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# Two finite difference schemes for the phase field crystal equation

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**Abstract** The phase field crystal (PFC) model is a nonlinear evolutionary equation that is of sixth order in space. In the first part of this work, we derive a three level linearized difference scheme, which is then proved to be energy stable, uniquely solvable and second order convergent in $L_2$ norm by the energy method combining with the inductive method. In the second part of the work, we analyze the unique solvability and convergence of a two level nonlinear difference scheme, which was developed by Zhang et al. in 2013. Some numerical results with comparisons are provided.

**Keywords** phase field crystal model, nonlinear evolutionary equation, finite difference scheme, solvability, convergence

**MSC(2010)** 65M06, 65M12, 65M15

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## 1 Introduction

The phase field crystal (PFC) model has been recently proposed to describe the elastic and plastic deformations, multiple crystal orientations and many other observable phenomena, see, e.g., [5, 6, 11]. The PFC model is as follows:

$$\phi_t = \nabla \cdot (M(\phi)\nabla\mu), \quad (x, y) \in \Omega, \quad 0 < t \leqslant T, \tag{1.1}$$

$$\mu = \phi^3 + (1-\epsilon)\phi + 2\Delta\phi + \Delta^2\phi, \quad (x, y) \in \Omega, \quad 0 < t \leqslant T, \tag{1.2}$$

$$\phi(x, y, 0) = \psi(x, y), \quad (x, y) \in \Omega, \tag{1.3}$$

where $\epsilon$ is a positive constant assumed to be less than 1, $\nabla$ and $\Delta$ are the gradient and Laplacian operators, respectively, $M(\phi) > 0$ is a mobility, $\mu$ is the chemical potential. Suppose that $\Omega = (0, L_1) \times (0, L_2)$ and that $\phi$ and $\Delta\phi$ are defined on $\mathcal{R}^2$ with the periodic box $\Omega$. Let

$$E(\phi(\cdot,\cdot,t)) = \int_\Omega \left[\frac{1}{4}\phi^4 + \frac{1-\epsilon}{2}\phi^2 - |\nabla\phi|^2 + \frac{1}{2}(\Delta\phi)^2\right] dxdy.$$

Then, we have the following energy identity:

$$\frac{d}{dt}E(\phi(\cdot,\cdot,t)) + \int_\Omega M(\phi)(\nabla\mu)^2 dxdy = 0.$$

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This means that the energy is monotonically non-increasing in time for PFC model. It follows easily that

$$\frac{d}{dt}E(\phi(\cdot,\cdot,t)) \leqslant 0, \quad 0 < t \leqslant T. \tag{1.4}$$

The PFC equation is a higher order (sixth order) nonlinear partial differential equation. The numerical methods play a very important rule since it is difficult to find the exact solution of the problem.

Wise et al. [14] presented the following difference scheme of first order in time,

$$\frac{1}{\tau}(\phi_{ij}^{n+1} - \phi_{ij}^{n}) = \nabla_h \cdot (M(\phi_{ij}^{n})\nabla_h \mu_{ij}^{n+1}), \tag{1.5}$$
$$\mu_{ij}^{n+1} = (\phi_{ij}^{n+1})^3 + (1-\epsilon)\phi_{ij}^{n+1} + 2\triangle_h \phi_{ij}^{n} + \triangle_h^2 \phi_{ij}^{n+1}. \tag{1.6}$$

They proved that the energy is always non-increasing in time, i.e.,

$$E_h(\phi^{k+1}) \leqslant E_h(\phi^k), \tag{1.7}$$

and thus the scheme is unconditionally energy stable. They also showed that the difference scheme is convergent with the order of one in time and of two in space in the $L_2$ norm. Wang and Wise [13] constructed a difference scheme with the convergence order one in time for the modified phase field crystal equation.

Zhang et al. [15] established the following difference scheme of second order in time:

$$\frac{1}{\tau}(\phi_{ij}^{n+1} - \phi_{ij}^{n}) = \nabla_h \cdot (M(\phi_{ij}^{n+\frac{1}{2}})\nabla_h \mu_{ij}^{n+\frac{1}{2}}), \tag{1.8}$$
$$\mu_{ij}^{n+\frac{1}{2}} = \phi_{ij}^{n+\frac{1}{2}} \cdot \frac{(\phi_{ij}^{n+1})^2 + (\phi_{ij}^{n})^2}{2} + (1-\epsilon)\phi_{ij}^{n+\frac{1}{2}} + 2\triangle_h \phi_{ij}^{n+\frac{1}{2}} + \triangle_h^2 \phi_{ij}^{n+\frac{1}{2}}, \tag{1.9}$$
$$\phi_{ij}^{0} = \psi(x_i, y_j), \quad 1 \leqslant i,j \leqslant M, \tag{1.10}$$

where $\phi_{ij}^{n+\frac{1}{2}} = \frac{1}{2}(\phi_{ij}^{n+1} + \phi_{ij}^{n})$ and the unconditional energy stability of the scheme was shown, however they did not analyze the solvability and the global convergence. Hu et al. [9] presented a fully second-order nonlinear three level difference scheme:

$$\frac{1}{\tau}(\phi_{ij}^{n+1} - \phi_{ij}^{n}) = \nabla_h \cdot (M(\phi_{ij}^{n+\frac{1}{2}})\nabla_h \mu_{ij}^{n+\frac{1}{2}}), \tag{1.11}$$
$$\mu_{ij}^{n+\frac{1}{2}} = \phi_{ij}^{n+\frac{1}{2}} \cdot \frac{(\phi_{ij}^{n+1})^2 + (\phi_{ij}^{n})^2}{2} + (1-\epsilon)\phi_{ij}^{n+\frac{1}{2}} + 3\triangle_h \phi_{ij}^{n} - \triangle_h \phi_{ij}^{n-1} + \triangle_h^2 \phi_{ij}^{n+\frac{1}{2}}, \tag{1.12}$$
$$\phi_{ij}^{0} = \psi(x_i, y_j), \quad \phi_{ij}^{-1} = \phi_{ij}^{0} \quad 1 \leqslant i,j \leqslant M, \tag{1.13}$$

which was proved energy bounded and uniquely solvable. The nonlinear difference scheme was solved by nonlinear multigrid methods. Numerical simulations confirmed the stability, efficiency and accuracy of the scheme. We can find the difference between the difference scheme (1.8)–(1.10) and the difference scheme (1.11)–(1.13).

Gomez and Nogueira [8] also presented some unconditionally energy-stable method with the truncation error of order two both in time and in space for PFC model, but did not discuss the global convergence. Galenko et al. [7] and Baskaran et al. [3] provided some energy stable second order nonlinear difference schemes without the convergence proof for the modified phase-field crystal (MPFC) equation. Baskaran et al. [4] gave a detailed convergence analysis for the nonlinear difference scheme derived in [3]. The analysis method in that paper can be easily modified to prove the convergence of the second-order scheme for the PFC scheme in Hu et al. [9].

To our knowledge, much work has been done on the unconditionally energy stable finite difference scheme for PFC and MPFC problem based on the convex splitting of a discrete energy. The established difference schemes are nonlinear, which is not convenient for the practical computation. In this paper, we will present a three-level second-order linearized difference scheme and prove its unconditional energy-stability and convergence.

This paper is organized as follows. Some discrete notations are introduced and two auxiliary lemmas are presented in the next section. A three-level linearized finite difference scheme is established and analyzed in Section 3. The energy stability, unique solvability and second order convergence are proved. Section 4 is devoted to the theoretical investigation of a two-level nonlinear finite difference scheme presented by Zhang et al. [15]. In Section 5, some numerical results with comparisons are provided. The paper ends with a brief conclusion in Section 6.

## 2 Some notation and lemmas

Throughout this article, we suppose that (1.1)–(1.3) is subject to periodic boundary conditions and has a smooth solution $u(x,y,t)\in C^{8,8,3}_{x,y,t}(\bar\Omega\times[0,T])$.

Without loss of generality, we suppose $\Omega=[0,2\pi]\times[0,2\pi]$. For simplicity as in [14], we assume that $M(\phi)=1$.

Take two positive integers $M$ and $N$. Let $h=2\pi/M$, $\tau=T/N$, $x_i=ih$, $y_j=jh$, $t_n=n\tau$, $t_{n+\frac12}=(t_n+t_{n+1})/2$. Denote

$$\begin{aligned}
&\Omega_h=\{(x_i,y_j)\mid 0\leqslant i\leqslant M, 0\leqslant j\leqslant M\},\\
&\Omega_\tau=\{t_n\mid 0\leqslant n\leqslant N\},\\
&\mathcal{V}_h=\{u\mid u=\{u_{ij}\}, u_{i+M,j}=u_{ij}, u_{i,j+M}=u_{ij}\}.
\end{aligned}$$

For $u\in\mathcal{V}_h$, denote

$$\begin{aligned}
&\delta_x u_{i+\frac12,j}=\frac1h(u_{i+1,j}-u_{i,j}),\quad \delta_y u_{i,j+\frac12}=\frac1h(u_{i,j+1}-u_{i,j}),\\
&\delta_x^2 u_{ij}=\frac{1}{h^2}(u_{i+1,j}-2u_{ij}+u_{i-1,j}),\quad \delta_y^2 u_{ij}=\frac{1}{h^2}(u_{i,j+1}-2u_{ij}+u_{i,j-1}),\\
&\Delta_h u_{ij}=\delta_x^2 u_{ij}+\delta_y^2 u_{ij}.
\end{aligned}$$

It is obvious that

$$\delta_x^2 u_{ij}=\frac1h(\delta_x u_{i+\frac12,j}-\delta_x u_{i-\frac12,j}),\quad \delta_y^2 u_{ij}=\frac1h(\delta_y u_{i,j+\frac12}-\delta_y u_{i,j-\frac12}).$$

For $u\in\mathcal{V}_h$ and $v\in\mathcal{V}_h$, define the inner product

$$(u,v)=h^2\sum_{i,j=1}^{M}u_{ij}\,v_{ij}$$

and Sobolev norms

$$\begin{aligned}
&\|u\|=\sqrt{(u,u)},\quad \|\nabla_h u\|=\sqrt{h^2\sum_{i,j=1}^{M}|\delta_x u_{i-\frac12,j}|^2+h^2\sum_{i,j=1}^{M}|\delta_y u_{i,j-\frac12}|^2},\\
&\|u\|_\infty=\max_{1\leqslant i,j\leqslant M}|u_{ij}|,\quad \|\Delta_h u\|=\sqrt{h^2\sum_{i,j=1}^{M}|\Delta_h u_{ij}|^2}\,.
\end{aligned}$$

For the grid function $v=(v^0,v^1,\ldots,v^N)$ on $\Omega_\tau$, denote

$$\begin{aligned}
&v^{n+\frac12}=\frac12(v^{n+1}+v^n),\quad v^{\bar n}=\frac12(v^{n+1}+v^{n-1}),\\
&\delta_t v^{n+\frac12}=\frac1\tau(v^{n+1}-v^n),\quad \Delta_t v^n=\frac{1}{2\tau}(v^{n+1}-v^{n-1}).
\end{aligned}$$

**Lemma 2.1** (See [14]). *Suppose $\phi \in \mathcal{V}_h$ and $\Delta_h\phi \in \mathcal{V}_h$. Then*

$$\|\Delta_h\phi\|^2 \leqslant \frac{1}{3\alpha^2}\|\phi\|^2 + \frac{2\alpha}{3}\|\nabla_h(\Delta_h\phi)\|^2 \tag{2.1}$$

*is valid for arbitrary $\alpha > 0$.*

We shall use the following Brouwer fixed-point theorem to show the existence of solution of the two-level nonlinear difference scheme.

