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Linear quadratic stochastic integral games and related topics

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Abstract This paper studies linear quadratic games problem for stochastic Volterra integral equations (SVIEs in short) where necessary and sufficient conditions for the existence of saddle points are derived in two different ways. As a consequence, the open problems raised by Chen and Yong (2007) are solved. To characterize the saddle points more clearly, coupled forward-backward stochastic Volterra integral equations and stochastic Fredholm-Volterra integral equations are introduced. Compared with deterministic game problems, some new terms arising from the procedure of deriving the later equations reflect well the essential nature of stochastic systems. Moreover, our representations and arguments are even new in the classical SDEs case.

Keywords stochastic integral games, backward stochastic Volterra integral equations, stochastic Fredholm-Volterra integral equations, saddle points, linear quadratic optimal control problems

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1 Introduction

Differential game is a classical problem, and there is a lot of literature in this field, see for example, [2, 5, 6, 12]. Nonetheless there is little literature (except [23] for deterministic Volterra integral equations (VIEs in short)) to demonstrate research on more general dynamic settings. In this paper, we will initiate a study on zero-sum linear quadratic (LQ in short) stochastic integral games. More precisely, we consider the state equation described by a controlled stochastic Volterra integral equation (SVIE in short),

$$\begin{aligned} X^u(t) = \varphi(t) &+ \int_0^t [A_1(t,s)X^u(s) + B_1(t,s)u_1(s) + C_1(t,s)u_2(s)]ds \\ &+ \int_0^t [A_2(t,s)X^u(s) + B_2(t,s)u_1(s) + C_2(t,s)u_2(s)]dW(s), \end{aligned} \tag{1.1}$$

where $u(\cdot) = (u_1(\cdot), u_2(\cdot))^{\mathrm{T}}$ are control variables for Players 1 and 2, $X^u(\cdot)$ is the corresponding state, and $\{W_t\}_{t\in[0,T]}$ is a scalar-valued Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, P)$ with $\mathbb{F} \equiv \{\mathcal{F}_t\}_{t\in[0,T]}$ being the natural filtration of Brownian motion $W(\cdot)$. Under certain conditions (see (H1) in Section 2), for any $u_i(\cdot) \in L^2_{\mathcal{F}}[0,T]$ by standard fixed point arguments (see [1]) one can obtain that

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$X^u(\cdot) \in L^2_{\mathcal{F}}[0,T]$ and $X^u(T) \in L^2_{\mathcal{F}_T}(\Omega)$. We also define the cost functional associated with (1.1) for the players as follows:

$$\begin{aligned} J(u_1(\cdot), u_2(\cdot)) &= \mathbb{E}\int_0^T [Q(t)|X^u(t)|^2 + 2X^u(t)S(t)\cdot u(t) + R(t)u(t)\cdot u(t)]dt + \mathbb{E}G|X^u(T)|^2 \\ &= \langle QX^u, X^u\rangle_2 + 2\langle X^u S, u\rangle_2 + \langle Ru, u\rangle_2 + \langle GX^u(T), X^u(T)\rangle_1, \end{aligned} \tag{1.2}$$

where $\langle \cdot,\cdot\rangle_1$ (or $\langle \cdot,\cdot\rangle_2$) is the inner product in $L^2(\Omega)$ (or $L^2_{\mathcal{F}}[0,T]$) which is the set of square integrable random variables (or $\mathbb{F}$-adapted processes), and

$$S(\cdot) = \begin{pmatrix} S_1(\cdot) \\ S_2(\cdot) \end{pmatrix}, \quad R(\cdot) = \begin{bmatrix} R_{11}(\cdot) & R_{12}(\cdot) \\ R_{21}(\cdot) & R_{22}(\cdot) \end{bmatrix}.$$

The problem that we are interested in is to find a saddle point for this cost functional, i.e., a pair of control $(\widehat{u}_1, \widehat{u}_2)$ such that $J(\widehat{u}_1, u_2) \leqslant J(\widehat{u}_1, \widehat{u}_2) \leqslant J(u_1, \widehat{u}_2)$ holds. Throughout this paper, we assume that Q, R_{ij} and S_i $(i,j=1,2)$ are bounded adapted processes and G is a bounded random variable. Later we will show that some coefficients can be non-positive, even though the Itô's formula and stochastic Riccati equations, which are indispensable in the SDEs case, are absent under our general framework. Before going further, let us give three important special cases of the state equation (1.1).

Example 1 (Stochastic delay equations). Given $h > 0$, suppose $X \doteq X^u(t) = k(t)$ with $t \in [-h, 0]$, A'_j, B'_i, C'_i, D', F'_i and k are bounded deterministic functions, $C'_1(t) = C'_2(t) \equiv 0$ with $t < h$. Let us consider a stochastic delay equation as follows:

$$\begin{aligned} dX(t) = \bigg[&A'_1(t)X(t) + A'_2(t)X(t-h) + \int_{t-h}^t A'_0(t,s)X(s)ds + C'_1(t)u_1(t-h) \\ &+ B'_1(t)u_1(t) + B'_2(t)u_2(t) + C'_2(t)u_2(t-h)\bigg]dt \\ &+ [D'(t) + A'_3(t)X(t) + F'_1(t)u_1(t) + F'_2(t)u_2(t)]dW(t). \end{aligned} \tag{1.3}$$

Note that X here can describe the amount of cash flow while u_1 and u_2 may represent consumption and investment, respectively. It was shown in [10] that (1.3) can be transformed as,

$$\begin{aligned} X(t) = X_0(t) &+ \int_0^t \Phi(t,s)D'(s)dW(s) + \sum_{i=1}^2 \int_0^t K_i(t,s)u_i(s)ds \\ &+ \int_0^t \Phi(t,s)A'_3(s)X(s)dW(s) + \sum_{i=1}^2 \int_0^t L_i(t,s)u_i(s)dW(s), \end{aligned} \tag{1.4}$$

where $L_i(t,s) = \Phi(t,s)F'_i(s)$, $\Phi(s,s) = 1$, $\Phi(t,s) = 0$ with $t < s$, and

$$\begin{aligned} &K_i(t,s) = \Phi(t,s)B'_i(s) + \Phi(t,s+h)C'_i(s+h), \\ &\frac{\partial \Phi}{\partial t}(t,s) = A'_1(t)\Phi(t,s) + A'_2(t)\Phi(t-h,s) + \int_{t-h}^t A'_0(t,u)\Phi(u,s)du, \\ &X_0(t) = \Phi(t,0)k(0) + \int_{-h}^0 \left[\Phi(t,s+h)A'_2(s+h) + \int_0^h \Phi(t,u)A'_0(u,s)du\right]k(s)ds. \end{aligned}$$

Example 2 (Stochastic advertising model). Given deterministic functions $\overline{l}(\cdot)$, $h(\cdot)$, $\overline{h}(\cdot)$, $f(\cdot)$, $\overline{f}(\cdot)$, $\varphi(\cdot)$, suppose $u_1(\cdot)$ represents the rate of advertising expenditures, $u_2(\cdot)$ represents the demand of consumer or the product price, see [8], and $X(\cdot)$ is the stock of goodwill. Let us consider the following stochastic advertising model (see also [7]),

$$X(t) = \varphi(t) + \int_0^t l(t-r)g(u_1(r))dr + \int_0^t \overline{l}(t-r)\overline{g}(u_2(r))dr$$

$$+\int_0^t h(t-r)f(u_1(r))dW(r)+\int_0^t \overline{h}(t-r)\overline{f}(u_2(r))dW(r), \tag{1.5}$$

which is a generalization of classical dynamic advertising model in [7]. We also refer to [18] for some related studies on such model. Note that we can also use (1.5) to describe stochastic capital replacement model (see [8] for the deterministic case).

Example 3 (Stochastic input-output model). Let $X(t)$ represent production outputs at time t, the evolution of which satisfies (see [4])

$$X^v(t)=X(0)+\sum_{i=1}^{2}\int_0^t h_i(t,s)v_i(s)ds+\sum_{i=1}^{2}\int_0^t k_i(t,s)v_i(s)dW(s), \tag{1.6}$$

where $v_i(s)$ $(i=1,2)$ denote production inputs and labor force, respectively, at time s. We can consider the drift term in (1.6) as the coefficient of productivity in terms of input v_1 and labor v_2, while the diffusion term may reflect the random effects of depreciation and stochastic growth. Such kind of equation can also describe some kind of investment or growth model, see [13, Example 3.1].

After obtaining the models above, it is then natural to introduce some suitable functional to minimize the risk/operation efforts, as well as, to maximize owned total wealth/assets/utility, etc. Hence it fits the framework of zero-sum games, the study of which constitutes the motivations for our paper. Now let us introduce the main contributions of this paper as follows:

(1) We will give the necessary and sufficient conditions for existence of saddle points by means of Hilbert operators and forward-backward stochastic Volterra integral equations (FBSVIEs in short) (the corresponding study on backward stochastic Volterra integral equations (BSVIEs in short) or FBSVIEs can be seen in [9, 14–22]). We solve the open problems left by Chen and Yong [3] under the framework of two-player games which demands more delicate manipulations of all the involved Hilbert operators. Our result also covers [12] as a special case, see for example, Remarks 2.1 and 2.2 below.

