SCIENCE CHINA
Mathematics

• ARTICLES •

December 2015 Vol. 58 No. 12: 2671–2682
doi: 10.1007/s11425-015-5022-4

Semisymmetric graphs admitting primitive groups of degree $9p$

HAN Hua[1,2] & LU ZaiPing[2,*]

[1] *College of Science, Tianjin University of Technology, Tianjin 300384, China;*
[2] *Center for Combinatorics, LPMC-TJKLC, Nankai University, Tianjin 300071, China*
Email: hh1204@mail.nankai.edu.cn, lu@nankai.edu.cn

Received December 27, 2013; accepted November 17, 2014; published online May 15, 2015

Abstract Let Γ be a connected regular bipartite graph of order $18p$, where p is a prime. Assume that Γ admits a group acting primitively on one of the bipartition subsets of Γ. Then, in this paper, it is shown that either Γ is arc-transitive, or Γ is isomorphic to one of 17 semisymmetric graphs which are constructed from primitive groups of degree $9p$.

Keywords edge-transitive graph, arc-transitive graph, semisymmetric graph, primitive permutation group, suborbit

MSC(2010) 05C25, 20B25

Citation: Han H, Lu Z P. Semisymmetric graphs admitting primitive groups of degree $9p$. Sci China Math, 2015, 58: 2671–2682, doi: 10.1007/s11425-015-5022-4

1 Introduction

All graphs in this paper are assumed to be finite, simple and undirected.

For a graph Γ, we use $V\Gamma$, $E\Gamma$ and $\mathrm{Aut}\Gamma$ to denote its vertex set, edge set and automorphism group, respectively. A graph Γ is said to be *vertex-transitive* or *edge-transitive* if $\mathrm{Aut}\Gamma$ acts transitively on $V\Gamma$ or $E\Gamma$, respectively. A regular edge-transitive graph is called *semisymmetric* if it is not vertex-transitive. An *arc* in a graph Γ is an ordered pair of adjacent vertices. A graph Γ is said to be *arc-transitive* if $\mathrm{Aut}\Gamma$ acts transitively on the set of arcs in Γ.

The class of semisymmetric graphs was first systematically studied by Folkman [10]. Afterwards, many authors have done much work on this topic, see [1, 2, 4, 15, 24–26] for references. In particular, lots of interesting examples of such graphs were found. For example, the Folkman graph on 20 vertices, the smallest semisymmetric graph, was constructed by Folkman [10]; the Gray graph **S54**, a cubic graph of order 54, was first observed to be semisymmetric by Bouwer [1] and proved to be the smallest cubic semisymmetric graph by Malnič et al. [22]. In 1985, Iofinova and Ivanov [15] classified all bi-primitive cubic semisymmetric graphs, they proved that there are only five such graphs. Tutte's 12-cage **S126** is one of those graphs, which is the unique cubic semisymmetric graph on 126 vertices and is the fifth smallest cubic semisymmetric graph, see [4]. The reader may consult [7–9, 12, 16, 21, 23] for more examples of semisymmetric graphs.

A recent work of Han and Lu [13] suggested a feasible construction of semisymmetric graphs from primitive permutation groups. In practice, it is plausible to consider the semisymmetric graphs associated

*Corresponding author

with those primitive permutation groups of special types or degrees. Let p be a prime and let k be a positive integer less than p. Then all primitive permutation groups of degree kp are explicitly known, see [19]. This inspires us to consider the classification problem about semisymmetric graphs (of order $2kp$) arising from primitive permutation groups of degree kp, where k is a composite number. (Note that a classification was given in [8] for the semisymmetric graphs of order $2pq$, where p and q are distinct primes.) As an attempt towards the mentioned problem, we deal with in this paper the case where $k = 9$.

Let Γ be a connected regular bipartite graph of order $18p$. Assume that Γ admits a group acting transitively on $E\Gamma$ and primitively on one of the bipartition subsets of Γ. We shall prove that either Γ is arc-transitive, or Γ is isomorphic to one of 17 semisymmetric graphs. These 17 semisymmetric graphs are either unworthy [30] or constructed from the distance partitions of several known graphs.

Let Γ be a connected graph with diameter d. For an integer $0 \leqslant i \leqslant d$, the *distance i graph*, denoted by $\partial_i(\Gamma)$, is defined as the graph on $V\Gamma$ such that two vertices are adjacent if and only if they are at distance i from each other in Γ.

Our main result is stated as follows.

Theorem 1.1. *Let Γ be a connected regular graph of order $18p$, where p is a prime. Assume that a subgroup $G \leqslant \mathrm{Aut}\Gamma$ acts transitively on $E\Gamma$ but not on $V\Gamma$. If G acts primitively on one of G-orbits on $V\Gamma$, then Γ is either arc-transitive or isomorphic to one of the following semisymmetric graphs*:

(1) *Six graphs of order 54: the Gray graph* $\mathbf{S54}$, $\partial_3(\mathbf{S54})$, $\partial_5(\mathbf{S54})$, *the graph* Γ_1 *defined in Example 3.1, and the graphs* $\Sigma_0^{1,3}$ *and* $\Sigma_1^{1,3}$ *defined in Example 3.7.*

(2) *Three graphs of order 126: Tutte's 12-cage* $\mathbf{S126}$, $\partial_3(\mathbf{S126})$ *and* $\partial_5(\mathbf{S126})$.

(3) *Eight graphs of order 342: the graphs* $\Lambda_1^{1,9}$ *and* $\Lambda_2^{1,9}$ *defined in Example 3.8, and the graphs* $\Pi_i^{1,3}$ $(1 \leqslant i \leqslant 6)$ *defined in Example 3.10.*

2 Preliminaries

Let Γ be a graph and let $G \leqslant \mathrm{Aut}\Gamma$. The graph Γ is called *G-vertex-transitive*, *G-edge-transitive* or *G-arc-transitive* if G acts transitively on its vertex set, edge set or arc set, respectively. The graph Γ is called a *G-semisymmetric* graph if it is regular, G-edge-transitive but not G-vertex-transitive.

Assume that Γ is a G-edge-transitive but not G-vertex-transitive graph, where $G \leqslant \mathrm{Aut}\Gamma$. Then Γ is a bipartite graph with bipartition subsets being the G-orbits on $V\Gamma$. It follows that the vertices in a same bipartition subset of Γ have the same valency. For convenience, we call Γ an *$\{l, r\}$-semiregular* graph if the vertices in one of the bipartition subsets have valency l and the other vertices have valency r. For a given vertex $u \in V\Gamma$, denote by $\Gamma(u)$ the neighborhood of u, i.e., the set of vertices adjacent to u in Γ. Then the vertex-stabilizer G_u acts transitively on $\Gamma(u)$. Take $w \in \Gamma(u)$. Then each vertex of Γ can be written as u^g or w^h for some $g, h \in G$. Then, for two arbitrary vertices u^g and w^h, they are adjacent in Γ if and only if u and $w^{hg^{-1}}$ are adjacent, i.e., $hg^{-1} \in G_wG_u$. Moreover, it is well known and easily shown that Γ is connected if and only if $\langle G_u, G_w\rangle = G$.

Let Γ be a G-semisymmetric graph with bipartition $\{U, W\}$. Suppose that G has a subgroup R which is regular on both U and W. Take an edge $\{u, w\} \in E\Gamma$ with $u \in U$ and $w \in W$. Then each vertex in U (W, resp.) can be written uniquely as u^x (w^x, resp.) for some $x \in R$. Set $S = \{s \in R \mid w^s \in \Gamma(u)\}$. Then u^x and w^y are adjacent if and only if $yx^{-1} \in S$. If R is abelian, then it is easily shown that $u^x \mapsto w^{x^{-1}}, w^x \mapsto u^{x^{-1}}$, $\forall\, x \in R$ is an automorphism of Γ, which leads to the vertex-transitivity of Γ, refer to [8, 20].

Lemma 2.1. *Let Γ be a G-semisymmetric graph with bipartition $\{U, W\}$. Assume that G has an abelian subgroup which is regular on both U and W. Then Γ is arc-transitive.*

Let Γ be a G-semisymmetric graph. Suppose that G has a normal subgroup N which acts intransitively on at least one of the bipartition subsets of Γ. Then we define the *quotient graph* Γ_N to have vertices the N-orbits on $V\Gamma$, and two N-orbits B and B' are adjacent in Γ_N if and only if some $v \in B$ and some $v' \in B'$ are adjacent in Γ. It is easy to see that G induces an edge-transitive subgroup of $\mathrm{Aut}\Gamma_N$.

Let Γ be a connected G-semisymmetric graph with $G \leqslant \mathrm{Aut}\Gamma$. Denote by $\mathrm{soc}(G)$ the subgroup generated by all minimal normal subgroups of G, which is called the *socle* of G. Take an edge $\{u, w\} \in E\Gamma$ and let $U = u^G$ and $W = w^G$ be the G-orbits on $V\Gamma$. Denote respectively by G^U and G^W the restrictions of G on U and on W. The next lemma is quoted from [13].

