla_{w,r,T} \cdot \boldsymbol{v}, \varphi)_T = -(\boldsymbol{v}_0, \nabla\varphi)_T + \langle \boldsymbol{v}_b \cdot \boldsymbol{n}, \varphi\rangle_{\partial T}, \quad \forall\, \varphi \in P_r(T). \tag{3.7}$$

From the integration by parts, we have

$$(\boldsymbol{v}_0, \nabla\varphi)_T = -(\nabla \cdot \boldsymbol{v}_0, \varphi)_T + \langle \boldsymbol{v}_0 \cdot \boldsymbol{n}, \varphi\rangle_{\partial T}.$$

Substituting the above identity into (3.3) yields

$$(\nabla_{w,r,T} \cdot \boldsymbol{v}, \varphi)_T - (\nabla \cdot \boldsymbol{v}_0, \varphi)_T = \langle (\boldsymbol{v}_b - \boldsymbol{v}_0) \cdot \boldsymbol{n}, \varphi\rangle_{\partial T}, \tag{3.8}$$

for all $\varphi \in P_r(T)$.

4 A hybridized weak Galerkin formulation

The goal of this section is to introduce a hybridized formulation for the weak Galerkin finite element algorithm that was first designed in [23].

4.1 Notation

Let $\mathcal{T}_h$ be a partition of the domain Ω into polygons in 2D or polyhedra in 3D. Assume that $\mathcal{T}_h$ is shape regular in the sense as defined in [24]. Denote by $\mathcal{E}_h$ the set of all edges or flat faces in $\mathcal{T}_h$, and let $\mathcal{E}_h^0 = \mathcal{E}_h \setminus \partial\Omega$ be the set of all interior edges or flat faces. Denote by h_T the diameter of $T \in \mathcal{T}_h$ and $h = \max_{T\in\mathcal{T}_h} h_T$ the meshsize for the partition $\mathcal{T}_h$.

On each element $T \in \mathcal{T}_h$, there are spaces of weak functions $\mathcal{V}(T)$ and $V(T)$ defined as in (3.1) and (3.5), respectively. Denote by $\mathcal{V}$ and Λ the function space on $\mathcal{T}_h$ and $\mathcal{E}_h$ given respectively by

$$\mathcal{V} = \prod_{T\in\mathcal{T}_h} \mathcal{V}(T), \quad \Lambda = \prod_{T\in\mathcal{T}_h} [H^{\frac{1}{2}}(\partial T)]^d. \tag{4.1}$$

Note that the values of functions in the spaces $\mathcal{V}(T_1)$ and $\mathcal{V}(T_2)$ are not related for any elements T_1 and T_2, even if T_1 and T_2 share an interior edge or flat face $e \in \mathcal{E}_h^0$. The jump of $\boldsymbol{v} = \{\boldsymbol{v}_0, \boldsymbol{v}_b\}$ on e is given by

$$[\![\boldsymbol{v}]\!]_e = \begin{cases} \boldsymbol{v}_b|_{\partial T_1} - \boldsymbol{v}_b|_{\partial T_2}, & e \in \mathcal{E}_h^0, \\ \boldsymbol{v}_b, & e \in \partial\Omega, \end{cases} \tag{4.2}$$

where $\boldsymbol{v}_b|_{\partial T_i}$ is the value of $\boldsymbol{v}$ on e as seen from the element T_i. The order of T_1 and T_2 is non-essential as long as the difference is taken in a consistent way in all the formulas. Analogously, for any function $\boldsymbol{\lambda} \in \Lambda$, we define its similarity on $e \in \mathcal{E}_h$ by

$$\langle\!\langle \boldsymbol{\lambda} \rangle\!\rangle_e = \begin{cases} \boldsymbol{\lambda}|_{\partial T_1} + \boldsymbol{\lambda}|_{\partial T_2}, & e \in \mathcal{E}_h^0, \\ 0, & e \in \partial\Omega. \end{cases} \tag{4.3}$$

Denote by $\langle\langle \boldsymbol{\lambda} \rangle\rangle$ the similarity of $\boldsymbol{\lambda}$ in $\mathcal{E}_h$.

For any integer $k \geqslant 1$, denote by $W_k(T)$ the discrete function space as follows:

$$W_k(T) = \{q : q \in L_0^2(\Omega), q|_T \in P_{k-1}(T)\}.$$

Let $V_k(T)$ denote the discrete weak function space as follows:

$$V_k(T) = \{\boldsymbol{v} = \{\boldsymbol{v}_0, \boldsymbol{v}_b\} : \{\boldsymbol{v}_0, \boldsymbol{v}_b\}|_T \in [P_k(T)]^d \times [P_{k-1}(e)]^d, e \subset \partial T\}.$$

Let $\Lambda_k(\partial T)$ denote

$$\Lambda_k(\partial T) = \{\boldsymbol{\lambda} : \boldsymbol{\lambda}|_e \in [P_{k-1}(e)]^d, e \subset \partial T\}.$$

By patching $W_k(T)$, $V_k(T)$, and $\Lambda_k(\partial T)$ over all the elements $T \in \mathcal{T}_h$, we obtain three weak Galerkin finite element spaces W_h, V_h, and Λ_h given by

$$W_h = \prod_{T \in \mathcal{T}_h} W_k(T), \quad V_h = \prod_{T \in \mathcal{T}_h} V_k(T), \quad \Lambda_h = \prod_{T \in \mathcal{T}_h} \Lambda_k(\partial T). \tag{4.4}$$

Denote by V_h^0 the subspace of V_h consisting of discrete weak functions with vanishing boundary value

$$V_h^0 = \{\boldsymbol{v} = \{\boldsymbol{v}_0, \boldsymbol{v}_b\} \in V_h : \boldsymbol{v}_b = 0 \text{ on } \partial\Omega\}.$$

Furthermore, let $\mathcal{V}_h$ be the subspace of V_h consisting of functions without jump on each interior edge or flat face

$$\mathcal{V}_h = \{\boldsymbol{v} \in V_h : [\![\boldsymbol{v}]\!]_e = 0, e \in \mathcal{E}_h^0\}.$$

Denote by $\mathcal{V}_h^0$ a subspace of $\mathcal{V}_h$ consisting of functions with vanishing boundary values

$$\mathcal{V}_h^0 = \{\boldsymbol{v} \in \mathcal{V}_h : \boldsymbol{v}_b|_e = 0, e \in \partial\Omega\}.$$

Let Ξ_h be the subspace of Λ_h consisting of functions with similarity zero across each edge or flat face

$$\Xi_h = \{\boldsymbol{\lambda} \in \Lambda_h : \langle\langle \boldsymbol{\lambda} \rangle\rangle_e = 0, e \in \mathcal{E}_h\}.$$

The functions in the space Ξ_h serve as Lagrange multipliers in hybridization methods.

Denote by $\nabla_{w,k-1}$ and $(\nabla_{w,k-1}\cdot)$ the discrete weak gradient and the discrete weak divergence on the finite element space V_h. They can be computed by using (3.3) and (3.7) on each element T, respectively.

For each element $T \in \mathcal{T}_h$, denote by Q_0 the L^2 projection operator from $[L^2(T)]^d$ onto $[P_k(T)]^d$. For each edge or face $e \in \mathcal{E}_h$, denote by Q_b the L^2 projection from $[L^2(e)]^d$ onto $[P_{k-1}(e)]^d$. We shall combine Q_0 with Q_b by writing $Q_h = \{Q_0, Q_b\}$.

4.2 Algorithm

On each element $T \in \mathcal{T}_h$, we introduce four bilinear forms given below:

$$s_T(\boldsymbol{v}, \boldsymbol{w}) = h_T^{-1} \langle Q_b \boldsymbol{v}_0 - \boldsymbol{v}_b, Q_b \boldsymbol{w}_0 - \boldsymbol{w}_b \rangle_{\partial T}, \tag{4.5}$$

$$a_T(\boldsymbol{v}, \boldsymbol{w}) = (\nabla_w \boldsymbol{v}, \nabla_w \boldsymbol{w})_T + s_T(\boldsymbol{v}, \boldsymbol{w}), \tag{4.6}$$

$$b_T(\boldsymbol{v}, q) = (\nabla_w \cdot \boldsymbol{v}, q)_T, \tag{4.7}$$

$$c_T(\boldsymbol{v}, \boldsymbol{\lambda}) = \langle \boldsymbol{v}_b, \boldsymbol{\lambda} \rangle_{\partial T}, \tag{4.8}$$

for $\boldsymbol{v} = \{\boldsymbol{v}_0, \boldsymbol{v}_b\} \in V_k(T)$, $\boldsymbol{w} = \{\boldsymbol{w}_0, \boldsymbol{w}_b\} \in V_k(T)$, $q \in W_h(T)$ and $\boldsymbol{\lambda} \in \Lambda_k(\partial T)$.

Their sums over all $T \in \mathcal{T}_h$ yield four globally-defined bilinear forms:

$$s(\boldsymbol{v}, \boldsymbol{w}) = \sum_{T \in \mathcal{T}_h} s_T(\boldsymbol{v}, \boldsymbol{w}), \quad \boldsymbol{v}, \boldsymbol{w} \in V_h, \tag{4.9}$$

$$a(\boldsymbol{v}, \boldsymbol{w}) = \sum_{T \in \mathcal{T}_h} a_T(\boldsymbol{v}, \boldsymbol{w}), \quad \boldsymbol{v}, \boldsymbol{w} \in V_h, \tag{4.10}$$

$$b(\boldsymbol{v},q)=\sum_{T\in\mathcal{T}_h} b_T(\boldsymbol{v},q),\quad \boldsymbol{v}\in V_h,\quad q\in W_h, \tag{4.11}$$

$$c(\boldsymbol{v},\boldsymbol{\lambda})=\sum_{T\in\mathcal{T}_h} c_T(\boldsymbol{v},\boldsymbol{\lambda}),\quad \boldsymbol{v}\in V_h,\quad \boldsymbol{\lambda}\in\Lambda_h. \tag{4.12}$$

The following weak Galerkin finite element scheme for the Stokes equation (2.1) was introduced and analyzed in [23].