(2) As forward-backward stochastic differential equations (FBSDEs in short) in the stochastic differential games, the coupled FBSVIEs are also important in stochastic integral games. However, the solvability of such equations is more challenging than FBSDEs' case since many conventional and convenient approaches or conditions, such as Itô's formula and monotonicity conditions, do not work here. Therefore, in this paper we will provide new ideas and demonstrate some positive results on the solvability of FBSVIEs.

(3) A new kind of stochastic Fredholm-Volterra integral equations is introduced to represent the saddle points in a new way. Compared with deterministic Volterra games problem, some new terms appear which reflect the deep nature of stochastic systems. Moreover, our representation for the saddle points here is also new in the case of SDEs.

Note that one may also use maximum principle in [22] to study such games problem with $GX^2(T)$ appearing in the cost functional. However, the next two points illustrate the differences between the two approaches. Firstly, our way follows the basic ideas in [3, 12], which is obviously different from the maximum principle method in [22]. Secondly, our method here allows the derivation of necessary and sufficient conditions for the games problem while the maximum principle approach just shows the necessary conditions. At last, it is also worthy to point out some comparisons with the results in [3]. Firstly, the framework here is a natural extension of the linear quadratic control problems in [3], where more sophisticated arguments are demanded and new features can arise here. Secondly, the well-posedness of coupled FBSVIEs is discussed here while it is untouched in [3]. Lastly, but not the least important, a new class of stochastic Fredholm-Volterra integral equations is provided here which enables us to study the games problem, or even particular control problem from another point of view.

The remainder of this paper is organized as follows. In Section 2, we will obtain the necessary and sufficient conditions of the existence of saddle points in two different ways. In Section 3, by introducing forward-backward stochastic Volterra integral equations and stochastic Fredholm-Volterra integral equations, we obtain two representations for the saddle points. Section 4 concludes this paper.

To conclude this section, let us give several notations for later use.

$$L^2_{\mathcal{F}}[0,T] \doteq \left\{ X:[0,T]\times\Omega\to\mathbb{R} \,\middle|\, X(\cdot) \text{ is } \mathbb{F}\text{-adapted process such that } \mathbb{E}\int_0^T |X(s)|^2 ds<\infty \right\},$$

$$L^2(0,T;L^2_{\mathcal{F}}[0,T]) \doteq \left\{ X:[0,T]^2\times\Omega\to\mathbb{R} \,\middle|\, X(t,\cdot) \text{ is } \mathbb{F}\text{-adapted for almost } t\in[0,T] \text{ such that } \mathbb{E}\int_0^T\int_0^T |Z(t,s)|^2 dsdt<\infty \right\},$$

$$L^\infty[0,T] \doteq \left\{ X:[0,T]\to\mathbb{R} \,\middle|\, X \text{ is a deterministic function such that } \sup_{t\in[0,T]}|X(t)|<\infty \right\},$$

$$L^2(0,T;L^\infty[0,T]) \doteq \left\{ X:[0,T]^2\to\mathbb{R} \,\middle|\, X \text{ is a deterministic function and } \sup_{s\in[0,T]}|X(t,s)|<\infty \text{ for almost } t\in[0,T] \right\},$$

$$C_{\mathcal{F}}(0,T;L^2(\Omega)) = \left\{ X:[0,T]\to L^2(\Omega) \,\middle|\, X(\cdot) \text{ is } \mathbb{F}\text{-adapted and continuous in } L^2(\Omega) \text{ such that } \sup_{r\in[0,T]}\mathbb{E}|X(r)|^2<\infty \right\}.$$

After this, one can define a new Hilbert space as

$$\mathcal{H}^2[0,T] = L^2_{\mathcal{F}}[0,T]\times L^2(0,T;L^2_{\mathcal{F}}[0,T]),$$

which is a crucial notation in treating BSVIEs later. As to $L^2_{\mathcal{F}_T}(\Omega)$, $L^\infty_{\mathcal{F}}[0,T]$, $L^\infty(0,T;L^2_{\mathcal{F}}[0,T])$ and $L^\infty(0,T;L^\infty_{\mathcal{F}}[0,T])$, we can define them in a similar manner.

2 Necessary and sufficient conditions of existence of saddle points

In this section, we will give the necessary and sufficient conditions of existence of saddle points in two different ways. At first, let us investigate these conditions by means of operators defined in suitable Hilbert spaces. We need the following assumption:

(H1) Suppose $\varphi(\cdot)\in C_{\mathcal{F}}(0,T;L^2(\Omega))$, $A_1(\cdot,\cdot)$, $B_1(\cdot,\cdot)$, $C_1(\cdot,\cdot)\in L^\infty(0,T;L^2_{\mathcal{F}}[0,T])$, $A_2(\cdot,\cdot)$, $B_2(\cdot,\cdot)$, $C_2(\cdot,\cdot)\in L^\infty(0,T;L^\infty_{\mathcal{F}}[0,T])$. Moreover, there exists a modulus of continuity $\rho:[0,\infty)\to[0,\infty)$ (i.e., $\rho(\cdot)$ is continuous and strictly increasing with $\rho(0)=0$) such that

$$|A_i(t,s)-A_i(t',s)|+|B_i(t,s)-B_i(t',s)|+|C_i(t,s)-C_i(t',s)|\leqslant\rho(|t-t'|),\quad t,t',s\in[0,T].$$

Under (H1), (1.1) admits a unique solution $X^u(\cdot)\in C_{\mathcal{F}}(0,T;L^2(\Omega))$, from which one can ensure that $X^u(T)\in L^2_{\mathcal{F}_T}(\Omega)$. For any $x(\cdot)\in L^2_{\mathcal{F}}[0,T]$, let us define a bounded operator $\mathcal{A}$ from $L^2_{\mathcal{F}}[0,T]$ to itself,

$$(\mathcal{A}x)(t)=\int_0^t A_1(t,s)x(s)ds+\int_0^t A_2(t,s)x(s)dW(s),\quad t\in[0,T]. \tag{2.1}$$

At this moment, one can show that $(I-\mathcal{A})^{-1}$ is bounded under (H1) (see [3]). Similarly, we can define $\mathcal{B}_1$ ($\mathcal{C}_1$) with A_1, A_2 in (2.1) replaced by B_1, B_2 (C_1, C_2). Therefore the state equation can be rewritten as,

$$X^u(\cdot)=\varphi(\cdot)+(\mathcal{A}X^u)(\cdot)+(\mathcal{B}_1u_1)(\cdot)+(\mathcal{C}_1u_2)(\cdot). \tag{2.2}$$

To treat the terminal term, we also need to define bounded and linear operator Δ_T from $L^2_{\mathcal{F}}[0,T]$ to $L^2(\Omega)$, i.e., for any $p(\cdot)\in L^2_{\mathcal{F}}[0,T]$,

$$\Delta_T p=\int_0^T A_1(T,s)p(s)ds+\int_0^T A_2(T,s)p(s)dW(s). \tag{2.3}$$

Similarly, we can also define Λ_T, Π_T with A_1, A_2 in (2.3) replaced by B_1, B_2 (C_1, C_2). Hence like (2.2), we have

$$X^u(T) = \varphi(T) + (\Delta_T X^u) + (\Lambda_T u_1) + (\Pi_T u_2). \tag{2.4}$$

In what follows, A^* is denoted to be the adjoint operator of A, $u(\cdot) = (u_1(\cdot), u_2(\cdot))$, and

$$\begin{aligned}&(\mathcal{U}u) \doteq (\mathcal{B}_1 u_1) + (\mathcal{C}_1 u_2), \quad \Gamma_T u \doteq \Lambda_T u_1 + \Pi_T u_2, \quad (\mathcal{Q}X) = QX,\\ &(\mathcal{R}_{i,j}X) = R_{i,j}X, \quad (\mathcal{S}X) = SX, \quad (\mathcal{S}_i X) = S_i X, \quad i = 1, 2.\end{aligned}$$