**Lemma 2.2** (See [1,2]). *Let $(H, (\cdot,\cdot))$ be a finite dimensional inner product space, $\|\cdot\|$ be the associated norm, and $g : H \to H$ be continuous. Assume moreover that*

$$\exists\, \beta > 0, \quad \forall\, z \in H, \quad \|z\| = \beta, \quad (g(z), z) \geqslant 0.$$

*Then, there exists an element $z^* \in H$ such that $g(z^*) = 0$ and $\|z^*\| \leqslant \beta$.*

For the convenience, we present the simplified Grönwall's inequality.

**Lemma 2.3** (See [12]). *Let $c_1, c_2$ and $a_k, k = 1, 2, 3, \ldots,$ be positive and satisfy*

$$a_{k+1} \leqslant (1 + c_1\tau)a_k + c_2\tau, \quad k = 1, 2, 3, \ldots,$$

*then*

$$a_{k+1} \leqslant \exp(c_1 k\tau)\left(a_1 + \frac{c_2}{c_1}\right), \quad k = 1, 2, 3, \ldots$$

## 3 The three-level linearized difference scheme

### 3.1 The derivation of the finite difference scheme

Define the grid functions $\Phi^n$ and $U^n$ on $\mathcal{V}_h$ with

$$\Phi^n_{ij} = \phi(x_i, y_j, t_n), \quad U^n_{ij} = \mu(x_i, y_j, t_n).$$

Using the Taylor expansion, we have

$$\phi(x_i, y_j, t_{\frac{1}{2}}) = \phi(x_i, y_j, t_0) + \frac{\tau}{2}\phi_t(x_i, y_j, t_0) + O(\tau^2).$$

Denote

$$\hat{\phi}_{ij} = \phi(x_i, y_j, t_0) + \frac{\tau}{2}\phi_t(x_i, y_j, t_0), \quad 0 \leqslant i, j \leqslant M.$$

Considering (1.1) and (1.2) at the points $(x_i, y_j, t_{\frac{1}{2}})$ and $(x_i, y_j, t_n)$, then with the help of the Taylor expansion, we have

$$\delta_t\Phi^{\frac{1}{2}}_{ij} = \Delta_h U^{\frac{1}{2}}_{ij} + R^0_{ij}, \quad 1 \leqslant i, j \leqslant M, \tag{3.1}$$

$$U^{\frac{1}{2}}_{ij} = (\hat{\phi}_{ij})^2\Phi^{\frac{1}{2}}_{ij} + (1-\epsilon)\Phi^{\frac{1}{2}}_{ij} + 2\Delta_h\Phi^{\frac{1}{2}}_{ij} + \Delta_h^2\Phi^{\frac{1}{2}}_{ij} + S^0_{ij}, \quad 1 \leqslant i, j \leqslant M, \tag{3.2}$$

and

$$\Delta_t\Phi^n_{ij} = \Delta_h U^{\overline{n}}_{ij} + R^n_{ij}, \quad 1 \leqslant i, j \leqslant M, \quad 1 \leqslant n \leqslant N-1, \tag{3.3}$$

$$U^{\overline{n}}_{ij} = (\Phi^n_{ij})^2\Phi^{\overline{n}}_{ij} + (1-\epsilon)\Phi^{\overline{n}}_{ij} + 2\Delta_h\Phi^{\overline{n}}_{ij} + \Delta_h^2\Phi^{\overline{n}}_{ij} + S^n_{ij}, \quad 1 \leqslant i, j \leqslant M, \quad 1 \leqslant n \leqslant N-1, \tag{3.4}$$

and there exists a positive constant $c_1$ such that

$$|R^n_{ij}| \leqslant c_1(\tau^2 + h^2), \quad 1 \leqslant i, j \leqslant M, \quad 0 \leqslant n \leqslant N-1, \tag{3.5}$$

$$|S^n_{ij}| \leqslant c_1(\tau^2 + h^2), \quad 1 \leqslant i, j \leqslant M, \quad 0 \leqslant n \leqslant N-1. \tag{3.6}$$

Noticing the initial conditions

$$\Phi^0_{ij} = \psi(x_i, y_j), \quad 1 \leqslant i, j \leqslant M \tag{3.7}$$

and omitting the small terms in the system (3.1)–(3.2) and the system (3.3)–(3.4), the difference scheme is constructed for the system (1.1)–(1.3) as follows: Find $\phi^n \in \mathcal{V}_h, n = 1, 2, \ldots, N$ such that

$$\delta_t \phi_{ij}^{\frac{1}{2}} = \Delta_h u_{ij}^{\frac{1}{2}}, \quad 1 \leqslant i, j \leqslant M, \tag{3.8}$$

$$u_{ij}^{\frac{1}{2}} = (\hat{\phi}_{ij})^2 \phi_{ij}^{\frac{1}{2}} + (1-\epsilon)\phi_{ij}^{\frac{1}{2}} + 2\Delta_h \phi_{ij}^{\frac{1}{2}} + \Delta_h^2 \phi_{ij}^{\frac{1}{2}}, \quad 1 \leqslant i, j \leqslant M, \tag{3.9}$$

$$\Delta_t \phi_{ij}^n = \Delta_h u_{ij}^{\overline{n}}, \quad 1 \leqslant i, j \leqslant M, \quad 1 \leqslant n \leqslant N-1, \tag{3.10}$$

$$u_{ij}^{\overline{n}} = (\phi_{ij}^n)^2 \phi_{ij}^{\overline{n}} + (1-\epsilon)\phi_{ij}^{\overline{n}} + 2\Delta_h \phi_{ij}^{\overline{n}} + \Delta_h^2 \phi_{ij}^{\overline{n}}, \quad 1 \leqslant i, j \leqslant M, \quad 1 \leqslant n \leqslant N-1, \tag{3.11}$$

$$\phi_{ij}^0 = \psi(x_i, y_j), \quad 1 \leqslant i, j \leqslant M, \tag{3.12}$$

or,

$$\delta_t \phi_{ij}^{\frac{1}{2}} = \Delta_h[(\hat{\phi}_{ij})^2 \phi_{ij}^{\frac{1}{2}} + (1-\epsilon)\phi_{ij}^{\frac{1}{2}} + 2\Delta_h \phi_{ij}^{\frac{1}{2}} + \Delta_h^2 \phi_{ij}^{\frac{1}{2}}], \quad 1 \leqslant i, j \leqslant M, \tag{3.13}$$

$$\Delta_t \phi_{ij}^n = \Delta_h[(\phi_{ij}^n)^2 \phi_{ij}^{\overline{n}} + (1-\epsilon)\phi_{ij}^{\overline{n}} + 2\Delta_h \phi_{ij}^{\overline{n}} + \Delta_h^2 \phi_{ij}^{\overline{n}}], \quad 1 \leqslant i, j \leqslant M, \quad 1 \leqslant n \leqslant N-1, \tag{3.14}$$

$$\phi_{ij}^0 = \psi(x_i, y_j), \quad 1 \leqslant i, j \leqslant M. \tag{3.15}$$

At each time level, the difference scheme (3.13)–(3.14) is a linear system of algebraic equations with respect to the wanted solution, which can be solved by the iterative methods.

### 3.2 The energy stable of the finite difference scheme

We consider the discrete energy stability of the difference scheme (3.13)–(3.15).

**Theorem 3.1.** *The finite difference scheme* (3.13)–(3.15)*, or equivalently,* (3.8)–(3.12)*, is energy stable. More precisely, define*

$$G_h(\phi^n, \phi^{n+1}) \equiv \frac{1}{4}((\phi^n)^2, (\phi^{n+1})^2) + \frac{1-\epsilon}{2} \cdot \frac{\|\phi^n\|^2 + \|\phi^{n+1}\|^2}{2} - \frac{\|\nabla_h \phi^n\|^2 + \|\nabla_h \phi^{n+1}\|^2}{2} + \frac{1}{2} \cdot \frac{\|\Delta_h \phi^n\|^2 + \|\Delta_h \phi^{n+1}\|^2}{2}, \quad 0 \leqslant n \leqslant N-1,$$

*we have*

$$G_h(\phi^n, \phi^{n+1}) \leqslant G_h(\phi^{n-1}, \phi^n), \quad 1 \leqslant n \leqslant N-1. \tag{3.16}$$

*In addition, we have*

$$\frac{1}{4}((\hat{\phi})^2, (\phi^1)^2) + \frac{1-\epsilon}{2}\|\phi^1\|^2 - \|\nabla_h \phi^1\|^2 + \frac{1}{2}\|\triangle_h \phi^1\|^2 \leqslant \frac{1}{4}((\hat{\phi})^2, (\phi^0)^2) + \frac{1-\epsilon}{2}\|\phi^0\|^2 - \|\nabla_h \phi^0\|^2 + \frac{1}{2}\|\triangle_h \phi^0\|^2,$$

*or,*

$$\frac{1}{4}\left((\hat{\phi})^2, \frac{(\phi^0)^2 + (\phi^1)^2}{2}\right) + \frac{1-\epsilon}{2} \cdot \frac{\|\phi^0\|^2 + \|\phi^1\|^2}{2} - \frac{\|\nabla_h \phi^0\|^2 + \|\nabla_h \phi^1\|^2}{2} + \frac{1}{4}(\|\triangle_h \phi^0\|^2 + \|\triangle_h \phi^1\|^2) \leqslant \frac{1}{4}((\hat{\phi})^2, (\phi^0)^2) + \frac{1-\epsilon}{2}\|\phi^0\|^2 - \|\nabla_h \phi^0\|^2 + \frac{1}{2}\|\triangle_h \phi^0\|^2. \tag{3.17}$$

*Proof.* Taking the inner product of (3.10) with $u^{\overline{n}}$, we obtain

$$(\Delta_t \phi^n, u^{\overline{n}}) = (\Delta_h u^{\overline{n}}, u^{\overline{n}}) = -\|\nabla_h u^{\overline{n}}\|^2 \leqslant 0.$$

Inserting (3.11) into the inequality above, we have

$$(\Delta_t \phi^n, (\phi^n)^2 \phi^{\overline{n}} + (1-\epsilon)\phi^{\overline{n}} + 2\Delta_h \phi^{\overline{n}} + \Delta_h^2 \phi^{\overline{n}}) \leqslant 0,$$

or,

$$(\Delta_t\phi^n,(\phi^n)^2\phi^{\overline{n}})+(1-\epsilon)(\Delta_t\phi^n,\phi^{\overline{n}})+2(\Delta_t\phi^n,\Delta_h\phi^{\overline{n}})+(\Delta_t\phi^n,\Delta_h^2\phi^{\overline{n}})\leqslant 0,\quad 1\leqslant n\leqslant N-1. \tag{3.18}$$