Weak Galerkin Algorithm 1. A numerical approximation for (2.1)–(2.3) can be obtained by seeking $\bar{\boldsymbol{u}}_h=\{\bar{\boldsymbol{u}}_0,\bar{\boldsymbol{u}}_b\}\in V_h$ and $\bar{p}_h\in W_h$ such that $\bar{\boldsymbol{u}}_b=Q_b\boldsymbol{g}$ on $\partial\Omega$ and

$$a(\bar{\boldsymbol{u}}_h,\boldsymbol{v})-b(\boldsymbol{v},\bar{p}_h)=(\boldsymbol{f},\boldsymbol{v}_0), \tag{4.13}$$

$$b(\bar{\boldsymbol{u}}_h,q)=0, \tag{4.14}$$

for all $\boldsymbol{v}=\{\boldsymbol{v}_0,\boldsymbol{v}_b\}\in\mathcal{V}_h^0$ and $q\in W_h$.

The weak Galerkin finite element algorithm 1 can be hybridized in the finite element space $\mathcal{V}_h$ by using a Lagrange multiplier that shall enforce the continuity of the functions in V_h on interior element boundaries. The corresponding formulation can be described as follows.

Hybridized Weak Galerkin (HWG) Algorithm 1. A numerical approximation for (2.1)–(2.3) can be obtained by seeking $(\boldsymbol{u}_h;p_h;\boldsymbol{\lambda}_h)\in V_h\times W_h\times\Xi_h$ such that $\boldsymbol{u}_b=Q_b\boldsymbol{g}$ on $\partial\Omega$ and

$$a(\boldsymbol{u}_h,\boldsymbol{v})-b(\boldsymbol{v},p_h)-c(\boldsymbol{v},\boldsymbol{\lambda}_h)=(\boldsymbol{f},\boldsymbol{v}_0), \tag{4.15}$$

$$b(\boldsymbol{u}_h,q)+c(\boldsymbol{u}_h,\boldsymbol{\mu})=0, \tag{4.16}$$

for all $\boldsymbol{v}=\{\boldsymbol{v}_0,\boldsymbol{v}_b\}\in V_h^0$, $q\in W_h$ and $\boldsymbol{\mu}\in\Xi_h$.

Lemma 4.1. *The WG finite element scheme* (4.15)–(4.16) *has a unique solution.*

Proof. Let $\boldsymbol{f}=\boldsymbol{0}$, we shall show that the solution to (4.15)–(4.16) is trivial. To this end, taking $\boldsymbol{v}=\boldsymbol{u}_h$, $q=p_h$, and $\boldsymbol{\mu}=\boldsymbol{\lambda}_h$ and subtracting (4.16) from (4.15) we arrive at

$$a(\boldsymbol{u}_h,\boldsymbol{u}_h)=0.$$

By the definition of $a(\cdot,\cdot)$, we know $\nabla_w\boldsymbol{u}_h=0$ on each $T\in\mathcal{T}_h$, $\boldsymbol{u}_0=\boldsymbol{u}_b$ on each ∂T.

By (3.8) and the fact that $\boldsymbol{u}_b=\boldsymbol{u}_0$ on ∂T we have, for any $\tau\in[P_{k-1}(T)]^{d\times d}$,

$$0=(\nabla_w\boldsymbol{u}_h,\tau)_T=(\nabla\boldsymbol{u}_0,\tau)_T-\langle\boldsymbol{u}_0-\boldsymbol{u}_b,\tau\cdot\boldsymbol{n}\rangle_{\partial T}=(\nabla\boldsymbol{u}_0,\tau)_T,$$

which implies $\nabla\boldsymbol{u}_0=0$ on each $T\in\mathcal{T}_h$ and thus $\boldsymbol{u}_0$ is a constant. Since $\boldsymbol{u}_0=\boldsymbol{u}_b$ on each ∂T, we have

$$b(\boldsymbol{u}_h,q)=-(\boldsymbol{u}_0,\nabla q)+\langle\boldsymbol{u}_0\cdot\boldsymbol{n},q\rangle=(\nabla\cdot\boldsymbol{u}_h,q)=0.$$

From (4.16), we obtain

$$c(\boldsymbol{u}_h,\boldsymbol{\mu})=0.$$

Let $\boldsymbol{\mu}=[\![\boldsymbol{u}_b]\!]$ in the above equation, then it follows that

$$\sum_{T\in\mathcal{T}_h} h\|[\![\boldsymbol{u}_b]\!]\|_e^2=0.$$

Thus $[\![\boldsymbol{u}_b]\!]=0$, which implies that $\boldsymbol{u}_0$ is continuous and we arrive at $\boldsymbol{u}_h=\{\boldsymbol{0},\boldsymbol{0}\}$ in Ω.

Let $\boldsymbol{v}_b=0$ in (4.15), then it follows from $\boldsymbol{u}_h=\{\boldsymbol{0},\boldsymbol{0}\}$ and $\boldsymbol{f}=\boldsymbol{0}$ that

$$b(\boldsymbol{v},p_h)=(\nabla_w\cdot\boldsymbol{v},p_h)=-(\boldsymbol{v}_0,\nabla p_h)=0.$$

Hence, we have $\nabla p_h=0$ on each $T\in\mathcal{T}_h$. Thus p_h is a constant in T. Let $\boldsymbol{v}_b|_e=[\![p_h]\!]_e,\boldsymbol{v}_0=0$. Then

$$c(\boldsymbol{v},\boldsymbol{\lambda}_h)=0.$$

Thus

$$0 = b(\boldsymbol{v}, p_h) = \sum_{e\in\mathcal{E}_h} \|[\![p_h]\!]\|_e^2.$$

Hence p_h is continuous. From $p_h \in L_0^2(\Omega)$, we would obtain $p_h = 0$ in Ω.

Finally, letting $\boldsymbol{v}_b = \boldsymbol{\lambda}_h$, from $\boldsymbol{u}_h = \{\mathbf{0}, \mathbf{0}\}$ and $p_h = 0$ in Ω, we obtain

$$c(\boldsymbol{v}, \boldsymbol{\lambda}_h) = 0,$$

which means $\boldsymbol{\lambda}_h = 0$.

This completes the proof of the lemma. □

4.3 The relation between WG and HWG

The rest of this section will show that the above two schemes are equivalent in that the solutions $\bar{\boldsymbol{u}}_h, \bar{p}_h$ from (4.13)–(4.14) and $\boldsymbol{u}_h, p_h$ from (4.15)–(4.16) are identical, respectively.

For any $\boldsymbol{v} \in \mathcal{V}_h^0$, let

$$|\!|\!|\boldsymbol{v}|\!|\!|^2 = a(\boldsymbol{v}, \boldsymbol{v}) = \sum_{T\in\mathcal{T}_h} \|\nabla_w \boldsymbol{v}\|_T^2 + \sum_{T\in\mathcal{T}_h} h_T^{-1} \|Q_b \boldsymbol{v}_0 - \boldsymbol{v}_b\|_{\partial T}^2. \tag{4.17}$$

It has been verified in [23] that (4.17) defines a norm in the vector space $\mathcal{V}_h^0$.

Theorem 4.2. *Let $\boldsymbol{u}_h \in V_h$, $p_h \in W_h$ be the first two components of the solution of the hybridized WG algorithm* (4.15)–(4.16)*. Then, we have $[\![\boldsymbol{u}]\!]_e = 0$ for all $e \in \mathcal{E}_h^0$; i.e., $\boldsymbol{u} \in \mathcal{V}_h$ and $\boldsymbol{u}_b = Q_b \boldsymbol{g}$ on $\partial\Omega$. Furthermore, we have that $\boldsymbol{u}_h$ and p_h satisfy the equation* (4.13)–(4.14)*, i.e., $\boldsymbol{u}_h = \bar{\boldsymbol{u}}_h$ and $p_h = \bar{p}_h$.*

Proof. Let $e \in \mathcal{E}_h^0$ be an interior edge shared by T_1 and T_2. By letting $q = 0$, $\boldsymbol{\mu} = [\![\boldsymbol{u}_h]\!]_e$ on $e \in \partial T_1$, $\boldsymbol{\mu} = -[\![\boldsymbol{u}_h]\!]_e$ for $e \in \partial T_2$, and $\boldsymbol{\mu} = 0$ otherwise in (4.16), we have from (4.12) that

$$0 = c(\boldsymbol{u}_h, \boldsymbol{\mu}) = \sum_{T\in\mathcal{T}_h} \langle \boldsymbol{u}_h, \boldsymbol{\mu}\rangle_{\partial T} = \int_e [\![\boldsymbol{u}_h]\!]_e^2 ds,$$

which implies $[\![\boldsymbol{u}_h]\!]_e = 0$ for any $e \in \mathcal{E}_h^0$.

Next, by letting $\boldsymbol{\mu} = 0$, we obtain from (4.12) that

$$b(\boldsymbol{u}_h, q) = 0$$

for all $q \in W_h$.