Substituting (2.2) and (2.4) into the cost functional, we obtain that

$$\begin{aligned}J(u) = {}&\langle \Theta u + 2\Theta_1 \varphi, , u\rangle_2 + \langle \Theta_2 \varphi, \varphi\rangle_2\\ &+ 2\langle (I - \mathcal{A})^{-1}\varphi, \Delta_T^* G\varphi(T)\rangle_2 + \langle G\varphi(T), \varphi(T)\rangle_1,\end{aligned}$$

where

$$\begin{aligned}&\Theta = (\mathcal{U}^*(I - \mathcal{A}^*)^{-1}\mathcal{Q}' + \mathcal{S}')(I - \mathcal{A})^{-1}\mathcal{U} + \mathcal{U}^*(I - \mathcal{A}^*)^{-1}\mathcal{S}'^* + \mathcal{R}',\\ &\Theta_1\varphi = (\mathcal{U}^*(I - \mathcal{A}^*)^{-1}\mathcal{Q}' + \mathcal{S}')(I - \mathcal{A})^{-1}\varphi\\ &\qquad + \mathcal{U}^*(I - \mathcal{A}^*)^{-1}\Delta_T^* G\varphi(T) + \Gamma_T^* G\varphi(T),\\ &\Theta_2 = (I - \mathcal{A}^*)^{-1}\mathcal{Q}'(I - \mathcal{A})^{-1},\end{aligned} \tag{2.5}$$

and

$$\mathcal{Q}' \doteq \mathcal{Q} + \Delta_T^* G\Delta_T, \quad \mathcal{S}' \doteq \mathcal{S} + \Gamma_T^* G\Delta_T, \quad \mathcal{R}' \doteq \mathcal{R} + \Gamma_T^* G\Gamma_T.$$

Moreover, we can express Θ as

$$\Theta = \begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{pmatrix},$$

where

$$\begin{aligned}\Theta_{11} = {}&\mathcal{B}_1^*(I - \mathcal{A}^*)^{-1}\mathcal{Q}'(I - \mathcal{A})^{-1}\mathcal{B}_1 + \mathcal{S}_1(I - \mathcal{A})^{-1}\mathcal{B}_1 + \Lambda_T^* G\Delta_T(I - \mathcal{A})^{-1}\mathcal{B}_1\\ &+ \mathcal{B}_1^*(I - \mathcal{A}^*)^{-1}(\mathcal{S}_1^* + \Delta_T^* G\Lambda_T) + \mathcal{R}_{11} + \Lambda_T^* G\Lambda_T,\\ \Theta_{22} = {}&\mathcal{C}_1^*(I - \mathcal{A}^*)^{-1}\mathcal{Q}'(I - \mathcal{A})^{-1}\mathcal{C}_1 + \mathcal{S}_2(I - \mathcal{A})^{-1}\mathcal{C}_1 + \Pi_T^* G\Delta_T(I - \mathcal{A})^{-1}\mathcal{C}_1\\ &+ \mathcal{C}_1^*(I - \mathcal{A}^*)^{-1}(\mathcal{S}_2^* + \Delta_T^* G\Pi_T) + \mathcal{R}_{22} + \Pi_T^* G\Pi_T.\end{aligned} \tag{2.6}$$

Now with the help of the results in [3] or [12], we state the first main result of this section.

Theorem 2.1. *Let* (H1) *hold. For given* $\varphi(\cdot) \in L^2_{\mathcal{F}}[0, T]$, *the open-loop game admits a saddle point* $\widehat{u} \equiv (\widehat{u}_1, \widehat{u}_2)$ *if and only if*

$$\Theta_{11} \geqslant 0, \quad \Theta_{22} \leqslant 0, \quad \Theta_1\varphi \in \mathcal{R}(\Theta),$$

where $\mathcal{R}(\Theta)$ *is the range of* Θ, Θ_{11} *and* Θ_{22} *are defined by* (2.6). *In this case, any saddle point* $\widehat{u}$ *is a solution of equation* $\Theta\widehat{u} + \Theta_1\varphi = 0$ *with* $\Theta_1\varphi$ *defined in* (2.5).

Even though the above theorem gives one necessary and sufficient condition for the saddle point, it is still a little implicit. To furthermore overcome this problem, we will make use of BSVIEs or FBSVIEs aforementioned. Before that, let us recall one important notion for BSVIE

$$Y(t) = \psi(t) + \int_t^T g(t, s, Y(s), Z(t, s), Z(s, t))ds - \int_t^T Z(t, s)dW(s), \quad t \in [0, T]. \tag{2.7}$$

Definition 2.2. A pair of $(Y(\cdot), Z(\cdot, \cdot)) \in \mathcal{H}^2[0, T]$ is called an adapted M-solution of BSVIE (2.7) on $[0, T]$ if (2.7) holds in the usual Itô's sense for almost all $t \in [0, T]$ and in addition, the following holds:

$$Y(t) = \mathbb{E}Y(t) + \int_0^t Z(t, s)dW(s), \quad t \in [0, T].$$

The next lemma is concerned with the adjoint operators $\mathcal{A}^*$ and Δ_T^* above.

Lemma 2.3. *Let* (H1) *hold. Then for any* $\rho(\cdot) \in L^2_{\mathcal{F}}[0,T]$, $\nu(\cdot,\cdot) \in L^2(0,T;L^2_{\mathcal{F}}[0,T])$ *satisfying*

$$\rho(t) = \mathbb{E}\rho(t) + \int_0^t \nu(t,s)dW(s), \quad t \in [0,T], \tag{2.8}$$

and $\eta \in L^2(\Omega)$, $\theta \in L^2_{\mathcal{F}}[0,T]$ *with*

$$\eta = \mathbb{E}\eta + \int_0^T \theta(s)dW(s), \tag{2.9}$$

we have the following representation for $\mathcal{A}^*$ *and* Δ_T^* *as,*

$$\begin{aligned}(\mathcal{A}^*\rho)(t) &= \mathbb{E}^{\mathcal{F}_t}\int_t^T [A_1(s,t)\rho(s) + A_2(s,t)\nu(s,t)]ds, \quad t \in [0,T],\\ (\Delta_T^*\eta)(t) &= A_1(T,t)\mathbb{E}^{\mathcal{F}_t}\eta + A_2(T,t)\theta(t), \quad t \in [0,T].\end{aligned} \tag{2.10}$$

Proof. Since $\mathcal{A}$ is a bounded linear operator from $L^2_{\mathcal{F}}[0,T]$ to itself, the adjoint operator $\mathcal{A}^*$ of $\mathcal{A}$ is well-defined. For any $X(\cdot) \in L^2_{\mathcal{F}}[0,T]$,

$$\begin{aligned}\mathbb{E}\int_0^T (\mathcal{A}^*\rho)(t)X(t)dt &\equiv \mathbb{E}\int_0^T \rho(t)(\mathcal{A}X)(t)dt\\ &= \mathbb{E}\int_0^T \rho(t)dt\int_0^t A_1(t,s)X(s)ds + \mathbb{E}\int_0^T \rho(t)dt\int_0^t A_2(t,s)X(s)dW(s)\\ &= \mathbb{E}\int_0^T X(t)dt\int_t^T A_1(s,t)\rho(s)ds + \mathbb{E}\int_0^T X(t)dt\int_t^T A_2(s,t)\nu(s,t)ds\\ &= \mathbb{E}\int_0^T X(t)dt \cdot \mathbb{E}^{\mathcal{F}_t}\int_t^T [A_1(s,t)\rho(s) + A_2(s,t)\nu(s,t)]ds.\end{aligned}$$

Thus by the arbitrariness of $X(\cdot)$, we get the first result in (2.10). As to the second one, for any $\eta \in L^2(\Omega)$, $X \in L^2_{\mathcal{F}}[0,T]$, we have

$$\begin{aligned}\mathbb{E}\int_0^T (\Delta_T^*\eta)(s)X(s)ds &= \langle \Delta_T^*\eta, X\rangle_2 = \langle \eta, \Delta_T X\rangle_1 = \mathbb{E}\eta\Delta_T X\\ &= \mathbb{E}\int_0^T A_1(T,s)\eta X(s)ds + \mathbb{E}\int_0^T A_2(T,s)X(s)\eta dW(s)\\ &= \mathbb{E}\int_0^T A_1(T,s)\eta X(s)ds + \mathbb{E}\int_0^T A_2(T,s)\theta(s)X(s)ds\\ &= \mathbb{E}\int_0^T [A_1(T,s)\eta + A_2(T,s)\theta(s)]X(s)ds\\ &= \mathbb{E}\int_0^T [A_1(T,s)\mathbb{E}^{\mathcal{F}_s}\eta + A_2(T,s)\theta(s)]X(s)ds.\end{aligned} \tag{2.11}$$

Therefore the conclusion for Δ_T^* holds naturally. □

After this lemma, let us give another way to obtain the existence of saddle points. To this end, we have the following theorem.

