SCIENCE CHINA
Mathematics

• ARTICLES •

January 2017 Vol. 60 No. 1: 59–82
doi: 10.1007/s11425-015-0860-8

Multi-recurrence and van der Waerden systems

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Received December 26, 2015; accepted May 31, 2016; published online August 25, 2016

Abstract We explore recurrence properties arising from dynamical approach to the van der Waerden theorem and similar combinatorial problems. We describe relations between these properties and study their consequences for dynamics. In particular, we present a measure-theoretical analog of a result of Glasner on multi-transitivity of topologically weakly mixing minimal maps. We also obtain a dynamical proof of the existence of a C-set with zero Banach density.

Keywords multi-recurrent points, van der Waerden systems, multiple recurrence theorem, multiple IP-recurrence property, multi-non-wandering points

MSC(2010) 37B20, 37B05, 37A25, 05D10

Citation: Kwietniak D, Li J, Oprocha P, et al. Multi-recurrence and van der Waerden systems. Sci China Math, 2017, 60: 59–82, doi: 10.1007/s11425-015-0860-8

1 Introduction

We study multiple-recurrence properties of dynamical systems on compact metric spaces. We use topological dynamics to characterize selected classes of subsets of $\mathbb{N}$ (e.g., IP-sets, C-sets, etc.) and to gain a better understanding of some classes of transitive systems. The idea goes back to the work of Furstenberg in the 1970s.

Our starting point is the following result published in [35].

Van der Waerden theorem. *If $\mathbb{N}$ is partitioned into finitely many subsets, then one of these sets contains arithmetic progressions of arbitrary finite length.*

In 1978, Furstenberg and Weiss [14] obtained a dynamical proof of the van der Waerden theorem. They proved the topological multiple recurrence theorem and showed that it is equivalent to the van der Waerden theorem. "Equivalent" means here that any of these results may be proved by assuming the other is true.

Topological multiple recurrence theorem. *Let (X,T) be a compact dynamical system. Then there exists a point $x\in X$ such that for any $d\in\mathbb{N}$ there is a strictly increasing sequence $\{n_k\}_{k=1}^{\infty}$ in $\mathbb{N}$ with $T^{in_k}x\to x$ as $k\to\infty$ for every $i=1,2,\ldots,d$.*

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We call a point $x \in X$ fulfilling the conclusion of the topological multiple recurrence theorem a *multi-recurrent point*. In Section 3, we show that the set of all multi-recurrent points is a G_δ subset of X; it is a residual set if (X,T) is minimal; and when (X,T) is distal or uniformly rigid, then every point is multi-recurrent. We also provide an example of a substitution subshift with minimal points which are not multi-recurrent. Then we prove that multi-recurrent points can be lifted through a distal extension but this does not need to hold for a proximal extension (we strongly believe that it cannot be lifted by weakly mixing extension, but we do not have an example at this moment). Using ergodic theory, we show that the collection of multi-recurrent points which return to any of their neighborhoods with positive upper density has full measure for every invariant measure. If the invariant measure is weakly mixing and fully supported then for almost every $x \in X$ and every $d \geqslant 1$ the diagonal d-tuple $(x, x, \ldots, x)$ has a dense orbit under the action of $T \times T^2 \times \cdots \times T^d$, which can be viewed as a measure-theoretical version of a result of Glasner on topological weakly mixing minimal maps [15].

Let us mention another equivalent version of the topological multiple recurrence theorem which shows the relationship between these results and Furstenberg's multiple recurrence theorem for measure preserving systems (the so-called "ergodic Szemerédi theorem"). It also comes from [14, Theorem 1.5]. For a short and elegant proof see [16, Theorem 1.56].

Topological multiple recurrence theorem II. *If a dynamical system (X,T) is minimal, then for any $d \in \mathbb{N}$ and any non-empty open subset U of X, there exists a positive integer $n \geqslant 1$ with*

$$U \cap T^{-n}U \cap T^{-2n}U \cap \cdots \cap T^{-dn}U \neq \emptyset.$$

Inspired by this result, we introduce a new class of dynamical systems, which we call *van der Waerden systems*. That is system (X,T) such that for every non-empty open subset U of X and for every $d \in \mathbb{N}$, there exists an $n \in \mathbb{N}$ such that

$$U \cap T^{-n}U \cap T^{-2n}U \cap \cdots \cap T^{-dn}U \neq \emptyset$$

and we will study their basic properties in Section 4. By the second variant of topological multiple recurrence theorem every minimal system is a van der Waerden system and it is also not hard to see that (X,T) is a van der Waerden system if and only if its multi-recurrent points are dense in X.

A generalization of van der Waerden theorem is Szemerédi's theorem [34], proved in 1975.

Szemerédi theorem. *If $F \subset \mathbb{N}$ has positive upper density, then it contains arithmetic progressions of arbitrary finite length.*

Two years later, in 1977, Furstenberg [11] presented a new proof of Szemerédi theorem using dynamical systems approach. Furstenberg's proof is based on the equivalence of Szemerédi theorem and the following multiple recurrence theorem.

Multiple recurrence theorem. *If $(X,\mathcal{B},\mu)$ is a probability space and T is a measure preserving transformation of $(X,\mathcal{B},\mu)$, then for any $d \in \mathbb{N}$ and any set $A \in \mathcal{B}$ with $\mu(A) > 0$, there exists an integer $n \geqslant 1$ with*

$$\mu(A \cap T^{-n}A \cap \cdots \cap T^{-2n}A \cap \cdots \cap T^{-dn}A) > 0.$$

It follows that every compact dynamical system with a fully supported invariant measure is a van der Waerden system. We examine whether the converse is true. It turns out that there exists a topologically strongly mixing system which is a van der Waerden system, but the only invariant measure is a point mass on a fixed point (see Remark 5.6). We also provide an example of a strongly mixing system which is not a van der Waerden system.

While we were preparing this paper we found a work of Host et al. [20] which studies closely related problems, but from a different point of view which emphasizes the connection between recurrence properties and associated *sets of* (*multiple*) *recurrence* (see [20, Definitions 2.1 and 2.9]). Here we focus on recurrence of a single point in a concrete dynamical system, and this complements the approach of [20].

Our study of van der Waerden systems leads naturally to $\mathcal{AP}$-recurrent points. We say that a point x is *$\mathcal{AP}$-recurrent* if for every neighborhood U of x the set of return times of x to U contains arithmetic progressions of arbitrary finite length. It is clear that every multi-recurrent point is $\mathcal{AP}$-recurrent, but the converse is not true. It is a consequence of the following characterization: a point is $\mathcal{AP}$-recurrent if and only if the closure of its orbit is a van der Waerden system. A nice property of $\mathcal{AP}$-recurrent points is that they can be lifted through factor maps.

In [12], Furstenberg defined central subsets of $\mathbb{N}$ in terms of some notions from topological dynamics. He showed that any finite partition of $\mathbb{N}$ must contain a central set in one of its cells and proved the following central sets theorem [12, Proposition 8.21].

Central sets theorem. *Let C be a central set of $\mathbb{N}$. Let $d \in \mathbb{N}$ and for each $i \in \{1, 2, \dots, d\}$, let $\{p_n^{(i)}\}_{n=1}^{\infty}$ be a sequence in $\mathbb{Z}$. Then there exist a sequence $\{a_n\}_{n=1}^{\infty}$ in $\mathbb{N}$ and a sequence $\{H_n\}_{n=1}^{\infty}$ of finite subsets of $\mathbb{N}$ such that*

1. *for every $n \in \mathbb{N}$, $\max H_n < \min H_{n+1}$ and*
2. *for every finite subset F of $\mathbb{N}$ and every $i \in \{1, 2, \dots, d\}$,*

$$\sum_{n \in F} \left(a_n + \sum_{j \in H_n} p_j^{(i)} \right) \in C.$$

Central sets theorem has very strong combinatorial consequences, such as Rado's theorem [32]. De et al. [8] proved a stronger version of the central sets theorem valid for an arbitrary semigroup S and proposed to call a subset of S a C-set if it satisfies the conclusion of this version of the central sets theorem. A dynamical characterization of C-sets was obtained in [27] by introducing a class of dynamical systems satisfying the multiple IP-recurrence property. Note that C-sets considered in [27] are subsets of $\mathbb{Z}$, however Hindman [19] pointed out to the second author of this paper that a similar characterization also holds for C-sets in $\mathbb{N}$[1). A dynamical characterization of C-sets in an arbitrary semigroup S is provided in [24] .

We study the multiple IP-recurrence property in Section 5. We show that every transitive system with the multiple IP-recurrence property is either equicontinuous or sensitive. This result generalizes theorems of Akin et al. [1] and Glasner and Weiss [17]. We also provide an example of a strongly mixing system which is a van der Waerden system but does not have the multiple IP-recurrence property. We characterize bounded density shifts with the multiple IP-recurrent property. Combining this result with the dynamical characterization of C-sets we obtain a dynamical proof of the main result of [19]: there is a C-set in $\mathbb{N}$ with zero Banach density.

As seen above, the notion of a multi-recurrent point, which is parallel to the notion of a recurrent point provides some insight to the theory of dynamical systems. In the same spirit, we define the notion of a multi-non-wandering point parallel to the classical notion of a non-wandering point. In Section 6, we study the relations between multi-non-wandering points and the sets containing arithmetic progressions of arbitrary finite length. In particular, we provide a link between multi-non-wandering sets and $\mathcal{AP}$-recurrence.

By what we said above, it is easy to see that a transitive van der Waerden system can be viewed as a generalization of an E-system (transitive system with a full supported invariant measure). In a transitive van der Waerden system each transitive point is $\mathcal{AP}$-recurrent, and the set of multi-recurrent points is dense. Note that for an E-system, the return time set of a transitive point to its neighborhood has positive upper Banach density and at the same time, the set of recurrent points with positive lower density of return time sets is dense. For an M-system (transitive system with a dense set of minimal points), this can be explained using piecewise syndetic sets and syndetic sets.

2 Preliminaries

In this section, we present basic notation, definitions and results.

1) See also the review of [27] by Hindman in MathSciNet, MR2890544.

2.1 Subsets of positive integers

Denote by $\mathbb{N}$ ($\mathbb{Z}_+$ and $\mathbb{Z}$, respectively) the set of all positive integers (non-negative integers and integers, respectively).

A *Furstenberg family* or simply a *family* on $\mathbb{N}$ is any collection $\mathcal{F}$ of subsets of $\mathbb{N}$ which is hereditary upwards, i.e., if $A \in \mathcal{F}$ and $A \subset B \subset \mathbb{N}$ then $B \in \mathcal{F}$. A *dual family* for $\mathcal{F}$, denoted by $\mathcal{F}^*$, consists of sets that meet every element of $\mathcal{F}$, i.e., $A \in \mathcal{F}^*$ provided that $\mathbb{N}\backslash A \notin \mathcal{F}$. Clearly, $\mathcal{F}^{**} = \mathcal{F}$.

Given a sequence $\{p_i\}_{i=1}^{\infty}$ in $\mathbb{N}$, define the set of finite sums of $\{p_i\}_{i=1}^{\infty}$ as

$$\mathrm{FS}\{p_i\}_{i=1}^{\infty} = \left\{ \sum_{i\in\alpha} p_i \colon \alpha \text{ is a non-empty finite subset of } \mathbb{N} \right\}.$$

We say that a subset F of $\mathbb{N}$ is

1. an IP-set if there exists a sequence $\{p_i\}_{i=1}^{\infty} \subset \mathbb{N}$ such that $FS\{p_i\}_{i=1}^{\infty} \subset F$;
2. an AP-set if it contains arbitrarily long arithmetic progressions, i.e., for every $d \geqslant 1$, there are $a, n \in \mathbb{N}$ such that $\{a, a+n, \ldots, a+dn\} \subset F$. The family of all AP-sets is denoted by $\mathcal{AP}$;
3. thick if it contains arbitrarily long blocks of consecutive integers, i.e., for every $d \geqslant 1$ there is $n \in \mathbb{N}$ such that $\{n, n+1, \ldots, n+d\} \subset F$;
4. syndetic if it has bounded gaps, i.e., for some $N \in \mathbb{N}$ and every $k \in \mathbb{N}$ we have $\{k, k+1, \ldots, k + N\} \cap F \neq \emptyset$;
5. co-finite if it has finite complement, i.e., $\mathbb{N}\backslash F$ is finite;
6. an IP*-set (AP*-set, respectively) if it has non-empty intersection with every IP-set (AP-set, respectively), i.e., it belongs to an appropriate dual family.

It is easy to see that a subset F of $\mathbb{N}$ is syndetic if and only if it has non-empty intersection with every thick set, i.e., is in the family dual to all thick sets. Every thick set is an IP-set, hence every IP*-set is syndetic.

A family $\mathcal{F}$ has the *Ramsey property* if $F \in \mathcal{F}$ and $F = F_1 \cup F_2$ imply that $F_i \in \mathcal{F}$ for some $i \in \{1, 2\}$. It is not hard to see that the van der Waerden theorem is equivalent to the fact that the family $\mathcal{AP}$ has the Ramsey property.