Noticing that

$$\begin{aligned}
(\Delta_t\phi^n,(\phi^n)^2\phi^{\overline{n}}) &= \frac{1}{4\tau}[((\phi^{n+1})^2,(\phi^n)^2)-((\phi^n)^2,(\phi^{n-1})^2)],\\
(\Delta_t\phi^n,\phi^{\overline{n}}) &= \frac{1}{4\tau}(\|\phi^{n+1}\|^2-\|\phi^{n-1}\|^2)\\
&= \frac{1}{4\tau}[(\|\phi^{n+1}\|^2+\|\phi^n\|^2)-(\|\phi^n\|^2+\|\phi^{n-1}\|^2)],\\
(\Delta_t\phi^n,\Delta_h\phi^{\overline{n}}) &= -\frac{1}{4\tau}(\|\nabla_h\phi^{n+1}\|^2-\|\nabla_h\phi^{n-1}\|^2)\\
&= -\frac{1}{4\tau}[(\|\nabla_h\phi^{n+1}\|^2+\|\nabla_h\phi^n\|^2)-(\|\nabla_h\phi^n\|^2+\|\nabla_h\phi^{n-1}\|^2)],\\
(\Delta_t\phi^n,\Delta_h^2\phi^{\overline{n}}) &= \frac{1}{4\tau}(\|\triangle_h\phi^{n+1}\|^2-\|\triangle_h\phi^{n-1}\|^2)\\
&= \frac{1}{4\tau}[(\|\triangle_h\phi^{n+1}\|^2+\|\triangle_h\phi^n\|^2)-(\|\triangle_h\phi^n\|^2+\|\triangle_h\phi^{n-1}\|^2)],
\end{aligned}$$

it then follows from (3.18) that

$$\begin{aligned}
&\frac{1}{4}((\phi^{n+1})^2,(\phi^n)^2)+\frac{1-\epsilon}{4}(\|\phi^{n+1}\|^2+\|\phi^n\|^2)\\
&\quad-\frac{1}{2}(\|\nabla_h\phi^{n+1}\|^2+\|\nabla_h\phi^n\|^2)+\frac{1}{4}(\|\triangle_h\phi^{n+1}\|^2+\|\triangle_h\phi^n\|^2)\\
&\qquad\leqslant\frac{1}{4}((\phi^n)^2,(\phi^{n-1})^2)+\frac{1-\epsilon}{4}(\|\phi^n\|^2+\|\phi^{n-1}\|^2)\\
&\qquad\quad-\frac{1}{2}(\|\nabla_h\phi^n\|^2+\|\nabla_h\phi^{n-1}\|^2)+\frac{1}{4}(\|\triangle_h\phi^n\|^2+\|\triangle_h\phi^{n-1}\|^2),\quad 1\leqslant n\leqslant N-1,
\end{aligned}$$

or,

$$G_h(\phi^n,\phi^{n+1})\leqslant G_h(\phi^{n-1},\phi^n),\quad 1\leqslant n\leqslant N-1. \tag{3.19}$$

Similarly, taking the inner product of (3.8) with $u^{\frac{1}{2}}$, we can obtain (3.17). This completes the proof. □

**Remark 3.2.** The discrete energy inequalities (3.16) and (3.17) are the counterpart of the energy inequality (1.4) of (1.1)–(1.3). This energy stability is not enough to guarantee the uniform boundedness of $\|\phi^k\|_\infty$ since the appearance of the term $\frac{1}{4}((\phi^n)^2,(\phi^{n+1})^2)$. In this sense, the energy stability of Theorem 3.1 is weaker than that for the two level nonlinear difference scheme in [15].

### 3.3 The unique solvability and convergence of the finite difference scheme

Define the error grid functions $\tilde{u}^n,\tilde{\phi}^n$, for $0\leqslant n\leqslant N$, on $\Omega_h$ as follows:

$$\tilde{\phi}^n_{ij}=\Phi^n_{ij}-\phi^n_{ij},\quad \tilde{u}^n_{ij}=U^n_{ij}-u^n_{ij},\quad 0\leqslant i,j\leqslant M.$$

For the solvability and convergence of the difference scheme, we have the following theorem.

**Theorem 3.3.** *Assume the solution to* (1.1)–(1.3) *is sufficiently smooth. The difference scheme* (3.13)–(3.15) *is uniquely solvable and convergent with the convergence order of two both in time and in space when* $\tau^2=o(h)$. *In detail, let*

$$c_2=\max_{0\leqslant x,y\leqslant 2\pi,0\leqslant t\leqslant T}|\phi(x,y,t)|$$

*and*

$$c=2\sqrt{d}\pi c_1\exp\{3[2(c_2+1)^4+6+2(2c_2+1)^2c_2^2]T\},\quad d=1+\frac{1}{2(c_1+1)^4+6+2(2c_2+1)^2c_2^2},$$

*then if*

$$ch^{-1}(\tau^2+h^2) \leqslant 1, \tag{3.20}$$

*we have*

$$\|\tilde{\phi}^n\| \leqslant c(\tau^2+h^2), \quad 0 \leqslant n \leqslant N. \tag{3.21}$$

**Remark 3.4.** If the solution to the difference scheme (3.13)–(3.15) is uniformly bounded, then the condition $\tau^2 = o(h)$, or, equivalently, (3.20), is not needed and furthermore the proof can be simplified greatly. However, we have not proved at the present that the solution of the difference scheme (3.13)–(3.15) is uniformly bounded theoretically.

*Proof of Theorem* 3.3. Subtracting (3.8)–(3.12) from (3.1)–(3.4) and (3.7), we obtain the error system

$$\delta_t \tilde{\phi}_{ij}^{\frac{1}{2}} = \Delta_h \tilde{u}_{ij}^{\frac{1}{2}} + R_{ij}^0, \quad 1 \leqslant i,j \leqslant M, \tag{3.22}$$

$$\tilde{u}_{ij}^{\frac{1}{2}} = (\hat{\phi}_{ij})^2 \tilde{\phi}_{ij}^{\frac{1}{2}} + (1-\epsilon)\tilde{\phi}_{ij}^{\frac{1}{2}} + 2\Delta_h \tilde{\phi}_{ij}^{\frac{1}{2}} + \Delta_h^2 \tilde{\phi}_{ij}^{\frac{1}{2}} + S_{ij}^0, \quad 1 \leqslant i,j \leqslant M, \tag{3.23}$$

$$\Delta_t \tilde{\phi}_{ij}^n = \Delta_h \tilde{u}_{ij}^{\overline{n}} + R_{ij}^n, \quad 1 \leqslant i,j \leqslant M, \quad 1 \leqslant n \leqslant N-1, \tag{3.24}$$

$$\begin{aligned}\tilde{u}_{ij}^{\overline{n}} = {} & (\Phi_{ij}^n)^2 \Phi_{ij}^{\overline{n}} - (\phi_{ij}^n)^2 \phi_{ij}^{\overline{n}} + (1-\epsilon)\tilde{\phi}_{ij}^{\overline{n}} + 2\Delta_h \tilde{\phi}_{ij}^{\overline{n}} + \Delta_h^2 \tilde{\phi}_{ij}^{\overline{n}} + S_{ij}^n, \\ & 1 \leqslant i,j \leqslant M, \quad 1 \leqslant n \leqslant N-1,\end{aligned} \tag{3.25}$$

$$\tilde{\phi}_{ij}^0 = 0, \quad 1 \leqslant i,j \leqslant M. \tag{3.26}$$

Denote

$$c_3 = \max_{0 \leqslant x,y \leqslant 2\pi} |\phi_t(x,y,0)|.$$

Then, when

$$\frac{1}{2} c_3 \tau \leqslant 1,$$

we have

$$|\hat{\phi}_{ij}| \leqslant c_2 + \frac{\tau}{2} c_3 \leqslant c_2 + 1.$$

Next, we will prove the theorem by the inductive method.

**Step 1.** The unique solvability of $\phi^1$.

The $\{\phi_{ij}^0 \mid 0 \leqslant i,j \leqslant M\}$ is given in (3.15). The equation (3.13) is a linear system about $\{\phi_{ij}^1 \mid 1 \leqslant i,j \leqslant M\}$. Considering its homogeneous system, we have

$$\frac{\phi_{ij}^1}{\tau} = \Delta_h \left[ \frac{1}{2}(\hat{\phi}_{ij})^2 \phi_{ij}^1 + \frac{1-\epsilon}{2}\phi_{ij}^1 + \Delta_h \phi_{ij}^1 + \frac{1}{2}\Delta_h^2 \phi_{ij}^1 \right], \quad 1 \leqslant i,j \leqslant M. \tag{3.27}$$

Taking the inner product of (3.27) with $2\phi^1$, and using Lemma 2.1 with $\alpha = \frac{1}{2}$, we arrive at

$$\begin{aligned}\frac{2}{\tau}\|\phi^1\|^2 &= ((\hat{\phi})^2\phi^1 + (1-\epsilon)\phi^1 + 2\Delta_h\phi^1 + \Delta_h^2\phi^1,\ \Delta_h\phi^1) \\ &= ((\hat{\phi})^2\phi^1, \Delta_h\phi^1) - (1-\epsilon)\|\nabla_h\phi^1\|^2 + 2\|\Delta_h\phi^1\|^2 - \|\nabla_h(\Delta_h\phi^1)\|^2 \\ &\leqslant (c_2+1)^2\|\phi^1\| \cdot \|\Delta_h\phi^1\| + 2\|\Delta_h\phi^1\|^2 - \|\nabla_h(\Delta_h\phi^1)\|^2 \\ &= \frac{1}{4}(c_2+1)^4\|\phi^1\|^2 + 3\|\Delta_h\phi^1\|^2 - \|\nabla_h\Delta_h\phi^1\|^2 \\ &= \left[\frac{1}{4}(c_2+1)^4 + 4\right]\|\phi^1\|^2.\end{aligned}$$

Thus, when $\tau$ is small, $\|\phi^1\| = 0$ and (3.13) determines $\phi^1$ uniquely.

**Step 2.** The convergence of $\phi^1$.

Taking the inner product of (3.22) with $\tilde{\phi}^{\frac{1}{2}}$, we have

$$(\delta_t\tilde{\phi}^{\frac{1}{2}},\tilde{\phi}^{\frac{1}{2}})=(\Delta_h\tilde{u}^{\frac{1}{2}},\tilde{\phi}^{\frac{1}{2}})+(R^0,\tilde{\phi}^{\frac{1}{2}})=(\tilde{u}^{\frac{1}{2}},\Delta_h\tilde{\phi}^{\frac{1}{2}})+(R^0,\tilde{\phi}^{\frac{1}{2}}).$$

Inserting (3.23) into the right-hand side of the equality above, we obtain

$$\begin{aligned}(\delta_t\tilde{\phi}^{\frac{1}{2}},\tilde{\phi}^{\frac{1}{2}})&=((\hat{\phi})^2\tilde{\phi}^{\frac{1}{2}},\Delta_h\tilde{\phi}^{\frac{1}{2}})+(1-\epsilon)(\tilde{\phi}^{\frac{1}{2}},\Delta_h\tilde{\phi}^{\frac{1}{2}})+2(\Delta_h\tilde{\phi}^{\frac{1}{2}},\Delta_h\tilde{\phi}^{\frac{1}{2}})+(\Delta_h^2\tilde{\phi}^{\frac{1}{2}},\Delta_h\tilde{\phi}^{\frac{1}{2}})\\&\quad+(S^0,\Delta_h\tilde{\phi}^{\frac{1}{2}})+(R^0,\tilde{\phi}^{\frac{1}{2}})\\&\leqslant[(c_2+1)^2+1]\|\tilde{\phi}^{\frac{1}{2}}\|\cdot\|\Delta_h\tilde{\phi}^{\frac{1}{2}}\|+2\|\Delta_h\tilde{\phi}^{\frac{1}{2}}\|^2-\|\nabla_h(\triangle_h\tilde{\phi}^{\frac{1}{2}})\|\\&\quad+\|S^0\|\cdot\|\Delta_h\tilde{\phi}^{\frac{1}{2}}\|+\|R^0\|\cdot\|\tilde{\phi}^{\frac{1}{2}}\|\\&\leqslant\frac{1}{2}[(c_2+1)^2+1]^2\|\tilde{\phi}^{\frac{1}{2}}\|^2+\frac{1}{2}\|\Delta_h\tilde{\phi}^{\frac{1}{2}}\|^2+2\|\Delta_h\tilde{\phi}^{\frac{1}{2}}\|^2-\|\nabla_h(\triangle_h\tilde{\phi}^{\frac{1}{2}})\|^2\\&\quad+\frac{1}{2}\|S^0\|^2+\frac{1}{2}\|\Delta_h\tilde{\phi}^{\frac{1}{2}}\|^2+\frac{1}{2}\|R^0\|^2+\frac{1}{2}\|\tilde{\phi}^{\frac{1}{2}}\|^2.\end{aligned}$$