Theorem 2.4. *Let* (H1) *hold. Then for* $i = 1,2$, *and any* $u_i(\cdot) \in L^2_{\mathcal{F}}[0,T]$, $X^{u_1}(\cdot)$ *is a solution to* (1.1) *with* $\varphi(\cdot) = 0$, $u_2(\cdot) = 0$, *and* $(Y^{u_1}(\cdot), Z^{u_1}(\cdot,\cdot), \lambda^{u_1}(\cdot))$ *is a solution to the following systems*:

$$\begin{cases} Y^{u_1}(t) = Q(t)X^{u_1}(t) + S_1(t)u_1(t) + A_1(T,t)GX^{u_1}(T) + A_2(T,t)\theta_1(t)\\ \qquad\qquad + \displaystyle\int_t^T [A_1(s,t)Y^{u_1}(s) + A_2(s,t)Z^{u_1}(s,t)]ds - \int_t^T Z^{u_1}(t,s)dW(s),\\ \lambda^{u_1}(t) = \mathbb{E}^{\mathcal{F}_t}\displaystyle\int_t^T [B_1(s,t)Y^{u_1}(s) + B_2(s,t)Z^{u_1}(s,t)]ds,\end{cases} \tag{2.12}$$

where $GX^{u_1}(T)$ and $\theta_1(\cdot)$ satisfy similar relations to (2.9). Then $\Theta_{11} \geqslant 0$ is equivalent to

$$\begin{aligned}&\mathbb{E}\int_0^T [\lambda^{u_1}(s) + S_1(s)X^{u_1}(s) + R_{11}(s)u_1(s)]u_1(s)ds \\ &\quad + \mathbb{E}\int_0^T [B_1(T,s)GX^{u_1}(T) + B_2(T,s)\theta_1(s)]u_1(s)ds \geqslant 0. \end{aligned} \tag{2.13}$$

Proof. It is clear that for any $u_1(t) \in L^2_{\mathcal{F}}[0,T]$,

$$\begin{aligned}&\mathcal{B}_1^*(I-\mathcal{A}^*)^{-1}(\mathcal{Q}'(I-\mathcal{A})^{-1}\mathcal{B}_1u_1 + \mathcal{S}_1^*u_1 + \Delta_T^*G\Lambda_Tu_1) \\ &\quad = \mathcal{B}_1^*(I-\mathcal{A}^*)^{-1}(QX^{u_1} + S_1u_1 + \Delta_T^*GX^{u_1}(T)),\end{aligned}$$

and

$$\begin{aligned}&\mathcal{S}_1(I-\mathcal{A})^{-1}\mathcal{B}_1u_1 + \mathcal{R}_{11}u_1 + \Lambda_T^*G\Delta_T(I-\mathcal{A})^{-1}\mathcal{B}_1u_1 + \Lambda_T^*G\Lambda_Tu_1 \\ &\quad = S_1X^{u_1} + R_{11}u_1 + \Lambda_T^*GX^{u_1}(T),\end{aligned}$$

where $X^{u_1}(t)$ and $X^{u_1}(T)$ are defined above. Substituting them into Θ_{11} of (2.6), together with Lemma 2.3, we can obtain the conclusion naturally. □

In a same way, we can get the corresponding equivalent condition for $\Theta_{22} \leqslant 0$ as

$$\begin{aligned}&\mathbb{E}\int_0^T [\lambda^{u_2}(s) + S_2(s)X^{u_2}(s) + R_{22}(s)u_2(s)]u_2(s)ds \\ &\quad + \mathbb{E}\int_0^T [C_1(T,s)GX^{u_2}(T) + C_2(T,s)\theta_2(s)]u_2(s)ds \leqslant 0, \end{aligned} \tag{2.14}$$

with $(X^{u_2}, Y^{u_2}, \lambda^{u_2})$ satisfying similar FBSVIE as above. Note that both (2.13) and (2.14) hold if $Q(\cdot)$, $R_{11}(\cdot)$, $R_{22}(\cdot)$, G are non-negative and $S(\cdot) = R_{12}(\cdot) = R_{21}(\cdot) = 0$.

Remark 2.5. If (1.1) is a controlled linear SDE with two control variables, (2.13) degenerates into [12, Proposition 4.4]. Actually, consider the simple BSDE of the form

$$\widehat{Y^{u_1}}(t) = GX^{u_1}(T) + \int_t^T Y^{u_1}(s)ds - \int_t^T \widehat{Z^{u_1}}(s)dW(s), \quad t \in [0,T].$$

After some basic calculations, it is easy to see that,

$$\widehat{Y^{u_1}}(t) = \mathbb{E}^{\mathcal{F}_t}\bigg\{GX^{u_1}(T) + \int_t^T Y^{u_1}(s)ds\bigg\}, \quad \widehat{Z^{u_1}}(t) = \mathbb{E}^{\mathcal{F}_t}\bigg\{\theta_1(t) + \int_t^T Z^{u_1}(s,t)ds\bigg\}.$$

Putting these two expressions into (2.12), we get that

$$Y^{u_1}(\cdot) = Q(\cdot)X^{u_1}(\cdot) + A_1(\cdot)\widehat{Y^{u_1}}(\cdot) + A_2(\cdot)\widehat{Z^{u_1}}(\cdot),$$

therefore we have the following BSDE satisfied by $(\widehat{Y^{u_1}}(\cdot), \widehat{Z^{u_1}}(\cdot))$,

$$\begin{aligned}\widehat{Y^{u_1}}(t) &= GX^{u_1}(T) + \int_t^T [Q(s)X^{u_1}(s) + A_1(s)\widehat{Y^{u_1}}(s) + A_2(s)\widehat{Z^{u_1}}(s)]ds \\ &\quad + \int_t^T S_1(s)u_1(s)ds - \int_t^T \widehat{Z^{u_1}}(s)dW(s), \quad t \in [0,T], \quad \text{a.e.}\end{aligned}$$

Moreover, (2.13) becomes

$$\mathbb{E}\int_0^T [B_1(s)\widehat{Y^{u_1}}(s) + C_1(s)\widehat{Z^{u_1}}(s) + S_1(s)X^{u_1}(s) + R_{11}(s)u_1(s)] \cdot u_1(s)ds \geqslant 0,$$

which covers the result in [12].

The next theorem is concerned about the remaining part of the necessary and sufficient condition in Theorem 2.1 via BSVIEs.

Theorem 2.6. *Let* (H1) *hold,* $\varphi(\cdot) \in L^2_{\mathcal{F}}[0,T]$*,* $X^u(\cdot)$ *be the unique solution to* (1.1) *and* $\lambda^u(\cdot)$ *be defined as*

$$\lambda^u(t) = \mathbb{E}^{\mathcal{F}_t}\int_t^T [U_1^{\mathrm{T}}(s,t)Y^u(s) + U_2^{\mathrm{T}}(s,t)Z^u(s,t)]ds, \tag{2.15}$$

where $(Y^u(\cdot), Z^u(\cdot))$ *is the adapted M-solution to*

$$\begin{aligned} Y^u(t) &= Q(t)X^u(t) + S(t)^{\mathrm{T}}u(t) + A_1(T,t)GX^u(T) + \int_t^T A_1(s,t)Y^u(s)ds \\ &\quad + A_2(T,t)\theta^u(t) + \int_t^T A_2(s,t)Z^u(s,t)ds - \int_t^T Z^u(t,s)dW(s), \quad t\in[0,T], \ a.e. \end{aligned} \tag{2.16}$$

$U_1^{\mathrm{T}}(\cdot,\cdot)$ *and* $U_2^{\mathrm{T}}(\cdot,\cdot)$ *are defined as*

$$U_1^{\mathrm{T}}(s,t) = \begin{pmatrix} B_1(s,t) \\ C_1(s,t) \end{pmatrix}, \quad U_2^{\mathrm{T}}(s,t) = \begin{pmatrix} B_2(s,t) \\ C_2(s,t) \end{pmatrix}, \quad s,t\in[0,T], \quad a.e.$$

$GX^u(T)$ *and* θ^u *also satisfy similar relation as* (2.9)*. Then for* $t\in[0,T]$*, a.e.*

$$(\Theta u)(t) + (\Theta_1\varphi)(t) = \lambda^u(t) + (SX^u)(t) + (Ru)(t) + U_1^{\mathrm{T}}(T,t)\mathbb{E}^{\mathcal{F}_t}GX^u(T) + U_2^{\mathrm{T}}(T,t)\theta^u(t).$$

Consequently, the condition $\Theta_1\varphi \in \mathcal{R}(\Theta)$ *holds if and only if there is a* $\widehat{u}(\cdot)$ *such that*

$$\lambda^{\widehat{u}}(\cdot) + (SX^{\widehat{u}})(\cdot) + (R\widehat{u})(\cdot) + U_1^{\mathrm{T}}(T,\cdot)\mathbb{E}^{\mathcal{F}_\cdot}GX^{\widehat{u}}(T) + U_2^{\mathrm{T}}(T,\cdot)\theta^{\widehat{u}}(\cdot) = 0. \tag{2.17}$$

Proof. It follows from (2.5) that

$$\begin{aligned} (\Theta_1\varphi)(\cdot) &= \mathcal{U}^{*\mathrm{T}}(I-\mathcal{A}^*)^{-1}[(QX^\varphi)(\cdot) + (\Delta_T^* G\Delta_T X^\varphi)(\cdot) + (\Delta_T^* G\varphi(T))(\cdot)] \\ &\quad + (SX^\varphi)(\cdot) + (\Gamma_T^* G\Delta_T X^\varphi)(\cdot) + (\Gamma_T^* G\varphi(T))(\cdot), \end{aligned}$$

where for $t\in[0,T]$, a.e.