For any $\boldsymbol{v} \in V_h^0$, it follows from $[\![\boldsymbol{v}]\!]_e = 0$ on any $e \in \mathcal{E}_h^0$ and $\boldsymbol{v} = 0$ on $\partial\Omega$ that

$$c(\boldsymbol{v}, \boldsymbol{\lambda}_h) = \sum_{T\in\mathcal{T}_h} \langle \boldsymbol{v}_b, \boldsymbol{\lambda}_h\rangle_{\partial T} = \sum_{T\in\mathcal{T}_h} \langle [\![\boldsymbol{v}]\!], \boldsymbol{\lambda}\rangle_e = 0.$$

Thus, we arrive at

$$a(\boldsymbol{u}_h, \boldsymbol{v}) - b(\boldsymbol{v}, p_h) = (\boldsymbol{f}, \boldsymbol{v}_0),$$

which is the same as (4.13). It implies that $(\boldsymbol{u}_h; p_h)$ is a solution of the WG scheme (4.13)–(4.14). It follows from the uniqueness of solution of (4.13)–(4.14) that $\boldsymbol{u}_h = \bar{\boldsymbol{u}}_h$ and $p_h = \bar{p}_h$, which completes the proof. □

5 Stability conditions for HWG

It is easy to see that the following defines a norm in the finite element space Ξ_h,

$$\|\boldsymbol{\lambda}\|_{\Xi_h} = \left(\sum_{e\in\mathcal{E}_h^0} h_e \|\boldsymbol{\lambda}\|_e^2 \right)^{\frac{1}{2}}. \tag{5.1}$$

As to V_h^0, for any $\boldsymbol{v} \in V_h^0$, let

$$\|\boldsymbol{v}\|_{V_h^0} = \left(⦀\boldsymbol{v}⦀^2 + \sum_{e \in \mathcal{E}_h^0} h_e^{-1} \|[\![\boldsymbol{v}]\!]_e\|_e^2 \right)^{\frac{1}{2}}. \tag{5.2}$$

We claim that $\|\cdot\|_{V_h^0}$ defines a norm in V_h^0. In fact, if $\|\boldsymbol{v}\|_{V_h^0} = 0$, then $[\![\boldsymbol{v}]\!]_e = 0$ on each interior edge or flat face $e \in \mathcal{E}_h^0$, and hence $\boldsymbol{v} \in \mathcal{V}_h^0$. Since $⦀\cdot⦀$ defines a norm in the vector space $\mathcal{V}_h^0$, then $\boldsymbol{v} = 0$. This verifies the positivity property of $\|\cdot\|_{V_h^0}$. The other properties for a norm can be checked trivially.

Remark 5.1. Similarly, for any $\boldsymbol{\phi} = (p; \boldsymbol{\lambda}) \in M_h$, we can define

$$\|\boldsymbol{\phi}\|_{M_h} = \|p\| + \|\boldsymbol{\lambda}\|_{\Xi_h}. \tag{5.3}$$

Lemma 5.2 (Trace inequality, see [24]). *Let $\mathcal{T}_h$ be a partition of the domain Ω into polygons in* 2D *or polyhedra in* 3D. *Assume that the partition $\mathcal{T}_h$ satisfies the assumptions* A1–A3 *as specified in* [24]. *Let $p > 1$ be any real number. Then, there exists a constant C such that for any $T \in \mathcal{T}_h$ and edge/face $e \in \partial T$, we have*

$$\|\theta\|_{L^p(e)}^p \leqslant C h_T^{-1} (\|\theta\|_{L^p(T)}^p + h_T^p \|\nabla \theta\|_{L^p(T)}^p), \tag{5.4}$$

where $\theta \in W^{1,p}(T)$ is any function.

Lemma 5.3 (Inverse inequality, see [24]). *Let $\mathcal{T}_h$ be a partition of the domain Ω into polygons or polyhedra. Assume that $\mathcal{T}_h$ satisfies all the assumptions* A1–A4 *in* [24] *and $p \geqslant 1$ be any real number. Then, there exists a constant $C(n)$ such that*

$$\|\nabla \varphi\|_{T,p} \leqslant C(n) h_T^{-1} \|\varphi\|_{T,p}, \quad \forall\, T \in \mathcal{T}_h \tag{5.5}$$

for any piecewise polynomial φ of degree n on $\mathcal{T}_h$.

Lemma 5.4 (Boundedness). *There exists a constant $C > 0$ such that*

$$|a(\boldsymbol{w}, \boldsymbol{v})| \leqslant C \|\boldsymbol{w}\|_{V_h^0} \|\boldsymbol{v}\|_{V_h^0}, \quad \forall\, \boldsymbol{w}, \boldsymbol{v} \in V_h^0, \tag{5.6}$$

$$|b(\boldsymbol{v}, q)| \leqslant C \|\boldsymbol{v}\|_{V_h^0} \|q\|, \quad \forall\, \boldsymbol{v} \in V_h^0, q \in W_h, \tag{5.7}$$

$$|c(\boldsymbol{v}, \boldsymbol{\lambda})| \leqslant C \|\boldsymbol{v}\|_{V_h^0} \|\boldsymbol{\lambda}\|_{\Xi_h}, \quad \forall\, \boldsymbol{v} \in V_h^0, \boldsymbol{\lambda} \in \Xi_h. \tag{5.8}$$

Proof. To derive (5.6), we use the Cauchy-Schwarz inequality to obtain

$$\begin{aligned}
|a(\boldsymbol{w}, \boldsymbol{v})| &= \left| \sum_{T \in \mathcal{T}_h} (\nabla_w \boldsymbol{w}, \nabla_w \boldsymbol{v})_T + h_T^{-1} \langle Q_b \boldsymbol{w}_0 - \boldsymbol{w}_b, Q_b \boldsymbol{v}_0 - \boldsymbol{v}_b \rangle_{\partial T} \right| \\
&\leqslant \left(\sum_{T \in \mathcal{T}_h} \|\nabla_w \boldsymbol{w}\|_T^2 \right)^{\frac{1}{2}} \left(\sum_{T \in \mathcal{T}_h} \|\nabla_w \boldsymbol{v}\|_T^2 \right)^{\frac{1}{2}} \\
&\quad + \left(\sum_{T \in \mathcal{T}_h} h_T^{-1} \|Q_b \boldsymbol{w}_0 - \boldsymbol{w}_b\|_{\partial T}^2 \right)^{\frac{1}{2}} \left(\sum_{T \in \mathcal{T}_h} h_T^{-1} \|Q_b \boldsymbol{v}_0 - \boldsymbol{v}_b\|_{\partial T}^2 \right)^{\frac{1}{2}} \\
&\leqslant C \|\boldsymbol{w}\|_{V_h^0} \|\boldsymbol{v}\|_{V_h^0}.
\end{aligned}$$

As to (5.7), we use (3.6), trace inequality (5.4), and inverse inequality (5.5) to obtain

$$\begin{aligned}
|b(\boldsymbol{v}, q)| &= \left| \sum_{T \in \mathcal{T}_h} (\nabla_w \cdot \boldsymbol{v}, q)_T \right| \\
&= - \sum_{T \in \mathcal{T}_h} (\boldsymbol{v}_0, \nabla p)_T + \sum_{T \in \mathcal{T}_h} \langle \boldsymbol{v}_b, p \boldsymbol{n} \rangle_{\partial T} \\
&= \sum_{T \in \mathcal{T}_h} (\nabla \cdot \boldsymbol{v}_0, p)_T - \sum_{T \in \mathcal{T}_h} \langle \boldsymbol{v}_0 - \boldsymbol{v}_b, p \boldsymbol{n} \rangle_{\partial T}
\end{aligned}$$

$$\leqslant \bigg(\sum_{T\in\mathcal{T}_h}\|\nabla\cdot\boldsymbol{v}_0\|_T\bigg)^{\frac{1}{2}}\bigg(\sum_{T\in\mathcal{T}_h}\|p\|_T^2\bigg)^{\frac{1}{2}}+\bigg(\sum_{T\in\mathcal{T}_h}\|\boldsymbol{v}_0-\boldsymbol{v}_b\|_{\partial T}\bigg)^{\frac{1}{2}}\bigg(\sum_{T\in\mathcal{T}_h}\|p\|_{\partial T}^2\bigg)^{\frac{1}{2}}$$
$$\leqslant C\bigg(\sum_{T\in\mathcal{T}_h}\|\nabla\boldsymbol{v}_0\|_T\bigg)^{\frac{1}{2}}\|p\|+Ch^{-\frac{1}{2}}\bigg(\sum_{T\in\mathcal{T}_h}\|\boldsymbol{v}_0-\boldsymbol{v}_b\|_{\partial T}\bigg)^{\frac{1}{2}}(\|p\|+h\|\nabla p\|)$$
$$\leqslant C\|\boldsymbol{v}\|_{V_h^0}\|p\|.$$

As to (5.8), it follows from the Cauchy-Schwarz inequality that

$$\begin{aligned}|c(\boldsymbol{v},\boldsymbol{\lambda})|&=\bigg|\sum_{T\in\mathcal{T}_h}\langle\boldsymbol{v}_b,\boldsymbol{\lambda}\rangle_{\partial T}\bigg|=\bigg|\sum_{e\in\mathcal{E}_h^0}\langle[\![\boldsymbol{v}]\!]_e,\boldsymbol{\lambda}\rangle_e\bigg|\\&\leqslant\bigg(\sum_{e\in\mathcal{E}_h^0}h_e^{-1}\|[\![\boldsymbol{v}]\!]_e\|_e^2\bigg)^{\frac{1}{2}}\bigg(\sum_{e\in\mathcal{E}_h^0}h_e\|\boldsymbol{\lambda}\|_e^2\bigg)^{\frac{1}{2}}\\&\leqslant C\|\boldsymbol{v}\|_{V_h^0}\|\boldsymbol{\lambda}\|_{\Xi_h},\end{aligned}$$

which completes the proof. □

Lemma 5.5 (Coercivity). *For any $\boldsymbol{v}\in\mathcal{V}_h^0$, we have*

$$|a(\boldsymbol{v},\boldsymbol{v})|\geqslant C\|\boldsymbol{v}\|_{V_h^0}^2. \tag{5.9}$$

Proof. For any $\boldsymbol{v}\in\mathcal{V}_h^0$, we have $\|\boldsymbol{v}\|_{V_h^0}^2=|\!|\!|\boldsymbol{v}|\!|\!|$, which means the estimate (5.9) holds true. This completes the proof. □

Lemma 5.6 (See [23]). *There exists a positive constant β independent of h such that*

$$\sup_{\boldsymbol{v}\in\mathcal{V}_h^0}\frac{b(\boldsymbol{v},\rho)}{|\!|\!|\boldsymbol{v}|\!|\!|}\geqslant\beta\|\rho\|, \tag{5.10}$$

for all $\rho\in W_h$.