Lemma 2.2. *Let Γ be a connected G-semisymmetric graph with bipartition $\{U, W\}$, where $G \leqslant \mathrm{Aut}\Gamma$. Assume that G^U is quasiprimitive, i.e., each minimal normal subgroup of G^U is transitive on U. Then one of the following statements hold*:

(1) Γ *is isomorphic to the complete bipartite graph* $\boldsymbol{K}_{|U|,|U|}$;

(2) G *is faithful on both U and W, and if G^U is of affine type then Γ is semisymmetric if and only if* $\mathrm{soc}(G)$ *is intransitive on W*;

(3) G *is faithful on W but not faithful on U, $G/K \cong G^U \cong G^{\bar{W}}$, and Γ is semisymmetric if further G^U is primitive, where K is the kernel of G acting on U and $\bar{W}$ is the set of K-orbits on W.*

Let G be a finite transitive permutation group on a set Ω. The orbits of G on the cartesian product $\Omega\times\Omega$ are the *orbitals* of G, and the diagonal orbital $\{(\alpha,\alpha)^g \mid g \in G\}$ is said to be *trivial*. For a G-orbital Δ and $\alpha \in \Omega$, the set $\Delta(\alpha) = \{\beta \mid (\alpha, \beta) \in \Delta\}$ is a G_α-orbit on Ω and called a *suborbit* of G at α. The *rank* of G on Ω is the number of G-orbitals, which equals to the number of G_α-orbits on Ω for any given $\alpha \in \Omega$. For a G-orbital Δ, the *paired orbital* Δ^* is defined as $\{(\beta, \alpha) \mid (\alpha, \beta) \in \Delta\}$, and Δ is said to be *self-paired* if $\Delta^* = \Delta$. For a self-paired G-orbital Δ, the suborbit $\Delta(\alpha)$ is called *self-paired*. For a non-trivial G-orbital Δ, the *orbital bipartite graph* $B(\Omega, \Delta)$ is the graph on two copies of Ω, say $\Omega\times\{1, 2\}$, such that $\{(\alpha, 1), (\beta, 2)\}$ is an edge if and only if $(\alpha, \beta) \in \Delta$. Then $B(\Omega, \Delta)$ is G-semisymmetric, where G acts on $\Omega\times\{1, 2\}$ as follows:

$$(\alpha, i)^g = (\alpha^g, i), \quad g \in G, \quad i = 1, 2.$$

If Δ is self-paired, then $(\alpha, 1) \leftrightarrow (\alpha, 2)$, $\alpha \in \Omega$ gives an automorphism of $B(\Omega, \Delta)$, which yields that $B(\Omega, \Delta)$ is G-arc-transitive. Moreover, the next lemma is easily shown, see also [11].

Lemma 2.3. *Assume that Γ is a connected G-semisymmetric graph of valency at least 2 with bipartition subsets U and W, and that, for an edge $\{u, w\} \in E\Gamma$, the two stabilizers G_u and G_w are conjugate in G. Then there is a bijection $\iota : U \leftrightarrow W$ such that $G_u = G_{\iota(u)}$ and $\{u, \iota(u)\} \notin E\Gamma$ for all $u \in U$. Moreover,*

$$\Delta = \{(u, \iota^{-1}(w)) \mid \{u, w\} \in E\Gamma, u \in U, w \in W\}$$

is a G-orbital on U. In particular, $\Gamma \cong B(U, \Delta)$, and ι extends to an automorphism of Γ if and only if Δ is self-paired.

Remark 2.4. Let Γ and $G \leqslant \mathrm{Aut}\Gamma$ be as in Lemma 2.3. Then $\{G_u \mid u \in U\} = \{G_w \mid w \in W\}$, and so $\bigcap_{u\in U} G_u = \bigcap_{w\in W} G_w = 1$ as $G \leqslant \mathrm{Aut}\Gamma$. Thus G is faithful on both parts of Γ. Take $u \in U$ and $w \in W$ with $G_u = G_w$. Then $u^g \leftrightarrow w^g$, $g \in G$ gives a bijection meeting the requirement of Lemma 2.3. Thus one can define l^2 bijections ι, where l is the number of the points in U fixed by a stabilizer G_u. By [6, Theorem 4.2A], $l = |\boldsymbol{N}_G(G_u) : G_u|$.

Let G be a finite transitive permutation group on Ω and Δ be a G-orbital. If Δ is self-paired, then $B(\Omega, \Delta)$ is arc-transitive. The next lemma indicates it is possible that $B(\Omega, \Delta)$ is arc-transitive even if Δ is not self-paired.

Lemma 2.5. *Let X be a permutation group on Ω and let G be a transitive subgroup of X with index $|X : G| = 2$. Let Δ be a G-orbital. If $\Delta \cup \Delta^*$ is an X-orbital, then $B(\Omega, \Delta)$ is arc-transitive.*

Proof. Assume that $\Delta \cup \Delta^*$ is an X-orbital. To show $\Gamma := B(\Omega, \Delta)$ is arc-transitive, it suffices to find an automorphism of Γ which interchanges two bipartition subsets of Γ. Take $x \in X \setminus G$. It is easily shown that $\Delta^x = \Delta^*$ and $(\Delta^*)^x = \Delta$. Define $\hat{x} : \Omega\times\{1, 2\} \to \Omega\times\{1, 2\}$, $(\alpha, 1) \mapsto (\alpha^x, 2)$, $(\beta, 2) \mapsto (\beta^x, 1)$. It is easy to check that $\hat{x} \in \mathrm{Aut}\Gamma$, and so the lemma follows. □

The next result is a special version of [8, Lemma 2.6].

Lemma 2.6. *Let Γ be a G-semisymmetric graph with bipartition $\{U, W\}$. Assume that G has an automorphism σ of order 2 such that $G_u^\sigma = G_w$ for some $u \in U$ and $w \in W$. If all G_u-orbits on W have distinct lengths, then Γ is arc-transitive.*

3 Some semisymmetric graphs of order 18p

In this section, we construct the semisymmetric graphs involved in Theorem 1.1.

We first give several semisymmetric graphs arising from the distance partitions of **S54** and **S126**. In particular, we shall show that $\partial_3(\mathbf{S54})$, $\partial_5(\mathbf{S54})$, $\partial_3(\mathbf{S126})$ and $\partial_5(\mathbf{S126})$ are (non-isomorphic) semisymmetric graphs.

For a prime power q and a positive integer d, we denote respectively by $\mathbb{F}_q$ and $\mathbb{F}_q^d$ the field of order q and the d-dimensional vector space over $\mathbb{F}_q$.

Example 3.1. Let $U = \mathbb{F}_3^3$ and $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ a basis of U. Let $W = \{\boldsymbol{y} + \langle \boldsymbol{e}_i \rangle \mid \boldsymbol{y} \in U, 1 \leqslant i \leqslant 3\}$, which consists of 27 1-dimensional affine subspaces of U. Take 5 subsets of W as follows:

$$
\begin{aligned}
&\Delta_0 := \{\langle \boldsymbol{e}_i \rangle \mid 1 \leqslant i \leqslant 3\},\\
&\Delta_1 := \{\pm \boldsymbol{e}_2 + \langle \boldsymbol{e}_1 \rangle, \pm \boldsymbol{e}_3 + \langle \boldsymbol{e}_2 \rangle, \pm \boldsymbol{e}_1 + \langle \boldsymbol{e}_3 \rangle\},\\
&\Delta_2 := \{\pm \boldsymbol{e}_3 + \langle \boldsymbol{e}_1 \rangle, \pm \boldsymbol{e}_1 + \langle \boldsymbol{e}_2 \rangle, \pm \boldsymbol{e}_2 + \langle \boldsymbol{e}_3 \rangle\},\\
&\Delta_3 := \{\pm \boldsymbol{e}_i \pm \boldsymbol{e}_j + \langle \boldsymbol{e}_k \rangle \mid \{i, j, k\} = \{1, 2, 3\}\},\\
&\Delta_4 := \Delta_1 \cup \Delta_2.
\end{aligned}
$$

For each s with $0 \leqslant s \leqslant 4$, define a bipartite graph Γ_s with bipartition $\{U, W\}$ such that $\boldsymbol{x} \in U$ and $\boldsymbol{y} + \langle \boldsymbol{e}_i \rangle \in W$ are adjacent in Γ_s if and only if $\boldsymbol{y} - \boldsymbol{x} + \langle \boldsymbol{e}_i \rangle \in \Delta_s$. Clearly, these graphs have valency 3, 6, 6, 12 and 12, respectively. Moreover, Γ_4 is the edge-disjoint union of Γ_1 and Γ_2, and it is easy to check that $\Gamma_3 = \partial_5(\Gamma_0)$ and $\Gamma_4 = \partial_3(\Gamma_0)$.

Lemma 3.2. *The graphs given in Example* 3.1 *are all semisymmetric. Moreover,* $\Gamma_1 \cong \Gamma_2$, $\Gamma_0 \cong \mathbf{S54}$ *and* $\Gamma_3 \ncong \Gamma_4$.