$$X^\varphi(t) = \varphi(t) + \int_0^t A_1(t,s)X^\varphi(s)ds + \int_0^t A_2(t,s)X^\varphi(s)dW(s).$$

So we have

$$\begin{aligned} (\Theta u)(t) + (\Theta_1\varphi)(t) &= [\mathcal{U}^*(I-\mathcal{A}^*)^{-1}(QX^u + S^{\mathrm{T}}u + \Delta_T^* GX^u(T))](t) \\ &\quad + (SX^u)(t) + (Ru)(t) + U_1^{\mathrm{T}}(T,t)E^{\mathcal{F}_t}GX^u(T) + U_2^{\mathrm{T}}(T,t)\theta^u(t) \\ &= \lambda^u(t) + (SX^u)(t) + (Ru)(t) + U_1^{\mathrm{T}}(T,t)\mathbb{E}^{\mathcal{F}_t}GX^u(T) + U_2^{\mathrm{T}}(T,t)\theta^u(t), \end{aligned}$$

where $\lambda^u(\cdot)$ is defined above. As to (2.17), it is obvious. □

Remark 2.7. If (1.1) degenerates into a linear SDE, (2.17) becomes the result in [12, Proposition 4.3]. In fact, in this case, (2.17) becomes

$$\Xi_1(t)\overline{Y^{\widehat{u}}}(t) + \Xi_2(t)\overline{Z^{\widehat{u}}}(t) + (SX^{\widehat{u}})(t) + (R\widehat{u})(t) = 0,$$

where $(\overline{Y^{\widehat{u}}}, \overline{Z^{\widehat{u}}})$ defined as

$$\overline{Y^{\widehat{u}}}(t) = \mathbb{E}^{\mathcal{F}_t}\left\{GX^{\widehat{u}}(T) + \int_t^T Y^{\widehat{u}}(s)ds\right\}, \quad \overline{Z^{\widehat{u}}}(t) = \mathbb{E}^{\mathcal{F}_t}\left\{\theta^{\widehat{u}}(t) + \int_t^T Z^{\widehat{u}}(s,t)ds\right\},$$

satisfy a BSDE of

$$\begin{aligned} \overline{Y^{\widehat{u}}}(t) &= GX^{\widehat{u}}(T) + \int_t^T [Q(s)X^{\widehat{u}}(s) + A_1(s)\overline{Y^{\widehat{u}}}(s) + A_2(s)\overline{Z^{\widehat{u}}}(s)]ds \\ &\quad + \int_t^T S_1(s)\widehat{u}(s)ds - \int_t^T \overline{Z^{\widehat{u}}}(s)dW(s), \quad t\in[0,T]. \end{aligned}$$

3 Some related stochastic equations to the games problem

In this section, let us introduce and discuss two different kinds of stochastic equations involved in our games problem.

3.1 A class of coupled forward-backward stochastic Volterra equation

Suppose $R^{-1}(\cdot)$ exists and is bounded, from (2.17) we then represent the saddle point as,

$$\widehat{u}(t) = -R^{-1}(t)[S(t)X^{\widehat{u}}(t) + \lambda^{\widehat{u}}(t) + U_1^{\mathrm{T}}(T,t)\mathbb{E}^{\mathcal{F}_t}[GX^{\widehat{u}}(T)] + U_2^{\mathrm{T}}(T,t)\theta^{\widehat{u}}(t)], \quad t \in [0,T]. \tag{3.1}$$

In this case, the related forward-backward equation with $t \in [0,T]$ should be,

$$\begin{cases} X^{\widehat{u}}(t) = \varphi(t) + \displaystyle\int_0^t [(A_1(t,s) - U_1(t,s)R^{-1}(s)S(s))X^{\widehat{u}}(s) - U_1(t,s)R^{-1}(s)\lambda^{\widehat{u}}(s)]ds \\ \qquad - \displaystyle\int_0^t U_1(t,s)R^{-1}(s)[U_1^{\mathrm{T}}(T,s)\mathbb{E}^{\mathcal{F}_s}[GX^{\widehat{u}}(T)] + U_2^{\mathrm{T}}(T,s)\theta^{\widehat{u}}(s)]ds \\ \qquad + \displaystyle\int_0^t [(A_2(t,s) - U_2(t,s)R^{-1}(s)S(s))X^{\widehat{u}}(s) - U_2(t,s)R^{-1}(s)\lambda^{\widehat{u}}(s)]dW(s) \\ \qquad - \displaystyle\int_0^t U_2(t,s)R^{-1}(s)[U_1^{\mathrm{T}}(T,s)\mathbb{E}^{\mathcal{F}_s}[GX^{\widehat{u}}(T)] + U_2^{\mathrm{T}}(T,s)\theta^{\widehat{u}}(s)]dW(s), \\ Y^{\widehat{u}}(t) = [Q(t) - S^{\mathrm{T}}(t)R^{-1}(t)S(t)]X^{\widehat{u}}(t) - S^{\mathrm{T}}(t)R^{-1}(t)\lambda^{\widehat{u}}(t) + \displaystyle\int_t^T A_1(s,t)Y^{\widehat{u}}(s)ds \\ \qquad - S^{\mathrm{T}}(t)R^{-1}(t)[U_1^{\mathrm{T}}(T,t)\mathbb{E}^{\mathcal{F}_t}[GX^{\widehat{u}}(T)] + U_2^{\mathrm{T}}(T,t)\theta^{\widehat{u}}(t)] \\ \qquad + \displaystyle\int_t^T A_2(s,t)Z^{\widehat{u}}(s,t)ds - \int_t^T Z^{\widehat{u}}(t,s)dW(s), \\ \lambda^{\widehat{u}}(t) = \mathbb{E}^{\mathcal{F}_t}\displaystyle\int_t^T [U_1^{\mathrm{T}}(s,t)Y^{\widehat{u}}(s) + U_2^{\mathrm{T}}(s,t)Z^{\widehat{u}}(s,t)]ds, \quad t \in [0,T], \end{cases} \tag{3.2}$$

where $Y^{\widehat{u}}(t)$, $Z^{\widehat{u}}(s,t)$ and $GX^{\widehat{u}}(T)$, $\theta^{\widehat{u}}$ satisfy the following relationships:

$$Y^{\widehat{u}}(t) = \mathbb{E}Y^{\widehat{u}}(t) + \int_0^t Z^{\widehat{u}}(t,s)dW(s), \quad GX^{\widehat{u}}(T) = \mathbb{E}GX^{\widehat{u}}(T) + \int_0^T \theta^{\widehat{u}}(s)dW(s). \tag{3.3}$$

Inspired by Definition 2.1, we call $(X^{\widehat{u}}(\cdot), Y^{\widehat{u}}(\cdot), Z^{\widehat{u}}(\cdot,\cdot), \lambda^{\widehat{u}}(\cdot))$ an M-solution to (3.2) if they satisfy (3.2) under the constraints of (3.3). Next let us discuss the solvability of M-solution to (3.2), which is important in providing the existence of saddle point $\widehat{u}(\cdot)$ above. To begin with, let us first consider the case with $G = 0$. In this case, (3.2) can be simplified as,

$$\begin{cases} X^{\widehat{u}}(t) = \varphi(t) + \displaystyle\int_0^t [(A_1(t,s) - U_1(t,s)R^{-1}(s)S(s))X^{\widehat{u}}(s) - U_1(t,s)R^{-1}(s)\lambda^{\widehat{u}}(s)]ds \\ \qquad + \displaystyle\int_0^t [(A_2(t,s) - U_2(t,s)R^{-1}(s)S(s))X^{\widehat{u}}(s) - U_2(t,s)R^{-1}(s)\lambda^{\widehat{u}}(s)]dW(s), \\ Y^{\widehat{u}}(t) = [Q(t) - S^{\mathrm{T}}(t)R^{-1}(t)S(t)]X^{\widehat{u}}(t) - S^{\mathrm{T}}(t)R^{-1}(t)\lambda^{\widehat{u}}(t) + \displaystyle\int_t^T A_1(s,t)Y^{\widehat{u}}(s)ds \\ \qquad + \displaystyle\int_t^T A_2(s,t)Z^{\widehat{u}}(s,t)ds - \int_t^T Z^{\widehat{u}}(t,s)dW(s), \\ \lambda^{\widehat{u}}(t) = \mathbb{E}^{\mathcal{F}_t}\displaystyle\int_t^T [U_1^{\mathrm{T}}(s,t)Y^{\widehat{u}}(s) + U_2^{\mathrm{T}}(s,t)Z^{\widehat{u}}(s,t)]ds, \quad t \in [0,T]. \end{cases} \tag{3.4}$$