Using Lemma 2.1 with $\alpha=\frac{1}{2}$ and then using (3.5) and (3.6), we get

$$\begin{aligned}(\delta_t\tilde{\phi}^{\frac{1}{2}},\tilde{\phi}^{\frac{1}{2}})&\leqslant\left\{\frac{1}{2}[(c_2+1)^2+1]^2+\frac{9}{2}\right\}\|\tilde{\phi}^{\frac{1}{2}}\|^2+\frac{1}{2}\|S^0\|^2+\frac{1}{2}\|R^0\|^2\\&\leqslant\left\{\frac{1}{2}[(c_2+1)^2+1]^2+\frac{9}{2}\right\}\|\tilde{\phi}^{\frac{1}{2}}\|^2+4\pi^2c_1^2(\tau^2+h^2)^2.\end{aligned}$$

Noticing $\tilde{\phi}^0_{ij}=0$, we have

$$\frac{1}{4\tau}\|\tilde{\phi}^1\|^2\leqslant\frac{1}{4}\left\{\frac{1}{2}[(c_2+1)^2+1]^2+\frac{9}{2}\right\}\|\phi^1\|^2+4\pi^2c_1^2(\tau^2+h^2)^2,$$

or,

$$\|\tilde{\phi}^1\|^2\leqslant\left\{\frac{1}{2}[(c_2+1)^2+1]^2+\frac{9}{2}\right\}\tau\|\phi^1\|^2+16\pi^2c_1^2\tau(\tau^2+h^2)^2.$$

When $\{[(c_2+1)^2+1]^2+9\}\tau\leqslant1$,

$$\|\tilde{\phi}^1\|^2\leqslant32\pi^2c_1^2\tau(\tau^2+h^2)^2\leqslant4\pi^2c_1^2(\tau^2+h^2)^2,$$

or,

$$\|\tilde{\phi}^1\|\leqslant2\pi c_1(\tau^2+h^2),\tag{3.28}$$

i.e., the estimate (3.21) is valid for $n=1$.

**Step 3.** The unique solvability of $\phi^{m+1}$.

Now suppose that $\{\phi^n\mid1\leqslant n\leqslant m\}$ $(m\leqslant N-1)$ has been determined and the estimate (3.21) is valid for $1\leqslant n\leqslant m$. Noticing (3.20), we have

$$|\tilde{\phi}^n_{ij}|\leqslant h^{-1}\|\tilde{\phi}^n\|\leqslant ch^{-1}(\tau^2+h^2)\leqslant1,\quad1\leqslant i,j\leqslant M,\quad1\leqslant n\leqslant m.$$

Consequently,

$$|\phi^n_{ij}|\leqslant|\Phi^n_{ij}|+|\tilde{\phi}^n_{ij}|\leqslant c_2+1,\quad1\leqslant i,j\leqslant M,\quad1\leqslant n\leqslant m,\tag{3.29}$$

where $c_2$ is defined in the theorem.

(3.14) is a linear system with respect to the unknowns $\{\phi^{m+1}_{ij}\mid1\leqslant i,j\leqslant M\}$. Considering its homogenous system, we have

$$\frac{1}{2\tau}\phi^{m+1}_{ij}=\Delta_h\left[\frac{1}{2}(\phi^m_{ij})^2\phi^{m+1}_{ij}+\frac{1-\epsilon}{2}\phi^{m+1}_{ij}+\Delta_h\phi^{m+1}_{ij}+\frac{1}{2}\Delta_h^2\phi^{m+1}_{ij}\right],\quad1\leqslant i,j\leqslant M.\tag{3.30}$$

Taking the inner product of (3.30) with $2\phi^{m+1}$ and using (3.29), we obtain

$$\begin{aligned}\frac{1}{\tau}\|\phi^{m+1}\|^2 &= ((\phi^m)^2\phi^{m+1} + (1-\epsilon)\phi^{m+1} + 2\Delta_h\phi^{m+1} + \Delta_h^2\phi^{m+1}, \Delta_h\phi^{m+1})\\ &\leqslant [(c_2+1)^2+1]\|\phi^{m+1}\|\cdot\|\Delta_h\phi^{m+1}\| + 2\|\Delta_h\phi^{m+1}\|^2 - \|\nabla_h(\Delta_h\phi^{m+1})\|^2\\ &\leqslant \frac{1}{4}[(c_2+1)^2+1]^2\|\phi^{m+1}\|^2 + 3\|\Delta_h\phi^{m+1}\|^2 - \|\nabla_h(\Delta_h\phi^{m+1})\|^2.\end{aligned}$$

By Lemma 2.1 with $\alpha = \frac{1}{2}$, we have

$$\frac{1}{\tau}\|\phi^{m+1}\|^2 \leqslant \left\{\frac{1}{4}[(c_2+1)^2+1]^2 + 4\right\}\|\phi^{m+1}\|^2.$$

When

$$\left\{\frac{1}{4}[(c_2+1)^2+1]^2 + 4\right\}\tau < 1,$$

we get

$$\|\phi^{m+1}\| = 0.$$

Consequently, (3.14) determines the $\phi^{m+1}$ uniquely.

**Step 4.** The convergence of $\phi^{m+1}$.

Now we prove that (3.21) is valid for $n = m+1$.

Taking the inner product of (3.24) with $\tilde{\phi}^{\overline{n}}$, we obtain

$$(\Delta_t\tilde{\phi}^n, \tilde{\phi}^{\overline{n}}) = (\Delta_h\tilde{u}^{\overline{n}}, \tilde{\phi}^{\overline{n}}) + (R^n, \tilde{\phi}^{\overline{n}}) = (\tilde{u}^{\overline{n}}, \Delta_h\tilde{\phi}^{\overline{n}}) + (R^n, \tilde{\phi}^{\overline{n}}), \quad 1 \leqslant n \leqslant N-1.$$

Inserting (3.25) into the equality above, we have

$$\begin{aligned}(\Delta_t\tilde{\phi}^n, \tilde{\phi}^{\overline{n}}) &= ((\Phi^n)^2\Phi^{\overline{n}} - (\phi^n)^2\phi^{\overline{n}}, \Delta_h\tilde{\phi}^{\overline{n}}) + (1-\epsilon)(\tilde{\phi}^{\overline{n}}, \Delta_h\tilde{\phi}^{\overline{n}})\\ &\quad + 2(\Delta_h\tilde{\phi}^{\overline{n}}, \Delta_h\tilde{\phi}^{\overline{n}}) + (\Delta_h^2\tilde{\phi}^{\overline{n}}, \Delta_h\tilde{\phi}^{\overline{n}}) + (S^n, \Delta_h\tilde{\phi}^{\overline{n}}) + (R^n, \tilde{\phi}^{\overline{n}}), \quad 1 \leqslant n \leqslant N-1. \end{aligned} \tag{3.31}$$

Noticing

$$\begin{aligned}(\Phi_{ij}^n)^2\Phi_{ij}^{\bar{n}} - (\phi_{ij}^n)^2\phi_{ij}^{\bar{n}} &= [(\Phi_{ij}^n)^2 - (\phi_{ij}^n)^2]\Phi_{ij}^{\bar{n}} + (\phi_{ij}^n)^2(\Phi_{ij}^{\bar{n}} - \phi_{ij}^{\bar{n}})\\ &= [(\Phi_{ij}^n + \phi_{ij}^n)\Phi_{ij}^{\bar{n}} + (\phi_{ij}^n)^2](\Phi_{ij}^{\bar{n}} - \phi_{ij}^{\bar{n}})\end{aligned}$$

and (3.29), we have

$$|(\Phi_{ij}^n)^2\Phi_{ij}^{\bar{n}} - (\phi_{ij}^n)^2\phi_{ij}^{\bar{n}}| \leqslant [(2c_2+1)c_2 + (c_2+1)^2]|\tilde{\phi}_{ij}^{\bar{n}}|, \quad 1 \leqslant n \leqslant m.$$

Then it follows from (3.31) that

$$\begin{aligned}&(\Delta_t\tilde{\phi}^n, \tilde{\phi}^{\bar{n}})\\ &\quad\leqslant [(2c_2+1)c_2\|\tilde{\phi}^n\| + (c_2+1)^2\|\tilde{\phi}^{\bar{n}}\|]\cdot\|\Delta_h\tilde{\phi}^{\bar{n}}\| + (1-\epsilon)\|\tilde{\phi}^{\bar{n}}\|\cdot\|\Delta_h\tilde{\phi}^{\bar{n}}\|\\ &\qquad + 2\|\Delta_h\tilde{\phi}^{\bar{n}}\|^2 - \|\nabla_h(\Delta_h\tilde{\phi}^{\bar{n}})\|^2 + \|S^n\|\cdot\|\Delta_h\tilde{\phi}^{\bar{n}}\| + \|R^n\|\cdot\|\tilde{\phi}^{\bar{n}}\|\\ &\quad\leqslant [(2c_2+1)c_2\|\tilde{\phi}^n\| + (c_2+1)^2\|\tilde{\phi}^{\bar{n}}\|]^2 + \frac{1}{4}\|\Delta_h\tilde{\phi}^{\bar{n}}\|^2 + (1-\epsilon)\left[\|\tilde{\phi}^{\bar{n}}\|^2 + \frac{1}{4}\|\Delta_h\tilde{\phi}^{\bar{n}}\|^2\right]\\ &\qquad + 2\|\Delta_h\tilde{\phi}^{\bar{n}}\|^2 - \|\nabla_h(\Delta_h\tilde{\phi}^{\bar{n}})\|^2 + \frac{1}{2}\|S^n\|^2 + \frac{1}{2}\|\Delta_h\tilde{\phi}^{\bar{n}}\|^2 + \frac{1}{2}\|R^n\|^2 + \frac{1}{2}\|\tilde{\phi}^{\bar{n}}\|^2, \quad 1 \leqslant n \leqslant m,\end{aligned}$$

or,

$$\begin{aligned}&\frac{1}{4\tau}(\|\tilde{\phi}^{n+1}\|^2 - \|\tilde{\phi}^{n-1}\|^2)\\ &\quad\leqslant 2(2c_2+1)^2c_2^2\|\tilde{\phi}^n\|^2 + \left[2(c_2+1)^4 + \frac{3}{2} - \epsilon\right]\|\tilde{\phi}^{\bar{n}}\|^2 + 3\|\Delta_h\tilde{\phi}^{\bar{n}}\|^2\end{aligned}$$

$$-\|\nabla_h(\Delta_h\tilde{\phi}^{\bar{n}})\|^2+\frac{1}{2}(\|S^n\|^2+\|R^n\|^2),\quad 1\leqslant n\leqslant m.$$

Using Lemma 2.1 with $\alpha=\frac{1}{2}$ and then using (3.5)–(3.6), we get

$$\begin{aligned}&\frac{1}{4\tau}(\|\tilde{\phi}^{n+1}\|^2-\|\tilde{\phi}^{n-1}\|^2)\\&\quad\leqslant 2(2c_2+1)^2c_2^2\|\tilde{\phi}^n\|^2+[2(c_2+1)^4+6]\|\tilde{\phi}^{\bar{n}}\|^2+4\pi^2c_1^2(\tau^2+h^2)^2\\&\quad\leqslant 2(2c_2+1)^2c_2^2\|\tilde{\phi}^n\|^2+\frac{1}{2}[2(c_2+1)^4+6](\|\tilde{\phi}^{n+1}\|^2+\|\tilde{\phi}^{n-1}\|^2)\\&\qquad+4\pi^2c_1^2(\tau^2+h^2)^2,\quad 1\leqslant n\leqslant m.\end{aligned}$$