Proof. We continue the notation used in Example 3.1. Take $h_0, h_1, h_2 \in \mathrm{GL}(3, 3)$ such that

$$
\begin{aligned}
&\boldsymbol{e}_1^{h_0} = \boldsymbol{e}_2, \quad &&\boldsymbol{e}_2^{h_0} = \boldsymbol{e}_3, \quad &&\boldsymbol{e}_3^{h_0} = \boldsymbol{e}_1,\\
&\boldsymbol{e}_1^{h_1} = -\boldsymbol{e}_1, \quad &&\boldsymbol{e}_2^{h_1} = \boldsymbol{e}_2, \quad &&\boldsymbol{e}_3^{h_1} = \boldsymbol{e}_3,\\
&\boldsymbol{e}_1^{h_2} = \boldsymbol{e}_1, \quad &&\boldsymbol{e}_2^{h_2} = \boldsymbol{e}_3, \quad &&\boldsymbol{e}_3^{h_2} = \boldsymbol{e}_2.
\end{aligned}
\tag{3.1}
$$

Set $H = \langle h_1, h_2, h_0 \rangle$ and $H_1 = \langle h_1, h_0 \rangle$. Then both H and H_1 are irreducible subgroups of $\mathrm{GL}(3, 3)$. Let N be the group consisting of all affine transformations of the form $\tau_{\boldsymbol{x}} : \mathbb{F}_3^3 \to \mathbb{F}_3^3$, $\boldsymbol{y} \mapsto \boldsymbol{y} + \boldsymbol{x}$. Then we get two primitive permutation groups $G = N{\rtimes}H$ and $G_1 = N{\rtimes}H_1$ (on U). Define an action of G on W by

$$
(\boldsymbol{y} + \langle \boldsymbol{e}_i \rangle)^{\tau_{\boldsymbol{x}}} = \boldsymbol{y} + \boldsymbol{x} + \langle \boldsymbol{e}_i \rangle, \quad (\boldsymbol{y} + \langle \boldsymbol{e}_i \rangle)^h = \boldsymbol{y}^h + \langle \boldsymbol{e}_i^h \rangle, \quad \boldsymbol{x}, \boldsymbol{y} \in U, \quad h \in H. \tag{3.2}
$$

It is easily shown that G is transitive on $E\Gamma_0$, $E\Gamma_3$ and $E\Gamma_4$, and that G_1 is transitive $E\Gamma_0$, $E\Gamma_1$, $E\Gamma_2$ and $E\Gamma_3$. Note that $\mathrm{soc}(G_1) = \mathrm{soc}(G) = N$ and N is intransitive on W. By Lemma 2.2(2), every graph Γ_s is semisymmetric. Moreover, it is easily shown that h_2 gives an isomorphism from Γ_1 to Γ_2.

It is known that, up to isomorphism, the Gray graph **S54** is the unique cubic semisymmetirc graph of order 54 (see [4]). Thus $\Gamma_0 \cong \mathbf{S54}$. Finally, since Γ_3 and Γ_4 have different diameters (confirmed by Magma), Γ_3 and Γ_4 are not isomorphic to each other. This completes bipartite the proof. □

Remark 3.3. Let Γ_0, Γ_1, Γ_2, Γ_3 and Γ_4 be defined as in Example 3.1.

(1) The graphs Γ_0, Γ_1, Γ_2 and Γ_3 give a factorization of complete graph $\boldsymbol{K}_{27,27}$.

(2) By the argument given in Section 4, we conclude that $\mathrm{Aut}\Gamma_0 = \mathrm{Aut}\Gamma_3 = \mathrm{Aut}\Gamma_4 \cong \mathbb{Z}_3^2 \rtimes (\mathbb{Z}_2{\times}\boldsymbol{S}_4)$, and $\mathrm{Aut}\Gamma_1 \cong \mathbb{Z}_3^2 \rtimes (\mathbb{Z}_2{\times}\boldsymbol{A}_4)$. (Confirmed also by Magma.)

It is well know that Tutte's 12-cage **S126** is a cubic semisymmetric graph with automorphism group isomorphic to $\mathrm{P\Gamma U}(3,3)$. In Example 3.4, we give a construction for **S126** based on the argument in [3, p. 383, Subsection 12.4].

Example 3.4. Equip $V=\mathbb{F}_9^3$ with the standard unitary inner product

$$(\boldsymbol{x},\boldsymbol{y})=x_1y_1^3+x_2y_2^3+x_3y_3^3,\quad \boldsymbol{x},\boldsymbol{y}\in V.$$

A non-zero vector $\boldsymbol{x}\in V$ is called non-isotropic if $(\boldsymbol{x},\boldsymbol{x})\neq 0$. Then V has 504 non-isotropic vectors. These vectors span 63 1-dimensional subspaces (non-isotropic points in $\mathrm{PG}(2,3)$). Let U be the set of these subspaces. Define a graph Φ on U such that $\langle\boldsymbol{x}\rangle,\langle\boldsymbol{y}\rangle\in U$ are adjacent if and only if $(\boldsymbol{x},\boldsymbol{y})=0$. Then $\mathrm{Aut}\Phi=\mathrm{P\Gamma U}(3,3)$, and Φ is a distance-transitive graph with valency 6 and diameter 3. Moreover, Φ has exactly 63 triangles. Note that the vertex set of each triangle consists of three mutually orthogonal members in U, which is called an *orthogonal frame* of V. Let W be the set of these orthogonal frames. Then Tutte's 12-cage **S126** can be construct on $U\cup W$ such that $u\in U$ and $w\in W$ are adjacent if and only if $u\in w$.

Lemma 3.5. *Let $\Sigma=$ **S126** be constructed as in Example 3.4. Then $\partial_3(\Sigma)$ and $\partial_5(\Sigma)$ are semisymmetric graphs of valency 12 and 48, respectively. In particular,* $\mathrm{Aut}\Sigma=\mathrm{Aut}\partial_3(\Sigma)=\mathrm{Aut}\partial_5(\Sigma)\cong\mathrm{P\Gamma U}(3,3)$.

Proof. We continue the notation used in Example 3.4 and, without loss of generality, write $\mathrm{Aut}\Sigma=\mathrm{P\Gamma U}(3,3)$. Note that Σ has valency 3, diameter 6 and girth 12. It is easily shown that for $1\leqslant i\leqslant 5$ the distance i graph $\partial_i(\Sigma)$ has valency $3\cdot 2^{i-1}$, and it is connected if and only if i is odd. Clearly, $\mathrm{Aut}\Sigma\leqslant\mathrm{Aut}\partial_i(\Sigma)$. Let $A=\mathrm{Aut}\Sigma$.

By the information given in [4] for the distance partitions of $\Sigma=$ **S126**, we know that Σ is locally distance transitive, i.e., for every $v\in V\Sigma$ and $1\leqslant i\leqslant 6$, the stabilizer A_v acts transitively on the vertices at distance i from v. It follows that both $\partial_3(\Sigma)$ and $\partial_5(\Sigma)$ are A-edge-transitive.

Let $\Gamma=\partial_3(\Sigma)$ or $\partial_5(\Sigma)$, and let X be the subgroup of $\mathrm{Aut}\Gamma$ which preserves the bipartition of Γ. Then $X\geqslant A=\mathrm{P\Gamma U}(3,3)$. Checking the subgroups of $\mathrm{P\Gamma U}(3,3)$ (see [5]), we know that, for $u\in U$ and $w\in W$, the stabilizers A_u and A_w are non-conjugate maximal subgroups in A. In particular, A and hence X acts primitively on both U and W. Since Σ is not a complete bipartite graph, it is easily shown that X acts faithfully on both U and W. Note that all primitive permutation groups of degree 63 are listed in Table 1. It follows that $X=A$.

Suppose that $\mathrm{Aut}\Gamma\neq A$. Then $|\mathrm{Aut}\Gamma:A|=2$, and so $\mathrm{Aut}\Gamma=A.\mathbb{Z}_2$. Note that $\mathrm{soc}(A)=\mathrm{PSU}(3,3)$ is a characteristic subgroup of A. It follows that $\mathrm{soc}(A)$ is normal in $\mathrm{Aut}\Gamma$. Then

$$\mathrm{Aut}\Gamma/\boldsymbol{C}_{\mathrm{Aut}\Gamma}(\mathrm{soc}(A))=\boldsymbol{N}_{\mathrm{Aut}\Gamma}(\mathrm{soc}(A))/\boldsymbol{C}_{\mathrm{Aut}\Gamma}(\mathrm{soc}(A))$$

is isomorphic to a subgroup of $\mathrm{Aut}(\mathrm{soc}(A))$. Since $\mathrm{Aut}(\mathrm{soc}(A))\cong A$ by the Atlas [5], it follows that $\boldsymbol{C}_{\mathrm{Aut}\Gamma}(\mathrm{soc}(A))\neq 1$. Since $\mathrm{soc}(A)$ is a non-abelian simple group, we know that

$$\boldsymbol{C}_{\mathrm{Aut}\Gamma}(\mathrm{soc}(A))\cap\mathrm{soc}(A)=1.$$

It implies that $\boldsymbol{C}_{\mathrm{Aut}\Gamma}(\mathrm{soc}(A))\cong\mathbb{Z}_2$ and $\mathrm{Aut}\Gamma=\mathrm{soc}(A)\times\boldsymbol{C}_{\mathrm{Aut}\Gamma}(\mathrm{soc}(A))$. It follows that there is an involution $g\in\mathrm{Aut}\Gamma$ which centralizes A and interchanges U and W. For $u\in U$, we have that $w:=u^g\in W$ and $A_w=(A_u)^g=A_u$, which is a contradiction.