Since the generator in the second equation of (3.4) is independent of $Z^{\widehat{u}}(t,s)$ with $t \geqslant s$, we can drop the last stochastic integral by taking conditional expectation. Due to the ad hoc relation between $\lambda^{\widehat{u}}(\cdot)$ and $(X^{\widehat{u}}(\cdot), Y^{\widehat{u}}(\cdot), Z^{\widehat{u}}(\cdot,\cdot))$, by substituting $\lambda^{\widehat{u}}(\cdot)$ into the equation for $(Y^{\widehat{u}}(\cdot), Z^{\widehat{u}}(\cdot,\cdot))$ we can also transform

(3.4) into a new form which is easier to be treated,

$$\begin{cases} X(t) = \varphi(t) + \int_0^t A_1'(t,s)X(s)ds + \int_0^t A_2'(t,s)X(s)dW(s) \\ \qquad + \int_0^t B_1'(t,s)\Big(\mathbb{E}^{\mathcal{F}_s}\int_s^T [D_1'(u,s)Y(u)du + D_2'(u,s)Z(u,s)]du\Big)ds \\ \qquad + \int_0^t B_2'(t,s)\Big(\mathbb{E}^{\mathcal{F}_s}\int_s^T [D_1'(u,s)Y(u)du + D_2'(u,s)Z(u,s)]du\Big)dW(s), \\ Y(t) = \phi_1'(t)X(t) + \mathbb{E}^{\mathcal{F}_t}\int_t^T C_1'(s,t)Y(s)ds + \mathbb{E}^{\mathcal{F}_t}\int_t^T C_2'(s,t)Z(s,t)ds, \quad t\in[0,T], \end{cases} \tag{3.5}$$

where A_i', C_i', and ϕ_1' are scalars, B_i' and ϕ_2' are 1×2 matrices, D_i' $(i=1,2)$ are 2×1 matrices, and

$$\begin{aligned} &A_1'(t,s) = A_1(t,s) - U_1(t,s)R^{-1}(s)S(s), \quad A_2'(t,s) = A_2(t,s) - U_2(t,s)R^{-1}(s)S(s), \\ &B_1'(t,s) = -U_1(t,s)R^{-1}(s), \quad B_2'(t,s) = -U_2(t,s)R^{-1}(s), \quad D_1'(t,s) = U_1^{\mathrm{T}}(t,s), \\ &D_2'(t,s) = U_2^{\mathrm{T}}(t,s), \quad \phi_1'(t) = Q(t) - S^{\mathrm{T}}(t)R^{-1}(t)S(t), \quad \phi_2'(t) = -S^{\mathrm{T}}(t)R^{-1}(t), \\ &C_1'(t,s) = A_1(t,s) + \phi_2'(s)D_1'(t,s), \quad C_2'(t,s) = A_2(t,s) + \phi_2(s)D_2'(t,s). \end{aligned}$$

The next result is concerned with the solvability for (3.5) under a general framework.

Theorem 3.1. *For $i=1,2$, let A_i', B_i', C_i', D_i' belong to $L^\infty(0,T;L^\infty_{\mathcal{F}}[0,T])$ with K being the upper bound, $\phi_i'(\cdot)$ be adapted processes such that*

$$|\phi_2'(\cdot)| < K, \quad |\phi_1'(t)| \leqslant \frac{1}{2}\mathrm{e}^{-\beta t},$$

with $\beta > 1$ being a constant depending on K and T. Then FBSVIE (3.5) *admits a unique M-solution.*

Proof. The basic idea here is to introduce a new equivalent norm with parameter β and use the fixed point theorem. Such skills have been used to treat BSVIEs in [15, 17], hence we just give a sketch for readers' convenience.

Let $\mathcal{M}^2[0,T]$ be one closed subspace of $\mathcal{H}^2[0,T]$, the element of which satisfies (2.8). Given one element $(x(\cdot),y(\cdot),z(\cdot,\cdot))$ in $L^2_{\mathcal{F}}[0,T]\times\mathcal{M}^2[0,T]$, let us define a mapping as

$$\Theta(x(\cdot),y(\cdot),z(\cdot,\cdot)) = (X(\cdot),Y(\cdot),Z(\cdot,\cdot)),$$

where (X,Y,Z) is the M-solution of FBSVIE

$$\begin{cases} X(t) = \varphi(t) + \int_0^t A_1'(t,s)x(s)ds + \int_0^t A_2'(t,s)x(s)dW(s) \\ \qquad + \int_0^t B_1'(t,s)\Big(\mathbb{E}^{\mathcal{F}_s}\int_s^T [D_1'(u,s)y(u)du + D_2'(u,s)z(u,s)]du\Big)ds \\ \qquad + \int_0^t B_2'(t,s)\Big(\mathbb{E}^{\mathcal{F}_s}\int_s^T [D_1'(u,s)y(u)du + D_2'(u,s)z(u,s)]du\Big)dW(s), \\ Y(t) = \phi_1'(t)x(t) + \mathbb{E}^{\mathcal{F}_t}\int_t^T C_1'(s,t)y(s)ds + \mathbb{E}^{\mathcal{F}_t}\int_t^T C_2'(s,t)z(s,t)ds. \end{cases} \tag{3.6}$$

Similarly, one can also obtain $(\overline{X}(\cdot),\overline{Y}(\cdot),\overline{Z}(\cdot,\cdot))$ corresponding to $(\overline{x}(\cdot),\overline{y}(\cdot),\overline{z}(\cdot,\cdot))$. Next, we will prove that Θ is contractive. For later convenience, let us make the convention of

$$\widehat{f} = f - \overline{f}, \quad f = X, x, Y, y, Z, z.$$

As to the forward equation in (3.6),

$$\mathbb{E}\int_0^T \mathrm{e}^{-\beta t}|\widehat{X}(t)|^2dt \leqslant \frac{8K}{\beta}\mathbb{E}\int_0^T \mathrm{e}^{-\beta s}|\widehat{x}(s)|^2ds + \frac{8K}{\beta}\mathbb{E}\int_0^T \mathrm{e}^{\beta s}|\widehat{p}(s)|^2ds,$$

where for $s \in [0,T]$, a.e.

$$\widehat{p}(s) := p(s) - \overline{p}(s) = \mathbb{E}^{\mathcal{F}_s}\int_s^T D_1'(u,s)\widehat{y}(u)du + \mathbb{E}^{\mathcal{F}_s}\int_s^T D_2'(u,s)\widehat{z}(u,s)du.$$

As to $\widehat{p}(\cdot)$, we can obtain the following estimates:

$$\mathbb{E}\int_0^T \mathrm{e}^{\beta t}|\widehat{p}(t)|^2dt \leqslant \frac{2K}{\beta}\mathbb{E}\int_0^T \mathrm{e}^{\beta s}|\widehat{y}(s)|^2ds + 2K\mathbb{E}\int_0^T \mathrm{e}^{\beta s}|\widehat{y}(s)|^2ds,$$

which leads to the fact that,

$$\mathbb{E}\int_0^T \mathrm{e}^{-\beta t}|\widehat{X}(t)|^2dt \leqslant \frac{8K}{\beta}\mathbb{E}\int_0^T \mathrm{e}^{-\beta s}|\widehat{x}(s)|^2ds + \frac{32K^2}{\beta}\mathbb{E}\int_0^T \mathrm{e}^{\beta s}|\widehat{y}(s)|^2ds. \tag{3.7}$$

As to the backward equation in (3.6), we can obtain,

$$\begin{aligned}\mathbb{E}\int_0^T \mathrm{e}^{\beta t}|\widehat{Y}(t)|^2dt &\leqslant 2\mathbb{E}\int_0^T \mathrm{e}^{\beta t}|\phi_1'(t)\widehat{x}(t)|^2dt + 4K^2(1+K)^2\mathbb{E}\int_0^T \mathrm{e}^{\beta t}\bigg|\int_t^T|\widehat{y}(s)|ds\bigg|^2dt\\ &\quad + 4K^2(1+K)^2\mathbb{E}\int_0^T \mathrm{e}^{\beta t}\bigg|\int_t^T|\widehat{z}(s,t)|ds\bigg|^2dt\\ &\leqslant \frac{4K^2(1+K)^2[T+1]}{\beta}\mathbb{E}\int_0^T \mathrm{e}^{\beta s}|\widehat{y}(s)|^2ds + \frac{1}{2}\mathbb{E}\int_0^T \mathrm{e}^{-\beta t}|\widehat{x}(t)|^2dt,\end{aligned} \tag{3.8}$$

where we use stochastic Fubini's theorem and the following inequality:

$$\mathrm{e}^{\beta t}\bigg[\int_t^T|f(s)|ds\bigg]^2 \leqslant \frac{1}{\beta}\int_t^T \mathrm{e}^{\beta s}|f(s)|^2ds, \quad t\in[0,T],$$

with f being a square integrable process. By (3.7) and (3.8), we obtain

$$\begin{aligned}&\mathbb{E}\int_0^T \mathrm{e}^{-\beta t}|\widehat{X}(t)|^2dt + \mathbb{E}\int_0^T \mathrm{e}^{\beta t}|\widehat{Y}(t)|^2dt\\ &\quad\leqslant \bigg[\frac{8K}{\beta}+\frac{1}{2}\bigg]\mathbb{E}\int_0^T|\widehat{x}(s)|^2\mathrm{e}^{-\beta s}ds + \frac{4K^2(1+K)^2[T+1]+32K^2}{\beta}\mathbb{E}\int_0^T \mathrm{e}^{\beta s}|\widehat{y}(s)|^2ds.\end{aligned}$$