Multiplying the inequality above by $4\tau$, we have

$$\begin{aligned}\{1-2[2(c_2+1)^4+6]\tau\}\|\tilde{\phi}^{n+1}\|^2&\leqslant\{1+2[2(c_2+1)^4+6]\tau\}\|\tilde{\phi}^{n-1}\|^2\\&\quad+8(2c_2+1)^2c_2^2\tau\|\tilde{\phi}^n\|^2+16\pi^2c_1^2\tau(\tau^2+h^2)^2,\quad 1\leqslant n\leqslant m.\end{aligned}\tag{3.32}$$

Notice that if $0<\alpha\leqslant\frac{1}{3}$, we have $\frac{1+\alpha}{1-\alpha}\leqslant 1+3\alpha$ and $\frac{1}{1-\alpha}\leqslant\frac{3}{2}$. Then,

$$2[2(c_2+1)^4+6]\tau\leqslant\frac{1}{3}.$$

It follows from (3.32) that

$$\begin{aligned}\|\tilde{\phi}^{n+1}\|^2&\leqslant\{1+6[2(c_2+1)^4+6]\tau\}\|\tilde{\phi}^{n-1}\|^2+12(2c_2+1)^2c_2^2\tau\|\tilde{\phi}^n\|^2+24\pi^2c_1^2\tau(\tau^2+h^2)^2\\&\leqslant\{1+6[2(c_2+1)^4+6]\tau+12(2c_2+1)^2c_2^2\tau\}\max\{\|\tilde{\phi}^{n-1}\|^2\|\tilde{\phi}^n\|^2\}\\&\quad+24\pi^2c_1^2\tau(\tau^2+h^2)^2,\quad 1\leqslant n\leqslant m.\end{aligned}$$

Consequently,

$$\begin{aligned}\max\{\|\tilde{\phi}^{n+1}\|^2,\|\tilde{\phi}^n\|^2\}&\leqslant\{1+[12(c_2+1)^4+36+12(2c_2+1)^2c_2^2]\tau\}\max\{\|\tilde{\phi}^n\|^2,\|\tilde{\phi}^{n-1}\|^2\}\\&\quad+24\pi^2c_1^2\tau(\tau^2+h^2)^2,\quad 1\leqslant n\leqslant m.\end{aligned}$$

The Grönwall's inequality (see Lemma 2.3) yields

$$\begin{aligned}\max\{\|\tilde{\phi}^{m+1}\|^2,\|\tilde{\phi}^m\|^2\}&\leqslant\exp\{[12(c_2+1)^4+36+12(2c_2+1)^2c_2^2]T\}\bigg[\max\{\|\tilde{\phi}^1\|^2,\|\tilde{\phi}^0\|^2\}\\&\quad+\frac{4\pi^2c_1^2}{2(c_2+1)^4+6+2(2c_2+1)^2c_2^2}(\tau^2+h^2)^2\bigg].\end{aligned}$$

Combining with (3.28), we have

$$\begin{aligned}\|\tilde{\phi}^{m+1}\|^2&\leqslant\exp\{6[2(c_2+1)^4+6+2(2c_2+1)^2c_2^2]T\}\\&\quad\times\bigg[4\pi^2c_1^2(\tau^2+h^2)^2+\frac{4\pi^2c_1^2}{2(c_2+1)^4+6+2(2c_2+1)^2c_2^2}(\tau^2+h^2)^2\bigg]\\&\leqslant\exp\{6[2(c_2+1)^4+6+2(2c_2+1)^2c_2^2]T\}\\&\quad\times4\pi^2c_1^2\bigg[1+\frac{1}{2(c_2+1)^4+6+2(2c_2+1)^2c_2^2}\bigg](\tau^2+h^2)^2\\&\leqslant c^2(\tau^2+h^2)^2,\end{aligned}$$

or,

$$\|\tilde{\phi}^{m+1}\|\leqslant c(\tau^2+h^2),$$

i.e., the estimate (3.21) is valid for $n=m+1$. This completes the proof. $\square$

# 4 Analysis of a two-level nonlinear difference scheme

## 4.1 The difference scheme and the truncation errors

Considering (1.1) and (1.2) at the point $(x_i, y_j, t_{n+\frac{1}{2}})$ and applying the Taylor expansion, we have

$$\delta_t \Phi_{ij}^{n+\frac{1}{2}} = \Delta_h U_{ij}^{n+\frac{1}{2}} + P_{ij}^n, \quad 1 \leqslant i,j \leqslant M, \quad 0 \leqslant n \leqslant N-1, \tag{4.1}$$

$$\begin{aligned} U_{ij}^{n+\frac{1}{2}} &= \Phi_{ij}^{n+\frac{1}{2}} \frac{(\Phi_{ij}^n)^2 + (\Phi_{ij}^{n+1})^2}{2} + (1-\epsilon)\Phi_{ij}^{n+\frac{1}{2}} + 2\Delta_h \Phi_{ij}^{n+\frac{1}{2}} + \Delta_h^2 \Phi_{ij}^{n+\frac{1}{2}} + Q_{ij}^n, \\ & 1 \leqslant i,j \leqslant M, \quad 0 \leqslant n \leqslant N-1, \end{aligned} \tag{4.2}$$

and there exists a constant $c_4$ such that

$$|P_{ij}^n| \leqslant c_4(\tau^2 + h^2), \quad |Q_{ij}^n| \leqslant c_4(\tau^2 + h^2), \quad 1 \leqslant i,j \leqslant M, \quad 0 \leqslant n \leqslant N-1. \tag{4.3}$$

Noticing

$$\Phi_{ij}^0 = \psi(x_i, y_j), \quad 1 \leqslant i,j \leqslant M, \tag{4.4}$$

and omitting the small terms $P_{ij}^n, Q_{ij}^n$ in (4.1) and (4.2), we obtain the difference scheme: For $0 \leqslant n \leqslant N$, find $\phi^n \in \mathcal{V}_h$ such that

$$\delta_t \phi_{ij}^{n+\frac{1}{2}} = \Delta_h u_{ij}^{n+\frac{1}{2}}, \quad 1 \leqslant i,j \leqslant M, \quad 0 \leqslant n \leqslant N-1, \tag{4.5}$$

$$\begin{aligned} u_{ij}^{n+\frac{1}{2}} &= \phi_{ij}^{n+\frac{1}{2}} \frac{(\phi_{ij}^n)^2 + (\phi_{ij}^{n+1})^2}{2} + (1-\epsilon)\phi_{ij}^{n+\frac{1}{2}} + 2\Delta_h \phi_{ij}^{n+\frac{1}{2}} + \Delta_h^2 \phi_{ij}^{n+\frac{1}{2}}, \\ & 1 \leqslant i,j \leqslant M, \quad 0 \leqslant n \leqslant N-1, \end{aligned} \tag{4.6}$$

$$\phi_{ij}^0 = \psi(x_i, y_j), \quad 1 \leqslant i,j \leqslant M. \tag{4.7}$$

This is just (1.8)–(1.10).

Inserting (4.6) into (4.5), we get: For $0 \leqslant n \leqslant N$, find $\phi^n \in \mathcal{V}_h$ such that

$$\begin{aligned} \delta_t \phi_{ij}^{n+\frac{1}{2}} &= \Delta_h \left[ \phi_{ij}^{n+\frac{1}{2}} \frac{(\phi_{ij}^n)^2 + (\phi_{ij}^{n+1})^2}{2} + (1-\epsilon)\phi_{ij}^{n+\frac{1}{2}} + 2\Delta_h \phi_{ij}^{n+\frac{1}{2}} + \Delta_h^2 \phi_{ij}^{n+\frac{1}{2}} \right], \\ & 1 \leqslant i,j \leqslant M, \quad 0 \leqslant n \leqslant N-1, \end{aligned} \tag{4.8}$$

$$\phi_{ij}^0 = \psi(x_i, y_j), \quad 1 \leqslant i,j \leqslant M. \tag{4.9}$$

Zhang et al. [15] have proved the following energy stability.

**Theorem 4.1** (Energy stability). *Let $\{\phi_{ij}^n\}$ be the solution to the difference scheme* (4.8)–(4.9). *Then it holds that*

$$F^{n+1} \leqslant F^n, \quad 0 \leqslant n \leqslant N-1, \tag{4.10}$$

*where*

$$F^n = \frac{1}{4}((\phi^n)^4, 1) + \frac{1-\epsilon}{2}\|\phi^n\|^2 - \|\nabla_h \phi^n\|^2 + \frac{1}{2}\|\Delta_h \phi^n\|^2.$$

Using the idea in [14], we can prove the uniform pointwise boundedness of the discrete solution to the PFC scheme.

**Theorem 4.2.** *Let $\phi_{ij}^n$ be the solution of the difference scheme* (4.8)–(4.9). *Then there exists a constant $c_5$ independent of $\tau$ or $h$ such that*

$$\|\phi^n\|_\infty \leqslant c_5, \quad 0 \leqslant n \leqslant N. \tag{4.11}$$

### 4.2 The solvability of the difference scheme

In this section, we consider the unique solvability of the difference scheme.

**Theorem 4.3.** *The difference scheme* (4.8)–(4.9) *is solvable if* $(\frac{25}{4}c_5^4+4)\tau \leqslant 1$.

*Proof.* Suppose that $\phi^n$ has been determined. Then difference scheme (4.8) can be written as

$$\frac{\phi_{ij}^{n+\frac{1}{2}}-\phi_{ij}^n}{\tau/2}=\Delta_h\bigg[\phi_{ij}^{n+\frac{1}{2}}\frac{(\phi_{ij}^n)^2+(2\phi_{ij}^{n+\frac{1}{2}}-\phi_{ij}^n)^2}{2}+(1-\epsilon)\phi_{ij}^{n+\frac{1}{2}}+2\Delta_h\phi_{ij}^{n+\frac{1}{2}}+\Delta_h^2\phi_{ij}^{n+\frac{1}{2}}\bigg],\quad 1\leqslant i,j\leqslant M,$$

or

$$w_{ij}-\phi_{ij}^n-\frac{\tau}{2}\Delta_h\bigg[w_{ij}\frac{(\phi_{ij}^n)^2+(2w_{ij}-\phi_{ij}^n)^2}{2}+(1-\epsilon)w_{ij}+2\Delta_h w_{ij}+\Delta_h^2 w_{ij}\bigg]=0,\quad 1\leqslant i,j\leqslant M, \tag{4.12}$$

with $w_{ij}=\phi_{ij}^{n+\frac{1}{2}}$. If $w_{ij}$ is determined, then $\phi_{ij}^{n+1}=2w_{ij}-\phi_{ij}^n$.