Lemma 5.7. *For any given $\rho\in W_h$, there exist a positive constant β independent of h and a $\boldsymbol{v}\in V_h^0$ such that*

$$\frac{b(\boldsymbol{v},\rho)}{\|\boldsymbol{v}\|_{V_h^0}}\geqslant\beta\|\rho\|. \tag{5.11}$$

Proof. From $\mathcal{V}_h^0\subset V_h^0$ and Lemma 5.6, we have, for any $\rho\in W_h$, there exists a $\boldsymbol{v}\in\mathcal{V}_h^0$,

$$\frac{b(\boldsymbol{v},\rho)}{\|\boldsymbol{v}\|_{V_h^0}}=\frac{b(\boldsymbol{v},\rho)}{|\!|\!|\boldsymbol{v}|\!|\!|}\geqslant\beta\|\rho\|, \tag{5.12}$$

which completes the proof of the lemma. □

Lemma 5.8. *For any given $\boldsymbol{\tau}\in\Xi_h$, there exist a $\boldsymbol{v}\in V_h^0$ with $\boldsymbol{v}_0=\mathbf{0}$ and a constant $C>0$ such that*

$$\frac{c(\boldsymbol{v},\boldsymbol{\tau})}{\|\boldsymbol{v}\|_{V_h^0}}\geqslant C\|\boldsymbol{\tau}\|_{\Xi_h}. \tag{5.13}$$

Proof. For any $\boldsymbol{\tau}\in\Xi_h$, we have $\langle\!\langle\boldsymbol{\tau}\rangle\!\rangle_e=0$ or equivalently $\boldsymbol{\tau}^1+\boldsymbol{\tau}^2=0$ on each interior edge $e\in\mathcal{E}_h^0$ and $\boldsymbol{\tau}=0$ on all boundary edges. By letting $\boldsymbol{v}=\{\mathbf{0},h_e\boldsymbol{\tau}\}\in V_h^0$ in $c(\boldsymbol{v},\boldsymbol{\tau})$ and $s(\boldsymbol{v},\boldsymbol{v})$, we obtain

$$c(\boldsymbol{v},\boldsymbol{\tau})=\sum_{e\in\mathcal{E}_h^0}\langle\boldsymbol{v}_b^1,\boldsymbol{\tau}^1\rangle_e+\langle\boldsymbol{v}_b^2,\boldsymbol{\tau}^2\rangle_e=2\sum_{e\in\mathcal{E}_h^0}h_e\|\boldsymbol{\tau}\|_e^2, \tag{5.14}$$

and

$$s(\boldsymbol{v},\boldsymbol{v})=2\sum_{e\in\mathcal{E}_h^0}h_T^{-1}h_e^2(\|\boldsymbol{\tau}^1\|_e^2+\|\boldsymbol{\tau}^2\|_e^2)\leqslant 2\sum_{e\in\mathcal{E}_h^0}h_e\|\boldsymbol{\tau}\|_e^2. \tag{5.15}$$

It follows from (3.3), Cauchy-Schwarz inequality, the trace inequality (5.4), and the inverse inequality (5.5) that

$$(\nabla_w \boldsymbol{v}, \nabla_w \boldsymbol{v})_T = \sum_{e\in\partial T} \langle \boldsymbol{v}_b^*, \nabla_w \boldsymbol{v}\rangle_e \leqslant \sum_{e\in\partial T} h_e \|\boldsymbol{\tau}^*\|_e \|\nabla_w \boldsymbol{v}\|_e \leqslant C \sum_{e\in\partial T} h_e^{\frac{1}{2}} \|\boldsymbol{\tau}^*\|_e \|\nabla_w \boldsymbol{v}\|_T, \tag{5.16}$$

where $\boldsymbol{v}_b^*$ is chosen to be $\boldsymbol{v}_b^1$ or $\boldsymbol{v}_b^2$ according to the relative position of $\boldsymbol{v}_b$ and e, which implies that

$$\|\nabla_w \boldsymbol{v}\|_T \leqslant C \sum_{e\in\partial T} h_e^{\frac{1}{2}} \|\boldsymbol{\tau}^*\|_e. \tag{5.17}$$

By summing over all elements, we obtain

$$(\nabla_w \boldsymbol{v}, \nabla_w \boldsymbol{v})_h \leqslant C \sum_{e\in\mathcal{E}_h^0} h_e \|\boldsymbol{\tau}^*\|_e^2. \tag{5.18}$$

It follows from (5.15) and (5.18) that

$$|\!|\!|\boldsymbol{v}|\!|\!|^2 \leqslant C \sum_{e\in\mathcal{E}_h^0} h_e \|\boldsymbol{\tau}^*\|_e^2 = C\|\boldsymbol{\tau}\|_{\Xi_h}^2. \tag{5.19}$$

By combining (5.14) and (5.19), we obtain that there exists a constant $C>0$ such that

$$\frac{c(\boldsymbol{v},\boldsymbol{\tau})}{\|\boldsymbol{v}\|_{V_h^0}} \geqslant C\|\boldsymbol{\tau}\|_{\Xi_h}, \tag{5.20}$$

which completes the proof. □

6 Error estimates

The goal of this section is to derive an error equation for the HWG finite element solution obtained from (4.15)–(4.16). This error equation is critical in convergence analysis.

In addition to the projection $Q_h = \{Q_0, Q_b\}$ defined in the previous section, let $\mathbb{Q}_h$ and $\boldsymbol{Q}_h$ be two local L^2 projections onto $P_{k-1}(T)$ and $[P_{k-1}(T)]^{d\times d}$, respectively.

Lemma 6.1 (See [23]). *The projection operators Q_h, $\boldsymbol{Q}_h$, and $\mathbb{Q}_h$ satisfy the following commutative properties*

$$\nabla_w(Q_h\boldsymbol{v}) = \boldsymbol{Q}_h(\nabla\boldsymbol{v}), \quad \forall\, \boldsymbol{v}\in[H^1(\Omega)]^d, \tag{6.1}$$

$$\nabla_w\cdot(Q_h\boldsymbol{v}) = \mathbb{Q}_h(\nabla\cdot\boldsymbol{v}), \quad \forall\, \boldsymbol{v}\in H(\mathrm{div},\Omega). \tag{6.2}$$

Denote by $(\boldsymbol{u};p)$ the exact solution of (2.1)–(2.3). Let $(\boldsymbol{u}_h;p_h;\boldsymbol{\lambda}_h)\in V_h\times W_h\times\Xi_h$ be the solutions to (4.15)–(4.16). Let $\boldsymbol{\lambda} = \nabla\boldsymbol{u}\cdot\boldsymbol{n} - p\boldsymbol{n}$. Define error functions as follows:

$$\mathbf{e}_h = \{Q_0\boldsymbol{u}-\boldsymbol{u}_0, Q_b\boldsymbol{u}-\boldsymbol{u}_b\}, \quad \varepsilon_h = \mathbb{Q}_h p - p_h, \quad \boldsymbol{\delta}_h = Q_b\boldsymbol{\lambda}-\boldsymbol{\lambda}_h. \tag{6.3}$$

Lemma 6.2. *Let $(\boldsymbol{u};p)$ be the exact solution to* (2.1)–(2.3)*, and $(\boldsymbol{u}_h;p_h;\boldsymbol{\lambda}_h)\in V_h\times W_h\times\Xi_h$ be the solutions of* (4.15)–(4.16)*. Then, the error functions $\mathbf{e}_h$, ε_h, and δ_h satisfy the following equations*:

$$a(\mathbf{e}_h,\boldsymbol{v}) - b(\boldsymbol{v},\varepsilon_h) - c(\boldsymbol{v},\boldsymbol{\delta}_h) = \ell_{\boldsymbol{u},p}(\boldsymbol{v}), \quad \forall\, \boldsymbol{v}\in V_h^0, \tag{6.4}$$

$$b(\mathbf{e}_h,q) + c(\mathbf{e}_h,\boldsymbol{\mu}) = 0, \quad \forall\, q\in W_h, \boldsymbol{\mu}\in\Xi_h, \tag{6.5}$$

where

$$\ell_{\boldsymbol{u},p}(\boldsymbol{v}) = \sum_{T\in\mathcal{T}_h} \langle \boldsymbol{v}_0-\boldsymbol{v}_b, (\nabla\boldsymbol{u}-\boldsymbol{Q}_h(\nabla\boldsymbol{u}))\cdot\boldsymbol{n}\rangle_{\partial T} - \sum_{T\in\mathcal{T}_h} \langle \boldsymbol{v}_0-\boldsymbol{v}_b, (p-\mathbb{Q}_h p)\boldsymbol{n}\rangle_{\partial T} + s(Q_h\boldsymbol{u},\boldsymbol{v}). \tag{6.6}$$