Therefore, $A=\mathrm{Aut}\Gamma$, and hence Γ is semisymmetric. This completes the proof. □

We remark that Tutte's 12-cage **S126**, $\partial_3($**S126**$)$ and $\partial_5($**S126**$)$ form a factorization of the complete bipartite graph $\boldsymbol{K}_{63,63}$.

A graph is said to be worthy if no two vertices have the same neighborhood [30]. If Γ is a worthy connected bipartite graph, then it is easily shown that $\mathrm{Aut}\Gamma$ acts faithfully on both bipartition subsets of Γ. For the rest of this section, we shall construct several unworthy semisymmetric graphs.

Let Σ be a connected bipartite graph with bipartition $\{U,\bar{W}\}$. Let $\Omega=\{1,2,\ldots,m\}$, where $m\geqslant 1$. Construct a bipartite graph $\Sigma^{1,m}$ with bipartition $\{U,\bar{W}\times\Omega\}$ such that $u\in U$ and $(\bar{w},i)\in\bar{W}\times\Omega$ are

Table 1 Primitive permutation groups of degree $9p$

Line	Degree $9p$	$T := \mathrm{soc}(X)$	Actions of T	Remark
1	45	PSL(2, 9)	cosets of $\boldsymbol{D}_8$	$\boldsymbol{S}_6 \ncong X \leqslant T.\mathbb{Z}_2^2$
2	153	PSL(2, 17)	cosets of $\boldsymbol{D}_{16}$	
3	$\frac{(c-1)c}{2}$	$\boldsymbol{A}_c$	2-subsets	$c \in \{10, 18, 19\}$
4	27	PSU(4, 2)	isotropic lines	$T \cong \mathrm{O}^-(6, 2)$
5	45	PSU(4, 2)	isotropic points	
6	63	Sp(6, 2)	points	
7	171	PSL(2, 19)	cosets of $\boldsymbol{D}_{20}$	
8	369	PSL$(2, 3^4)$	cosets of PGL(2, 9)	
9	117	PSL(3, 3)	anti-flags of PG(2, 3)	$X = T.\mathbb{Z}_2$
10	657	PSL(3, 8)	flags of PG(2, 8)	$X = T.\mathbb{Z}_2, T.\mathbb{Z}_6$
11	63	PSU(3, 3)	non-isotropic points;	
			bases	
12	117	$\mathrm{O}^+(6, 3)$	one of T-orbits on	$X = T, \mathrm{PGO}^+(6, 3)$
			non-isotropic points	$T \cong \mathrm{PSL}(4, 3)$
13	$9p$	$\boldsymbol{A}_{9p}$	natural action	2-transitive
14	18	PSL(2, 17)	points	2-transitive
15	$2^e + 1$	PSL$(2, 2^e)$	points	$e = 3r$, odd prime r
				2-transitive
16	63	PSL(6,2)	points	2-transitive
			hyperplanes	2-transitive

adjacent if and only if $\{u, \bar{w}\} \in E\Sigma$. Such a construction was used in [8, 28] to construct semisymmetric graphs. Let $g \in \mathrm{Aut}\Sigma$. Then g can be extended to an automorphism of $\Sigma^{1,m}$, which acts on U in the same way as g on U and acts on $\bar{W}\times\Omega$ as follows: $(\bar{w}, i)^g = (\bar{w}^g, i)$, $\bar{w} \in \bar{W}, i \in \Omega$. For each $\sigma \in \boldsymbol{S}_m$, we may define an automorphism of $\Sigma^{1,m}$, which acts on U trivially and acts on $\bar{W}\times\Omega$ as follows: $(\bar{w}, i) \mapsto (\bar{w}, i^\sigma)$, $\bar{w} \in \bar{W}, i \in \Omega$. Then $\mathrm{Aut}\Sigma^{1,m}$ contains a subgroup $\mathrm{Aut}\Sigma\times\boldsymbol{S}_m$ which acts on $U \cup (\bar{W}\times\Omega)$ by

$$u^{(g,\sigma)} = u^g, \quad (\bar{w}, i)^{(g,\sigma)} = (\bar{w}^g, i^\sigma), \quad g \in \mathrm{Aut}\Sigma, \quad \sigma \in \boldsymbol{S}_m, \quad u \in U, \quad \bar{w} \in \bar{W}, \quad i \in \Omega.$$

Thus $\Sigma^{1,m}$ is edge-transitive provided that Σ is edge-transitive. Moreover, if Σ is worthy then it is easily shown that $\mathrm{Aut}\Sigma^{1,m} = \mathrm{Aut}\Sigma\times\boldsymbol{S}_m$. Then we may formulate a result as follows from the observations made in [8, 28].

Lemma 3.6. *Let Σ be a connected bipartite graph with bipartition $\{U, \bar{W}\}$.*

(1) *If Σ is edge-transitive then $\Sigma^{1,m}$ is edge-transitive.*

(2) *If $m > 1$ and no two vertices in U have the same neighborhood, then $\Sigma^{1,m}$ is not vertex-transitive.*

(3) *If Σ is worthy then* $\mathrm{Aut}\Sigma^{1,m} = \mathrm{Aut}\Sigma\times\boldsymbol{S}_m$.

Example 3.7. Let U and $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ be as in Example 3.1. Set $U_1 = \langle\boldsymbol{e}_2, \boldsymbol{e}_3\rangle$, $U_2 = \langle\boldsymbol{e}_1, \boldsymbol{e}_3\rangle$ and $U_3 = \langle\boldsymbol{e}_1, \boldsymbol{e}_2\rangle$. Let

$$\bar{W} = \bigcup_{i=1}^{3}\{U_i, \boldsymbol{e}_i + U_i, -\boldsymbol{e}_i + U_i\}.$$

Define a bipartite graph Σ_0 with bipartition $\{U, \bar{W}\}$ such that $\boldsymbol{x} \in U$ and $\boldsymbol{y} + U_i \in \bar{W}$ are adjacent if and only if $\boldsymbol{y} - \boldsymbol{x} \in U_i$. Let Σ_1 be the complement graph of Σ_0 in the complete bipartite graph with bipartition $\{U, \bar{W}\}$. Then Σ_0 is $\{3, 9\}$-semiregular and Σ_0 is $\{6, 18\}$-semiregular.

By a similar argument as in the proof of Lemma 3.2, we know that both Σ_0 and Σ_1 admit an edge-transitive group isomorphic to $\mathbb{Z}_3^3\rtimes(\mathbb{Z}_2\times\boldsymbol{S}_4)$, which acts primitively on U. Moreover,

(1) both Σ_0 and Σ_1 are worthy, and $\mathrm{Aut}\Sigma_0 = \mathrm{Aut}\Sigma_1 \cong \mathbb{Z}_3^3\rtimes(\mathbb{Z}_2\times\boldsymbol{S}_4)$ (confirmed by Magma); and

(2) by Lemma 2.2 or Lemma 3.6, the graphs $\Sigma_0^{1,3}$ and $\Sigma_1^{1,3}$ are semisymmetric and of order 54, which have valency 9 and 18, respectively; and

(3) $\mathrm{Aut}\Sigma_0^{1,3} = \mathrm{Aut}\Sigma_1^{1,3} \cong \mathbb{Z}_3^3{\rtimes}(\mathbb{Z}_2{\times}\boldsymbol{S}_4){\times}\boldsymbol{S}_3$ by Lemma 3.6.

Example 3.8. Let $\bar{W} = \{1, 2, \ldots, 19\}$, and let U be the set of 2-subsets of $\bar{W}$. Let $T = \boldsymbol{A}_{19}$. Then T acts primitively on both U and $\bar{W}$. For $\{i, j\} \in U$, the stabilizer $T_{\{i,j\}}$ has exactly two orbits on $\bar{W}$, which are $\{i, j\}$ and $\bar{W} \setminus \{i, j\}$.

Define a bipartite graph Λ_1 with bipartition $\{U, \bar{W}\}$ such that $u \in U$ and $w \in \bar{W}$ are adjacent in Λ_1 if and only if $w \in u$. (Note that Λ_1 is just the vertex-edge incidence graph of the complete graph $\boldsymbol{K}_{19}$.) Let Λ_2 be the complement graph of Λ_1 in the complete bipartite graph $\boldsymbol{K}_{171,19}$. Then both Λ_1 and Λ_2 are worthy T-edge-transitive graphs. Moreover, the following statements hold.

(1) By Lemma 2.2 or Lemma 3.6, the graphs $\Lambda_1^{1,9}$ and $\Lambda_2^{1,9}$ are semisymmetric and of order 342, whcih have valency 18 and 153, respectively.

(2) $\mathrm{Aut}\Lambda_1 = \mathrm{Aut}\Lambda_2 \cong \boldsymbol{S}_{19}$, and so $\mathrm{Aut}\Lambda_1^{1,9} = \mathrm{Aut}\Lambda_2^{1,9} \cong \boldsymbol{S}_{19}{\times}\boldsymbol{S}_9$ by Lemma 3.6.