Define the map $g:\mathcal{V}_h\to\mathcal{V}_h$ by

$$g(w)_{ij}=w_{ij}-\phi_{ij}^n-\frac{\tau}{2}\Delta_h\bigg[w_{ij}\frac{(\phi_{ij}^n)^2+(2w_{ij}-\phi_{ij}^n)^2}{2}+(1-\epsilon)w_{ij}+2\Delta_h w_{ij}+\Delta_h^2 w_{ij}\bigg],\quad 1\leqslant i,j\leqslant M.$$

Then

$$\begin{aligned}(g(w),w)&=(w,w)-(\phi^n,w)-\frac{\tau}{2}\bigg(\Delta_h\bigg[w\frac{(\phi^n)^2+(2w-\phi^n)^2}{2}+(1-\epsilon)w+2\Delta_h w+\Delta_h^2 w\bigg],w\bigg)\\&=(w,w)-(\phi^n,w)-\frac{\tau}{2}\bigg(w\frac{(\phi^n)^2+(2w-\phi^n)^2}{2}+(1-\epsilon)w+2\Delta_h w+\Delta_h^2 w,\Delta_h w\bigg).\end{aligned}$$

For the third term on the right-hand side, we have

$$\begin{aligned}A_n&\equiv\bigg(w\frac{(\phi^n)^2+(2w-\phi^n)^2}{2}+(1-\epsilon)w+2\Delta_h w+\Delta_h^2 w,\Delta_h w\bigg)\\&=\bigg(w\frac{(\phi^n)^2+(2w-\phi^n)^2}{2},\Delta_h w\bigg)+(1-\epsilon)(w,\Delta_h w)+2(\Delta_h w,\Delta_h w)+(\Delta_h^2 w,\Delta_h w)\\&\leqslant\frac{1}{4}\bigg\|w\frac{(\phi^n)^2+(2w-\phi^n)^2}{2}\bigg\|^2+3\|\Delta_h w\|^2-(1-\epsilon)\|\nabla_h w\|^2-\|\nabla_h(\Delta_h w)\|^2.\end{aligned}$$

It follows from Theorem 4.2 that $|\phi_{ij}^n|\leqslant c_5$ and $|w_{ij}|\leqslant c_5$. Consequently,

$$\frac{(\phi^n)^2+(2w-\phi^n)^2}{2}\leqslant 5c_5^2.$$

Using Lemma 2.1 and taking $\alpha=\frac{1}{2}$, we have

$$\begin{aligned}A_n&\leqslant\frac{1}{4}\bigg\|w\frac{(\phi^n)^2+(2w-\phi^n)^2}{2}\bigg\|^2+\frac{1}{\alpha^2}\|w\|^2+2\alpha\|\nabla_h(\Delta_h w)\|^2\\&\quad-(1-\epsilon)\|\nabla_h w\|^2-\|\nabla_h(\Delta_h w)\|^2\\&\leqslant\bigg(\frac{25}{4}c_5^4+4\bigg)\|w\|^2.\end{aligned}$$

When

$$\bigg(\frac{25}{4}c_5^4+4\bigg)\tau\leqslant 1,$$

it follows that

$$(g(w),w) \geqslant \left[1-\frac{\tau}{2}\left(\frac{25}{4}c_5^4+4\right)\right]\|w\|^2-\|\phi^n\|\|w\| \\ \geqslant \frac{1}{2}\|w\|^2-\|\phi^n\|\|w\| \\ =\frac{1}{2}(\|w\|-2\|\phi^n\|)\|w\|.$$

If $\|w\| = 2\|\phi^n\|$, we have $(g(w),w) \geqslant 0$. By Lemma 2.2, there is at last one solution $w$ satisfying $\|w\| \leqslant 2\|\phi^n\|$. This completes the proof. □

Now we consider the uniqueness of the solution.

**Theorem 4.4.** *If* $\frac{1}{8}(121c_5^4+16)\tau<1$, *then the difference scheme* (4.8)–(4.9) *has at most one solution.*

*Proof.* Now suppose that (4.12) has an another solution $z_{ij}$, which satisfies

$$z_{ij}-\phi_{ij}^n-\frac{\tau}{2}\Delta_h\left[z_{ij}\frac{(\phi_{ij}^n)^2+(2z_{ij}-\phi_{ij}^n)^2}{2}+(1-\epsilon)z_{ij}+2\Delta_h z_{ij}+\Delta_h^2 z_{ij}\right]=0, \\ 1\leqslant i,j\leqslant M. \tag{4.13}$$

Let

$$\rho_{ij}=w_{ij}-z_{ij},\quad 1\leqslant i,j\leqslant M.$$

Subtracting (4.13) from (4.12), we have

$$\rho_{ij}-\frac{\tau}{2}\Delta_h\Bigg[w_{ij}\frac{(\phi_{ij}^n)^2+(2w_{ij}-\phi_{ij}^n)^2}{2}-z_{ij}\frac{(\phi_{ij}^n)^2+(2z_{ij}-\phi_{ij}^n)^2}{2} \\ +(1-\epsilon)\rho_{ij}+2\Delta_h\rho_{ij}+\Delta_h^2\rho_{ij}\Bigg]=0,\quad 1\leqslant i,j\leqslant M,$$

or,

$$\rho_{ij}-\frac{\tau}{2}\Delta_h\Bigg[\frac{w_{ij}(2w_{ij}-\phi_{ij}^n)^2-z_{ij}(2z_{ij}-\phi_{ij}^n)^2}{2}+\rho_{ij}\frac{(\phi_{ij}^n)^2}{2} \\ +(1-\epsilon)\rho_{ij}+2\Delta_h\rho_{ij}+\Delta_h^2\rho_{ij}\Bigg]=0,\quad 1\leqslant i,j\leqslant M. \tag{4.14}$$

Taking the inner product of (4.14) with $\rho$, we have

$$\|\rho\|^2=\frac{\tau}{2}\left(\frac{w(2w-\phi^n)^2-z(2z-\phi^n)^2}{2}+\rho\frac{(\phi^n)^2}{2}+(1-\epsilon)\rho+2\Delta_h\rho+\Delta_h^2\rho,\Delta_h\rho\right). \tag{4.15}$$

For the first term on the right-hand side, we have

$$w(2w-\phi^n)^2-z(2z-\phi^n)^2 \\ =(w-z)(2w-\phi^n)^2+z[(2w-\phi^n)^2-(2z-\phi^n)^2] \\ =(w-z)(2w-\phi^n)^2+4z[(w+z)(w-z)-(w-z)\phi^n] \\ =\rho(2w-\phi^n)^2+4\rho z(w+z-\phi^n).$$

Thus, we have

$$|w(2w-\phi^n)^2-z(2z-\phi^n)^2|\leqslant 21c_5^2|\rho|.$$

Using the inequality above in (4.15), we get

$$\|\rho\|^2\leqslant\frac{\tau}{2}(11c_5^2\|\rho\|\|\Delta_h\rho\|-(1-\epsilon)\|\nabla_h\rho\|^2+2\|\Delta_h\rho\|^2-\|\nabla_h(\Delta_h\rho)\|^2).$$

Applying the Cauchy inequality and Lemma 2.1 with $\alpha = \frac{1}{2}$, we obtain

$$\begin{aligned}\|\rho\|^2 &\leqslant \frac{\tau}{2}\bigg(\frac{121c_5^4}{4}\|\rho\|^2 + 3\|\Delta_h\rho\|^2 - (1-\epsilon)\|\nabla_h\rho\|^2 - \|\nabla_h(\Delta_h\rho)\|^2\bigg)\\ &\leqslant \frac{\tau}{2}\bigg(\frac{121c_5^4}{4}\|\rho\|^2 + 3\|\Delta_h\rho\|^2 - \|\nabla_h(\Delta_h\rho)\|^2\bigg)\\ &\leqslant \frac{\tau}{2}\bigg(\frac{121c_5^4}{4} + 4\bigg)\|\rho\|^2.\end{aligned}$$

When $\frac{1}{8}(121c_5^4 + 16)\tau < 1$, it follows that $\|\rho\| = 0$. Consequently, $\rho_{ij} = 0,\ 1 \leqslant i, j \leqslant M$. This completes the proof. □

Combining Theorems 4.3 and 4.4, we conclude that the difference scheme (4.8)–(4.9) has a unique solution.

### 4.3 The convergence of the difference scheme

Define the error grid functions $\tilde{\phi}^n, \tilde{u}^n$, for $0 \leqslant n \leqslant N$, on $\Omega_h$ as follows:

$$\tilde{\phi}_{ij}^n = \Phi_{ij}^n - \phi_{ij}^n, \quad \tilde{u}_{ij}^n = U_{ij}^n - u_{ij}^n, \quad 0 \leqslant i, j \leqslant M.$$

**Theorem 4.5.** *Let $\phi^n = \{\phi_{ij}^n \mid 0 \leqslant i, j \leqslant M\}$ be the solution of the difference scheme* (4.8)–(4.9)*, or, equivalently,* (4.5)–(4.7)*. Then there exists a constant $\hat{c}$ independent of the grid $h$ and $\tau$ such that*

$$\|\tilde{\phi}^n\| \leqslant \hat{c}(\tau^2 + h^2), \quad 1 \leqslant n \leqslant N. \tag{4.16}$$

*Proof* Subtracting (4.5)–(4.7) from (4.1), (4.2) and (4.4), respectively, we obtain the error system:

$$\delta_t\tilde{\phi}_{ij}^{n+\frac{1}{2}} = \Delta_h\tilde{u}_{ij}^{n+\frac{1}{2}} + P_{ij}^n, \quad 1 \leqslant i, j \leqslant M, \quad 0 \leqslant n \leqslant N-1, \tag{4.17}$$

$$\begin{aligned}\tilde{u}_{ij}^{n+\frac{1}{2}} &= \bigg[\Phi_{ij}^{n+\frac{1}{2}}\frac{(\Phi_{ij}^n)^2 + (\Phi_{ij}^{n+1})^2}{2} - \phi_{ij}^{n+\frac{1}{2}}\frac{(\phi_{ij}^n)^2 + (\phi_{ij}^{n+1})^2}{2}\bigg]\\ &\quad + (1-\epsilon)\tilde{\phi}_{ij}^{n+\frac{1}{2}} + 2\Delta_h\tilde{\phi}_{ij}^{n+\frac{1}{2}} + \Delta_h^2\tilde{\phi}_{ij}^{n+\frac{1}{2}} + Q_{ij}^n, \quad 1 \leqslant i, j \leqslant M, \quad 0 \leqslant n \leqslant N-1,\end{aligned} \tag{4.18}$$

$$\tilde{\phi}_{ij}^0 = 0, \quad 1 \leqslant i, j \leqslant M. \tag{4.19}$$

Taking the product of (4.17) with $\tilde{\phi}^{n+\frac{1}{2}}$, we have

$$\begin{aligned}(\delta_t\tilde{\phi}^{n+\frac{1}{2}}, \tilde{\phi}^{n+\frac{1}{2}}) &= (\Delta_h\tilde{u}^{n+\frac{1}{2}}, \tilde{\phi}^{n+\frac{1}{2}}) + (P^n, \tilde{\phi}^{n+\frac{1}{2}})\\ &= (\tilde{u}^{n+\frac{1}{2}}, \Delta_h\tilde{\phi}^{n+\frac{1}{2}}) + (P^n, \tilde{\phi}^{n+\frac{1}{2}}), \quad 0 \leqslant n \leqslant N-1.\end{aligned} \tag{4.20}$$

Inserting (4.18) into (4.20), we obtain

$$\begin{aligned}(\delta_t\tilde{\phi}^{n+\frac{1}{2}}, \tilde{\phi}^{n+\frac{1}{2}}) &= \bigg(\Phi^{n+\frac{1}{2}}\frac{(\Phi^n)^2 + (\Phi^{n+1})^2}{2} - \phi^{n+\frac{1}{2}}\frac{(\phi^n)^2 + (\phi^{n+1})^2}{2}, \Delta_h\tilde{\phi}^{n+\frac{1}{2}}\bigg)\\ &\quad + (1-\epsilon)(\tilde{\phi}^{n+\frac{1}{2}}, \Delta_h\tilde{\phi}^{n+\frac{1}{2}}) + 2\|\Delta_h\tilde{\phi}^{n+\frac{1}{2}}\|^2 + (\Delta_h^2\tilde{\phi}^{n+\frac{1}{2}}, \Delta_h\tilde{\phi}^{n+\frac{1}{2}})\\ &\quad + (P^n, \tilde{\phi}^{n+\frac{1}{2}}) + (Q^n, \Delta_h\tilde{\phi}^{n+\frac{1}{2}}), \quad 0 \leqslant n \leqslant N-1.\end{aligned} \tag{4.21}$$