Proof. First, applying (3.2), Lemma 6.1, and the integration by parts, we have

$$\begin{aligned}(\nabla_w(Q_h\boldsymbol{u}),\nabla_w\boldsymbol{v})_T &= (\boldsymbol{Q}_h(\nabla\boldsymbol{u}),\nabla_w\boldsymbol{v})_T\\ &= -(\boldsymbol{v}_0,\nabla\cdot\boldsymbol{Q}_h(\nabla\boldsymbol{u}))_T+\langle\boldsymbol{v}_b,\boldsymbol{Q}_h(\nabla\boldsymbol{u})\cdot\boldsymbol{n}\rangle_{\partial T}\\ &= (\nabla\boldsymbol{v}_0,\boldsymbol{Q}_h(\nabla\boldsymbol{u}))_T-\langle\boldsymbol{v}_0-\boldsymbol{v}_b,\boldsymbol{Q}_h(\nabla\boldsymbol{u})\cdot\boldsymbol{n}\rangle_{\partial T}\\ &= (\nabla\boldsymbol{v}_0,\nabla\boldsymbol{u})_T-\langle\boldsymbol{v}_0-\boldsymbol{v}_b,\boldsymbol{Q}_h(\nabla\boldsymbol{u})\cdot\boldsymbol{n}\rangle_{\partial T}\\ &= -(\Delta\boldsymbol{u},\boldsymbol{v}_0)_T+\langle\boldsymbol{v}_0-\boldsymbol{v}_b,(\nabla\boldsymbol{u}-\boldsymbol{Q}_h(\nabla\boldsymbol{u}))\cdot\boldsymbol{n}\rangle_{\partial T}+\langle\boldsymbol{v}_b,\nabla\boldsymbol{u}\cdot\boldsymbol{n}\rangle_{\partial T}.\end{aligned}$$

Summing over all $T\in\mathcal{T}_h$ reaches

$$-(\Delta\boldsymbol{u},\boldsymbol{v}_0)=(\nabla_w(Q_h\boldsymbol{u}),\nabla_w\boldsymbol{v})-\sum_{T\in\mathcal{T}_h}\langle\boldsymbol{v}_0-\boldsymbol{v}_b,(\nabla\boldsymbol{u}-\boldsymbol{Q}_h(\nabla\boldsymbol{u}))\cdot\boldsymbol{n}\rangle_{\partial T}-\sum_{T\in\mathcal{T}_h}\langle\boldsymbol{v}_b,\nabla\boldsymbol{u}\cdot\boldsymbol{n}\rangle_{\partial T}. \tag{6.7}$$

Similarly, by using (3.6) and the integration by parts, we have

$$\begin{aligned}(\nabla_w\cdot\boldsymbol{v},\mathbb{Q}_hp)_T &= -(\boldsymbol{v}_0,\nabla(\mathbb{Q}_hp))_T+\langle\boldsymbol{v}_b,(\mathbb{Q}_hp)\boldsymbol{n}\rangle_{\partial T}\\ &= (\nabla\cdot\boldsymbol{v}_0,\mathbb{Q}_hp)_T-\langle\boldsymbol{v}_0-\boldsymbol{v}_b,(\mathbb{Q}_hp)\boldsymbol{n}\rangle_{\partial T}\\ &= (\nabla\cdot\boldsymbol{v}_0,p)_T-\langle\boldsymbol{v}_0-\boldsymbol{v}_b,(\mathbb{Q}_hp)\boldsymbol{n}\rangle_{\partial T}\\ &= -(\boldsymbol{v}_0,\nabla p)_T+\langle\boldsymbol{v}_0,p\boldsymbol{n}\rangle_{\partial T}-\langle\boldsymbol{v}_0-\boldsymbol{v}_b,(\mathbb{Q}_hp)\boldsymbol{n}\rangle_{\partial T}\\ &= -(\boldsymbol{v}_0,\nabla p)_T+\langle\boldsymbol{v}_0-\boldsymbol{v}_b,(p-\mathbb{Q}_hp)\boldsymbol{n}\rangle_{\partial T}+\langle\boldsymbol{v}_b,\nabla\boldsymbol{u}\cdot\boldsymbol{n}\rangle_{\partial T}.\end{aligned}$$

Summing over all T leads to

$$(\nabla p,\boldsymbol{v}_0)=(\nabla_w\cdot\boldsymbol{v},\mathbb{Q}_hp)+\sum_{T\in\mathcal{T}_h}\langle\boldsymbol{v}_0-\boldsymbol{v}_b,(p-\mathbb{Q}_hp)\boldsymbol{n}\rangle_{\partial T}+\sum_{T\in\mathcal{T}_h}\langle\boldsymbol{v}_b,p\boldsymbol{n}\rangle_{\partial T}. \tag{6.8}$$

By using the identity $-(\Delta\boldsymbol{u},\boldsymbol{v}_0)+(\nabla p,\boldsymbol{v}_0)=(\boldsymbol{f},\boldsymbol{v}_0)$ and noticing that

$$\sum_{T\in\mathcal{T}_h}\langle\boldsymbol{v}_b,\nabla\boldsymbol{u}\cdot\boldsymbol{n}-p\boldsymbol{n}\rangle_{\partial T}=c(\boldsymbol{v},\boldsymbol{\lambda}),$$

we obtain

$$a(\boldsymbol{v},\boldsymbol{Q}_h\boldsymbol{u})-b(\boldsymbol{v},\mathbb{Q}_hp)-c(\boldsymbol{v},\boldsymbol{\lambda})=(\boldsymbol{f},\boldsymbol{v}_0)+\ell_{\boldsymbol{u},p}(\boldsymbol{v}). \tag{6.9}$$

Combining with the scheme (4.15) as follows:

$$a(\boldsymbol{v},\boldsymbol{u}_h)-b(\boldsymbol{v},p_h)-c(\boldsymbol{v},\boldsymbol{\lambda}_h)=(\boldsymbol{f},\boldsymbol{v}_0),$$

we obtain $a(\mathbf{e}_h,\boldsymbol{v})-b(\boldsymbol{v},\varepsilon_h)-c(\boldsymbol{v},\boldsymbol{\delta}_h)=\ell_{\boldsymbol{u},p}(\boldsymbol{v})$.

As to (6.5), from Theorem 4.2 we know that $[\![\mathbf{e}_h]\!]=0$, which leads to

$$c(\mathbf{e}_h,\boldsymbol{\mu})=0,\quad \forall\,\boldsymbol{\mu}\in\Xi_h.$$

Moreover, for any $q\in W_h$, we have

$$b(\mathbf{e}_h,q)=b(\boldsymbol{Q}_h\boldsymbol{u},q)=\sum_{T\in\mathcal{T}_h}(\nabla_w\cdot(\boldsymbol{Q}_h\boldsymbol{u}),q)_T=\sum_{T\in\mathcal{T}_h}(\mathbb{Q}_h(\nabla\cdot\boldsymbol{u}),q)_T=(\nabla\cdot\boldsymbol{u},q)=0.$$

This completes the proof. □

Next, we shall establish some error estimates for the hybridized WG finite element solution $(\boldsymbol{u}_h;p_h;\boldsymbol{\lambda}_h)$ arising from (4.15)–(4.16). The error equations (6.4)–(6.5) imply

$$\begin{aligned}&a(Q_h\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{v})-b(\boldsymbol{v},\mathbb{Q}_hp-p_h)-c(\boldsymbol{v},Q_b\boldsymbol{\lambda}-\boldsymbol{\lambda}_h)=\ell_{\boldsymbol{u},p}(\boldsymbol{v}),\quad\forall\,\boldsymbol{v}\in V_h^0,\\ &b(Q_h\boldsymbol{u}-\boldsymbol{u}_h,q)+c(Q_h\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{\mu})=0,\quad\forall\,q\in W_h,\boldsymbol{\mu}\in\Xi_h,\end{aligned}$$

where $\ell_{\boldsymbol{u},p}(\boldsymbol{v})$ is given by (6.6). The above is a saddle point problem for which the Brezzi's theorem [4] can be applied for an analysis on its stability and solvability. Note that all the conditions of Brezzi's theorem have been verified in Section 5 (see Lemmas 5.4, 5.5 and 5.8). The following error estimate can be proved similarly with [23, Theorem 7.1].

Theorem 6.3. *Let $(\boldsymbol{u};p)$ be the exact solution to (2.1)–(2.3) and $(\boldsymbol{u}_h;p_h;\boldsymbol{\lambda}_h)\in V_h\times W_h\times\Xi_h$ be the solutions to (4.15)–(4.16). Then, there exists a constant C such that*

$$\|Q_h\boldsymbol{u}-\boldsymbol{u}_h\|_{V_h^0}+\|\mathbb{Q}_hp-p_h\|+\|Q_b\boldsymbol{\lambda}-\boldsymbol{\lambda}_h\|_{\Xi_h}\leqslant Ch^k(\|\boldsymbol{u}\|_{k+1}+\|p\|_k). \tag{6.10}$$

Theorem 6.4. *Let $(\boldsymbol{u};p)$ be the exact solution to (2.1)–(2.3) and $\boldsymbol{\lambda}_h\in\Xi_h$ be the last component of the solution to (4.15)–(4.16). On the set of interior edges $\mathcal{E}_h^0$, let $\boldsymbol{\lambda}=\nabla\boldsymbol{u}\cdot\boldsymbol{n}-p\boldsymbol{n}$. Then, there exists a constant C such that*

$$\|\boldsymbol{\lambda}-\boldsymbol{\lambda}_h\|_{\Xi_h}\leqslant Ch^k(\|\boldsymbol{u}\|_{k+1}+\|p\|_k). \tag{6.11}$$

Proof. From the triangle inequality and Theorem 6.3, we have

$$\|\boldsymbol{\lambda}-\boldsymbol{\lambda}_h\|_{\Xi_h}\leqslant\|\boldsymbol{\lambda}-Q_b\boldsymbol{\lambda}\|_{\Xi_h}+\|Q_b\boldsymbol{\lambda}-\boldsymbol{\lambda}_h\|_{\Xi_h},$$
$$\|Q_b\boldsymbol{\lambda}-\boldsymbol{\lambda}_h\|_{\Xi_h}\leqslant Ch^k(\|u\|_{k+1}+\|p\|_k).$$

Thus we just need to concentrate on $\|\boldsymbol{\lambda}-Q_b\boldsymbol{\lambda}\|_{\Xi_h}$.