By [5], the simple group $\mathrm{PSL}(2, 19)$ has two conjugacy classes of maximal subgroups isomorphic to $\boldsymbol{A}_5$. Take $\bar{W}$ as one of these two conjugacy classes. Define a graph on $\bar{W}$ by letting $\bar{w}_1, \bar{w}_2 \in \bar{W}$ be adjacent whenever $\bar{w}_1 \cap \bar{w}_2 \cong \mathrm{D}_{10}$. Then this graph, called the Perkel graph, is a distance-transitive graph with automorphism group $\mathrm{PSL}(2, 19)$, order 57 and intersection array $\{6, 5, 2; 1, 1, 3\}$, refer to [3, p. 401, Subsection 13.3].

Lemma 3.9. *Assume that Σ is the Perkel graph constructed as above. Let $G = \mathrm{Aut}\Sigma$ and $\bar{w} \in V\Sigma$. Then $G_{\bar{w}}$ has exactly* 6 *orbits on $E\Sigma$: one has length* 6*, one has length* 15*, three have length* 30 *and one has length* 60*.*

Proof. For $1 \leqslant i \leqslant 3$, denote by $\Sigma_i(\bar{w})$ the set of vertices at distance i from $\bar{w}$. Then $|\Sigma_1(\bar{w})| = 6$, $|\Sigma_2(\bar{w})| = 30$ and $|\Sigma_3(\bar{w})| = 20$. For $1 \leqslant i \leqslant 3$ and $j = i$ or $i-1$, denote by $\Sigma_{j,i}$ the subgraph of Σ induced by $\Sigma_j(\bar{w}) \cup \Sigma_i(\bar{w})$, where $\Sigma_0(\bar{w}) = \{\bar{w}\}$. Let $E_{j,i}$ be the edge set of $\Sigma_{j,i}$. Then $E_{0,1}$, $E_{1,2}$, $E_{2,2}$, $E_{2,3}$ and $E_{3,3}$ form a partition of $E\Sigma$. It is easily shown that $|E_{0,1}| = 6$, $|E_{1,2}| = 30$, $|E_{2,2}| = 45$, $|E_{2,3}| = 60$ and $|E_{3,3}| = 30$.

Let $H = G_{\bar{w}}$. Then $H \cong \boldsymbol{A}_5$. Since Σ is distance-transitive, H acts transitively on each $\Sigma_i(\bar{w})$, where $1 \leqslant i \leqslant 3$. In particular, H is transitive on $E_{0,1}$. Note that H is 2-transitive on $\Sigma_1(\bar{w})$. It follows that G acts transitively on the directed 2-paths of Σ. Then H is transitive on those 2-paths from $\bar{w}$, and hence H is transitive on $E_{1,2}$. By the construction of Σ, we know that, for an edge $\{\bar{w}_1, \bar{w}_2\}$ of Σ, the arc-stabilizer $G_{\bar{w}_1\bar{w}_2}$ is isomorphic to D_{10}. Thus, for an element $h \in H$ of order 3, if $h \in G_{\bar{w}_1}$ then h does not fix $\bar{w}_2$. Using such an observation, it is easily shown that H is transitive on each of $E_{2,3}$ and $E_{3,3}$. Then we get 4 H-orbits on $E\Sigma$, which have length 6, 30, 60 and 30, respectively.

Consider that action of H on $E_{2,2}$. Take $\bar{v} \in \Sigma_2(\bar{w})$. Since H is transitive on $\Sigma_2(\bar{w})$, we know that $|H_{\bar{v}}| = \frac{|H|}{|\Sigma_2(\bar{w})|} = 2$. Note that $\Sigma_1(\bar{w})$ contains a unique vertex, say $\bar{u}$, adjacent to $\bar{v}$. Then $H_{\bar{v}}$ fixes $\bar{u}$, and so $H_{\bar{v}} < G_{\bar{u}\bar{v}} \cong \mathrm{D}_{10}$. Set $H_{\bar{v}} = \langle k \rangle$. Then $G_{\bar{u}\bar{v}} = \langle h, k \rangle$ for some h of order 5, and $khk = h^{-1}$. Since $G_{\bar{v}}$ is faithful on $\Sigma_1(\bar{v})$, writing h and k as permutations on $\Sigma_1(\bar{v})$, we know that h is a 5-cycle and k is a product of two disjoint transpositions. It follows that k interchanges two of the three vertices contained in $\Sigma_1(\bar{v}) \cap \Sigma_2(\bar{w})$. It implies that one of the H-orbits on $E_{2,2}$ has length at least 30. Since 45 is not a divisor of $|H|$, we know that H has at least two orbits on $E_{2,2}$. Note that a vertex-transitive non-empty graph of order 30 has at least 15 edges. It follows that H has exactly two orbits on $E_{2,2}$, which have length 30 and 15, respectively. This completes the proof. □

Example 3.10. Let Σ be the Perkel graph. Set $G = \mathrm{Aut}\Sigma$ and take $\bar{w} \in V\Sigma$. Then, for an edge $\{\bar{w}, \bar{v}\}$, the edge-stabilizer $G_{\{\bar{w},\bar{v}\}} \cong G_{\bar{w}\bar{v}}.\mathbb{Z}_2 \cong \mathrm{D}_{20}$, which is a maximal subgroup of $G = \mathrm{PSL}(2, 19)$. Thus G acts primitively on $E\Sigma$. Assume that $\Delta_i(\bar{w})$, $1 \leqslant i \leqslant 6$, are the six $G_{\bar{w}}$-orbits on $E\Sigma$. Without loss of generality, let $|\Delta_1(\bar{w})| = 6$, $|\Delta_2(\bar{w})| = 15$, $|\Delta_3(\bar{w})| = |\Delta_4(\bar{w})| = |\Delta_5(\bar{w})| = 30$ and $|\Delta_6(\bar{w})| = 60$.

Let $U = E\Sigma$ and $\bar{W} = V\Sigma$. Then $\bar{W} = \{\bar{w}^g \mid g \in G\}$. For each i with $1 \leqslant i \leqslant 6$, define a worthy bipartite graph Π_i with bipartition $\{U, \bar{W}\}$ such $u \in U$ and $\bar{w}^g \in \bar{W}$ are adjacent if and only if $u^{g^{-1}} \in \Delta_i(\bar{w})$. Then every graph Π_i is G-edge-transitive. The graph Π_1 is the vertex-edge incidence graph of the Perkel graph, which has girth 10 and diameter 8. Two of the three $\{10,30\}$-semiregular graphs, say Π_4 and Π_5, are respectively the distance 3 and distance 7 graphs of Π_1, and the graphs Π_2, Π_3 and Π_6 form a factorization of the distance 5 graph of Π_1. Moreover, we have the following statements.

(1) The three $\{10,30\}$-semiregular graphs Π_3, Π_4 and Π_5 are not isomorphic to every other (confirmed by Magma); and

(2) By Lemma 2.2 or 3.6, the six graphs $\Pi_i^{1,3}$ are semisymmetric and of order 342, which have valency 6, 15, 30, 30, 30 and 60, respectively; and

(3) $\mathrm{Aut}\Pi_i = \mathrm{PSL}(2,9)$, and so $\mathrm{Aut}\Pi_1^{1,3} = \mathrm{PSL}(2,9)\times \boldsymbol{S}_3$ by Lemma 3.6, where $1 \leqslant i \leqslant 6$.

4 The proof of Theorem 1.1

In this section, we give a proof Theorem 1.1. Our argument is based on analyzing the primitive permutation groups of degrees $9p$ and $3p$.

For a positive integer $k < p$, all primitive permutation groups of degree kp are explicitly known by [18,19]. Let X be a primitive permutation group of degree $9p$ or $3p$. Combining with [6, Appendix B], either $p = 3$ and X is of affine type, or X is one of almost simple groups listed in Tables 1 and 2.

In the following, we assume that Γ is a connected G-semisymmetric graph of order $18p$, where $G \leqslant \mathrm{Aut}\Gamma$ and p is a prime. Let U and W be the orbits of G acting on $V\Gamma$. Assume that one of G^U and G^W is primitive. Without loss of generality, we assume further that G^U is primitive and that Γ is not a complete bipartite graph. By Lemma 2.2, G is faithful on W, i.e., $G^W \cong G$.

Lemma 4.1 says that Theorem 1.1 holds while G^U is of affine type.

Lemma 4.1. *Assume that G^U is an affine primitive group. Then Γ is either arc-transitive or isomorphic to one of the graphs given in Examples* 3.1 *and* 3.7.

Proof. Since G^U is of affine type, $\mathrm{soc}(G^U) \cong \mathbb{Z}_3^3$. Identify U with the 3-dimensional vector space over $\mathbb{F}_3$. Write $G^U = N \rtimes H$, where H is an irreducible subgroup of $\mathrm{GL}(3,3)$, and $N = \mathrm{soc}(G^U)$ consists of the affine transformations of the form $\tau_{\boldsymbol{x}} : \mathbb{F}_3^3 \to \mathbb{F}_3^3$, $\boldsymbol{y} \mapsto \boldsymbol{y} + \boldsymbol{x}$. Let u be the vertex corresponding to the zero vector. Then $H = (G^U)_u$.