Let F be a subset of $\mathbb{Z}_+$. Define the *upper density* $\overline{d}(F)$ of F by

$$\overline{d}(F) = \limsup_{n\to\infty} \frac{\#(F \cap [0, n-1])}{n},$$

where $\#(\cdot)$ is the number of elements of a set. Similarly, $\underline{d}(F)$, the *lower density* of F, is defined by

$$\underline{d}(F) = \liminf_{n\to\infty} \frac{\#(F \cap [0, n-1])}{n}.$$

The *upper Banach density* $\mathrm{BD}^*(F)$ and *lower Banach density* $\mathrm{BD}_*(F)$ are defined by

$$\mathrm{BD}^*(F) = \limsup_{N-M\to\infty} \frac{\#(F \cap [M, N])}{N - M + 1}, \quad \mathrm{BD}_*(F) = \liminf_{N-M\to\infty} \frac{\#(F \cap [M, N])}{N - M + 1}.$$

2.2 Topological dynamics

By a (*topological*) *dynamical system* we mean a pair (X, T) consisting of a compact metric space (X, ρ) and a continuous map $T \colon X \to X$. If X is a singleton, then we say that (X, T) is *trivial*. If $K \subset X$ is a non-empty closed subset satisfying $T(K) \subset K$, then we say that (K, T) is a *subsystem* of (X, T) and (X, T) is *minimal* if it has no proper subsystems. The (*positive*) *orbit of x under T* is the set $\mathrm{Orb}(x, T) = \{T^n x \colon n \in \mathbb{Z}_+\}$. Clearly, $(\overline{\mathrm{Orb}(x, T)}, T)$ is a subsystem of (X, T) and (X, T) is minimal if $\overline{\mathrm{Orb}(x, T)} = X$ for every $x \in X$.

We say that a point $x \in X$ is the following:

1. *minimal*, if x belongs to some minimal subsystem of (X, T);
2. *recurrent*, if $\liminf_{n\to\infty} \rho(T^n x, x) = 0$;

3. *transitive*, if $\overline{\mathrm{Orb}(x,T)} = X$.

For a point $x \in X$ and subsets $U, V \subset X$, we define the following sets of *transfer times*:

$$N(U,V) = \{n \in \mathbb{N} \colon T^n U \cap V \neq \emptyset\} = \{n \in \mathbb{N} \colon U \cap T^{-n} V \neq \emptyset\},$$
$$N(x,U) = \{n \in \mathbb{N} \colon T^n x \in U\}.$$

To emphasize that we are calculating the above sets using transformation T we will sometimes write $N_T(x,U)$ and $N_T(U,V)$.

We say that a dynamical system (X,T) is the following:

1. *transitive* if $N(U,V) \neq \emptyset$ for every two non-empty open subsets U and V of X;
2. *totally transitive* if (X,T^n) is transitive for every $n \in \mathbb{N}$;
3. (*topologically*) *weakly mixing* if the product system $(X \times X, T \times T)$ is transitive;
4. (*topologically*) *strongly mixing* if for every two non-empty open subsets U and V of X, the set of transfer times $N(U,V)$ is cofinite.

Denote by $\mathrm{Tran}(X,T)$ the set of all transitive points of (X,T). It is easy to see that if a dynamical system (X,T) is transitive then $\mathrm{Tran}(X,T)$ is a dense G_δ subset of X. It is also clear that a dynamical system (X,T) is minimal if and only if $\mathrm{Tran}(X,T) = X$, and a point $x \in X$ is minimal if and only if $(\overline{\mathrm{Orb}(x,T)}, T)$ is a minimal system.

The following characterizations of recurrent points and minimal points are well-known (see, e.g., [12]).

Lemma 2.1. *Let (X,T) be a dynamical system. A point $x \in X$ is the following*:

1. *recurrent if and only if for every open neighborhood U of x the set $N(x,U)$ contains an IP-set*;
2. *minimal if and only if for every open neighborhood U of x the set $N(x,U)$ is syndetic.*

A dynamical system (X,T) is *equicontinuous* if for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $x, y \in X$ with $\rho(x,y) < \delta$ then $\rho(T^n x, T^n y) < \varepsilon$ for $n = 0, 1, 2, \ldots$ A point $x \in X$ is *equicontinuous* if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for every $y \in X$ with $\rho(x,y) < \delta$, $\rho(T^n x, T^n y) < \varepsilon$ for all $n \in \mathbb{Z}_+$. By compactness, (X,T) is equicontinuous if and only if every point in X is equicontinuous.

We say that a dynamical system (X,T) has *sensitive dependence on initial condition* or briefly (X,T) is *sensitive* if there exists a $\delta > 0$ such that for every $x \in X$ and every neighborhood U of x there exist $y \in U$ and $n \in \mathbb{N}$ such that $\rho(T^n x, T^n y) > \delta$.

A transitive system is *almost equicontinuous* if there is at least one equicontinuous point. It is known that if (X,T) is almost equicontinuous then the set of equicontinuous points coincides with the set of all transitive points and additionally (X,T) is *uniformly rigid*, i.e., for every $\varepsilon > 0$ there exists an $n \in \mathbb{N}$ such that $\rho(T^n x, x) < \varepsilon$ for all $x \in X$. We also have the following dichotomy: if a dynamical system (X,T) is transitive, then it is either almost equicontinuous or sensitive. See [1, 17] for proofs and more details.

A pair $(x,y) \in X^2$ is *proximal* if $\liminf_{n\to\infty} \rho(T^n x, T^n y) = 0$, and *distal* if it is not proximal, i.e., $\liminf_{n\to\infty} \rho(T^n x, T^n y) > 0$. A point x is *distal* if (x,y) is distal for any $y \in \overline{\mathrm{Orb}(x,T)}$ with $y \neq x$. If every point in X is distal then we say that (X,T) is *distal*.

Let (X,T) and (Y,S) be two dynamical systems. If there is a continuous surjection $\pi : X \to Y$ with $\pi \circ T = S \circ \pi$, then we say that π is a *factor map* and the system (Y,S) is a *factor* of (X,T) or (X,T) is an *extension* of (Y,S).

A factor map $\pi : X \to Y$ is

1. *proximal* if $(x_1, x_2) \in X^2$ is proximal provided $\pi(x_1) = \pi(x_2)$;
2. *distal* if $(x_1, x_2) \in X^2$ is distal provided $\pi(x_1) = \pi(x_2)$ with $x_1 \neq x_2$;
3. *almost one-to-one* if there exists a residual subset G of X such that $\pi^{-1}(\pi(x)) = \{x\}$ for any $x \in G$.

Let $M(X)$ be the set of Borel probability measures on X. We are interested in those members of $M(X)$ that are invariant measures for T. Therefore, denote by $M(X,T)$ the set consisting of all $\mu \in M(X)$ making T a measure-preserving transformation of $(X, \mathcal{B}(X), \mu)$, where $\mathcal{B}(X)$ is the Borel σ-algebra of X. By the Krylov-Bogolyubov theorem, $M(X,T)$ is non-empty .

The *support* of a measure $\mu \in M(X)$, denoted by $\mathrm{supp}(\mu)$, is the smallest closed subset C of X such that $\mu(C) = 1$. We say that a measure has *full support* or is *fully supported* if $\mathrm{supp}(\mu) = X$. We say that (X,T) is an *E-system* if it is transitive and admits a T-invariant Borel probability measure with full support.

2.3 Symbolic dynamics

Below we have collected some basic facts from symbolic dynamics. The standard reference here is the book of Lind and Marcus [29].

Let $\{0,1\}^{\mathbb{Z}_+}$ be the space of infinite sequence of symbols in $\{0,1\}$ indexed by the non-negative integers. Equip $\{0,1\}$ with the discrete topology and $\{0,1\}^{\mathbb{Z}_+}$ with the product topology. The space $\{0,1\}^{\mathbb{Z}_+}$ is compact and metrizable. A compatible metric ρ is given by

$$\rho(x,y)=\begin{cases}0, & x=y,\\ 2^{-J(x,y)}, & x\neq y,\end{cases}$$

where $J(x,y)=\min\{i\in\mathbb{Z}_+ : x_i\neq y_i\}$.

A *word* of length n is a sequence $w=w_1w_2\cdots w_n\in\{0,1\}^n$ and its *length* is denoted by $|w|=n$. The *concatenation* of words $w=w_1w_2\cdots w_n$ and $v=v_1v_2\cdots v_m$ is the word $wv=w_1w_2\cdots w_nv_1v_2\cdots v_m$. If u is a word and $n\in\mathbb{N}$, then u^n is the concatenation of n copies of u and u^∞ is the sequence in $\{0,1\}^{\mathbb{Z}_+}$ obtained by infinite concatenation of the word u. We say that a word $u=u_1u_2\cdots u_k$ appears in $x=(x_i)\in\{0,1\}^{\mathbb{Z}_+}$ at position t if $x_{t+j-1}=u_j$ for $j=1,2,\ldots,k$. For $x\in\{0,1\}^{\mathbb{Z}_+}$ and $i,j\in\mathbb{Z}_+$, $i\leqslant j$ write $x_{[i,j]}=x_ix_{i+1}\cdots x_j$. Words $x_{[i,j)}$, $x_{(i,j]}$ and $x_{(i,j)}$ are defined in the same way.

The *shift map* $\sigma\colon\{0,1\}^{\mathbb{Z}_+}\to\{0,1\}^{\mathbb{Z}_+}$ is defined by $\sigma(x)_n=x_{n+1}$ for $n\in\mathbb{Z}_+$. It is clear that σ is a continuous surjection. The dynamical system $(\{0,1\}^{\mathbb{Z}_+},\sigma)$ is called the *full shift*. If X is non-empty, closed and σ-invariant (i.e., $\sigma(X)\subset X$), then (X,σ) is called a *subshift*.

Given any collection $\mathcal{F}$ of words over $\{0,1\}$, we define a *subshift specified by* $\mathcal{F}$, denoted by $X_{\mathcal{F}}$, as the set of all sequences from $\{0,1\}^{\mathbb{Z}_+}$ which do not contain any words from $\mathcal{F}$. We say that $\mathcal{F}$ is a collection of *forbidden words for* $X_{\mathcal{F}}$ as words from $\mathcal{F}$ are forbidden to occur in $X_{\mathcal{F}}$.

A *cylinder* in $\{0,1\}^{\mathbb{Z}_+}$ is any set $[u]=\{x\in X : x_0x_1\cdots x_{n-1}=u\}$, where u is a word of length n. Note that the family of cylinders in $\{0,1\}^{\mathbb{Z}_+}$ is a base of the topology of $\{0,1\}^{\mathbb{Z}_+}$. Let X be a subshift of $\{0,1\}^{\mathbb{Z}_+}$. The *language of* X, denoted by $\mathcal{L}(X)$, consists of all words that can appear in some $x\in X$, i.e., $\mathcal{L}(X)=\{x_{[i,j]} : x\in X, i\leqslant j\}$.

For every word $u\in\mathcal{L}(X)$, let $[u]_X=X\cap[u]$. Then $\{[u]_X\colon u\in\mathcal{L}(X)\}$ forms a base of the topology of X. Let $\mathcal{F}=\{0,1\}^*\backslash\mathcal{L}(X)$, where $\{0,1\}^*$ is the collection of all finite words over $\{0,1\}$. Then $X=X_{\mathcal{F}}$, i.e., $\mathcal{F}$ is the set of forbidden words for X.

Remark 2.2. In some examples, we will consider sequences indexed by positive integers $\mathbb{N}$ instead of $\mathbb{Z}_+$, i.e., we identify $\{0,1\}^{\mathbb{N}}$ with $\{0,1\}^{\mathbb{Z}_+}$. It will simplify some calculations.

3 Multi-recurrent points

3.1 Definition and basic properties

Definition 3.1. Let (X,T) be a dynamical system. A point $x\in X$ is called *multi-recurrent* if for every $d\geqslant 1$, there exists a strictly increasing sequence $\{n_k\}_{k=1}^\infty$ in $\mathbb{N}$ such that for each $i=1,2,\ldots,d$ we have $T^{in_k}x\to x$ as $k\to\infty$.

In other words, a point $x\in X$ is multi-recurrent if and only if for every $d\geqslant 1$ the point $(x,\ldots,x)\in X^d$ is recurrent for $T\times T^2\times\cdots\times T^d$. Equivalently, x is multi-recurrent if and only if for every $d\geqslant 1$ and every neighborhood U of x there exists $k\in\mathbb{N}$ such that $k,2k,\ldots,dk\in N(x,U)$.

While we do not need such generality in the present paper, observe that Definition 3.1 can be stated for $\mathbb{Z}^d$-actions in a similar manner. A proof of the following observation is straightforward, thus we leave it to the reader.

Lemma 3.2. *Let (X,T) be a dynamical system and $x \in X$. Then the following conditions are equivalent*:

1. *x is a multi-recurrent point of (X,T);*
2. *x is a multi-recurrent point of (X,T^n) for some $n \in \mathbb{N}$;*
3. *x is a multi-recurrent point of (X,T^n) for any $n \in \mathbb{N}$.*

The following fact implies that every dynamical system contains a multi-recurrent point, because every dynamical system has a minimal subsystem. Note that Lemma 3.3 can also be deduced from properties of sets of multiple recurrence provided by [20, Lemma 2.5]. Results in [20] allow further analysis of return times of multi-recurrent points.