So by choosing a suitable β we know that Θ is contractive, and the desired result follows. □

Before going further, we will make several points here. Firstly, the parameter β is a finite constant which can be determined completely by T and the upper bound K. Secondly, one can relax the bounded assumption by imposing certain integrable conditions (see [22]). Thirdly, as to the LQ games problem for SDEs with $G \neq 0$, in some cases it can be transformed into another new, yet equivalent case with $G=0$. Actually, the state equation becomes an SDE of

$$X^u(t) = x_0 + \int_0^t [A_1(s)X^u(s) + U_1(s)u(s)]ds + \int_0^t [A_2(s)X^u(s) + U_2(s)u(s)]dW(s), \tag{3.9}$$

while the cost functional (1.2) keeps the same. Note that $U_i(\cdot) = (B_i(\cdot), C_i(\cdot))$. Firstly, given $G \in L^2_{\mathcal{F}_T}(\Omega)$ being bounded such that the following BSDE admits a unique bounded solution $(P(\cdot),\Lambda(\cdot))$,

$$P(t) = G - \int_t^T \Lambda(s)dW(s), \quad t\in[0,T], \tag{3.10}$$

by using Itô's formula to $P(\cdot)|X^u(\cdot)|^2$ one can obtain that

$$\mathbb{E}G|X^u(T)|^2 - P(0)|x_0|^2 = \mathbb{E}\int_0^T [K_1(t)|X^u(t)|^2 + 2K_2(t)u(t)X^u(t) + \langle K_3(t)u(t),u(t)\rangle]dt, \tag{3.11}$$

where for $t \in [0,T]$, a.e.

$$K_1(t) = 2P(t)A_1(t) + 2\Lambda(t)A_2(t) + |A_2(t)|^2 P(t),$$
$$K_2(t) = P(t)U_1(t) + \Lambda(t)U_2(t) + A_2(t)P(t)U_2(t), \quad K_3(t) = U_2^{\mathrm{T}}(t)P(t)U_2(t). \tag{3.12}$$

Therefore, we can rewrite cost functional (1.2) above as

$$J(u_1(\cdot), u_2(\cdot)) = \langle \bar{Q}X^u, X^u \rangle_2 + 2\langle X^u \bar{S}, u \rangle_2 + \langle \bar{R}u, u \rangle_2 + P(0)|x_0|^2, \tag{3.13}$$

with

$$\bar{Q}(\cdot) = Q(\cdot) + K_1(\cdot), \quad \bar{S}(\cdot) = S(\cdot) + K_2(\cdot), \quad \bar{R}(\cdot) = R(\cdot) + K_3(\cdot).$$

Therefore under this new cost functional (3.13), one can use the above fixed point arguments to discuss the related forward-backward system. Returning to the SVIEs case with $G \neq 0$, the case becomes much complicated. On one hand, the above procedures from (3.9) to (3.13) may not work well under SVIEs framework due to the dependence on t for $A_i(t,\cdot)$, $B_i(t,\cdot)$, $C_i(t,\cdot)$. On the other hand, if we try to extend the ideas in Theorem 3.1 into the general case with $G \neq 0$, additional sophisticated conditions like T or the upper bound for the coefficients to be small enough are needed. In other words, such a situation prompts us to look at the LQ games problem from another way which in some sense shows the necessity of Subsection 3.2 next. At last we claim that due to the complicated form of (3.2), it is quite challenging to adapt the four-step method in [11] or monotonicity condition in [20] to our framework here, which explains the reason of introducing the above method. It is observed that one cost of our approach is the strict requirement on $\phi_1'(\cdot)$ which leads to *weak coupling* between forward and backward equations. However, in our games problem such a strong requirement can be transformed into certain constraint among related coefficients.

Theorem 3.2. *For $i = 1, 2$, suppose the coefficients of* (3.2)*, $A_i(\cdot)$, $B_i(\cdot)$, $C_i(\cdot)$, $Q(\cdot)$, $S(\cdot)$ and $R^{-1}(\cdot)$ are bounded by K,*

$$[Q(t) - S^{\mathrm{T}}(t)R^{-1}(t)S(t)] < \frac{1}{2}\mathrm{e}^{-\beta t}, \quad t \in [0,T], \tag{3.14}$$

where β is a constant determined by T and K. Then there exists a unique M-solution to (3.2)*. Moreover,* (2.13) *and* (2.14) *hold. Consequently, the quadratic integral game admits an open-loop saddle point as*

$$\widehat{u}(t) = -R(t)^{-1}[S(t)X^{\widehat{u}}(t) + \lambda^{\widehat{u}}(t) + U_1^{\mathrm{T}}(T,t)\mathbb{E}^{\mathcal{F}_t}[GX^{\widehat{u}}(T)] + U_2^{\mathrm{T}}(T,t)\theta^{\widehat{u}}(t)], \quad t \in [0,T].$$

3.2 Some stochastic equations of Fredholm-Volterra type

In this subsection, let us introduce another kind of stochastic equations of Fredholm-Volterra type to show some open-loop representations for the saddle point. At first let us make the following assumption.

(H2) All the coefficients in the state equation (1.1) and cost functional (1.2) are deterministic, $A_i(t,s) = 0$ with $t,\ s \in [0,T]$, a.e.

Within such a framework, the optimal state equation becomes, for $t \in [0,T]$,

$$X^{\widehat{u}}(t) = \varphi(t) + \int_0^t U_1(t,s)\widehat{u}(s)ds + \int_0^t U_2(t,s)\widehat{u}(s)dW(s). \tag{3.15}$$

As to the backward equation in (3.2),

$$Y^{\widehat{u}}(t) = Q(t)X^{\widehat{u}}(t) + S^{\mathrm{T}}(t)\widehat{u}(t), \quad Z^{\widehat{u}}(t,s) = 0, \quad 0 \leqslant t \leqslant s \leqslant T. \tag{3.16}$$

Given $\widehat{u}(\cdot)$ and $X^{\widehat{u}}(\cdot)$, similarly to (3.3), there exist two processes $K_{\widehat{u}}(\cdot,\cdot)$ and $K_{\widehat{x}}(\cdot,\cdot)$ such that

$$\widehat{u}(t) = \mathbb{E}\widehat{u}(t) + \int_0^t K_{\widehat{u}}(t,s)dW(s), \quad X^{\widehat{u}}(t) = \mathbb{E}X^{\widehat{u}}(t) + \int_0^t K_{\widehat{x}}(t,s)dW(s), \tag{3.17}$$

with $t \in [0,T]$. Under (H2), one can express $Z^{\widehat{u}}(t,s)$ with $t \geqslant s$ by

$$Z^{\widehat{u}}(t,s) = Q(t)K_{\widehat{x}}(t,s) + S^{\mathrm{T}}(t)K_{\widehat{u}}(t,s). \tag{3.18}$$

On the other hand, as to $GX^{\widehat{u}}(T)$ given by

$$GX^{\widehat{u}}(T) = G\varphi(T) + \int_0^T GU_1(T,s)\widehat{u}(s)ds + \int_0^T GU_2(T,s)\widehat{u}(s)dW(s), \tag{3.19}$$

we obtain the expression for $\theta^{\widehat{u}}(\cdot)$ with the help of stochastic Fubini's theorem as follows:

$$\theta^{\widehat{u}}(s) = \int_s^T GU_1(T,u)K_{\widehat{u}}(r,s)dr + GU_2(T,s)\widehat{u}(s), \tag{3.20}$$

with $s \in [0,T]$. Similar ideas also hold for $K_{\widehat{x}}(\cdot,\cdot)$ in (3.17),

$$K_{\widehat{x}}(t,s) = \int_s^t U_1(t,u)K_{\widehat{u}}(r,s)dr + U_2(t,s)\widehat{u}(s), \quad t \geqslant s.$$

Plugging $K_{\widehat{x}}$ into (3.18), we then obtain

$$Z^{\widehat{u}}(t,s) = Q(t)\int_s^t U_1(t,u)K_{\widehat{u}}(r,s)dr + Q(t)U_2(t,s)\widehat{u}(s) + S^{\mathrm{T}}(t)K_{\widehat{u}}(t,s), \quad t \geqslant s. \tag{3.21}$$