Let K be the kernel of G acting on U. Then K is faithful on W. Consider the quotient graph $\Sigma := \Gamma_K$ with respect to K. Identifying G^U with a subgroup of $\mathrm{Aut}\Sigma$, the graph Σ is G^U-edge-transitive. Since G is transitive on W and K is normal in G, all K-orbits on W have the same length, say m. Then either $K = 1$ or $\Gamma \cong \Sigma^{1,m}$.

Let $\bar{W}$ be the set of K-orbits on W. Since Γ is not a complete bipartite graph, G^U is faithful on $\bar{W}$ by Lemma 2.2. Suppose that N is transitive on $\bar{W}$. It is easily shown that N is regular on $\bar{W}$, and hence $K = 1$. By Lemma 2.1, $\Gamma \cong \Sigma$ is arc-transitive. Thus we assume further that N is intransitive on $\bar{W}$; in this case, Γ must be semisymmetric by Lemma 2.2.

Table 2 Primitive permutation groups of degree $3p$ (refer to [12])

Degree $3p$	X	Action or Remark
6	$\boldsymbol{A}_5, \boldsymbol{S}_5$	cosets of $\boldsymbol{D}_{10}$ in $\boldsymbol{A}_5$
15	$\boldsymbol{A}_6, \boldsymbol{S}_6$	2-subsets
21	$\boldsymbol{A}_7, \boldsymbol{S}_7$	2-subsets
21	$\mathrm{PSL}(3,2).\mathbb{Z}_2$	point-line incedent pairs
57	$\mathrm{PSL}(2,19)$	cosets of $\boldsymbol{A}_5$ (two actions)
15	$\boldsymbol{A}_7$	cosets of $\mathrm{PSL}(2,7)$ (two actions)
$3p$	$\boldsymbol{A}_{3p}, \boldsymbol{S}_{3p}$	
15	$\mathrm{PSL}(4,2)$	points, hyperplanes
2^e+1	$\mathrm{PSL}(2,2^e)$, $\mathrm{P\Gamma L}(2,2^e)$	points; odd prime e
q^2+q+1	$\mathrm{PSL}(3,q).O$	points, hyperplanes; $q \equiv 1 \pmod 3$, $q = r^e$, prime r, $\lvert O\rvert \mid 3e$

Let l be the number of N-orbits on $\bar{W}$. Then l is a proper divisor of $|\bar{W}| = \frac{27}{m}$ as N is intransitive on $\bar{W}$. Let p be an arbitrary prime divisor of H and let $h \in H$ be of order p. Since G^U acts faithfully on $\bar{W}$, we know that either $\langle h\rangle$ is faithful on the set of N-orbits on $\bar{W}$, or $\langle h\rangle$ fixes every N-orbit set-wise and acts faithfully on at least one of N-orbits. It follows that p is a divisor of $l!$ or $(\frac{27}{lm})!$, and hence $p<9$. Since $H \leqslant \mathrm{GL}(3,3) \cong \mathbb{Z}_2{\times}\mathrm{PSL}(3,3)$, checking the subgroups of $\mathrm{PSL}(3,3)$ in the Atlas [5], we conclude that H is isomorphic to a subgroup of $\mathbb{Z}_2{\times}\boldsymbol{S}_4$.

Since G^U acts transitively on $\bar{W}$, we know that H acts transitively on the l orbits of N acting on $\bar{W}$. Recall that l is a proper divisor of $\frac{27}{m}$. Then $l=3$ or 9. Since $|H|$ is not divisible by 9, we know that $l=3$. Let $\bar{W}_1$, $\bar{W}_2$ and $\bar{W}_3$ be the N-orbits on $\bar{W}$. Then $|\bar{W}_1|=|\bar{W}_2|=|\bar{W}_3|=\frac{9}{m}$, and $m=1$ or 3. For each i with $1\leqslant i\leqslant 3$, considering the action of N on $\bar{W}_i$, there is a subspace U_i of U such that $\langle \tau_x \mid x\in U_i\rangle$ is the kernel of N acting on $\bar{W}_i$. Recall that H is transitive on $\{\bar{W}_1,\bar{W}_2,\bar{W}_3\}$. It is easily shown that $U_i\neq U_j$ for all $i\neq j$ as N is faithful on $\bar{W}$, and that H acts transitively on $\{U_1,U_2,U_3\}$. Noting that $|\bar{W}_i|=|U:U_i|$, we have $|U_i|=3m$.

Case 1. Let $m=1$. Then $K=1$, $\Gamma\cong\Sigma$ and $|U_i|=3$ for $1\leqslant i\leqslant 3$. In particular, each U_i is a 1-dimensional subspace of U, and so we may let $U_i=\langle \boldsymbol{e}_i\rangle$ for a non-zero vector $\boldsymbol{e}_i\in U$. Recall that H is transitive on $\{\langle \boldsymbol{e}_1\rangle,\langle \boldsymbol{e}_2\rangle,\langle \boldsymbol{e}_3\rangle\}$. Then, since H is an irreducible subgroup of $\mathrm{GL}(3,3)$, we know that $\{\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3\}$ is a basis of U. Identifying W with the set $\{\boldsymbol{y}+\langle \boldsymbol{e}_i\rangle \mid \boldsymbol{y}\in U, 1\leqslant i\leqslant 3\}$ of 27 1-dimensional affine subspaces of U, the action of G on W is given by

$$(\boldsymbol{y}+\langle \boldsymbol{e}_i\rangle)^{\tau_{\boldsymbol{x}}}=\boldsymbol{y}+\boldsymbol{x}+\langle \boldsymbol{e}_i\rangle,\quad (\boldsymbol{y}+\langle \boldsymbol{e}_i\rangle)^h=\boldsymbol{y}^h+\langle \boldsymbol{e}_i^h\rangle,\quad \boldsymbol{x},\boldsymbol{y}\in U,\quad h\in H.$$

Take $h_0,h_1,h_2\in\mathrm{GL}(3,3)$ satisfying (3.1). Then $\langle h_1,h_2,h_0\rangle\cong\mathbb{Z}_2{\times}\boldsymbol{S}_4$. Without of generality, we assume that H contains h_0. Since H fixes $\{\langle \boldsymbol{e}_1\rangle,\langle \boldsymbol{e}_2\rangle,\langle \boldsymbol{e}_3\rangle\}$ set-wise, it is easily shown that $H\leqslant\langle h_1,h_2,h_0\rangle$. Analyzing the irreducible subgroups of $\langle h_1,h_2,h_0\rangle$, we conclude that H is one of the following groups:

$$\langle h_1,h_0\rangle,\ \langle h_1h_1^{h_0},h_0\rangle,\ \langle h_1,h_2,h_0\rangle,\ \langle h_1h_1^{h_0},h_2,h_0\rangle,\ \langle h_1h_1^{h_0},h_1h_1^{h_0}h_1^{h_0^2}h_2,h_0\rangle.$$

Consider the orbits of H on W. If $H=\langle h_1,h_0\rangle$ or $\langle h_1h_1^{h_0},h_0\rangle$ then H has 4 orbits on W, which are

$$\begin{aligned}
&\Delta_0:=\{\langle \boldsymbol{e}_i\rangle \mid 1\leqslant i\leqslant 3\},\\
&\Delta_1:=\{\pm\boldsymbol{e}_2+\langle \boldsymbol{e}_1\rangle,\pm\boldsymbol{e}_3+\langle \boldsymbol{e}_2\rangle,\pm\boldsymbol{e}_1+\langle \boldsymbol{e}_3\rangle\},\\
&\Delta_2:=\{\pm\boldsymbol{e}_3+\langle \boldsymbol{e}_1\rangle,\pm\boldsymbol{e}_1+\langle \boldsymbol{e}_2\rangle,\pm\boldsymbol{e}_2+\langle \boldsymbol{e}_3\rangle\},\\
&\Delta_3:=\{\pm\boldsymbol{e}_i\pm\boldsymbol{e}_j+\langle \boldsymbol{e}_k\rangle \mid \{i,j,k\}=\{1,2,3\}\}.
\end{aligned}$$

If H is one of $\langle h_1,h_2,h_0\rangle$, $\langle h_1h_1^{h_0},h_2,h_0\rangle$ and $\langle h_1h_1^{h_0},h_1h_1^{h_0}h_1^{h_0^2}h_2,h_0\rangle$ then H has 3 orbits on W, which are Δ_0, Δ_3 and $\Delta_4:=\Delta_1\cup\Delta_2$. It follows that Γ is isomorphic to one of the semisymmetric graphs given in Example 3.1.