Lemma 3.3. *Let (X,T) be a dynamical system.*

(1) *The set of all multi-recurrent points of (X,T) is a G_δ subset of X.*

(2) *If (X,T) is minimal, then the set of all multi-recurrent points is residual in X.*

Proof. (1) Given $d \geqslant 1$, let

$$R_d = \left\{ y \in X : \exists\, n \geqslant 1 \text{ such that } \rho(y, T^{in}y) < \frac{1}{d} \text{ for } i = 0, 1, \ldots, d \right\}.$$

It is clear that every R_d is open, hence $R = \bigcap_{d=1}^{\infty} R_d$ is a G_δ subset of X. It is easy to see that $R = \bigcap_{d=1}^{\infty} R_d$ is the set of all multi-recurrent points.

(2) If (X,T) is minimal, then it follows from the topological multiple recurrence theorem II that R_d is dense in X for every $d \geqslant 1$. Thus $R = \bigcap_{d=1}^{\infty} R_d$ is residual in X. □

Lemma 3.4. *If a dynamical system (X,T) is uniformly rigid, then every point in X is multi-recurrent.*

Proof. Fix $d \geqslant 1$. Since (X,T) is uniformly rigid, for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $\rho(T^n x, x) < \varepsilon/d$ for all $x \in X$. Then

$$\rho(x, T^n x) < \varepsilon/d, \quad \rho(T^n x, T^{2n} x) < \varepsilon/d, \ldots, \rho(T^{(d-1)n} x, T^{dn} x) < \varepsilon/d,$$

which shows that the diameter of $\{x, T^n x, T^{2n} x, \ldots, T^{dn} x\}$ is less than ε. It follows that $(x, \ldots, x) \in X^d$ is recurrent for $T \times T^2 \times \cdots \times T^d$. But d is arbitrary, hence x is multi-recurrent. □

Remark 3.5. It is shown in [12, Proposition 9.16] that if a point is distal then it is multi-recurrent. In particular, in a distal system every point is multi-recurrent.

Remark 3.6. Notice that there exist minimal as well as non-minimal weakly mixing and uniformly rigid systems (see, respectively, [18] and [10]). By Lemma 3.4, every point in those systems is multi-recurrent. None of these examples can be a subshift. Furthermore, a non-trivial strongly mixing dynamical system can never be uniformly rigid by Glasner and Maon [18].

One of the referees of this paper, motivated by the above remark, suggested the following problem.

Question 3.7. *Is there a non-trivial weakly mixing subshift or any mixing dynamical system for which each point is multi-recurrent? Can such a system be minimal?*

In [36], it is proved that if each pair in a dynamical (X,T) is positively recurrent under $T \times T$, then it has zero topological entropy (it is also a consequence of a result in [6]). Distal or uniformly rigid systems are examples of pointwise multi-recurrent systems which have zero topological entropy. But pointwise multi-recurrence does not imply zero topological entropy in general as shown below.

Remark 3.8. A dynamical system (X,T) is *multi-minimal* if for every $d \geqslant 1$ $(X^d, T \times T^2 \times \cdots \times T^d)$ is minimal [30]. Clearly, every point in a multi-minimal system is multi-recurrent. Note that by the proof of [23, Proposition 3.5] there exists a multi-minimal system with positive topological entropy.

The existence of a system constructed in the following theorem is probably a folklore, but we were unable to find it in the literature.

Theorem 3.9. *For every $d \geqslant 1$, there is a minimal point x in the full shift $(\{0,1\}^{\mathbb{Z}_+}, \sigma)$ such that $(x, x, \ldots, x) \in X^d$ is recurrent under $\sigma \times \sigma^2 \times \cdots \times \sigma^d$ and $(x, x, \ldots, x) \in X^{d+1}$ is not recurrent under $\sigma \times \sigma^2 \times \cdots \times \sigma^d \times \sigma^{d+1}$.*

Proof. First, we consider the case $d = 1$ and then the general case. For $d = 1$, we define the local rule of a substitution by

$$\tau\colon 1 \to 1101, \quad 0 \to 0101,$$

and then extend it to all finite words over $\{0, 1\}$ putting inductively $\tau(uv) = \tau(u)\tau(v)$. Let

$$x = (x_i)_{i=0}^{\infty} = \lim_{k\to\infty} \tau^k(1)0^{\infty}$$

be a fixed point of τ. It is easy to check that $x \in \{0, 1\}^{\mathbb{Z}_+}$ is a minimal point.

We claim that $x_i = 1$ if and only if $i = 0$ or $i = 4^m(2n + 1)$ for some $n, m \in \mathbb{Z}_+$. It will follow that $x_i = 0$ if and only if $i = 2 \cdot 4^m(2n + 1)$ for some $n, m \in \mathbb{Z}_+$.

These conditions are clearly true for $i = 0, 1, 2, 3$. Now fix any $i \geqslant 0$ and assume that our claim holds for i. We will show that the claim also holds for $4i, 4i + 1, 4i + 2, 4i + 3$. We have two cases to consider.

If $x_i = 1$, then by the claim $i = 4^m(2n + 1)$ for some $m, n \in \mathbb{Z}_+$. By the definition of substitution $x_{[4i,4i+3]} = \tau(x_i) = \tau(1)$, so

- $x_{4i} = 1$ and $4i = 4^{m+1}(2n + 1)$;
- $x_{4i+1} = 1$ and $4i + 1 = 4^{m+1}(2n + 1) + 1 = 2(2 \cdot 4^m(2n + 1)) + 1$;
- $x_{4i+2} = 0$ and $4i + 2 = 4^{m+1}(2n + 1) + 2 = 2(2 \cdot 4^m(2n + 1) + 1)$;
- $x_{4i+3} = 1$ and $4i + 3 = 4^{m+1}(2n + 1) + 3 = 2(2 \cdot 4^m(2n + 1) + 1) + 1$.

If $x_i = 0$, then $i = 2 \cdot 4^m \cdot n$ for some $m, n \in \mathbb{Z}_+$. Then $x_{[4i,4i+3]} = \tau(0)$ and we have

- $x_{4i} = 0$ and $4i = 2 \cdot 4^{m+1} \cdot n$;
- $x_{4i+1} = 1$ and $4i + 1 = 2 \cdot 4^{m+1} \cdot n + 1 = 2(4^{m+1} \cdot n) + 1$;
- $x_{4i+2} = 0$ and $4i + 2 = 2 \cdot 4^{m+1} \cdot n + 2 = 2(4^{m+1} \cdot n + 1)$;
- $x_{4i+3} = 1$ and $4i + 3 = 2 \cdot 4^{m+1} \cdot n + 3 = 2(4^{m+1} \cdot n + 1) + 1$.

This ends the proof of the claim.

The point x is minimal, hence it is recurrent under σ. By the claim, it is clear that if $i \in \mathbb{N}$ and $x_i = 1$ then $x_{2i} = 0$. So (x, x) is not recurrent under $\sigma \times \sigma^2$, because it will never return to $[1] \times [1]$.

For the case $d \geqslant 2$, we extend the above idea. We define a local rule of a substitution by

$$\tau\colon 1 \to 1a_1 \cdots a_{(d+1)^2-1}, \quad 0 \to 0a_1 \cdots a_{(d+1)^2-1},$$

where $a_j = 0$ for $j \equiv 0 \bmod (d+1)$ and $a_j = 1$ otherwise. Let $x = \lim_{k\to\infty} \tau^k(1)0^{\infty}$ be a fixed point of τ. As above, x is a minimal point.

For every $k \in \mathbb{N}$, x can be expressed as $x = [\tau^k(1)]^{d+1}\tau^k(0)\cdots$, so $(x, x, \ldots, x) \in X^d$ is recurrent under $\sigma \times \sigma^2 \times \cdots \times \sigma^d$. Analogously to the case $d = 1$, we prove that if $j \in \mathbb{N}$ and $x_j = 1$ then $x_{(d+1)j} = 0$. The details are left to the reader. So $(x, x, \ldots, x) \in X^{d+1}$ is not recurrent under $\sigma \times \sigma^2 \times \cdots \times \sigma^d \times \sigma^{d+1}$. □

3.2 Multi-recurrent points and factor maps

Let $\pi\colon (X, T) \to (Y, S)$ be a factor map. It is well known that if $y \in Y$ is a recurrent point of S, then there is a recurrent point $x \in X$ of T with $\pi(x) = y$. In this subsection, we investigate if this result holds for multi-recurrent points. It turns out that it is still the case for distal extensions but may fail for proximal extensions.

Proposition 3.10. *Let $\pi\colon (X, T) \to (Y, S)$ be a factor map.*

(1) *If $x \in X$ is multi-recurrent, then so is $\pi(x)$.*

(2) *If $y \in Y$ is multi-recurrent and $\pi^{-1}(y)$ consists of a single point x, then x is also multi-recurrent.*

Proof. (1) It is a direct consequence of continuity of π.

(2) Since $\pi^{-1}(y) = \{x\}$, for every neighborhood U of x there exists a neighborhood V of y such that $\pi^{-1}(V) \subset U$. Therefore $N(y, V) \subset N(x, U)$. It follows that if y is multi-recurrent, then so is x. □

By Remark 3.5 every distal system is multi-recurrent. In particular, every equicontinuous system is multi-recurrent. Therefore, the projection of minimal dynamical system onto its maximal equicontinuous factor maps every point onto a multi-recurrent point. It turns out that the system presented in Theorem 3.9 is a proximal extension of its maximal equicontinuous factor and there is a fiber not containing any multi-recurrent points.

Proposition 3.11. *There exist two dynamical systems (X,T) and (Y,S), a proximal factor map $\pi\colon (X,T)\to(Y,S)$ and a point $y\in Y$ which is multi-recurrent but $\pi^{-1}(y)$ does not contain any multi-recurrent points.*

Proof. Let τ be a local rule of a substitution defined by

$$\tau\colon 1\to 1101,\quad 0\to 0101,$$

i.e., τ is the substitution from the proof of Theorem 3.9. Let

$$x=\lim_{n\to\infty}\tau^n(1)0^\infty \quad\text{and}\quad z=\lim_{n\to\infty}\tau^n(0)0^\infty$$

be fixed points of τ. Let $X=\overline{\mathrm{Orb}(x,\sigma)}$. Then X is a minimal set and $z\in X$.

Observe that $z_0=0$, $z_k=1$ for $k=4^m(2n+1)$ and $z_k=0$ for $k=2\cdot 4^m(2n+1)$. In particular one has $z_i=x_i$ for $i>0$ (see the proof of Theorem 3.9). Note that if $z_j=0$ for some $j>0$ then $z_{2j}=1$ and if $z_j=1$ then $z_{2j}=0$. Neither (x,x) nor (z,z) is recurrent under $\sigma\times\sigma^2$.

Denote $k_n=|\tau^n(1)|=4^n$ and observe that position of 11 uniquely identifies position of $\tau(1)$ in $x=\tau(x)$. By the same argument $\tau(1)\tau(1)$ identifies uniquely beginning of $\tau^2(1)$ in x, etc. In other words, blocks $\tau^n(0)$ and $\tau^n(1)$ form a code for every $n\geqslant 1$ and hence there is a unique decomposition of x into blocks from $\{\tau^n(0),\tau^n(1)\}$. But X is the closure of the orbit of x which yields that for any $v\in X$ and any $n\geqslant 1$ there is a uniquely determined infinite concatenation $\{w^{(n)}\}_{j=1}^\infty$ of blocks over $\{\tau^n(0),\tau^n(1)\}$ and a block u_n of length $0\leqslant|u_n|<k_n$ such that $v=u_nw_1^{(n)}w_2^{(n)}w_3^{(n)}\cdots$.

With every n associate a natural projection $\xi_n\colon\mathbb{Z}_{k_{n+1}}\to\mathbb{Z}_{k_n}$, $\xi_n(x)=x \pmod{k_n}$. Then we obtain a well-defined inverse limit

$$Y=\varprojlim(\mathbb{Z}_{k_n},\xi_n)=\{(j_1,j_2,\ldots):\xi_n(j_{n+1})=j_n\}\subset\prod\mathbb{Z}_{k_n}.$$

In addition, Y is coordinatewise, modulo k_n on each coordinate n. Endowed with the product topology over the discrete topologies in Z_{k_n} space Y becomes a topological group satisfying the four properties characterizing odometers (see [9]). Let $S\colon Y\to Y$ be defined by $S(j_1,j_2,\ldots)=(j_1+1,j_2+1,\ldots)$. Then $Y=\overline{\mathrm{Orb}((0,0,\ldots),S)}$ and (Y,S) is a minimal dynamical system (an odometer).

With every $v\in X$ we can associate a sequence $j^{(v)}=(j_1^{(v)},j_2^{(v)},\ldots)\in Y$ given by $j_n^{(v)}=k_n-|u_n| \pmod{k_n}$. This way we obtain a natural factor map $\pi\colon(X,\sigma)\to(Y,S)$, $v\mapsto j^{(v)}$. Note that if $\pi(u)=\pi(u')$ then for every $n\geqslant 1$ taking $k=|\tau^{n+1}(0)|-j_{n+1}$ provides a decomposition $\sigma^k(u),\sigma^k(u')\in\{\tau^n(0),\tau^n(1)\}^{\mathbb{Z}_+}$, which in turn implies that u,u' share arbitrarily long common word of symbols (e.g., $\tau^n(0)$), and as a consequence u,u' form a proximal pair. This proves that π is a proximal extension. Denote $y=\pi(x)$.