Applying the definition of Lagrange multiplier, trace inequality, and the property of L^2 projection, yields

$$\begin{aligned}\|\boldsymbol{\lambda}-Q_b\boldsymbol{\lambda}\|_{\Xi_h}^2&=\sum_{e\in\mathcal{E}_h^0}h_e\|\boldsymbol{\lambda}-Q_b\boldsymbol{\lambda}\|_e^2\\&\leqslant\sum_{e\in\mathcal{E}_h^0}h\|\nabla\boldsymbol{u}-\boldsymbol{Q}_h\nabla\boldsymbol{u}\|_e^2+\sum_{e\in\mathcal{E}_h^0}h\|p-\mathbb{Q}_hp\|_e^2\\&\leqslant C\sum_{T\in\mathcal{T}_h}(\|\nabla\boldsymbol{u}-\boldsymbol{Q}_h\nabla\boldsymbol{u}\|_T^2+h^2\|\nabla\boldsymbol{u}-\boldsymbol{Q}_h\nabla\boldsymbol{u}\|_{1,T}^2)\\&\quad+C\sum_{T\in\mathcal{T}_h}(\|p-\mathbb{Q}_hp\|_T^2+h^2\|p-\mathbb{Q}_hp\|_{1,T}^2)\\&\leqslant Ch^{2k}(\|u\|_{k+1}+\|p\|_k)^2,\end{aligned}$$

which completes the proof. □

The following L^2-error estimate for $Q_0\boldsymbol{u}-\boldsymbol{u}_0$ follows from Theorem 4.2 and [23, Theorem 7.2].

Theorem 6.5 (See [23])**.** *Let $(\boldsymbol{u};p)$ with $k\geqslant1$ and $(\boldsymbol{u}_h;p_h;\boldsymbol{\lambda}_h)\in V_h\times W_h\times\Xi_h$ be the exact solution to (2.1)–(2.3) and be the solutions to (4.15)–(4.16), respectively. Then, the following optimal order error estimate holds true*

$$\|Q_0\boldsymbol{u}-\boldsymbol{u}_0\|\leqslant Ch^{k+1}(\|\boldsymbol{u}\|_{k+1}+\|p\|_k). \tag{6.12}$$

7 Efficient implementation via variable reduction

The degrees of freedom in the WG algorithm (4.13)–(4.14) can be divided into two classes: (1) the interior variables representing $\boldsymbol{u}_0$, and (2) the interface variables for $\boldsymbol{u}_b$. For the HWG algorithm (4.15)–(4.16), more unknowns must be added to the picture from the Lagrange multiplier $\boldsymbol{\lambda}_h$. Thus, the size of the discrete system arising from either (4.13)–(4.14) or (4.15)–(4.16) is enormously large.

The goal of this section is to present a Schur complement formulation for the WG algorithm (4.13)–(4.14) based on the hybridized formulation (4.15)–(4.16). The method shall eliminate all the interior unknowns associated with $\boldsymbol{u}_0$ and the interface unknow $\boldsymbol{\lambda}_h$, and produce a much reduced system of linear equations involving only the unknowns representing the interface variables $\boldsymbol{u}_b$.

7.1 Theory of variable reduction

Denote by B_h the interface finite element space defined as the restriction of $\mathcal{V}_h$ on the set of edges $\mathcal{E}_h$; i.e.,

$$B_h=\{\boldsymbol{v}=\{\boldsymbol{\mu};p\}:\boldsymbol{\mu}\in[P_{k-1}(e)]^d,p|_e\in P_{k-1}(e),e\in\mathcal{E}_h\}.$$

B_h is a Hilbert space with the following inner product

$$\langle \boldsymbol{w}_b, \boldsymbol{q}_b\rangle_{\mathcal{E}_h} = \sum_{e\in\mathcal{E}_h} \langle \boldsymbol{w}_b, \boldsymbol{q}_b\rangle_e, \quad \forall\, \boldsymbol{w}_b, \boldsymbol{q}_b \in B_h.$$

Let B_h^0 be the subspace of B_h consisting of functions with vanishing boundary value on $\partial\Omega$. It is not hard to see that the interface finite element space B_h is isomorphic to the space of Lagrange multiplier Ξ_h. The Schur complement through an elimination of the Lagrange multiplier $\boldsymbol{\lambda}_h$ and the interior unknown $\boldsymbol{u}_0$ can be implemented through a map, denoted by $S_{\boldsymbol{f}}$.

We define the map $S_{\boldsymbol{f}} : B_h \to B_h^0$ as follows: for a fixed p_h and any given function $\boldsymbol{w}_b \in B_h$, the image $S_{\boldsymbol{f}}(\boldsymbol{w}_b; p_h)$ can be obtained by

Step 1. On each element $T \in \mathcal{T}_h$, solve for $\boldsymbol{w}_0$ in term of $\boldsymbol{w}_b$ and p_h from the following local problem:

$$a_T(\boldsymbol{w}_h, \boldsymbol{v}) - b_T(\boldsymbol{v}, p_h) = (\boldsymbol{f}, \boldsymbol{v}_0)_T, \quad \forall\, \boldsymbol{v} = \{\boldsymbol{v}_0, \boldsymbol{0}\} \in V_k(T), \tag{7.1}$$

where $\boldsymbol{w}_h = \{\boldsymbol{w}_0, \boldsymbol{w}_b\} \in V_k(T), p_h \in W_k(T)$. Denote $\boldsymbol{w}_0 = D_{\boldsymbol{f}}(\boldsymbol{w}_b; p_h)$.

Step 2. On each element $T \in \mathcal{T}_h$, solve for $\boldsymbol{\zeta}_{h,T} \in \Lambda_k(\partial T)$ in term of $\boldsymbol{w}_h = \{\boldsymbol{w}_0, \boldsymbol{w}_b\}$ and p_h from the following local problem:

$$c_T(\boldsymbol{v}, \boldsymbol{\zeta}_{h,T}) = a_T(\boldsymbol{w}_h, \boldsymbol{v}) - b_T(\boldsymbol{v}, p_h), \quad \forall\, \boldsymbol{v} = \{\boldsymbol{0}, \boldsymbol{v}_b\} \in V_k(T). \tag{7.2}$$

Thus we obtain a function $\boldsymbol{\zeta}_{h,T} \in \Lambda_h$. Denote $\boldsymbol{\zeta}_{h,T} = L_{\boldsymbol{f}}(\boldsymbol{w}_b; p_h)$.

Step 3. Define $S_{\boldsymbol{f}}(\boldsymbol{w}_b; p_h)$ by the similarity of $\boldsymbol{\zeta}_h$ on interior edges and zero on boundary edges, i.e.,

$$S_{\boldsymbol{f}}(\boldsymbol{w}_b; p_h) = \langle\!\langle \boldsymbol{\zeta}_h \rangle\!\rangle. \tag{7.3}$$

It follows from (7.3) that $S_{\boldsymbol{f}}(\boldsymbol{w}_b; p_h) \in B_h^0$. The following are two properties regarding the operator $S_{\boldsymbol{f}}$ and the related terms.

(1) The sum of (7.1) and (7.2) yields

$$c_T(\boldsymbol{v}, \boldsymbol{\zeta}_{h,T}) = a_T(\boldsymbol{w}_h, \boldsymbol{v}) - b_T(\boldsymbol{v}, p_h) - (\boldsymbol{f}, \boldsymbol{v}_0)_T, \quad \forall\, \boldsymbol{v} = \{\boldsymbol{v}_0, \boldsymbol{v}_b\} \in V_k(T). \tag{7.4}$$

(2) It follows from the superposition principle that

$$S_{\boldsymbol{f}}(\boldsymbol{w}_b; p_h) = S_{\boldsymbol{0}}(\boldsymbol{w}_b; p_h) + S_{\boldsymbol{f}}(\boldsymbol{0}; 0), \quad \forall\, \boldsymbol{w}_b \in B_h, p_h \in W_h, \tag{7.5}$$

where $S_{\boldsymbol{0}}$ is the operator with respect to $\boldsymbol{f} = \boldsymbol{0}$.

Lemma 7.1. *The following identity holds true for the operator* S_0:

$$\langle S_{\boldsymbol{0}}(\boldsymbol{w}_b; p_h), \boldsymbol{q}_b\rangle_{\mathcal{E}_h} = a(\boldsymbol{w}_h, \boldsymbol{q}_h) - b(\boldsymbol{q}_h, p_h), \quad \forall\, \boldsymbol{w}_b, \boldsymbol{q}_b \in B_h^0, \tag{7.6}$$

where $\boldsymbol{w}_h = \{D_{\boldsymbol{0}}(\boldsymbol{w}_b; p_h), \boldsymbol{w}_b\}$ *and* $\boldsymbol{q}_h = \{D_{\boldsymbol{0}}(\boldsymbol{q}_b; p_h), \boldsymbol{q}_b\}$.