Case 2. Let $m=3$. Then $\Gamma\cong\Sigma^{1,3}$. In this case, every U_i is a 2-dimensional subspace of U, and hence for $i\neq j$ the intersection $U_i\cap U_j$ is 1-dimensional. Set $\langle \boldsymbol{e}_1\rangle=U_2\cap U_3$, $\langle \boldsymbol{e}_2\rangle=U_1\cap U_3$ and $\langle \boldsymbol{e}_3\rangle=U_1\cap U_2$. Then $\langle \boldsymbol{e}_i\rangle\neq\langle \boldsymbol{e}_j\rangle$ for all $i\neq j$; otherwise, $\langle \boldsymbol{e}_1\rangle=\langle \boldsymbol{e}_2\rangle=\langle \boldsymbol{e}_3\rangle$ is H-invariant, a contradiction. Noting that H is transitive on $\{U_1,U_2,U_3\}$, it follows that H acts transitively on $\{\langle \boldsymbol{e}_1\rangle,\langle \boldsymbol{e}_2\rangle,\langle \boldsymbol{e}_3\rangle\}$. Thus $\{\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3\}$ is a basis of U. To determine Σ, we identify $\bar{W}$ with the set $\{\boldsymbol{y}+U_i \mid \boldsymbol{y}\in U, 1\leqslant i\leqslant 3\}$. Then $|\bar{W}|=9$ and the action of G^U on W is given by

$$(\boldsymbol{y}+U_i)^{\tau_{\boldsymbol{x}}}=\boldsymbol{y}+\boldsymbol{x}+U_i,\quad (\boldsymbol{y}+U_i)^h=\boldsymbol{y}^h+U_i^h,\quad \boldsymbol{x},\boldsymbol{y}\in U,\quad h\in H.$$

A similar argument as in Case 1 yields that H is one of

$$\langle h_1,h_0\rangle,\ \langle h_1h_1^{h_0},h_0\rangle,\ \langle h_1,h_2,h_0\rangle,\ \langle h_1h_1^{h_0},h_2,h_0\rangle,\ \langle h_1h_1^{h_0},h_1h_1^{h_0}h_1^{h_0^2}h_2,h_0\rangle,$$

where $h_0,h_1,h_2\in\mathrm{GL}(3,3)$ satisfying (3.1). It is easy to check that H has exactly 2 orbits on $\bar{W}$, say $\{U_i \mid 1\leqslant i\leqslant 3\}$ and $\{\pm\boldsymbol{e}_i+U_i \mid 1\leqslant i\leqslant 3\}$. It follows that Σ is isomorphic one of the graphs Σ_0 and Σ_1 described as in Example 3.7. This completes the proof. □

Next, we deal with the case where G^U is almost simple, and then finish the proof of Theorem 1.1.

Lemma 4.2. *Assume that G^U is almost simple. Then Γ is either arc-transitive or isomorphic to one of Tutte's* 12*-cage* **S126***,* ∂_3(**S126**)*,* ∂_5(**S126**) *and the graphs defined in Examples* 3.8(1) *and* 3.10(2).

Proof. Recall that G is faithful on W. We shall discuss in two cases according to whether or not G acts faithfully on U.

Case 1. Assume that G is faithful on U. Then $T := \mathrm{soc}(G)$ is listed in Table 1.

Assume that G is described as in lines 13–16 of Table 1. Then G is 2-transitive on U. Moreover, G has no faithful permutation representations of degree less than $9p$ (refer to [17, p. 175]). Thus G is also 2-transitive on W. It follows that either one of Γ and its complement in $\boldsymbol{K}_{9p,9p}$ is the point-hyperplane incidence graph of the projective geometry $\mathrm{PG}(5,2)$, or Γ is the standard double cover of the complete graph $\boldsymbol{K}_{9p}$. Therefore, Γ is arc-transitive.

Assume that G is described as in line 3 of Table 1. Then $T = \mathrm{soc}(G) = \boldsymbol{A}_c$ with $c \in \{10, 18, 19\}$. Note that G has no faithful permutation representations of degree less than c (see [17, p. 175]). Suppose that G is imprimitive on W. Let B be a maximal block of G acting on W. Then $|B| = 3$ or 9, and G acts faithfully and primitively on $\Omega := \{B^g \mid g \in G\}$. Note that Table 2 gives all primitive permutation group of degree $3p$. It follows that $|\Omega| = p$, and hence $T = \boldsymbol{A}_{19}$ and $p = 19$. Then $T_B \cong \boldsymbol{A}_{18}$. It is easily shown that T is transitive on W. Then for $u \in B$ we have $|T_B : T_u| = 9$; however, $\boldsymbol{A}_{18}$ has no subgroups of index 9, a contradiction. Thus G is primitive on W. Moreover, the actions of G on U and W are equivalent, i.e., G_u and G_w are conjugate in G for $u \in U$ and $w \in W$. Then $\Gamma \cong B(U, \Delta)$ by Lemma 2.3, where Δ is an orbital of G on U. It is easy to check that G has exactly three orbitals on U, which are self-paired. It follows Γ is arc-transitive.

Now let G be one of the groups described as in lines 1, 2, 4–12 of Table 1.

Suppose that the actions of G on U and W are equivalent. Then $\Gamma \cong B(U, \Delta)$ by Lemma 2.3, where Δ is an orbital of G on U. Checking one by one the possible participants of G, the lengths of suborbits $|\Delta(u)|$ (for a given $u \in U$) are listed in Table 3, where the non-self-paired suborbits are marked by *. (Note that, for line 1, the action of G on U is equivalent to that on the edge set of Tutte's 8-cage.)

If Δ is self-paired, then Γ is arc-transitive. Thus we assume that $G = \mathrm{PSL}(2,p)$ with $p = 17$ or 19. It is easily shown that any two paired suborbits of G^U are merged into some self-paired suborbit of $\mathrm{PGL}(2,p)$ (acting on U), we know that $\Gamma \cong B(U, \Delta)$ is arc-transitive by Lemma 2.5.

Table 3 Suborbits of some primitive groups of degree $9p$

Line	Degree	$T = \mathrm{soc}(G)$	Suborbits $\lvert\Delta(u)\rvert$	Remark	references
1	45	$\mathrm{PSL}(2,9)$	4, 8, 16 (two)		
2	153	$\mathrm{PSL}(2,17)$	4 (two), 8^* (two)		
			8 (four), 16 (six),	$G = \mathrm{PSL}(2,17)$	[27, Subsection 4.4]
			8, 16 (seven), 32	$G = \mathrm{PGL}(2,17)$	
4	27	$\mathrm{PSU}(4,2)$	10, 16		[29]
5	45	$\mathrm{PSU}(4,2)$	12, 32		[29]
6	63	$\mathrm{Sp}(6,2)$	30, 32		[29]
7	171	$\mathrm{PSL}(2,19)$	5^* (two), 10 (four),	$G = \mathrm{PSL}(2,19)$	[27, Subsection 4.4]
			10^* (four), 20 (four)		
	171		10, 20 (eight)	$G = \mathrm{PGL}(2,19)$	
8	369	$\mathrm{PSL}(2,3^4)$	36, 72, 80, 90 (two)		[27, Subsection 4.1]
9	117	$\mathrm{PSL}(3,3)$	12, 16 (two), 24, 48		[18, Subsection 2.3]
10	657	$\mathrm{PSL}(3,8)$	16, 128, 512		[18, Subsection 2.2]
11	63	$\mathrm{PSU}(3,3)$	6, 16 (two), 24	bases	[29]
			6, 24, 32	non-isotropic points	
12	117	$\mathrm{O}^+(6,3)$	36, 80		[18, Subsection 2.12]

Suppose that the actions of G on U and W are not equivalent. Check the subgroups of G (see [14, Chapter II, Theorem 8.27] for $\mathrm{soc}(G) = \mathrm{PSL}(2, 3^4)$ and to [5] for others). Then we conclude that every subgroup of index $9p$ is maximal in G. In particular, G_w is maximal in G, where $w \in W$. Thus G acts primitively on W. Then G has two inequivalent faithful primitive permutation representations. Checking Table 1, we have $T = \mathrm{soc}(G) = \mathrm{O}^+(6, 3)$ or $\mathrm{PSU}(3, 3)$.

Assume that $T = \mathrm{PSU}(3, 3)$. Let V be a non-degenerate 3-dimensional unitary space over $\mathbb{F}_9$. Identify U with the set of 63 non-isotropic 1-dimensional subspaces of V and W with the set of 63 orthogonal frames of V. By Example 3.4 and Lemma 3.5, Γ is isomorphic to one of Tutte's 12-cage and its distance 3 and distance 5 graphs.