To finish the proof observe that if $u\in X$ and $\pi(u)_n=0$ then $u\in\{\tau_2^n(0),\tau_2^n(1)\}^{\mathbb{Z}_+}$ by the definition of π. But if $\tau^n(0)$ is a prefix of u (the same for $\tau^n(1)$ and $\pi(u)_{n+1}=0$ then $\tau^{n+1}(0)$ must be a prefix of u (resp. $\tau^{n+1}(1)$ is a prefix). Therefore, if we put $y=(0,0,0,\ldots)$ then $\pi^{-1}(y)=\{x,z\}$ and every point in (Y,S) is multi-recurrent (it is a distal system and so Remark 3.5 applies). $\square$

To prove that multi-recurrent points can be lifted by distal extensions, we apply the theory of enveloping semigroup. Let (X,T) be a dynamical system. Endow X^X with the product topology. By the Tychonoff theorem, X^X is a compact Hausdorff space. The *enveloping semigroup* of (X,T), denoted by $E(X,T)$, is defined as the closure of the set $\{T^n:n\in\mathbb{Z}_+\}$ in X^X. We refer the reader to the book [3] for more details (see also [2]).

Theorem 3.12. *Let $\pi\colon (X,T)\to(Y,S)$ be a factor map, let $d\geqslant 1$ and assume that $y\in Y$ is recurrent under $S\times S^2\times\cdots\times S^d$. If $x\in\pi^{-1}(y)$ is such that the pair (x,z) is distal for any $z\in\pi^{-1}(y)$ with $z\neq x$, then x is recurrent under $T\times T^2\times\cdots\times T^d$. In particular, if y is multi-recurrent then so is x.*

Proof. Let $\pi_d=\pi\times\pi\times\cdots\times\pi\colon (X^d,T\times T^2\times\cdots\times T^d)\to(Y^d,S\times S^2\times\cdots\times S^d)$. Then π_d is a factor map. There exists a unique onto homomorphism $\theta\colon E(X^d,T\times T^2\times\cdots\times T^d)\to E(Y^d,S\times S^2\times\cdots\times S^d)$ such that $\pi_d(pz)=\theta(p)\pi_d(z)$ for any $p\in E(X^d,T\times\cdots\times T^d)$ and $z\in X^d$ (see [3, Theorem 3.7]). Since $(y,\dots,y)$ is recurrent under the action of $S\times S^2\times\cdots\times S^d$, by [2, Proposition 2.4] there is an idempotent $u\in E(Y^d,S\times S^2\times\cdots\times S^d)$ such that $u(y,\dots,y)=(y,\dots,y)$. If we denote $J=\theta^{-1}(u)$ then clearly it is a closed subsemigroup of $E(X^d,T\times T^2\times\cdots\times T^d)$ and so by Ellis-Numakura Lemma there is an idempotent $v\in J$.

Observe that

$$\pi_d(v(x,\dots,x))=\theta(v)\pi_d(x,\dots,x)=u(y,\dots,y)=(y,\dots,y),$$

hence each coordinate of $v(x,\dots,x)$ belongs to $\pi^{-1}(y)$. Furthermore, since v is an idempotent, we have $v(v(x,\dots,x))=v(x,\dots,x)$, thus again by [2, Proposition 2.4] we obtain that $v(x,\dots,x)$ and $(x,\dots,x)$ are proximal under $T\times T^2\times\cdots\times T^d$, and therefore each coordinate of $v(x,\dots,x)$ is proximal with x (under the action of T). But the pair (x,z) is distal for any $z\in\pi^{-1}(y)$ with $z\neq x$, which immediately implies that $v(x,\dots,x)=(x,\dots,x)$. Since v is an idempotent, it is equivalent to saying that $(x,\dots,x)$ is recurrent under $T\times T^2\times\cdots\times T^d$ which ends the proof. □

Corollary 3.13. *Let $\pi\colon (X,T)\to(Y,S)$ be a factor map. If π is distal, then a point $x\in X$ is multi-recurrent if and only if so is $\pi(x)$.*

3.3 The measure of multi-recurrent points

It follows from the Poincaré recurrence theorem that almost every point is recurrent for any invariant measure (see [12, Theorem 3.3]). A similar connection holds between multi-recurrent points and multiple recurrence in ergodic theory.

Theorem 3.14. *Let (X,T) be a dynamical system and μ be a T-invariant Borel probability measure on X. Then μ-almost every point of X is multi-recurrent for T.*

Proof. Choose a countable base $\{B_i\}_{i=1}^\infty$ for topology of X. For every $i\in\mathbb{N}$, let

$$A_i=\bigcup_{d=1}^\infty\bigg(B_i\Big\backslash\bigcup_{n=1}^\infty B_i\cap T^{-n}B_i\cap T^{-2n}B_i\cap\cdots\cap T^{-dn}B_i\bigg).$$

Note that a point x is not multi-recurrent if and only if there exist $d\geqslant 1$ and $i\in\mathbb{N}$ such that $x\in B_i$ but $x\notin B_i\cap T^{-n}B_i\cap\cdots\cap T^{-dn}B_i$ for all $n\in\mathbb{N}$. Therefore $\bigcup_{i=1}^\infty A_i$ is the collection of non-multi-recurrent points of (X,T). By the multiple recurrence theorem, $\mu(A_i)=0$ for every $i\geqslant 1$. Then $\mu(\bigcup_{i=1}^\infty A_i)=0$. □

Corollary 3.15. *If a dynamical system (X,T) admits an ergodic invariant Borel probability measure μ with full support, then there exists a dense G_δ subset X_0 of X with full μ-measure such that every point in X_0 is both transitive and multi-recurrent.*

Proof. Since μ is ergodic, then the set of all transitive points is a dense G_δ subset of X and has full μ-measure. By Lemma 3.3 and Theorem 3.14, the set of all multi-recurrent point is also a dense G_δ subset of X and has full μ-measure. Then the intersection of those two sets is as required. □

Using results on multiple recurrence developed by Furstenberg [11], we strengthen Theorem 3.14 as follows.

Theorem 3.16. *Let (X,T) be a dynamical system. For every T-invariant Borel probability measure μ on X, there exists a Borel subset X_0 of X with $\mu(X_0)=1$ such that for every $x\in X_0$, every $d\in\mathbb{N}$ and every neighborhood U of x the set $N_{T\times T^2\times\cdots\times T^d}((x,\dots,x),U\times\cdots\times U)$ has positive upper density.*

Proof. For every $d \in \mathbb{N}$ and every $\delta > 0$, let $A_{d,\delta}$ be the collection of all points $x \in X$ for which there exists a neighborhood U of x with $\operatorname{diam}(U) < \delta$ such that the set

$$N_{T\times T^2\times\cdots\times T^d}((x,\dots,x), U\times\cdots\times U)$$

has positive upper density.

Let μ be an ergodic T-invariant Borel probability measure on X. We are going to show that $\mu(A_{d,\delta}) = 1$ for every $d \in \mathbb{N}$ and every $\delta > 0$. First we show that $A_{d,\delta}$ is Borel measurable. To this end, for every $t > 0$ and every $n, m \in \mathbb{N}$, let $A_{d,\delta}(t, n, m)$ be the collection of all points $x \in X$ such that there exists an neighborhood U of x with $\operatorname{diam}(U) < \delta$ satisfying

$$\frac{1}{n}\#(N_{T\times T^2\times\cdots\times T^d}((x,\dots,x), U\times\cdots\times U)\cap[0,n-1]) > t - \frac{1}{m}.$$

It is clear that $A_\delta(t, n, m)$ is an open subset of X and

$$A_{d,\delta} = \bigcup_{k=1}^{\infty}\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty} A_{d,\delta}\Big(\frac{1}{k}, n, m\Big).$$

It follows that $A_{d,\delta}$ is Borel measurable.

If $\mu(A_{d,\delta}) < 1$, then we can choose a Borel subset $B \subset X\backslash A_\delta$ with $\operatorname{diam}(B) < \delta/3$ and $\mu(B) > 0$. For any $x \in X$, let

$$g(x) = \limsup_{N\to\infty}\frac{1}{N}\sum_{i=0}^{N-1}\mathbf{1}_{B\cap T^{-i}B\cap\cdots\cap T^{-id}B}(x).$$

Then g is also Borel measurable and $0 \leqslant g(x) \leqslant 1$ for any $x \in X$. By the Fatou lemma and [12, Theorem 7.14], we have

$$\begin{aligned}\int_X g(x)d\mu(x) &\geqslant \limsup_{N\to\infty}\frac{1}{N}\int_X\sum_{i=0}^{N-1}\mathbf{1}_{B\cap T^{-i}B\cap\cdots\cap T^{-id}B}(x)d\mu(x)\\ &\geqslant \liminf_{N\to\infty}\frac{1}{N}\sum_{i=0}^{N-1}\mu(B\cap T^{-i}B\cap\cdots\cap T^{-id}B) > 0.\end{aligned}$$

Clearly $g(x) = 0$ for any $x \notin B$, hence there exists some $x \in B$ such that $g(x) > 0$. Let $U = B(x, \frac{2}{3}\delta)$. Then $B \subset U$ and the upper density of $N_{T\times T^2\times\cdots\times T^d}((x,\dots,x), U\times\cdots\times U)$ is not less than $g(x)$. We obtain that $x \in A_{d,\delta}$, which leads to a contradiction.

Therefore $\mu(A_{d,\delta}) = 1$ for every ergodic measure μ, every $d \in \mathbb{N}$ and every $\delta > 0$. Let

$$X_0 = \bigcap_{d=1}^{\infty}\bigcap_{k=1}^{\infty} A_{d,\frac{1}{k}}.$$

Then $\mu(X_0) = 1$ for every ergodic measure, and by the ergodic decomposition the same holds for any T-invariant measure. Therefore X_0 is as required. □

Remark 3.17. Assume that pointwise convergence of multiple averages holds for μ, i.e., for every $d \in \mathbb{N}$ and $f_1, f_2, \dots, f_d \in L^\infty(\mu)$,

$$\frac{1}{N}\sum_{n=0}^{N-1} f_1(T^n x)f_2(T^{2n}x)\cdots f_d(T^{dn}x) \quad \text{converges} \quad \mu \quad \text{a.e.}$$

Then the proof of Theorem 3.16 can be modified by replacing limsup in the definition of g by liminf, and the modified proof yields that for every $x \in X_0$, every $d \in \mathbb{N}$ and every neighborhood U of x the set $N_{T\times T^2\times\cdots\times T^d}((x,\dots,x), U\times\cdots\times U)$ has positive lower density. Unfortunately, the pointwise convergence of multiple averages for general ergodic measures is still an open problem. It was proved recently that the pointwise convergence of multiple averages holds for distal measures (see [22]).

Glasner [15] proved that if a minimal system (X,T) is topologically weakly mixing, then there is a dense G_δ subset X_0 such that for each $x \in X_0$, the orbit of $(x,\dots,x)$ is dense in X^d under $T\times T^2\times\cdots\times T^d$. Below we present an analogous result for systems possessing a fully weakly mixing invariant measure. Note that Lehrer [25] proved a variant of the Jewett-Krieger theorem, which implies that there are topologically weakly mixing minimal systems without weakly mixing invariant measures. Therefore our result complements Glasner's theorem.

Theorem 3.18. *Let (X,T) be a dynamical system. If there exists a weakly mixing, fully supported T-invariant Borel probability measure μ on X, then there exists a Borel subset X_0 of X with $\mu(X_0)=1$ such that for every $x\in X_0$, every $d\in\mathbb{N}$, and every non-empty open subsets $U_1,U_2,\dots,U_d$ of X the set*

$$N_{T\times T^2\times\cdots\times T^d}((x,x,\dots,x),U_1\times U_2\times\cdots\times U_d)$$

has positive upper density.

Proof. For every $d\in\mathbb{N}$ and every $\delta>0$, let $A_{d,\delta}$ be the collection of all points $x\in X$ such that there exists an open cover $\{U_i\}_{i=1}^{\ell}$ of X with $\operatorname{diam}(U_i)<\delta$ for $i=1,\dots,\ell$ and such that for every $\alpha\in\{1,2,\dots,\ell\}^d$ the set $N_{T\times T^2\times\cdots\times T^d}((x,x,\dots,x),U_{\alpha(1)}\times U_{\alpha(2)}\times\cdots\times U_{\alpha(d)})$ has positive upper density.

Following the same lines as in the proof of Theorem 3.16 we obtain that $A_{d,\delta}$ is Borel measurable. We are going to show that $\mu(A_{d,\delta})=1$.

If $\mu(A_{d,\delta})<1$, there exists a Borel set $W_0\subset X\backslash A_\delta$ with $\operatorname{diam}(W_0)<\delta/2$ and $\mu(W_0)>0$. Fix an open cover $\{U_i\}_{i=1}^{p}$ of X with $\operatorname{diam}(U_i)<\delta$ for $i=1,\dots,\ell$. Enumerate $\{1,2,\dots,p\}^d$ as $\{\alpha_1,\alpha_2,\dots,\alpha_k\}$ with $k=p^d$.