For the left-hand side of (4.21), we have

$$(\delta_t\tilde{\phi}^{n+\frac{1}{2}}, \tilde{\phi}^{n+\frac{1}{2}}) = \frac{1}{2\tau}(\|\tilde{\phi}^{n+1}\|^2 - \|\tilde{\phi}^n\|^2).$$

For the first term on the right-hand side of (4.21), we have

$$\bigg(\Phi^{n+\frac{1}{2}}\frac{(\Phi^n)^2 + (\Phi^{n+1})^2}{2} - \phi^{n+\frac{1}{2}}\frac{(\phi^n)^2 + (\phi^{n+1})^2}{2}, \Delta_h\tilde{\phi}^{n+\frac{1}{2}}\bigg)$$

$$
\begin{aligned}
&= \bigg(\Phi^{n+\frac{1}{2}}\bigg[\frac{(\Phi^{n+1})^2-(\phi^{n+1})^2}{2}+\frac{(\Phi^{n})^2-(\phi^{n})^2}{2}\bigg]\\
&\quad +(\Phi^{n+\frac{1}{2}}-\phi^{n+\frac{1}{2}})\frac{(\phi^{n})^2+(\phi^{n+1})^2}{2},\Delta_h\tilde{\phi}^{n+\frac{1}{2}}\bigg)\\
&= \bigg(\Phi^{n+\frac{1}{2}}\bigg[\frac{\Phi^{n+1}+\phi^{n+1}}{2}\tilde{\phi}^{n+1}+\frac{\Phi^{n}+\phi^{n}}{2}\tilde{\phi}^{n}\bigg]+\frac{(\phi^{n})^2+(\phi^{n+1})^2}{2}\tilde{\phi}^{n+\frac{1}{2}},\Delta_h\tilde{\phi}^{n+\frac{1}{2}}\bigg)\\
&\leqslant \frac{1}{2}[(c_5+c_2)c_2+c_5^2](\|\tilde{\phi}^{n+1}\|+\|\tilde{\phi}^{n}\|)\|\Delta_h\tilde{\phi}^{n+\frac{1}{2}}\|\\
&\leqslant \frac{1}{8}[(c_5+c_2)c_2+c_5^2]^2(\|\tilde{\phi}^{n+1}\|+\|\tilde{\phi}^{n}\|)^2+\frac{1}{2}\|\Delta_h\tilde{\phi}^{n+\frac{1}{2}}\|^2.
\end{aligned}
$$

For the second term of the right-hand side of (4.21), we have

$$(1-\epsilon)(\tilde{\phi}^{n+\frac{1}{2}},\Delta_h\tilde{\phi}^{n+\frac{1}{2}})=-(1-\epsilon)\|\nabla_h\tilde{\phi}^{n+\frac{1}{2}}\|^2.$$

For the forth term of the right-hand side of (4.21), we have

$$(\Delta_h^2\tilde{\phi}^{n+\frac{1}{2}},\Delta_h\tilde{\phi}^{n+\frac{1}{2}})=-\|\nabla_h(\Delta_h\tilde{\phi}^{n+\frac{1}{2}})\|^2.$$

For the fifth term and the sixth term of the right-hand side of (4.21), we have

$$
\begin{aligned}
&(Q^n,\Delta_h\tilde{\phi}^{n+\frac{1}{2}})\leqslant\frac{1}{2}\|Q^n\|^2+\frac{1}{2}\|\Delta_h\tilde{\phi}^{n+\frac{1}{2}}\|^2,\\
&(P^n,\tilde{\phi}^{n+\frac{1}{2}})\leqslant\frac{1}{2}\|P^n\|^2+\frac{1}{2}\|\tilde{\phi}^{n+\frac{1}{2}}\|^2.
\end{aligned}
$$

Inserting the expressions above to (4.21) and using Lemma 2.1 with $\alpha=\frac{1}{2}$, we obtain

$$
\begin{aligned}
&\frac{1}{2\tau}(\|\tilde{\phi}^{n+1}\|^2-\|\tilde{\phi}^{n}\|^2)\\
&\quad\leqslant \frac{1}{8}[(c_5+c_2)c_2+c_5^2]^2(\|\tilde{\phi}^{n+1}\|+\|\tilde{\phi}^{n}\|)^2+\frac{1}{2}\|\Delta_h\tilde{\phi}^{n+\frac{1}{2}}\|^2\\
&\qquad-(1-\epsilon)\|\nabla_h\tilde{\phi}^{n+\frac{1}{2}}\|^2+2\|\Delta_h\tilde{\phi}^{n+\frac{1}{2}}\|^2-\|\nabla_h(\Delta_h\tilde{\phi}^{n+\frac{1}{2}})\|^2\\
&\qquad+\frac{1}{2}\|Q^n\|^2+\frac{1}{2}\|\Delta_h\tilde{\phi}^{n+\frac{1}{2}}\|^2+\frac{1}{2}\|P^n\|^2+\frac{1}{2}\|\tilde{\phi}^{n+\frac{1}{2}}\|^2\\
&\quad\leqslant \frac{1}{8}(c_5^2+c_5c_2+c_2^2)^2(\|\tilde{\phi}^{n+1}\|+\|\tilde{\phi}^{n}\|)^2+\frac{1}{2}\|\tilde{\phi}^{n+\frac{1}{2}}\|^2\\
&\qquad+3\|\Delta_h\tilde{\phi}^{n+\frac{1}{2}}\|^2-\|\nabla_h(\Delta_h\tilde{\phi}^{n+\frac{1}{2}})\|^2+\frac{1}{2}(\|P^n\|^2+\|Q^n\|^2)\\
&\quad\leqslant \frac{1}{8}(c_5^2+c_5c_2+c_2^2)^2(\|\tilde{\phi}^{n+1}\|+\|\tilde{\phi}^{n}\|)^2+\frac{1}{2}\|\tilde{\phi}^{n+\frac{1}{2}}\|^2+4\|\tilde{\phi}^{n+\frac{1}{2}}\|^2+\frac{1}{2}(\|P^n\|^2+\|Q^n\|^2)\\
&\quad\leqslant \frac{1}{4}[(c_5^2+c_5c_2+c_2^2)^2+9](\|\tilde{\phi}^{n+1}\|^2+\|\tilde{\phi}^{n}\|^2)+\frac{1}{2}(\|P^n\|^2+\|Q^n\|^2).
\end{aligned}
$$

Noticing (4.3), we have

$$
\begin{aligned}
&\bigg\{1-\frac{1}{2}[(c_5^2+c_5c_2+c_2^2)^2+9]\tau\bigg\}\|\tilde{\phi}^{n+1}\|^2\\
&\quad\leqslant\bigg\{1+\frac{1}{2}[(c_5^2+c_5c_2+c_2^2)^2+9]\tau\bigg\}\|\tilde{\phi}^{n}\|^2+8\pi^2c_4^2\tau(\tau^2+h^2)^2.
\end{aligned}
$$

When

$$\frac{1}{2}[(c_5^2+c_5c_2+c_2^2)^2+9]\tau\leqslant\frac{1}{3},$$

it follows that

$$\|\tilde{\phi}^{n+1}\|^2\leqslant\bigg\{1+\frac{3}{2}[(c_5^2+c_5c_2+c_2^2)^2+9]\tau\bigg\}\|\tilde{\phi}^{n}\|^2+12\pi^2c_4^2\tau(\tau^2+h^2)^2,\quad 0\leqslant n\leqslant N-1.$$

The Grönwall's lemma (see Lemma 2.3) yields

$$\|\tilde{\phi}^{n+1}\|^2 \leqslant \frac{8\pi^2 c_4^2}{(c_5^2+c_5c_2+c_2^2)^2+9}\exp\left\{\frac{3}{2}\big[(c_5^2+c_5c_2+c_2^2)^2+9\big]T\right\}\cdot(\tau^2+h^2)^2,$$
$$0\leqslant n\leqslant N-1.$$

The proof of the theorem is completed. □

## 5 Numerical examples

Consider (1.1)–(1.3) with $T=0.5, \epsilon=0.025$, $\Omega=[0,32]\times[0,32]$, and initial condition $\psi(x,y)=0.5\sin(\frac{2\pi x}{32})\sin(\frac{2\pi y}{32})$ and the periodic boundary condition as in [15].

Firstly, we compute the numerical solutions to this problem by the difference scheme (3.13)–(3.15).

Denote the solution of the difference scheme (3.13)–(3.15) with the step sizes $(h,\tau)$ by $u_{ij}^n(h,\tau)$. Let

$$\begin{aligned}
N_m &= \{(p,q)\mid |p|+|q|\leqslant m\},\\
N_m^1 &= \{(p,q)\mid (p,q)\in N_m; (p,q) \text{ with } q=-m,-m+1,\ldots,-1,\\
&\qquad \text{or, } (p,q) \text{ with } -m\leqslant p\leqslant -1, q=0\},\\
N_m^2 &= \{(p,q)\mid (p,q)\in N_m; (p,q) \text{ with } q=1,2,\ldots,m, \text{ or, } (p,q) \text{ with } 1\leqslant p\leqslant m, q=0\}.
\end{aligned}$$

At each time level, the difference scheme (3.13)–(3.15) can be written as

$$\sum_{(p,q)\in N_3} c_{p,q}\phi_{i+p,j+q}=f_{ij},\quad 1\leqslant i,j\leqslant M.$$

We solve the system of the above linear algebraic equation by the Gauss-Seidel iterative method: For $k=0,1,2,\ldots,$

$$\phi_{i,j}^{(k+1)}=\left[f_{i,j}-\sum_{(p,q)\in N_3^1}c_{p,q}\phi_{i+p,j+q}^{(k+1)}-\sum_{(p,q)\in N_3^2}c_{p,q}\phi_{i+p,j+q}^{(k)}\right]\Big/ c_{0,0},\quad 1\leqslant i,j\leqslant M.$$

The tolerance of the Gauss Seidel iteration is set to be $10^{-12}$. The initial guess at each time step is taken as the numerical solution at the previous time level.

Suppose $h$ and $\tau$ are sufficiently small. Denote

$$H_\infty(h,\tau)=\max_{0\leqslant k\leqslant N}\max_{0\leqslant i,j\leqslant M}\left|u_{i,j}^k(h,\tau)-u_{2i,2j}^{2k}\left(\frac{h}{2},\frac{\tau}{2}\right)\right|,$$

and

$$\text{order}=\log_2\frac{H_\infty(2h,2\tau)}{H_\infty(h,\tau)}.$$

Some numerical results are presented in Table 1. From this table, we see that the convergence order of our scheme (3.13)–(3.15) is $O(\tau^2+h^2)$.

Secondly, we plot the energy evolution pictures by different constant steps. Figure 1 shows the decrease of the discrete energy $G_h(\phi^n,\phi^{n+1})$ defined as Theorem 3.1. This is accordance with Theorem 3.1. Next, we show the solution contours at different time steps in Figure 2, where the horizontal axis and the vertical axis represent $x$ and $y$, respectively, and the mesh grid is set as $100\times 100$.

Finally, we compare our scheme (3.13)–(3.15) with the scheme (1.8)–(1.10) (see [15]) and the scheme (1.11)–(1.13) (see [9]).