In the following, for the sake of clarity, let us make the convention of

$$f_{ij}(r,t,s) = U_i^{\mathrm{T}}(r,t)U_j(r,s), \quad r,t,s \in [0,T], \quad i,j = 1,2,$$

where

$$U_i(t,s) = (B_i(t,s), C_i(t,s)), \quad f_{ij}(r,t,s), \quad i,j = 1,2,$$

are 2×2 matrices. Substituting (3.15), (3.16) and (3.19)–(3.21) into the following equality:

$$\begin{aligned} 0 &= R(t)\widehat{u}(t) + U_1^{\mathrm{T}}(T,t)G\mathbb{E}^{\mathcal{F}_t}X^{\widehat{u}}(T) + U_2^{\mathrm{T}}(T,t)\theta^{\widehat{u}}(t) + S(t)X^{\widehat{u}}(t) \\ &\quad + \mathbb{E}^{\mathcal{F}_t}\int_t^T [U_1^{\mathrm{T}}(s,t)Y^{\widehat{u}}(s) + U_2^{\mathrm{T}}(s,t)Z^{\widehat{u}}(s,t)]ds, \end{aligned} \tag{3.22}$$

together with some necessary calculations, one has

$$\begin{aligned} 0 &= R(t)\widehat{u}(t) + \Sigma_1(t) + \mathbb{E}^{\mathcal{F}_t}\int_0^T \Sigma_2'(t,s)\widehat{u}(s)ds + \int_0^t \Sigma_3'(t,s)\widehat{u}(s)dW(s) \\ &\quad + \mathbb{E}^{\mathcal{F}_t}\int_t^T \Sigma_4(t,s)K_{\widehat{u}}(s,t)ds + \Sigma_7(t)\widehat{u}(t), \quad t \in [0,T], \end{aligned} \tag{3.23}$$

where for any $t,s \in [0,T]$,

$$\Sigma_2'(t,s) = \Sigma_2(t,s) + \Sigma_9(t,s)\cdot I_{[0,t]}(s) + \Sigma_5(t,s)\cdot I_{[t,T]}(s), \quad \Sigma_3'(t,s) = \Sigma_3(t,s) + \Sigma_8(t,s),$$

and all the other coefficients can be shown explicitly as follows:

$$\begin{aligned} \Sigma_1(t) &= \int_t^T U_1^{\mathrm{T}}(s,t)Q(s)\varphi(s)ds + U_1^{\mathrm{T}}(T,t)G\varphi(T) + S(t)\varphi(t), \\ \Sigma_2(t,s) &= Gf_{11}(T,t,s), \quad \Sigma_3(t,s) = Gf_{12}(T,t,s), \\ \Sigma_4(t,s) &= \int_s^T f_{21}(u,t,s)Q(u)du + Gf_{21}(T,t,s) + U_2^{\mathrm{T}}(s,t)S^{\mathrm{T}}(s), \\ \Sigma_5(t,s) &= \int_s^T f_{11}(u,t,s)Q(u)du + U_1^{\mathrm{T}}(s,t)S(s), \end{aligned}$$

$$\Sigma_7(t) = \int_t^T f_{22}(s,t,t)Q(s)ds + f_{22}(T,t,t)G,$$

$$\Sigma_8(t,s) = \int_t^T f_{12}(u,t,s)Q(u)du + S(t)U_2(t,s),$$

$$\Sigma_9(t,s) = \int_t^T f_{11}(u,t,s)Q(u)du + S(t)U_1(t,s), \quad t,s \in [0,T].$$

Note that $Q(\cdot)$ and $S(\cdot)$ are scalars, $R(\cdot)$ is a 2×2 matrix, and $S(\cdot)$ is a 2×1 vector. Due to the appearance of $\Sigma_7(\cdot)$ in (3.23), we need to reorganize it in a new way. More precisely, firstly let us introduce two notations of $\widehat{\lambda}(\cdot)$ and $\widehat{\pi}(\cdot,\cdot)$ satisfying

$$\widehat{\lambda}(t) = -(R(t)+\Sigma_7(t))\widehat{u}(t), \quad \widehat{\lambda}(t) = \mathbb{E}\widehat{\lambda}(t) + \int_0^t \widehat{\pi}(t,s)dW(s), \quad t\in[0,T]. \tag{3.24}$$

Since $R(\cdot)$ and $\Sigma_7(\cdot)$ are deterministic, we can deduce from (3.24) the following relation between $\widehat{\pi}(\cdot)$ and $K_{\widehat{u}}(\cdot,\cdot)$,

$$\widehat{\pi}(t,s) = -(R(t)+\Sigma_7(t))K_{\widehat{u}}(t,s), \quad t \geqslant s.$$

As a result, (3.23) can be rewritten as

$$\begin{aligned}\widehat{\lambda}(t) &= \Sigma_1(t) - \mathbb{E}^{\mathcal{F}_t}\int_0^T \widehat{\Sigma}_2(t,s)\widehat{\lambda}(s)ds \\ &\quad - \int_0^t \widehat{\Sigma}_3(t,s)\widehat{\lambda}(s)dW(s) - \mathbb{E}^{\mathcal{F}_t}\int_t^T \widehat{\Sigma}_4(t,s)\widehat{\pi}(s,t)ds,\end{aligned} \tag{3.25}$$

where the coefficients are defined as

$$\widehat{\Sigma}_2(t,s) = \Sigma_2'(t,s)(R(s)+\Sigma_7(s))^{-1}, \quad \widehat{\Sigma}_3(t,s) = \Sigma_3'(t,s)(R(s)+\Sigma_7(s))^{-1},$$

$$\widehat{\Sigma}_4(t,s) = \Sigma_4(t,s)(R(s)+\Sigma_7(s))^{-1}, \quad t,s\in[0,T].$$

(3.25) also yields a new expression for saddle point $\widehat{u}(\cdot)$ as

$$\widehat{u}(t) = -(R(t)+\Sigma_7(t))^{-1}\widehat{\lambda}(t), \quad t\in[0,T]. \tag{3.26}$$

Since the equation (3.25) has both the character of Volterra and Fredholm equations, we call it a linear stochastic Fredholm-Volterra integral equation. As to the solvability of (3.25), like Theorem 3.1 above, we can use the contraction method by imposing some requirement on T. To sum up, we have the following theorem.

Theorem 3.3. *Let* (H2)*,* (2.13) *and* (2.14) *hold. If* (3.25) *admits a solution* $\widehat{\lambda}(\cdot)$*, then there exists a saddle point* $\widehat{u}$*, and the open-loop representation can be given in* (3.26).

To conclude this section let us make several comments here. Firstly following the above arguments, one can relax the deterministic assumption on $\varphi(\cdot)$ by allowing, for example,

$$\varphi(t) = a_1(t) + \int_0^t a_2(t,s)dW(s)$$

with $t\in[0,T]$, $a_i(\cdot)$ being deterministic functions. Secondly, as to coefficient $\Sigma_7(\cdot)$ in (3.23), it only depends on $B_2(\cdot)$ and $C_2(\cdot)$ and vanishes if the control variables do not enter the diffusion term. Therefore, compared with the games or control problems for deterministic Volterra equation, such a new term can character well the deep nature in stochastic case. Thirdly, it seems like (3.26) is more suitable than (3.1) above. On the one hand, the assumption of $R^{-1}(\cdot)$ being bounded is replaced with the boundedness of

$$\frac{1}{R(t)+\Sigma_7(t)} = \left[R(t) + \int_t^T U_2^{\mathrm{T}}(s,t)U_2(s,t)Q(s)ds + GU_2^{\mathrm{T}}(T,t)U_2(T,t)\right]^{-1}, \quad t\in[0,T], \quad \text{a.e.} \tag{3.27}$$

Taking into account of non-negativeness of $Q(\cdot)$ and G, this later requirement is slightly weak which allows $R(\cdot) = 0$. On the other hand, inspired by (3.27), we believe that the assumption of 2×2 matrix $R(\cdot)$ being definite could be relaxed in some cases due to the non-negativeness of $Q(\cdot)$ and G. Fourthly, it is remarkable to see that the representation (3.26) is still new in the literature if the state equation is a classical controlled SDE. Since some more related discussions along this, such as the exploration of closed-loop representation via certain stochastic Fredholm-Volterra equation, require more things involved, we hope to demonstrate more results in the forthcoming papers.

4 Concluding remarks

In this paper, the linear quadratic stochastic integral game problem for linear SVIEs is discussed for the first time where the necessary and sufficient conditions of existence of saddle points are derived in two different ways. Some open problems left by Chen and Yong [3] are solved here as well, which can cover the SDEs case in [12]. Two new descriptions for the saddle points are represented via coupled FBSVIEs and certain stochastic equations of Fredholm-Volterra type. To the best of our knowledge, both kinds of equations are new in the literature.

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