First note that $\mu(U_j)>0$ for $i=1,2,\dots,\ell$ since μ has the full support. For every $x\in X$, let

$$g_1(x)=\limsup_{N\to\infty}\frac{1}{N}\sum_{l=0}^{N-1}\mathbf{1}_{W_0\cap T^{-l}U_{\alpha_1(1)}\cap\cdots\cap T^{-ld}U_{\alpha_1(d)}}(x).$$

Then g_1 is also Borel measurable and $0\leqslant g_1(x)\leqslant 1$ for any $x\in X$. The measure μ is weakly mixing, hence we can apply [11, Theorem 2.2] obtaining that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{l=1}^{N}\mathbf{1}_{W_0\cap T^{-l}U_{\alpha_1(1)}\cap\cdots\cap T^{-ld}U_{\alpha_1(d)}}(x)=\mathbf{1}_{W_0}(x)\prod_{l=1}^{d}\mu(U_{\alpha_1(l)})$$

in $L^2(X)$. In particular $\int_X g_1(x)d\mu>0$. Clearly $g_1(x)=0$ for any $x\notin W_0$. Then there exists a Borel set $W_1\subset W_0$ with $\mu(W_1)>0$ and $g_1(x)>0$ for any $x\in W_1$. Note that for every $x\in W_1$ the upper density of $N_{T\times T^2\times\cdots\times T^d}((x,x,\dots,x),U_{\alpha_1(1)}\times U_{\alpha_1(2)}\times\cdots\times U_{\alpha_1(d)})$ is not less than $g_1(x)$.

Working by induction, for every $i=1,2,\dots,k$, we can construct a Borel set $W_i\subset W_{i-1}$ with $\mu(W_i)>0$ such that for every $x\in W_i$ the set $N_{T\times T^2\times\cdots\times T^d}((x,x,\dots,x),U_{\alpha_i(1)}\times U_{\alpha_i(2)}\times\cdots\times U_{\alpha_i(d)})$ has positive upper density. This implies that for every $x\in W_k$ and every $\alpha\in\{1,2,\dots,\ell\}^d$ the set $N_{T\times T^2\times\cdots\times T^d}((x,x,\dots,x),U_{\alpha(1)}\times U_{\alpha(2)}\times\cdots\times U_{\alpha(d)})$ has positive upper density. Then $W_k\subset A_{d,\delta}$, which leads to a contradiction, hence $\mu(A_{d,\delta})=1$.

To finish the proof, it is enough to put

$$X_0=\bigcap_{d=1}^{\infty}\bigcap_{k=1}^{\infty}A_{d,\frac{1}{k}},$$

since $\mu(X_0)=1$ and X_0 is as required. □

Remark 3.19. One can modify the proof of Theorem 3.18, by replacing $A_{d,\delta}$ by $A'_{d,\delta}$ defined as the collection of all points $x\in X$ such that there exists an open cover $\{U_i\}_{i=1}^{\ell}$ of X with $\operatorname{diam}(U_i)<\delta$ for $i=1,\dots,\ell$ for which the set

$$N_{T\times T^2\times\cdots\times T^d}((x,x,\dots,x),U_{\alpha(1)}\times U_{\alpha(2)}\times\cdots\times U_{\alpha(d)})$$

is not empty for every $\alpha \in \{1, 2, \ldots, \ell\}^d$. Then one obtains that $A'_{d,\delta}$ is a dense open subset of X and

$$X'_0 = \bigcap_{d=1}^{\infty} \bigcap_{k=1}^{\infty} A'_{d,\frac{1}{k}}$$

is a dense G_δ subset of X with full μ-measure. Moreover, for every $d \in \mathbb{N}$ and every $x \in X'_0$, the orbit of $(x, x, \ldots, x)$ is dense in X^d under $T \times T^2 \times \cdots \times T^d$. Since $(X^d, T \times T^2 \times \cdots \times T^d)$ is an E-system, by [21, Lemma 3.6] we know that for every $x \in X'_0$, every $d \in \mathbb{N}$ and every non-empty open subsets $U_1, U_2, \ldots, U_d$ of X the set $N_{T\times T^2\times\cdots\times T^d}((x, x, \ldots, x), U_1 \times U_2 \times \cdots \times U_d)$ has positive upper Banach density, but we cannot conclude that it has positive upper density. On the other hand, we do not know whether the set X_0 constructed in Theorem 3.18 is residual.

4 Van der Waerden systems and $\mathcal{AP}$-recurrent points

In this section, we introduce the concept of a van der Waerden system. We explore how this notion relates to the behaviour of multi-recurrent points and $\mathcal{AP}$-recurrent points.

Definition 4.1. We say that a dynamical system (X, T) is a *van der Waerden system* if it satisfies the topological multiple recurrence property, i.e., for every non-empty open set $U \subset X$ and every $d \in \mathbb{N}$ there exists an $n \in \mathbb{N}$ such that

$$U \cap T^{-n}U \cap T^{-2n}U \cap \cdots \cap T^{-dn}U \neq \emptyset.$$

By the topological multiple recurrence theorem, we know that every minimal system is a van der Waerden system. It follows from the ergodic multiple recurrence theorem that every E-system is a van der Waerden system.

It is easy to see that if (X, T) is a van der Waerden system, then the relation $R = \bigcap_{d=1}^{\infty} R_d$ is residual, where

$$R_d = \left\{ y \in X : \exists\, n \geqslant 1 \text{ such that } \rho(y, T^{in}y) < \frac{1}{d} \text{ for } i = 0, 1, \ldots, d \right\}.$$

As a corollary, we obtain the following (see Lemma 3.3).

Lemma 4.2. *A dynamical system (X, T) is a van der Waerden system if and only if it has a dense set of multi-recurrent points.*

By Lemmas 4.2 and 3.2, we have the following result.

Proposition 4.3. *Let (X, T) be a dynamical system. Then the following conditions are equivalent*:

1. (X, T) *is a van der Waerden system*;
2. (X, T^n) *is a van der Waerden system for some* $n \in \mathbb{N}$;
3. (X, T^n) *is a van der Waerden system for any* $n \in \mathbb{N}$.

Lemma 3.4 implies that every point in a uniformly rigid system is multi-recurrent. Then by Lemma 4.2 every uniformly rigid system is a van der Waerden system. By [1,17], every almost equicontinuous system is uniformly rigid. We have just proved the following.

Proposition 4.4. *Every almost equicontinuous system is also a van der Waerden system.*

Moothathu introduced Δ-transitive systems in [30]. Recall that a dynamical system (X, T) is Δ-*transitive* if for every $d \in \mathbb{N}$ there exists $x \in X$ such that the diagonal d-tuple $(x, x, \ldots, x)$ has a dense orbit under the action of $T \times T^2 \times \cdots \times T^d$.

Proposition 4.5. *If a dynamical system (X, T) is Δ-transitive, then it is a van der Waerden system.*

Proof. Let U be a non-empty open subset of X and fix any $d \in \mathbb{N}$. There exists $x \in X$ such that diagonal d-tuple $(x, x, \ldots, x)$ has a dense orbit under the action of $T \times T^2 \times \cdots \times T^d$. Then there exists $n \in \mathbb{N}$ such that $T^n x \in U, T^{2n}x \in U, \ldots, T^{dn}x \in U$ and thus

$$T^n x \in U \cap T^{-n}U \cap \cdots \cap T^{-(d-1)n}U.$$

This shows that (X,T) is a van der Waerden system. $\square$

By Proposition 3.11, multi-recurrent points may not be lifted through factor maps. To remove this disadvantage, we introduce the following slightly weaker notion of $\mathcal{AP}$*-recurrent* point. As we will see later, it is possible to characterize van der Waerden systems through $\mathcal{AP}$-recurrent points.

Definition 4.6. A point $x \in X$ is $\mathcal{AP}$*-recurrent* if $N(x,U)$ is an AP-set for every open neighborhood U of x .

Remark 4.7. It is clear that every multi-recurrent point is $\mathcal{AP}$-recurrent and every $\mathcal{AP}$-recurrent point is recurrent. The notion of $\mathcal{AP}$-recurrent points can be seen as an intermediate notion of recurrence. By Proposition 4.14, every minimal point is $\mathcal{AP}$-recurrent since minimal systems are van der Waerden systems. But by Theorem 3.9, there exist some minimal points which are not multi-recurrent. Those minimal points are $\mathcal{AP}$-recurrent but not multi-recurrent. Every transitive point of the dynamical system presented in the proof of Proposition 4.17 is not $\mathcal{AP}$-recurrent. So those transitive points are recurrent but not $\mathcal{AP}$-recurrent.

Lemma 4.8. *Let (X,T) be a dynamical system.*

(1) *The collection of all $\mathcal{AP}$-recurrent points of (X,T) is a G_δ subset of X.*

(2) *(X,T) is a van der Waerden system if and only if it has a dense set of $\mathcal{AP}$-recurrent points.*

Proof. (1) Given $d \geqslant 1$, let

$$Q_d = \left\{ y \in X \colon \exists\, n, a \geqslant 1 \text{ such that } \rho(y, T^{in+a}y) < \frac{1}{d} \text{ for } i = 0, 1, \dots, d \right\}.$$

It is clear that every Q_d is open, hence $Q = \bigcap_{d=1}^{\infty} Q_d$ is a G_δ subset of X. It is easy to see that $Q = \bigcap_{d=1}^{\infty} Q_d$ is the set of all $\mathcal{AP}$-recurrent points.

(2) First note that by Lemma 4.2 every van der Waerden system has dense set of multi-recurrent points, hence $\mathcal{AP}$-recurrent points are dense.

On the other hand, if x is $\mathcal{AP}$-recurrent and $x \in U$ then for every $d \geqslant 1$ there are $a, n \geqslant 1$ such that $T^{a+in}x \in U$ for every $i = 0, 1, \dots, d$ and so

$$T^a x \in U \cap T^{-n}U \cap T^{-2n}U \cap \cdots \cap T^{-dn}U$$

completing the proof. $\square$

We have the following connection between $\mathcal{AP}$-recurrent points and their orbit closures.

Proposition 4.9. *Let (X,T) be a dynamical system and $x \in X$. Then x is $\mathcal{AP}$-recurrent if and only if $(\overline{\mathrm{Orb}(x,T)}, T)$ is a van der Waerden system.*

Proof. If x is $\mathcal{AP}$-recurrent, then every point in the orbit of x is also $\mathcal{AP}$-recurrent. By Lemma 4.8, $(\overline{\mathrm{Orb}(x,T)}, T)$ is a van der Waerden system.

Now assume that $(\overline{\mathrm{Orb}(x,T)}, T)$ is a van der Waerden system. By Lemma 4.8, $(\overline{\mathrm{Orb}(x,T)}, T)$ has a dense set of $\mathcal{AP}$-recurrent points. Fix an open neighborhood U of x. It suffices to show that $N(x,U) \in \mathcal{AP}$. Choose an $\mathcal{AP}$-recurrent point y in U. For every $d \geqslant 1$, there exist $k, n \in \mathbb{N}$ such that

$$T^k y \in U, T^{k+n} y \in U, T^{k+2n} y \in U, \dots, T^{k+dn} y \in U.$$

By continuity of T, there exists an open neighborhood V of y such that for any $z \in V$ we have

$$T^k z \in U, \quad T^{k+n} z \in U, \quad T^{k+2n} z \in U, \dots, T^{k+dn} z \in U.$$

Since $y \in \overline{\mathrm{Orb}(x,T)}$, there exists $m \geqslant 0$ such that $T^m x \in V$. Then

$$T^{m+k} x \in U, \quad T^{m+k+n} x \in U, \quad T^{m+k+2n} x \in U, \dots, T^{m+k+dn} x \in U,$$

which implies that $N(x,U)$ is an AP-set. This ends the proof. $\square$

Proposition 4.10. *Let (X,T) be a dynamical system and $x \in X$. Then the following conditions are equivalent*:

(1) *x is an $\mathcal{AP}$-recurrent point in (X,T)*;

(2) *x is an $\mathcal{AP}$-recurrent point in (X,T^n) for some $n \in \mathbb{N}$*;

(3) *x is an $\mathcal{AP}$-recurrent point in (X,T^n) for any $n \in \mathbb{N}$.*

Proof. The implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are clear. We only need to show (1) $\Rightarrow$ (3). Fix $n \in \mathbb{N}$. Without loss of generality, we can assume that $X = \overline{\mathrm{Orb}(x,T)}$. Then (X,T) is a topologically transitive system, because x is a recurrent point. Moreover, as x is $\mathcal{AP}$-recurrent in (X,T), applying Proposition 4.9 we get that (X,T) is a van der Waerden system. Denote $X_0 = \overline{\mathrm{Orb}(x,T^n)}$. It is well known (see, for example, [27, Lemma 6.5]) that the interior of X_0 (with respect to the topology of X) is dense in X_0, i.e., X_0 is a regular closed subset of X. By Lemma 4.2, the collection of multi-recurrent points in (X,T) is dense in X. By Lemma 3.2, every point multi-recurrent under action of T is also multi-recurrent for T^n. Hence the set of multi-recurrent points of (X_0,T^n) is dense in X_0. By Lemma 4.2 again, (X_0,T^n) is a van der Waerden system. By Proposition 4.9 we obtain that every transitive point in (X_0,T^n) is $\mathcal{AP}$-recurrent. So x is also $\mathcal{AP}$-recurrent in (X_0,T^n). □

In the proof of the next result we will employ the technique developed in [27] and show that every $\mathcal{AP}$-recurrent point can be lifted through factor maps.