Proof. For any $\boldsymbol{w}_b, \boldsymbol{q}_b \in B_h^0$, from the definition of the operator $S_{\boldsymbol{f}}$ we obtain

$$\boldsymbol{w}_h = \{D_0(\boldsymbol{w}_b; p_h), \boldsymbol{w}_b\}, \quad \boldsymbol{\zeta}_h = L_{\boldsymbol{0}}(\boldsymbol{w}_b; p_h), \quad \boldsymbol{q}_h = \{D_0(\boldsymbol{q}_b; p_h), \boldsymbol{q}_b\}.$$

Letting $\boldsymbol{f} = \boldsymbol{0}$ in (7.4) yields

$$\begin{aligned}\langle S_{\boldsymbol{0}}(\boldsymbol{w}_b; p_h), \boldsymbol{q}_b\rangle_{\mathcal{E}_h} &= \sum_{e\in\mathcal{E}_h^0} \langle \langle\!\langle \boldsymbol{\zeta}_h \rangle\!\rangle_e, \boldsymbol{q}_b\rangle_e = \sum_{T\in\mathcal{T}_h} \langle \boldsymbol{\zeta}_{h,T}, \boldsymbol{q}_b\rangle_{\partial T} \\ &= \sum_{T\in\mathcal{T}_h} c_T(\boldsymbol{q}_h, \boldsymbol{\zeta}_{h,T}) = \sum_{T\in\mathcal{T}_h} a_T(\boldsymbol{w}_h, \boldsymbol{q}_h) - b_T(\boldsymbol{q}_h, p_h).\end{aligned}$$

This completes the identity (7.6). $\square$

Lemma 7.2. *Let $\boldsymbol{u}_h = \{\boldsymbol{u}_0, \boldsymbol{u}_b\} \in V_h$, $p_h \in W_h$, and $\boldsymbol{\lambda}_h \in \Xi_h$ be the unique solution of the hybridized WG algorithm* (4.15)–(4.16)*. Then, we have $\boldsymbol{u}_h \in \mathcal{V}_h$ and $\boldsymbol{u}_b \in B_h$ is a well-defined function. Furthermore, it satisfies the following equation*

$$S_{\boldsymbol{f}}(\boldsymbol{u}_b; p_h) = \langle\!\langle \boldsymbol{\zeta}_h \rangle\!\rangle = \mathbf{0}. \tag{7.7}$$

Proof. Since $(\boldsymbol{u}_h; p_h; \boldsymbol{\lambda}_h)$ is the unique solution to the HWG scheme (4.15)–(4.16), then from Theorem 4.2 we have $[\![\boldsymbol{u}_h]\!]_e = 0$ on each interior edge or flat face $e \in \mathcal{E}_h^0$ and $\boldsymbol{u}_b = Q_b \boldsymbol{g}$ on $\partial\Omega$. Thus, $\boldsymbol{u}_h \in \mathcal{V}_h$ and its restriction on $\mathcal{E}_h$ is a well-defined function in B_h.

In order to verify (7.7), we take $\boldsymbol{v} = \{\boldsymbol{v}_0, \mathbf{0}\} \in V_k(T)$ on T and zero elsewhere in (4.15), it follows that $\boldsymbol{u}_h$ satisfies the local equation

$$a_T(\boldsymbol{u}_h, \boldsymbol{v}) - b_T(\boldsymbol{v}, p_h) = (\boldsymbol{f}, \boldsymbol{v}_0)_T, \quad \forall\, \boldsymbol{v} = \{\boldsymbol{v}_0, \mathbf{0}\} \in V_k(T).$$

Next, taking $\boldsymbol{v} = \{\mathbf{0}, \boldsymbol{v}_b\} \in V_k(T)$ on T and zero elsewhere in (4.15), yields that $\boldsymbol{\lambda}_h$ satisfies the local equation

$$c_T(\boldsymbol{v}, \boldsymbol{\lambda}_{h,T}) = a_T(\boldsymbol{w}_h, \boldsymbol{v}) - b_T(\boldsymbol{v}, p_h), \quad \forall\, \boldsymbol{v} = \{\mathbf{0}, \boldsymbol{v}_b\} \in V_k(T),$$

where $\boldsymbol{\lambda}_{h,T}$ is the restriction of $\boldsymbol{\lambda}_h$ on ∂T. Thus, from the definition of the operator $S_{\boldsymbol{f}}$, we obtain

$$S_{\boldsymbol{f}}(\boldsymbol{u}_b; p_h) = \langle\!\langle \boldsymbol{\lambda}_h \rangle\!\rangle.$$

Combining with the fact that $\boldsymbol{\lambda}_h \in \Xi_h$, we have $\langle\!\langle \boldsymbol{\zeta}_h \rangle\!\rangle = \mathbf{0}$, which completes the proof of the lemma. □

Lemma 7.3. *Let $\bar{\boldsymbol{u}}_b \in B_h$ be a function satisfying $\bar{\boldsymbol{u}}_b = Q_b \boldsymbol{g}$ on $\partial\Omega$, $\bar{\boldsymbol{u}}_b$ and p_h satisfy the following operator equation*:

$$S_{\boldsymbol{f}}(\bar{\boldsymbol{u}}_b; p_h) = \mathbf{0}. \tag{7.8}$$

Then, $(\bar{\boldsymbol{u}}_h; p_h) \in V_h \times W_h$ is the solution to the WG finite element solution arising from (4.13)–(4.14)*. Here $\bar{\boldsymbol{u}}_0$ is the solution to the following local problems on each element $T \in \mathcal{T}_h$,*

$$a_T(\bar{\boldsymbol{u}}_h, \boldsymbol{v}) - b_T(\boldsymbol{v}, p_h) = (\boldsymbol{f}, \boldsymbol{v}_0)_T, \quad \forall\, \boldsymbol{v} = \{\boldsymbol{v}_0, \mathbf{0}\} \in V_k(T), \tag{7.9}$$

with $\bar{\boldsymbol{u}}_h = \{\bar{\boldsymbol{u}}_0, \bar{\boldsymbol{u}}_b\}$.

Proof. For each $T \in \mathcal{T}_h$, we solve for $\bar{\boldsymbol{\lambda}}_{h,T} \in \Lambda_k(\partial T)$ from the local problem

$$c_T(\boldsymbol{v}, \bar{\boldsymbol{\lambda}}_{h,T}) = a_T(\bar{\boldsymbol{u}}_h, \boldsymbol{v}) - b_T(\boldsymbol{v}, p_h), \quad \forall\, \boldsymbol{v} = \{\mathbf{0}, \boldsymbol{v}_b\} \in V_k(T). \tag{7.10}$$

Define a function $\bar{\boldsymbol{\lambda}}_h \in \Lambda_h$ by $\bar{\boldsymbol{\lambda}}_h|_{\partial T} = \bar{\boldsymbol{\lambda}}_{h,T}$. Since $(\bar{\boldsymbol{u}}_b; p_h) \in B_h \times W_h$ satisfies the operator equation (7.8), $\bar{\boldsymbol{u}}_b$ satisfies the boundary condition, and $\bar{\boldsymbol{u}}_0$ is given by (7.9), it follows from the definition of the operator $S_{\boldsymbol{f}}$ that

$$\langle\!\langle \bar{\boldsymbol{\lambda}}_h \rangle\!\rangle = S_{\boldsymbol{f}}(\bar{\boldsymbol{u}}_b; p_h) = 0, \tag{7.11}$$

which implies $\bar{\boldsymbol{\lambda}}_h \in \Xi_h$.

By subtracting (7.10) from (7.9), we have

$$a_T(\bar{\boldsymbol{u}}_h, \boldsymbol{v}) - b_T(\boldsymbol{v}, p_h) - c_T(\boldsymbol{v}, \bar{\boldsymbol{\lambda}}_{h,T}) = (\boldsymbol{f}, \boldsymbol{v}_0)_T, \quad \forall\, \boldsymbol{v} = \{\boldsymbol{v}_0, \boldsymbol{v}_b\} \in V_k(T). \tag{7.12}$$

By summing up the above equations over all $T \in \mathcal{T}_h$, we obtain

$$a(\bar{\boldsymbol{u}}_h, \boldsymbol{v}) - b(\boldsymbol{v}, p_h) - c(\boldsymbol{v}, \bar{\boldsymbol{\lambda}}_h) = (\boldsymbol{f}, \boldsymbol{v}_0), \quad \forall\, \boldsymbol{v} = \{\boldsymbol{v}_0, \boldsymbol{v}_b\} \in V_h. \tag{7.13}$$

By restricting $\boldsymbol{v}$ to the weak finite element space V_h^0 and using (7.11) we arrive at

$$c(\boldsymbol{v}, \bar{\boldsymbol{\lambda}}_h) = \sum_{e \in \mathcal{E}_h^0} \langle \langle\!\langle \bar{\boldsymbol{\lambda}}_h \rangle\!\rangle_e, \boldsymbol{v}_b \rangle_e = 0.$$

Thus we obtain

$$a(\bar{\boldsymbol{u}}_h, \boldsymbol{v}) - b(\boldsymbol{v}, p_h) = (\boldsymbol{f}, \boldsymbol{v}_0), \quad \forall\, \boldsymbol{v} = \{\boldsymbol{v}_0, \boldsymbol{v}_b\} \in V_h^0.$$

Recalling the assumption $\bar{\boldsymbol{u}}_b = Q_b \boldsymbol{g}$ on $\partial\Omega$ and Theorem 4.2, we have $\bar{\boldsymbol{u}}_h$ is the WG finite element solution to (4.13)–(4.14). This completes the proof of the lemma. □

Combining the above two lemmas yields the following result.