Let $T = \mathrm{O}^+(6, 3)$. Then $G = T$ or $\mathrm{PGO}^+(6, 3)$. Consider a non-degenerate 6-dimensional orthogonal space V over $\mathbb{F}_3$. Identify U and W respectively with two T-orbits on the 234 non-isotropic 1-dimensional subspaces of V:

$$U = \{\langle \boldsymbol{x}\rangle \mid \boldsymbol{x} \in V, Q(\boldsymbol{x}) = 1\}, \quad W = \{\langle \boldsymbol{x}\rangle \mid \boldsymbol{x} \in V, Q(\boldsymbol{x}) = -1\},$$

where Q is the associated quadratic form. Write

$$V = \langle \boldsymbol{e}_1, \boldsymbol{f}_1\rangle \perp \langle \boldsymbol{e}_2, \boldsymbol{f}_2\rangle \perp \langle \boldsymbol{e}_3, \boldsymbol{f}_3\rangle,$$

where $\{\boldsymbol{e}_i, \boldsymbol{f}_i\}$ are hyperbolic pairs. Set

$$\boldsymbol{e} = \boldsymbol{e}_1 + \boldsymbol{f}_1,\ \boldsymbol{f} = \boldsymbol{e}_1 - \boldsymbol{f}_1 \quad \text{and} \quad V_1 = \langle \boldsymbol{e}_2, \boldsymbol{f}_2\rangle \perp \langle \boldsymbol{e}_3, \boldsymbol{f}_3\rangle.$$

Then $\langle \boldsymbol{e}\rangle \in U$ and $\boldsymbol{e}^\perp = \langle \boldsymbol{f}\rangle \perp V_1$. Moreover, $G_{\langle \boldsymbol{e}\rangle} \cong \mathrm{O}(5, 3)$ or $\mathrm{GO}(5, 3)$, which has exactly two orbits on the 162 non-isotropic vectors of $\boldsymbol{e}^\perp$:

$$S_1 = \{\boldsymbol{x} \mid \boldsymbol{x} \in \boldsymbol{e}^\perp, Q(\boldsymbol{x}) = -1\} \quad \text{and} \quad S_2 = \{\boldsymbol{x} \mid \boldsymbol{x} \in \boldsymbol{e}^\perp, Q(\boldsymbol{x}) = 1\}.$$

An easy calculation implies that

$$S_1 = \{\boldsymbol{x} \mid \boldsymbol{x} \in V_1, Q(\boldsymbol{x}) = -1\} \cup \{\pm \boldsymbol{f} + \boldsymbol{x} \mid \boldsymbol{x} \in V_1, Q(\boldsymbol{x}) = 0\},$$
$$S_2 = \{\boldsymbol{x} \mid \boldsymbol{x} \in V_1, Q(\boldsymbol{x}) = 1\} \cup \{\pm \boldsymbol{f} + \boldsymbol{x} \mid \boldsymbol{x} \in V_1, Q(\boldsymbol{x}) = -1\},$$

which have size 90 and 72, respectively. Thus $G_{\langle \boldsymbol{e}\rangle}$ has exactly two orbits on W,

$$\{\langle \boldsymbol{x}\rangle \mid \boldsymbol{x} \in S_1\} \quad \text{and} \quad \{\langle \boldsymbol{e} + \boldsymbol{x}\rangle \mid \boldsymbol{x} \in S_2\}$$

with size 45 and 72, respectively. By the information about $T = \mathrm{O}^+(6, 3)$ given in the Atlas [5], we conclude that G has an automorphism σ of order 2 such that $G_{\langle \boldsymbol{e}\rangle}^\sigma = G_{\langle \boldsymbol{f}\rangle}$. It follows from Lemma 2.6 that Γ is arc-transitive.

Case 2. Assume that G is unfaithful on U. Then Γ is semisymmetric by Lemma 2.2(3). Let K be the kernel of G acting on U. Set $\Sigma = \Gamma_K$. Then $\Gamma \cong \Sigma^{1,m}$, where m is the length of a K-orbit on W. Thus it suffices to determine m and Σ.

Let $\bar{W}$ be the set of K-orbits on W. Then G^U is faithful on $\bar{W}$ and, since $K \neq 1$ is faithful on W, the size of $\bar{W}$ is a proper divisor of $|W| = 9p$. This observation helps us to determine G^U as follows.

The groups in lines 13–16 of Table 1 are excluded as each of them has no faithful permutation representations of degree less than $9p$ (see [17, p. 175]). If G^U is described as in line 3 of Table 1 then a similar argument as in Case 1 implies that $\mathrm{soc}(G^U) = \boldsymbol{A}_{19}$ and $|\bar{W}| = p = 19$. For the groups in lines 1,2 and 4–12 of Table 1, checking the subgroups of G (see [5] and [14, Chapter II, Theorem 8.27]), the only possible case is that $G^U = \mathrm{PSL}(2, 19)$ and $G^{\bar{W}}$ is described as in Table 2.

Let $\mathrm{soc}(G^U) = \boldsymbol{A}_{19}$. We may identify $\bar{W}$ with the set of positive integers no more than 19 and U with the set of 2-subsets of $\bar{W}$. Then $m = 9$, and $\Sigma = \Gamma_K$ is isomorphic to one of the graphs Λ_1 and Λ_2 defined in Example 3.8. Thus $\Gamma \cong \Lambda_1^{1,9}$ or $\Lambda_2^{1,9}$.

Let $G^U = \mathrm{PSL}(2, 19)$. We may identify $\bar{W}$ and U respectively with the vertex set and edge set of the Perkel graph. Then $m = 3$, and Σ is isomorphic to one of the six graphs Π_i defined in Example 3.10, and so $\Gamma \cong \Pi_i^{1,3}$, where $1 \leqslant i \leqslant 6$. Thus our lemma follows. $\square$

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant Nos. 11271267 and 11371204). The authors thank the referees for helpful comments.

References

1 Bouwer I Z. An edge but not vertex transitive cubic graphs. Canad Math Bull, 1968, 11: 533–535

2 Bouwer I Z. On edge but not vertex transitive graphs. J Combin Theory Ser B, 1972, 12: 32–40

3 Brouwer A E, Cohen A M, Neumaier A. Distance-Regular Graphs. Berlin-Heidelberg: Springer-Verlag, 1989

4 Conder M, Malnič A, Marušič D, et al. A census of semisymmetric cubic graphs on up to 768 vertices. J Algebraic Combin, 2006, 23: 255–294

5 Conway J H, Curtis R T, Norton S P, et al. Atlas of Finite Groups. Oxford: Clarendon Press, 1985

6 Dixon J D, Mortimer B. Permutation Groups. New York: Springer-Verlag, 1996

7 Du S F, Marušič D. Biprimitive graphs of smallest order. J Algebraic Combin, 1999, 9: 151–156

8 Du S F, Xu M Y. A classification of semisymmetric graphs of order $2pq$. Comm Algebra, 2000, 28: 2685–2714

9 Feng Y Q, Ghasemi M, Wang C Q. Cubic semisymmetric graphs of order $6p^3$. Discrete Math, 2010, 310: 2345–2355

10 Folkman J. Regular line-symmetric graphs. J Combin Theory Ser B, 1967, 3: 215–232

11 Giudici M, Li C H, Praeger C E. Analysing finite locally s-arc transitive graphs. Trans Amer Math Soc, 2004, 365: 291–317

12 Han H, Lu Z P. Semisymmetric graphs of order $6p^2$ and prime valency. Sci China Math, 2012, 55: 2579–2592

13 Han H, Lu Z P. Affine primitive permutation groups and semisymmetric graphs. Electronic J Combin, 2013, 20: Paper 39, 9pp

14 Huppert H. Endliche Gruppen I. Berlin: Springer-Verlag, 1967

15 Iofinova M E, Ivanov A A. Biprimitive cubic graphs (in Russian). In: Investigation in Algebraic Theory of Combinatorial Objects. Moscow: Institute for System Studies, 1985, 124–134

16 Ivanov A V. On edge but not vertex transitive regular graphs. Ann Discrete Math, 1987, 34: 273–286

17 Kleidman P, Liebeck M. The Subgroup Structure of the Finite Classical Groups. Cambridge: Cambridge University Press, 1990

18 Li H L, Wang J, Wang L Y, et al. Vertex primitive graphs of order containing a large prime factor. Comm Algebra, 1994, 22: 3449–3477

19 Liebeck M, Saxl J. Primitive permutation groups containing an element of large prime order. J London Math Soc, 1985, 31: 237–249

20 Lu Z P. On the automorphism groups of bi-Cayley graphs. Beijing Daxue Xuebao, 2003, 39: 1–5

21 Lu Z P, Wang C Q, Xu M Y. On semisymmetric cubic graphs of order $6p^2$. Sci China Ser A, 2004, 47: 1–17

22 Malnič A, Marušič D, Potočnik P, et al. An infinite family of Cubic edge- but not vertex-transitive graphs. Discrete Math, 2002, 280: 133–148

23 Malnič A, Marušič D, Wang C Q. Cubic edge-transitive graphs of order $2p^3$. Discrete Math, 2004, 274: 187–198

24 Marušič M, Potočnik P. Semisymmetry of generalized Folkman graphs. European J Combin, 2001, 22: 333–349

25 Monson B, Pisanski T, Schulte E, et al. Semisymmetric graphs from polytopes. J Combin Theory Ser A, 2007, 114: 421–435

26 Parker C W. Semisymmetric cubic graphs of twice odd order. European J Combin, 2007, 28: 572–591

27 Praeger C E, Xu M Y. Symmetric graphs of order a product of two distinct primes. J Combin Theory Ser B, 1993, 59: 245–266

28 Wang L, Du S F, Li X W. A class of semisymmetric graphs. Ars Math Contemp, 2014, 7: 40–53

29 Wilson R, Walsh P, Tripp J, et al. Atlas of finite group representation. http://brauer.maths.qmul.ac.uk/Atlas/v3

30 Wilson S. A worthy family of semisymmetric graphs. Discrete Math, 2003, 271: 283–94