Proposition 4.11. *Let $\pi\colon (X,T) \to (Y,S)$ be a factor map. If $y \in Y$ is an $\mathcal{AP}$-recurrent point, then there exists an $\mathcal{AP}$-recurrent point $x \in X$ such that $\pi(x) = y$.*

Proof. It is clear that for any $n \in \mathbb{Z}$ and any $F \in \mathcal{AP}$, the translation of F by n denoted by

$$n + F = \{n + k \in \mathbb{N}\colon k \in F\},$$

is also an AP-set. In other words, the family $\mathcal{AP}$ is translation invariant (see [27, p. 263]). Recall that the family $\mathcal{AP}$ has the Ramsey property. Then by [27, Lemma 3.4], all the assumptions of [27, Proposition 4.5] are satisfied by $\mathcal{AP}$. The result follows by application of [27, Proposition 4.5] to the family $\mathcal{AP}$. □

Remark 4.12. The proof of Proposition 4.11 which is short and compact, uses advanced machinery from [27]. Another more elementary proof will be given later in Section 6.

To characterize when a transitive system is a van der Waerden system, we need the following definition. It is a special case of a notion considered in [26].

Definition 4.13. We say that $x \in X$ is an *$\mathcal{AP}$-transitive point* if $N(x,U)$ is an AP-set for every non-empty open set $U \subset X$.

Proposition 4.14. *Let (X,T) be a transitive system. Then the following conditions are equivalent*:

(1) *(X,T) is a van der Waerden system*;

(2) *there exists an $\mathcal{AP}$-transitive point*;

(3) *every transitive point is an $\mathcal{AP}$-transitive point.*

Proof. The implication (3) $\Rightarrow$ (2) is obvious and (2) $\Rightarrow$ (1) follows from Proposition 4.9. We only need to show that (1) $\Rightarrow$ (3).

Let x be a transitive point. It follows from Proposition 4.9 that x is an $\mathcal{AP}$-recurrent point. Fix a non-empty open subset U of X. There exist a neighborhood V of x and $k \in \mathbb{N}$ such that $T^kV \subset U$. Then $k + N(x,V) \subset N(x,U)$. But $N(x,V)$ is an AP-set and so $N(x,U)$ is also an AP-set, which proves that x is an $\mathcal{AP}$-transitive point. □

Proposition 4.15. *Let (X,T) be a transitive system. If (X,T) is a van der Waerden system, then $(X^n,T^{(n)})$ is also a van der Waerden system for every $n \in \mathbb{N}$, where $T^{(n)}$ denotes n-times Cartesian product $T^{(n)} = T \times T \times \cdots \times T$.*

Proof. Let $U_1, U_2, \dots, U_n$ be non-empty open subsets in X. Pick a transitive point $x \in U_1$. Then there exist $k_1, k_2, \dots, k_{n-1} \in \mathbb{N}$ such that $T^{k_1}x \in U_2, T^{k_2}x \in U_3, \dots, T^{k_{n-1}}x \in U_n$. Since (X,T) is a van der Waerden system, x is $\mathcal{AP}$-recurrent. This immediately implies that $(x, T^{k_1}x, T^{k_2}x, \dots, T^{k_{n-1}}x)$

is $\mathcal{AP}$-recurrent in $(X^n, T^{(n)})$, hence $(X^n, T^{(n)})$ has a dense set of $\mathcal{AP}$-recurrent points. The proof is finished by application of Lemma 4.8. □

The following example shows that Proposition 4.15 is no longer true if we do not assume that (X, T) is transitive. As a byproduct, we obtain two van der Waerden systems whose product is not a van der Waerden system.

Example 4.16. Let $n_1 = 2$ and define inductively $n_{k+1} = (n_k)^3$. Put $A_k = [n_k, (n_k)^2] \cap \mathbb{N}$ and denote $S = \bigcup_{k=1}^{\infty} A_{2k}$ and $R = \bigcup_{k=1}^{\infty} A_{2k+1}$. Clearly, $S \cap R = \emptyset$. Denote by X_S and X_T the following subshifts (so-called spacing shifts, see [5]):

$$X_S = \{x \in \{0,1\}^{\mathbb{N}} \colon x_i = x_j = 1 \Rightarrow |i - j| \in S \cup \{0\}\},$$
$$X_R = \{x \in \{0,2\}^{\mathbb{N}} \colon x_i = x_j = 2 \Rightarrow |i - j| \in R \cup \{0\}\}.$$

We can consider X_S and X_R as subshifts of $\{0,1,2\}^{\mathbb{N}}$. Let $X = X_S \cup X_T \subset \{0,1,2\}^{\mathbb{N}}$. For a word w over $\{0,1,2\}^{\mathbb{N}}$, we write $[w]_S = [w] \cap X_S$ and $[w]_R = [w] \cap X_R$. First note that the product system $(X \times X, \sigma \times \sigma)$ is not a van der Waerden system. This is because

$$N_{\sigma\times\sigma}(([1] \times [2]) \cap X, ([1] \times [2]) \cap X) = N_\sigma([1]_S, [1]_S) \cap N_\sigma([2]_R, [2]_R) = S \cap R = \emptyset.$$

Now we show that (X, σ) is a van der Waerden system. It it enough to prove that both (X_S, σ) and (X_R, σ) are van der Waerden systems. We will consider only the case of (X_S, σ), since the proof for (X_R, σ) is the same.

Fix a word $w \in \mathcal{L}(X_S)$, take any positive integer k such that $n_{2k} > 2(d + |w|)$ and consider the following sequence $x = \left(w0^{n_{2k}}\right)^{d+1} 0^\infty$. We claim that $x \in X_S$. Take any integers $i < j$ with $x_i = x_j = 1$. If $j - i \leqslant |w|$, then $j - i \in S$ by the choice of w. In the remaining case $j - i > |w|$ we have

$$n_{2k} \leqslant j - i \leqslant (d+1)|w0^{n_{2k}}| = (d+1)(|w| + n_{2k}) \leqslant \frac{n_{2k}}{2}\left(\frac{n_{2k}}{2} + n_{2k}\right) < (n_{2k})^2,$$

therefore also in this case $j - i \in S$. Indeed, $x \in X_S$. Put $m = |w0^{n_{2k}}|$ and observe that

$$x, T^m x, T^{2m} x, \ldots, T^{dm} x \in [w]_S.$$

But for every non-empty open set $U \subset X_S$ we can find a word w such that $[w]_S \subset U$ and then there is m such that

$$x \in U \cap T^{-n} U \cap T^{-2n} U \cap \cdots \cap T^{-dn} U.$$

This shows that (X_S, σ) is a van der Waerden system.

By Proposition 4.5 every Δ-transitive system is a van der Waerden system. On the other hand, [30, Proposition 3] provides an example of a strongly mixing system which is not Δ-transitive. In fact, we will show that the example in [30, Proposition 3] is not even a van der Waerden system.

Proposition 4.17. *There exists a strongly mixing system which is not a van der Waerden system.*

Proof. Let $\mathcal{F}$ be a collection of finite words over $\{0,1\}$ satisfying the following two conditions: the word 11 is in $\mathcal{F}$ and if u and v are two finite words over $\{0,1\}$ such that $|u| = |v|$, then the word $1u1v1$ is in $\mathcal{F}$. Let $X = X_{\mathcal{F}}$ be the subshift specified by taking $\mathcal{F}$ as the collection of forbidden words. Note that X is non-empty since $0^\infty, 0^n 1 0^\infty \in X$ for every $n \geqslant 0$.

Put $W = [1]_X$ and assume that there exists $n \in \mathbb{N}$ such that $W \cap \sigma^{-n} W \cap \sigma^{-2n} W \neq \emptyset$. Then there exist two words u and v with length $n - 1$ such that $1u1v10^\infty \in X$, which leads to a contradiction. This shows that (X, σ) is not a van der Waerden system.

Now we show that (X, σ) is strongly mixing. Let u and v be two words in the language of X. Put $N = |u| + |v|$. For every $n \geqslant N$, one has $u0^n v0^\infty \in X$. This implies that $n \in N([u]_X, [v]_X)$ for every $n \geqslant N$, proving that (X, σ) is strongly mixing. □

Remark 4.18. In fact, one can show that the only $\mathcal{AP}$-recurrent point of (X,σ) in Proposition 4.17 is the fixed point 0^∞.

Proposition 4.19. *Let $\pi\colon (X,T)\to(Y,S)$ be a factor map.*

(1) *If (X,T) is a van der Waerden system, then so is (Y,S).*

(2) *If (Y,S) is a van der Waerden system, then there exists a van der Waerden subsystem (Z,T) of (X,T) such that $\pi(Z)=Y$.*

(3) *If π is almost one to one, then (X,T) is a van der Waerden system if and only if (Y,S) is a van der Waerden system.*

Proof. (1) It is a consequence of the definition of van der Waerden system.

(2) By Lemma 4.8, the set of $\mathcal{AP}$-recurrent point of (Y,S), denoted by Y_0, is a dense subset of Y. Then by Proposition 4.11, for every $y\in Y_0$, there exists $x_y\in X$ such that $\pi(x_y)=y$ and x_y is $\mathcal{AP}$-recurrent. Let $X_0=\{x_y\colon y\in Y_0\}$ and $Z=\overline{\bigcup_{x\in X_0}\mathrm{Orb}(x,T)}$. Clearly $\pi(Z)=Y$. For every $x\in X_0$, any point in $\mathrm{Orb}(x,T)$ is $\mathcal{AP}$-recurrent. So Z has a dense set of $\mathcal{AP}$-recurrent points and so (Z,T) is a van der Waerden system by Lemma 4.8.

(3) By (1), we only need to prove that when π is almost one-to-one and (Y,S) is a van der Waerden system then (X,T) is also a van der Waerden system.

If we put $X_0=\{x\in X\colon \pi^{-1}(\pi(x))=\{x\}\}$, then by the definition of an almost one-to-one factor, X_0 is residual in X. For every $x\in X_0$ and every neighborhood U of x there is a neighborhood V of $\pi(x)$ such that $\pi^{-1}(V)\subset U$. This implies that $\pi(X_0)$ is residual in Y. By Lemma 4.8, the set of $\mathcal{AP}$-recurrent points of (Y,S), denoted again by Y_0, is a residual subset of Y. Then $\pi(X_0)\cap Y_0$ is also residual in Y and $\pi^{-1}(\pi(X_0)\cap Y_0)$ is residual in X. By Proposition 4.11, every point in $\pi^{-1}(\pi(X_0)\cap Y_0)$ is $\mathcal{AP}$-recurrent. Thus (X,T) is a van der Waerden system by Lemma 4.8. □

5 Multiple IP-recurrence property

To get a dynamical characterization of C-sets, the second author of this paper introduced in [27] a class of dynamical system satisfying the multiple IP-recurrence property. In this section, we study this property and its relation to the van der Waerden systems.

Definition 5.1. We say that a dynamical system (X,T) has the *multiple IP-recurrence property* if for every non-empty open subset U of X, every $d\geqslant 1$ and every IP-sets

$$FS\{p_i^{(1)}\}_{i=1}^\infty,\quad FS\{p_i^{(2)}\}_{i=1}^\infty,\quad \ldots,\quad FS\{p_i^{(d)}\}_{i=1}^\infty$$

in $\mathbb{N}$, there exists a finite subset α of $\mathbb{N}$ such that

$$U\cap T^{-\sum_{i\in\alpha}p_i^{(1)}}U\cap T^{-\sum_{i\in\alpha}p_i^{(2)}}U\cap\cdots\cap T^{-\sum_{i\in\alpha}p_i^{(d)}}U\neq\emptyset.$$

It is clear that if a dynamical system (X,T) has the multiple IP-recurrent property, then it is a van der Waerden system.

By [13, Theorem A], we know that every E-system has the multiple IP-recurrent property. It is shown in [17] that every E-system is either equicontinuous or sensitive. We show that this dichotomy also holds for transitive systems with the multiple IP-recurrence property. This is an extension of the main result in [17] because there are transitive multiple IP-recurrent systems which are not E-systems (see Remark 5.6).

Theorem 5.2. *If (X,T) is a transitive system with the multiple IP-recurrence property, then (X,T) is either equicontinuous or sensitive.*

Proof. Every transitive system is either almost equicontinuous or sensitive (see [1]), so let us assume that (X,T) is almost equicontinuous. It suffices to show that (X,T) is minimal, since every minimal almost equicontinuous system is equicontinuous (see [4]).

Pick a transitive point x of (X,T). By [1, Theorem 2.4], the set of transitive points coincides with the set of equicontinuity points. Then x is also an equicontinuity point. Fix any open neighborhood U of x and take $\varepsilon > 0$ such that the open ε-ball around x is contained in U. By equicontinuity of x there is $\delta > 0$ such that if $\rho(x,y) < \delta$ then $\rho(T^i x, T^i y) < \varepsilon/2$ for every integer $i \geqslant 0$. Let V denote the open δ-ball around x. Since (X,T) has the multiple IP-recurrence property, for every IP-set $\mathrm{FS}\,\{p_i\}_{i=1}^{\infty}$ there exists a finite subset α of $\mathbb{N}$ such that $V \cap T^{-\sum_{i\in\alpha} p_i} V \neq \emptyset$. It follows that $N(V,V)$ is an IP*-set. In particular, $N(V,V)$ is a syndetic set. Next observe that if $y \in V$, then $\rho(x,y) < \delta$. Therefore if $y, T^n y \in V$, then $\rho(T^n x, T^n y) < \varepsilon/2$ and $\rho(T^n y, x) < \varepsilon/2$. It follows that $T^n x \in U$ and therefore $N(V,V) \subset N(x,U)$. So $N(x,U)$ is syndetic. This implies that x is a minimal point and hence (X,T) is minimal. □

Remark 5.3. It is shown in [1] that there exists an almost equicontinuous system (X,T) which is not equicontinuous. By Proposition 4.4, the system (X,T) is a van der Waerden systems. But it cannot have the multiple IP-recurrence property by Theorem 5.2.