Theorem 7.4. *Let $\bar{\boldsymbol{u}}_b \in B_h$ be any function such that $\bar{\boldsymbol{u}}_b = Q_b\boldsymbol{g}$ on $\partial\Omega$. Let $\bar{\boldsymbol{u}}_0$ be the solution to* (7.9). *Then $(\bar{\boldsymbol{u}}_h; p_h)$ is the solution to* (4.13)–(4.14) *if and only if $\bar{\boldsymbol{u}}_b$ satisfies the following operator equation*:

$$S_{\boldsymbol{f}}(\bar{\boldsymbol{u}}_b; p_h) = \mathbf{0}. \tag{7.14}$$

7.2 Computational algorithm with reduced variables

Together with the equation (7.5), (7.14) gives rise to

$$S_{\mathbf{0}}(\bar{\boldsymbol{u}}_b; p_h) = -S_{\boldsymbol{f}}(\mathbf{0}; 0). \tag{7.15}$$

Let $\boldsymbol{G}_b \in B_h$ be a finite element function such that $\boldsymbol{G}_b = Q_b\boldsymbol{g}$ on the boundary of Ω and zero elsewhere. From the linearity of operator $S_{\mathbf{0}}$, we have

$$S_{\mathbf{0}}(\bar{\boldsymbol{u}}_b; p_h) = S_{\mathbf{0}}(\bar{\boldsymbol{u}}_b - \boldsymbol{G}_b; p_h) + S_{\mathbf{0}}(\boldsymbol{G}_b; p_h).$$

Substituting this equation into (7.15), one obtains

$$S_{\mathbf{0}}(\bar{\boldsymbol{u}}_b - \boldsymbol{G}_b; p_h) = -S_{\boldsymbol{f}}(\mathbf{0}; 0) - S_{\mathbf{0}}(\boldsymbol{G}_b; p_h).$$

Note that the function $\boldsymbol{H}_b = \bar{\boldsymbol{u}}_b - \boldsymbol{G}_b$ has vanishing boundary value. Letting

$$\mathbf{r}_b = -S_{\boldsymbol{f}}(\mathbf{0}; 0) - S_{\mathbf{0}}(\boldsymbol{G}_b; p_h),$$

we have

$$S_{\mathbf{0}}(\boldsymbol{H}_b; p_h) = \mathbf{r}_b. \tag{7.16}$$

The reduced system of linear equations (7.16) is actually a Schur complement formulation for the WG finite element scheme (4.13)–(4.14). Note that (7.16) involves only the variables representing the value of the function on $\mathcal{E}_h^0$. This is clearly a significant reduction on the size of the linear system that has to be solved in the WG finite element method.

Variable Reduction Algorithm 1. The solution $(\boldsymbol{u}_h; p_h)$ to the WG algorithm (4.13)–(4.14) can be obtained in the following steps:

Step 1. On each element $T \in \mathcal{T}_h$, solve for $\mathbf{r}_b$ from the following equation:

$$\mathbf{r}_b = -S_{\boldsymbol{f}}(\mathbf{0}; 0) - S_{\mathbf{0}}(\boldsymbol{G}_b; p_h).$$

This step requires the inversion of local stiffness matrices and can be accomplished in parallel. The computational complexity is linear with respect to the number of unknowns.

Step 2. Solve for $\{\boldsymbol{H}_b, p_h\}$ by means of the operator equation (7.16).

Step 3. Compute $\boldsymbol{u}_b = \boldsymbol{G}_b + \boldsymbol{H}_b$ to get the solution on element boundaries. Then on each element T, compute $\boldsymbol{u}_0 = D_{\boldsymbol{f}}(\boldsymbol{u}_b; p_h)$ by solving the local problem (7.1). This task can also be implemented in parallel, and the computational complexity is proportional to the number of unknowns.

Note that, Step 2 in the Variable Reduction Algorithm 1 is the primary computational part of the implementation.

8 Numerical experiments

The goal of this section is to report some numerical results for the hybridization weak Galerkin finite element method proposed and analyzed in previous sections.

A Schur complement technique of the HWG method is utilized to decrease the degree of freedom. For example, if $\Omega = (0,1)^2$ and the uniform triangulation is used with mesh size $\sqrt{2}/n$, the number of elements is $N_T = 2n^2$ and the number of edges is $N_E = 3n^2 + 2n$. If $k = 1$, then the degree of freedom for usual weak Galerkin method is $7N_T + 2N_E = 20n^2 + 4n$, the degree of freedom for hybridized weak Galerkin method is $13N_T + 2N_E = 32n^2 + 4n$, while the degree of freedom can be reduced to $2N_E + N_T = 8n^2 + 4n$ by using the Schur complement.

Let $(\boldsymbol{u}; p)$ be the exact solution to (2.1)–(2.3) and $(\boldsymbol{u}_h; p_h)$ be the numerical solution to (4.13)–(4.14). Denote $\mathbf{e}_h = Q_h\boldsymbol{u} - \boldsymbol{u}_h$ and $\varepsilon_h = \widetilde{Q}_h p - p_h$. The error for the weak Galerkin solution is measured in four norms defined as follows:

$$
\begin{aligned}
|\!|\!|\mathbf{e}_h|\!|\!|^2 &= \sum_{T\in\mathcal{T}_h} \left(\int_T |\nabla_w \mathbf{e}_h|^2 dT + h_T^{-1} \int_{\partial T} (\mathbf{e}_0 - \mathbf{e}_b)^2 ds \right),\\
\|\mathbf{e}_h\|^2 &= \sum_{T\in\mathcal{T}_h} \int_T |\mathbf{e}_h|^2 dT,\\
\|\varepsilon_h\|^2 &= \sum_{T\in\mathcal{T}_h} \int_T |\varepsilon_h|^2 dT,\\
\|Q_b\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|^2 &= \sum_{e\in\mathcal{E}_h} h_e \int_e |Q_b\boldsymbol{\lambda} - \boldsymbol{\lambda}_h|^2 ds.
\end{aligned}
$$

Example 8.1. Consider the problem (2.1)–(2.3) in the square domain $\Omega = (0, 1)^2$. The HWG finite element space $k = 1$ is employed in the numerical discretization. It has the analytic solution

$$
\boldsymbol{u} = \begin{pmatrix} \sin(2\pi x)\cos(2\pi y) \\ -\cos(2\pi x)\sin(2\pi y) \end{pmatrix} \quad \text{and} \quad p = x^2y^2 - \frac{1}{9}.
$$

The right-hand side function $\boldsymbol{f}$ in (2.1) is computed to match the exact solution. The mesh size is denoted by h.

Table 1 shows that the errors and convergence rates of Example 8.1 in $|\!|\!|\cdot|\!|\!|$-norm and L^2-norm for the HWG-FEM solution $\boldsymbol{u}$ are of order $O(h)$ and $O(h^2)$ when $k = 1$, respectively.

Table 2 shows that the errors and orders of Example 8.1 in L^2-norm for pressure and $\boldsymbol{\lambda}$. The numerical results are also consistent with theory for these two cases.

Table 1 Numerical errors and orders for $\boldsymbol{u}$ of Example 8.1

h	$\|\!\|\!\|\mathbf{e}_h\|\!\|\!\|$	order	$\|\mathbf{e}_h\|$	order
1/4	5.8950e+00		1.3555e+00	
1/8	2.9253e+00	1.0109	2.3750e−01	2.5128
1/16	1.4552e+00	1.0074	4.9049e−02	2.2756
1/32	7.2651e−01	1.0022	1.1500e−02	2.0926
1/64	3.6312e−01	1.0006	2.8254e−03	2.0251
1/128	1.8154e−01	1.0001	7.0325e−04	2.0063

Table 2 Numerical errors and orders for p and $\boldsymbol{\lambda}$ of Example 8.1

h	$\|\varepsilon_h\|$	order	$\|Q_b\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|$	order
1/4	5.1609e−01		8.6908e−01	
1/8	2.9426e−01	0.8105	3.2951e−01	1.3992
1/16	1.4706e−01	1.0007	9.8958e−02	1.7354
1/32	7.2990e−02	1.0107	2.6698e−02	1.8901
1/64	3.6391e−02	1.0041	6.9159e−03	1.9487
1/128	1.8180e−02	1.0012	1.7681e−03	1.9677

Table 3 Numerical errors and orders for $\boldsymbol{u}$ of Example 8.2

h	$\|\|\|\mathbf{e}_h\|\|\|$	order	$\|\|\mathbf{e}_h\|\|$	order
1/4	2.8805e−01		4.2555e−02	
1/8	1.4913e−01	0.9498	1.0184e−02	2.0630
1/16	7.5883e−02	0.9747	2.5894e−03	1.9756
1/32	3.8233e−02	0.9890	6.5809e−04	1.9762
1/64	1.9173e−02	0.9957	1.6598e−04	1.9872
1/128	9.5963e−03	0.9985	4.1663e−05	1.9942

Table 4 Numerical errors and orders for p and $\boldsymbol{\lambda}$ of Example 8.2

h	$\|\|\varepsilon_h\|\|$	order	$\|\|Q_b\boldsymbol{\lambda}-\boldsymbol{\lambda}_h\|\|$	order
1/4	7.7802e−02		1.8028e−01	
1/8	3.7184e−02	1.0651	8.2876e−02	1.1212
1/16	1.4725e−02	1.3364	3.2275e−02	1.3605
1/32	5.1629e−03	1.5121	1.1179e−02	1.5297
1/64	1.6890e−03	1.6120	3.5619e−03	1.6500
1/128	5.4636e−04	1.6282	1.0720e−03	1.7324

Example 8.2. Consider the problem (2.1)–(2.3) in the square domain $\Omega = (0,1)^2$. The HWG finite element space $k = 1$ is employed in the numerical discretization. It has the analytic solution

$$\boldsymbol{u} = \begin{pmatrix} -2xy(x-1)(y-1)x(x-1)(2y-1) \\ 2xy(x-1)(y-1)y(y-1)(2x-1) \end{pmatrix}$$

and

$$p = x^4 + y^4 - \frac{2}{5}.$$

The right-hand side function $\boldsymbol{f}$ in (2.1) is computed to match the exact solution. The mesh size is denoted by h.

The numerical results are presented in Tables 3 and 4 which confirm the theory developed in previous sections.

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