At each time level, both schemes (1.8)–(1.10) and (1.11)–(1.13) are systems of nonlinear equations, which can be written as

$$\sum_{(p,q)\in N_3}c_{p,q}\phi_{i+p,j+q}+\sum_{(p,q)\in N_1}[d_{p,q}\phi_{i+p,j+q}^2+e_{p,q}\phi_{i+p,j+q}^3]=f_{ij},\quad 1\leqslant i,j\leqslant M.$$

**Table 1** The maximum norm errors and convergence orders of our scheme (3.13)–(3.15)

| $h$ | $\tau$ | $H_\infty(h,\tau)$ | order |
|---|---|---|---|
| 1/10 | 1/10 | 1.2329e−3 | * |
| 1/20 | 1/20 | 3.3400e−4 | 1.8842 |
| 1/40 | 1/40 | 8.5491e−5 | 1.9660 |
| 1/80 | 1/80 | 2.1755e−5 | 1.9744 |

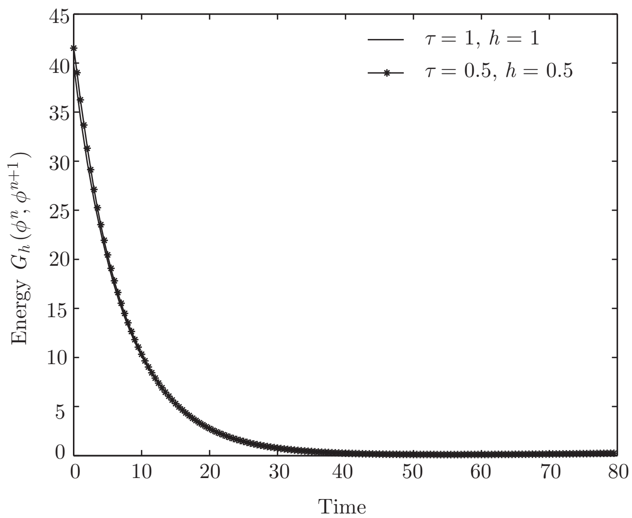


**Figure 1** The discrete energy of the numerical solution obtained by the scheme (3.13)–(3.15) with different constant time-space steps

We adopt the following quasi-Gauss Seidel iteration method (QGS): For $k = 0, 1, 2, \dots,$

$$\begin{aligned}
&c_{p,q}\phi_{i,j}^{(k+1)} + d_{0,0}(\phi_{i,j}^{(k+1)})^2 + e_{p,q}(\phi_{i,j}^{(k+1)})^3\\
&\quad = f_{ij} - \sum_{(p,q)\in N_3^1} c_{p,q}(\phi_{i+p,j+q}^{(k+1)}) - \sum_{(p,q)\in N_3^2} c_{p,q}(\phi_{i+p,j+q}^{(k)})\\
&\quad\quad - \sum_{(p,q)\in N_1^1} [d_{p,q}(\phi_{i+p,j+q}^{(k+1)})^2 + e_{p,q}(\phi_{i+p,j+q}^{(k+1)})^3]\\
&\quad\quad - \sum_{(p,q)\in N_1^2} [d_{p,q}(\phi_{i+p,j+q}^{(k)})^2 + e_{p,q}(\phi_{i+p,j+q}^{(k)})^3].
\end{aligned}$$

The tolerance of the quasi-Gauss Seidel iteration is set to be $10^{-12}$, and the initial guess at each time step is also taken as the numerical solution at the previous time level. We use the Newton iteration method to solve $\phi_{i,j}^{(k+1)}$, with the tolerance of $10^{-12}$. It needs lots of CPU time. Thus, we propose the following modified quasi-Gauss Seidel iteration (MQGS) to solve the systems of nonlinear equations of the schemes (1.8)–(1.10) and (1.11)–(1.13):

For $k = 0, 1, 2, \dots,$

$$\begin{aligned}
c_{p,q}\phi_{i,j}^{(k+1)} &= f_{ij} - \sum_{(p,q)\in N_3^1} c_{p,q}\phi_{i+p,j+q}^{(k+1)} - \sum_{(p,q)\in N_3^2} c_{p,q}\phi_{i+p,j+q}^{(k)}\\
&\quad - \sum_{(p,q)\in N_1^1} [d_{p,q}(\phi_{i+p,j+q}^{(k+1)})^2 + e_{p,q}(\phi_{i+p,j+q}^{(k+1)})^3]\\
&\quad - \sum_{(p,q)\in N_1^2\cup N_0} [d_{p,q}(\phi_{i+p,j+q}^{(k)})^2 + e_{p,q}(\phi_{i+p,j+q}^{(k)})^3]. \qquad (5.1)
\end{aligned}$$

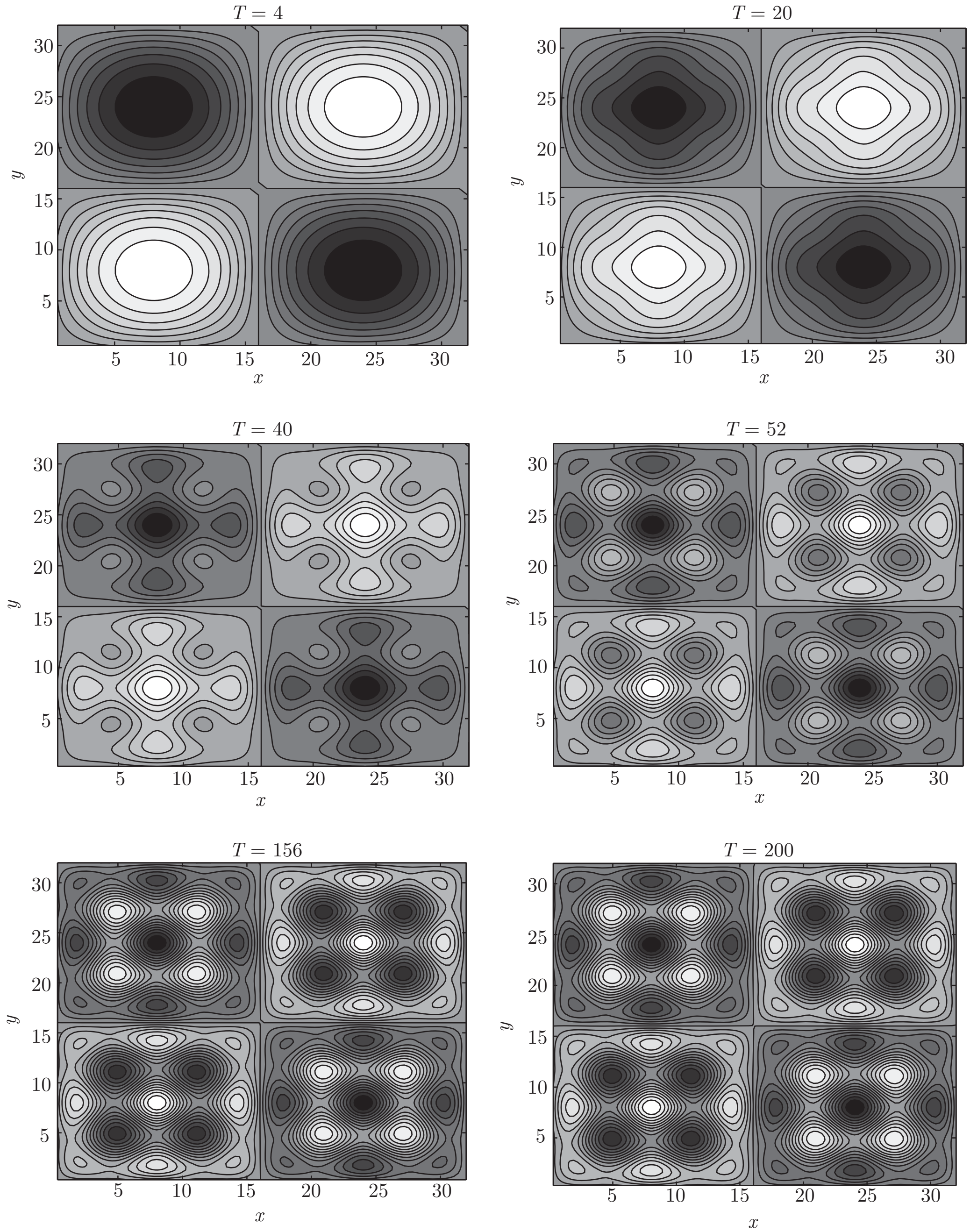


**Figure 2** The contours of the numerical solutions at different time with the same temporal grid numbers $N = 100$

**Table 2** The maximum norm errors and convergence orders of the difference scheme (1.8)–(1.10)

| $h$ | $\tau$ | $H_\infty(h,\tau)$ with QGS | order | $H_\infty(h,\tau)$ with MQGS | order |
|---|---|---|---|---|---|
| 1/10 | 1/10 | 1.2329e−3 | * | 1.2329e−3 | * |
| 1/20 | 1/20 | 3.3399e−4 | 1.8842 | 3.3399e−4 | 1.8842 |
| 1/40 | 1/40 | 8.5483e−5 | 1.9661 | 8.5483e−5 | 1.9661 |
| 1/80 | 1/80 | 2.1408e−5 | 1.9975 | 2.1408e−5 | 1.9975 |

However, the modified quasi-Gauss Seidel iteration is also time-consuming.

From Tables 2 and 3, we see that the convergence orders of those schemes are nearly $O(\tau^2 + h^2)$. From Table 4, we see that those schemes need more CPU time compared with our scheme.

**Table 3** The maximum norm errors and convergence orders of the (1.11)–(1.13)

| $h$ | $\tau$ | $H_\infty(h,\tau)$ with QGS | order | $H_\infty(h,\tau)$ with MQGS | order |
|---|---|---|---|---|---|
| 1/10 | 1/10 | 3.6044e−4 | * | 3.6044e−4 | * |
| 1/20 | 1/20 | 9.3916e−5 | 1.9403 | 9.3916e−5 | 1.9403 |
| 1/40 | 1/40 | 2.4769e−5 | 1.9228 | 2.4769e−5 | 1.9228 |
| 1/80 | 1/80 | 6.4775e−6 | 1.9350 | 6.4775e−6 | 1.9350 |

**Table 4** CPU time of (3.13)–(3.15), (1.8)–(1.10) with QGS, (1.8)–(1.10) with MQGS, (1.11)–(1.13) with QGS and (1.11)–(1.13) with MQGS

| $h$ | $\tau$ | (3.13)–(3.15) | (1.8)–(1.10) with QGS | (1.8)–(1.10) with MQGS | (1.11)–(1.13) with QGS | (1.11)–(1.13) with MQGS |
|---|---|---|---|---|---|---|
| 1/10 | 1/10 | 1.9753e−1 | 3.9588e−1 | 3.1301e−1 | 3.4651e−1 | 3.0890e−1 |
| 1/20 | 1/20 | 3.5283e+0 | 1.1034e+1 | 1.0483e+1 | 8.2384e+0 | 7.2819e+0 |
| 1/40 | 1/40 | 4.9852e+2 | 1.6544e+3 | 1.5770e+3 | 1.1379e+3 | 1.0092e+3 |
| 1/80 | 1/80 | 5.5163e+4 | 3.1962e+5 | 3.0499e+5 | 2.0593e+5 | 2.0571e+5 |

## 6 Conclusion

In this paper, we have developed a three-level linearized difference scheme for the phase crystal equation. The energy stability, unique solvability and second order global convergence both in time and in space in $L_2$ norm were strictly proved. In the second part of this work, we theoretically analyzed a two-level nonlinear difference scheme for the phase crystal equation, which was developed by Zhang et al. [15]. It was proved that the difference scheme is uniquely solvable by the Brouwer fixed-point theorem and second order global convergent in $L_2$ norm by the energy method. Our analysis method may be applicable to the modified phase field crystal model.

Li et al. [10] proposed a three-level linearized difference scheme for the Cahn-Hilliard equation and showed that the difference scheme converges in maximum norm. The phase crystal equation is a sixth order nonlinear partial differential equation. We only proved that the developed difference scheme is convergent in $L_2$ norm. It is our future work to prove that the difference scheme converges in maximum norm.

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