Next, we will modify the example constructed in Proposition 4.17, to obtain a strongly mixing van der Waerden system without the multiple IP-recurrence property.

Proposition 5.4. *There is a strongly mixing system which is a van der Waerden system but does not have the multiple IP-recurrence property.*

Proof. We are going to construct a subshift X and two IP-sets $\mathrm{FS}\{p_i\}_{i=1}^{\infty}, \mathrm{FS}\{q_i\}_{i=1}^{\infty}$ such that for every finite $\alpha \subset \mathbb{N}$ we have

$$[1]_X \cap T^{-\sum_{i\in\alpha} p_i}[1]_X \cap T^{-\sum_{i\in\alpha} q_i}[1]_X = \emptyset.$$

Let us take any sequences $\{p_i\}_{i=1}^{\infty}$ and $\{q_i\}_{i=1}^{\infty}$ satisfying

$$\sum_{j=1}^{n} p_j < p_{n+1} \quad \text{and} \quad q_n = 2^n p_{n+1} \quad \text{for every} \quad n \in \mathbb{N}.$$

Let $\mathcal{F}$ be a collection of finite words over $\{0,1\}$ satisfying the following two conditions: the words 11 is in $\mathcal{F}$, and if u and v are two finite words over $\{0,1\}$ such that $|u| = \sum_{i\in\alpha} p_i - 1$ and $|u|+|v| = \sum_{i\in\alpha} q_i - 2$ for some finite subset α of $\mathbb{N}$ then the word $1u1v1$ is in $\mathcal{F}$. Let X be the subshift specified by taking $\mathcal{F}$ as the collection of forbidden words. Note that X is non-empty since $0^\infty \in X$.

Let w' and w'' be two words in the language of X. Take any s such that

$$|w'| + |w''| + 2 < p_{s+1} < q_s < q_{s+1}.$$

It follows that if $\alpha \subset \mathbb{N}$ is a finite set such that

$$\sum_{i\in\alpha} p_i < |w'|$$

then $\max\alpha \leqslant s$. Let $N = q_{s+1}$. For any $n \geqslant N$, let $x_n = w'0^n w''0^\infty$. We will show that x_n is a point in X and hence X is a mixing subshift. We need to show that no word from $\mathcal{F}$ may appear in x_n. First note that the word 11 does not appear in x_n, since the word 11 appears neither in w' nor in w''. Suppose that for some non-empty words u and v over $\{0,1\}$ the word $1u1v1$ appears in x_n. If it is a subblock of w' or w'', then it does not belong to $\mathcal{F}$. Now assume that $1u1v1$ appears in x_n, but neither in w', nor in w''. Therefore either $1u1$ is a subword of w' or $1v1$ is a subword of w''. In the first case, if $\alpha \subset \mathbb{N}$ is a finite set such that

$$\sum_{i\in\alpha} p_i = |u| + 1 \leqslant |w'| < p_{s+1},$$

then $\max\alpha \leqslant s$, hence

$$\sum_{i\in\alpha} q_i \leqslant \sum_{j=1}^{s} q_j < q_{s+1}.$$

But on the other hand $|v| \geqslant n \geqslant q_{s+1}$ and therefore $|u| + |v| + 2 > \sum_{i\in\alpha} q_i$. It implies that $1u1v1 \notin \mathcal{F}$.

In the second case note that $|w''| \geqslant |v| + 2$. Now, if $\alpha \subset \mathbb{N}$ is a finite set such that

$$\sum_{i\in\alpha} p_i = |u| + 1 \geqslant n \geqslant q_{s+1},$$

then $\max\alpha > s$, hence

$$\sum_{i\in\alpha} q_i \geqslant q_{s+1} \geqslant \sum_{i\in\alpha} p_i + p_{s+1} > |u| + 1 + |w'| + |w''| > |u| + |v| + 2.$$

It implies that $1u1v1 \notin \mathcal{F}$. Hence $x_n \in X$ and therefore $n \in N([u]_X, [v]_X)$ and (X, σ) is strongly mixing.

By a similar argument, one can show that (X, σ) is a van der Waerden system.

Finally observe that if

$$[1]_X \cap T^{-\sum_{i\in\alpha} p_i}[1]_X \cap T^{-\sum_{i\in\alpha} q_i}[1]_X \neq \emptyset,$$

then there are two finite words u, v such that $1u1v1$ is in the language of X and $|u| = \sum_{i\in\alpha} p_i - 1$ and $|u| + |v| + 1 = \sum_{i\in\alpha} q_i - 1$. This contradicts the definition of X. Thus (X, σ) does not have multiple IP-recurrence property. □

In the rest of this section, we show that there is a large family of subshifts, with the multiple IP-recurrence property. For a function $f\colon \mathbb{Z}_+ \to [0, \infty)$, we define

$$\Psi_f = \left\{ x \in \{0, 1\}^{\mathbb{N}} : \forall\, p \in \mathbb{Z}_+, \forall\, i \in \mathbb{N}, \sum_{r=i}^{i+p-1} x_r \leqslant f(p) \right\}$$

and call it the *bounded density subshift* generated by f. Bounded density shifts were introduced by Stanley [33]. Stanley [33] proved also that to define Ψ_f we can consider only *canonical functions* $f\colon \mathbb{Z}_+ \to [0, \infty)$. By [33, Theorem 2.9], a function $f\colon \mathbb{Z}_+ \to [0, \infty)$ is *canonical* for the bounded density shift Ψ_f if and only if

1. $f(0) = 0$;
2. $f(m + 1) \in f(m) + \mathbb{Z}_+$ for any $m \in \mathbb{Z}_+$;
3. $f(m + n) \leqslant f(m) + f(n)$ for any $n, m \in \mathbb{Z}_+$.

Note that if $f(1) = 0$, then $\Psi_f = \{0^\infty\}$.

Theorem 5.5. *If f is an unbounded canonical function, then the bounded density subshift (Φ_f, σ) generated by f has the multiple IP-recurrent property.*

Proof. Fix a word w in the language of Ψ_f and let $U = [w] \cap \Psi_f$. Take any $d \geqslant 1$ and any IP-sets $FS\{p_i^{(1)}\}_{i=1}^{\infty}$, $FS\{p_i^{(2)}\}_{i=1}^{\infty}, \ldots, FS\{p_i^{(d)}\}_{i=1}^{\infty}$. For simplicity of notation, given a finite subset α of $\mathbb{N}$, we define $p_\alpha^{(i)} = \sum_{j\in\alpha} p_j^{(i)}$.

Without loss of generality, we may assume that for any $i \in \{1, \ldots, d\}$ and $j \in \mathbb{N}$ we have

$$p_j^{(i)} < p_{j+1}^{(i)} \quad \text{and} \quad p_j^{(i)} < p_j^{(i+1)} \quad (\text{provided } i < d).$$

Since f is unbounded, there exists $p \in \mathbb{N}$ such that $f(p) > (d + 1)|w|$ and $p \geqslant d|w|$. There is $N \in \mathbb{N}$ such that if $\alpha \subset \mathbb{N}$ is a finite set with $\max\alpha \geqslant N$, then $\sum_{j\in\alpha} p_j^{(i)} > p + |w|$ for every $i \in \{1, \ldots, d\}$. Note that for every $\alpha = \{a_1, \ldots, a_s\} \subset \mathbb{N}$ and any $1 \leqslant i < d$ we have

$$p_\alpha^{(i+1)} \geqslant \sum_{j=1}^{s} p_{a_j}^{(i+1)} \geqslant \sum_{j=1}^{s} (p_{a_j}^{(i)} + 1) \geqslant s + p_\alpha^{(i)}.$$

Denote $\beta = \{N + 1, \ldots, N + 2p + 1\}$ and observe that $p_\beta^{(i+1)} > p_\beta^{(i)} + 2p$ for any $1 \leqslant i < d$ and $p_\beta^{(1)} > p + |w|$. Let

$$x = w0^{p_\beta^{(1)} - |w|} w0^{p_\beta^{(2)} - p_\beta^{(1)} - 2|w|} w \cdots w0^{p_\beta^{(d)} - p_\beta^{(d-1)} - d|w|} w0^\infty.$$

It is easy to see that $x \in \Psi_f$ and

$$\sigma^{p_\beta^{(i)}}(x) \in [w] \quad \text{for} \quad i = 1, \ldots, d.$$

Therefore,

$$U \cap \sigma^{-p_\beta^{(1)}} U \cap \sigma^{-p_\beta^{(2)}} U \cap \cdots \cap \sigma^{-p_\beta^{(d)}} U \neq \emptyset. \qquad \square$$

Remark 5.6. By [33, Theorem 2.14], the bounded density shift (Φ_f, σ) in Theorem 5.5 is also strongly mixing. If the function f grows very slow, for example $f(n) = \log(n+1)$, then for any point $x \in \Phi_f$ one has

$$\lim_{n\to\infty} \frac{1}{n} \#(N(x,[1]) \cap [1,n]) \leqslant \lim_{n\to\infty} \frac{f(n)}{n} = 0.$$

It follows that the only invariant measure of (Φ_f, σ) is the point mass on $\{0^\infty\}$. But Ψ_f is uncountable, hence (Ψ_f, σ) is not an E-system. Let x be transitive point of (Φ_f, σ). By [27, Theorems 8.5 and 4.4], we know that $N(x, U)$ is a C-set for every neighborhood U of x. Since (Ψ_f, σ) is not an E-system and x is its transitive point, there exists a neighborhood V of x such that $N(x, V)$ has the Banach density zero. This gives a dynamical proof of a combinatorial result in [19] that there exists a C-set which has Banach density zero.

6 Multi-non-wandering points and van der Waerden center

We say that a point $x \in X$ is a *non-wandering point* if for every neighborhood U of x there exists an $n \in \mathbb{N}$ such that $U \cap T^{-n}U \neq \emptyset$. Denote by $\Omega(X,T)$ the set of all non-wandering points of (X,T). It is easy to see that $\Omega(X,T)$ is non-empty, closed and T-invariant. So $(\Omega(X,T),T)$ also forms a dynamical system, so we can consider non-wandering points of the subsystem $(\Omega(X,T),T)$. To introduce the notion of Birkhoff center, we define a (possibly transfinite) descending chain of non-empty closed and T-invariant subsets of X. We put inductively $\Omega_0(X,T) = X$, $\Omega_1(X,T) = \Omega(\Omega_0(X,T),T)$, and for every ordinal α we set $\Omega_{\alpha+1}(X,T) = \Omega(\Omega_\alpha(X,T),T)$. We continue this process by a transfinite induction: if λ is a limit ordinal we define

$$\Omega_\lambda(X,T) = \bigcap_{\alpha<\lambda} \Omega_\alpha(X,T).$$

In the compact metric space decreasing family of closed sets is always at most countable, hence there is a countable ordinal α such that

$$X = \Omega_0(X,T) \supset \Omega_1(X,T) \supset \cdots \supset \Omega_\alpha(X,T) = \Omega_{\alpha+1}(X,T) = \cdots.$$

We say that $\Omega_\alpha(X,T)$ is the *Birkhoff center* of (X,T) if $\Omega_{\alpha+1}(X,T) = \Omega_\alpha(X,T)$ and we define *depth* of (X,T) by

$$\mathrm{depth}(X,T) = \min\{\alpha \colon \Omega_\alpha(X,T) = \Omega_{\alpha+1}(X,T)\}.$$

Note that compactness of X implies that $\mathrm{depth}(X,T) < \omega_1$, where ω_1 is the first uncountable ordinal number.

Inspired by the notion of non-wandering points and the Birkhoff center, we introduce multi-non-wandering points and the van der Waerden center.

Definition 6.1. Let (X,T) be a dynamical system. A point $x \in X$ is *multi-non-wandering* if for every open neighborhood U of x and every $d \in \mathbb{N}$ there exists an $n \in \mathbb{N}$ such that

$$U \cap T^{-n}U \cap T^{-2n}U \cap \cdots \cap T^{-dn}U \neq \emptyset,$$

i.e., for every $d \in \mathbb{N}$, the diagonal d-tuple $(x, x, \ldots, x)$ is non-wandering in $(X^d, T \times T^2 \times \cdots \times T^d)$. Denote by $\Omega^{(\infty)}(X,T)$ the collection of all multi-non-wandering points.

First, we have the following characterization of multi-non-wandering points in a orbit closure of a point.

Proposition 6.2. *Let (X,T) be a dynamical system and $x \in X$. Suppose that $\overline{Orb(x,T)} = X$. Then y is a multi-non-wandering point if and only if $N(x,U)$ is an AP-set for every neighborhood U of y.*

Proof. First assume that y is a multi-non-wandering point. Fix a neighborhood U of y. For every $d \in \mathbb{N}$ there exists an $n \in \mathbb{N}$ such that the set $V = U \cap T^{-n}U \cap T^{-2n}U \cap \cdots \cap T^{-dn}U$ is non-empty and open. Since $\overline{\mathrm{Orb}(x,T)} = X$ there exists $m \geqslant 0$ such that $T^m x \in V \subset U$, and hence

$$T^{m+n}x \in U, \quad T^{m+2n}x \in U, \quad \ldots, \quad T^{m+dn}x \in U,$$

i.e., $\{m+n, m+2n, \ldots, m+dn\} \subset N(x,U)$. Thus $N(x,U)$ is an AP-set.

Fix a neighborhood U of y and assume that $N(x,U)$ is an AP-set. There exist $m, n \in \mathbb{N}$ such that $\{m, m+n, m+2n, \ldots, m+dn\} \in N(x,U)$. Put $z = T^m x$. Then $z \in U \cap T^{-n}U \cap T^{-2n}U \cap \cdots \cap T^{-dn}U$ and so y is a multi-non-wandering point. □

The proof of the following result is inspired by the set's forcing in [7] (see [27, Section 5] for more information on this topic).

Theorem 6.3. *A set $F \subset \mathbb{N}$ is an AP-set if and only if for every dynamical system (X,T) and every $x \in X$, there is a multi-non-wandering point in $\overline{T^F x}$, where $T^F x = \{T^n x \colon n \in F\}$.*

Proof. Assume that F is an AP-set. Let (X,T) be a dynamical system and $x \in X$. Without loss of generality, assume that $\overline{\mathrm{Orb}(x,T)} = X$. Set $K = \overline{T^F x}$. Cover K with closed balls with diameter less than 1 and let r_1 be the cardinality of a finite subcover of this cover. Then we can present

$$K = \bigcup_{i=1}^{r_1} K_{1,i},$$

where each $K_{1,i}$ is compact and has diameter less than 1. Since the family $\mathcal{AP}$ of AP-sets has the Ramsey property, there are an AP-set $F_1 \subset F$ and i_1 such that $T^{F_1}x \in K_{1,i_1}$. Set $K_1 = K_{1,i_1}$. Cover K_1 with closed balls with diameter less than $1/2$ and let r_2 be the cardinality of some finite subcover of this cover. Write

$$K_1 = \bigcup_{i=1}^{r_2} K_{2,i},$$

where each $K_{2,i}$ is compact and has diameter less than $1/2$. By induction we have a sequence of compact sets $\{K_i\}_{i=1}^{\infty}$ and a sequence of AP-sets $\{F_i\}_{i=1}^{\infty}$ such that $K_{i+1} \subset K_i$, $\mathrm{diam}(K_i) < 1/i$, $F_{i+1} \subset F_i$ and $T^{F_i}x \subset K_i$. By the compactness of X, there is $y \in X$ such that $\bigcap_{i=1}^{\infty} K_i = \{y\}$. For every neighborhood U of y, there exists i_0 such that $K_{i_0} \subset U$. Then $F_{i_0} \subset N(x,U)$, hence $N(x,U)$ is an AP-set. Thus y is a multi-non-wandering point by Proposition 6.2.

Now assume that for every dynamical system (X,T) and every $x \in X$ there is a multi-non-wandering point in $\overline{T^F x}$. Let x be the characteristic function of F. We can view x as a point in the full shift $(\{0,1\}^{\mathbb{Z}_+}, \sigma)$. Put $X = \overline{\mathrm{Orb}(x,\sigma)}$ and note that $N(x,[1] \cap X) = F$. By assumption, there exists a multi-non-wandering point $y \in \overline{T^F x} \subset [1] \cap X$. By Proposition 6.2, $F = N(x,[1] \cap X)$ is an AP-set, since $[1] \cap X$ is a neighborhood of y. □

Theorem 6.4. *Let (X,T) be a dynamical system and $x \in X$ be such that $\overline{\mathrm{Orb}(x,T)} = X$. Then*

(1) *If U is a neighborhood of $\Omega^{(\infty)}(X,T)$ and $y \in X$, then $N(y,U)$ is an AP*-set.*

(2) *If M is a non-empty closed subset X satisfying* (1)*, then $\Omega^{(\infty)}(X,T) \subset M$, i.e., $\Omega^{(\infty)}(X,T)$ is characterized as the smallest subset of X satisfying* (1)*.*

Proof. We first show that (1) holds. Take a neighborhood U of $\Omega^{(\infty)}(X,T)$. If there exists $z \in X$ such that $N(z,U)$ is not an AP*-set, then $F = N(z,U^c)$ is an AP-set. By Theorem 6.3, there exists a multi-non-wandering point in $\overline{T^F z} \subset U^c$. This contradicts $\Omega^{(\infty)}(X,T) \subset U$.

Assume that $M \subset X$ is non-empty, closed and satisfies (1). We show that $\Omega^{(\infty)}(X,T) \subset M$. Fix a multi-non-wandering point z. Let V be a neighborhood of z. It follows from Proposition 6.2 that $N(x,V)$ is an AP-set. But $N(x,U)$ is an AP*-set for every neighborhood U of M. Hence $N(x,V) \cap N(x,U) \neq \emptyset$. We get that $U \cap V \neq \emptyset$ for every neighborhood V of z and every neighborhood U of M. Thus $z \in M$, since M is closed. □

Using the characterization of the set of multi-non-wandering points (see Theorem 6.4), we can give another proof of Proposition 4.11 without using the advanced results on ultrafilters.

Proof of Proposition 4.11. Without loss of generality, assume that $Y = \overline{\mathrm{Orb}(y, S)}$. Let

$$\mathcal{A} = \{A \subset X : (A, T) \text{ is a subsystem of } (X, T) \text{ and } Y \subset \pi(A)\}.$$

It is clear that $\mathcal{A}$ is not empty since $X \in \mathcal{A}$. By Zorn lemma, there is a minimal (under the inclusion) element $Z \in \mathcal{A}$. Pick $x \in \pi^{-1}(y) \cap Z$. Note that $\overline{\mathrm{Orb}(x, T)} \subset Z$ and $Y \subset \pi(\overline{\mathrm{Orb}(x, T)})$. By the minimality of Z, we have $Z = \overline{\mathrm{Orb}(x, T)}$. Fix a neighborhood U of $\Omega^{(\infty)}(Z, T)$ and a neighborhood V of y. By Theorem 6.4, $N(z, U)$ is an AP*-set. But $N(x, V)$ is an AP-set. Then there exists $n \in \mathbb{N}$ such that $T^n z \in U$ and $T^n y \in V$. Thus $y \in \pi(\Omega^{(\infty)}(Z, T))$. By the minimality of Z again, one has $Z = \Omega^{(\infty)}(Z, T)$. Thus Z is a van der Waerden system and x is $\mathcal{AP}$-recurrent by Proposition 6.2 and Lemma 4.8. □

It is clear that $\Omega^{(\infty)}(X, T)$ is closed and T-invariant. So $(\Omega^{(\infty)}(X, T), T)$ also forms a dynamical system. We can consider multi-non-wandering points in $(\Omega^{(\infty)}(X, T), T)$. It is shown in Example 6.7 that $\Omega^{(\infty)}(\Omega^{(\infty)}(X, T), T)$ may not equal $\Omega^{(\infty)}(X, T)$. Similar to the Birkhoff center, we introduce the van der Waerden center. We put $\Omega_0^{(\infty)}(X, T) = X$, $\Omega_1^{(\infty)}(X, T) = \Omega^{(\infty)}(\Omega_0^{(\infty)}(X, T), T)$ and $\Omega_2^{(\infty)}(X, T) = \Omega^{(\infty)}(\Omega_1^{(\infty)}(X, T), T)$. We continue this process. Then

$$\begin{aligned} X &= \Omega_0^{(\infty)}(X, T) \supset \Omega_1^{(\infty)}(X, T) \supset \cdots \Omega_{\alpha+1}^{(\infty)}(X, T) \\ &= \Omega^{(\infty)}(\Omega_\alpha^{(\infty)}(X, T), T), \Omega_\lambda^{(\infty)}(X, T) \\ &= \bigcap_{\alpha<\lambda} \Omega_\alpha^{(\infty)}(X, T), \end{aligned}$$

where λ is a limit ordinal number. We say that $\Omega_\alpha^{(\infty)}(X, T)$ is the *van der Waerden center* of (X, T) if $\Omega_{\alpha+1}^{(\infty)}(X, T) = \Omega_\alpha^{(\infty)}(X, T)$.

Note that a dynamical system is a van der Waerden system if and only if every point is multi-non-wandering. The following result shows that the van der Waerden center is just the the maximal van der Waerden subsystem.

Proposition 6.5. *Let (X, T) be a dynamical system and $\Omega_\alpha^{(\infty)}(X, T)$ be the van der Waerden center of (X, T). Then $\Omega_\alpha^{(\infty)}(X, T)$ is the closure of the set of $\mathcal{AP}$-recurrent points of (X, T). Furthermore, $(\Omega_\alpha^{(\infty)}(X, T), T)$ is the maximal van der Waerden subsystem of (X, T).*

Proof. Let Z be the set of $\mathcal{AP}$-recurrent points of (X, T). It is not hard to see that $Z \subset \Omega_\gamma^{(\infty)}(X, T)$ for every ordinal number γ. So $\overline{Z} \subset \Omega_\alpha^{(\infty)}(X, T)$.

Since $\Omega_{\alpha+1}^{(\infty)}(X, T) = \Omega_\alpha^{(\infty)}(X, T)$, every point in the dynamical system $(\Omega_\alpha^{(\infty)}(X, T), T)$ is multi-non-wandering, and then $(\Omega_\alpha^{(\infty)}(X, T), T)$ is a van der Waerden system. By Lemma 4.2, the set of $\mathcal{AP}$-recurrent points of $(\Omega_\alpha^{(\infty)}(X, T), T)$ is dense in $\Omega_\alpha^{(\infty)}(X, T)$. Then $\Omega_\alpha^{(\infty)}(X, T) \subset \overline{Z}$. □

Proposition 6.6. *Let $\pi : (X, T) \to (Y, S)$ be a factor map. Then the image of van der Waerden center of (X, T) under π coincides with the van der Waerden center of (Y, S).*

Proof. Let X_0 and Y_0 be the set of all $\mathcal{AP}$-recurrent points in (X, T) and (Y, T) respectively. By Proposition 4.11, we have $\pi(X_0) = Y_0$. Then the result follows from Proposition 6.5. □

Example 6.7. There exists a dynamical system (X, T) such that $\Omega^{(\infty)}\big(\Omega^{(\infty)}(X, T), T\big) \neq \Omega^{(\infty)}(X, T)$.

Take any increasing sequence $\{z_n\}_{n\in\mathbb{Z}} \subset (0, 1)$ such that $\lim_{n\to-\infty} z_n = 0$ and $\lim_{n\to\infty} z_n = 1$. Let $X = \{0, 1\} \cup \{z_n : n \in \mathbb{Z}\} \pmod 1$, i.e., we view z_n as a sequence on the unit circle. Then we have $\lim_{n\to\infty} \rho(z_{-n}, z_n) = 0$, where ρ is the standard metric on the unit circle.

Define

$$Y = X \times \{0\} \cup \bigcup_{n=1}^{\infty} \bigcup_{j=2^n}^{2^{n+1}} \bigcup_{i=-n}^{n} \{(z_i, 4^{-n} - j2^{-n-1}4^{-n-1})\} \cup \bigcup_{n=0}^{\infty} (z_{-n}, 2) \cup (0, 2).$$

Clearly, if $j \neq s$ then $4^{-n} - j2^{-n-1}4^{-n-1} \neq 4^{-n} - s2^{-n-1}4^{-n-1}$ and $4^{-n} - 4^{-n-1} > 4^{-n-1}$. Therefore the coordinates like $(z_i, 4^{-n} - j2^{-n-1}4^{-n-1})$ uniquely determine a point in Y. The set Y is a closed subset of a product space $X \times [0,4]$. Therefore Y with the maximum metric is compact.

Let $g(z_n) = z_{n+1}$ for every $n \in \mathbb{Z}$ and $g(0) = 0 \in X$. For any integer $j \in [2^n, 2^{n+1}]$ denote $a_j = (z_{-n}, 4^{-n} - j2^{-n-1}4^{-n-1})$ and $b_j = (z_n, 4^{-n} - j2^{-n-1}4^{-n-1})$. Then we define a function $f\colon Y \to Y$ by putting

$$f(x,y) = \begin{cases} (g(x), y), & y = 0 \quad \text{or} \quad (y = 2 \quad \text{and} \quad x \neq z_0), \\ a_1, & y = 2 \quad \text{and} \quad x = z_0, \\ (g(x), y), & y \in (0,2) \quad \text{and} \quad (x,y) \neq b_j \quad \text{for every} \quad j, \\ a_{j+1}, & (x,y) = b_j. \end{cases}$$

Clearly f is a bijection and it is also not hard to verify that it is a homeomorphism. Observe that $\Omega(f) = \{(0,2)\} \cup X \times \{0\}$. We are going to show that $\Omega^{(\infty)}(f) = \Omega(f)$. Clearly both fixed points are in $\Omega^{(\infty)}(f)$. Now let us take any $m \in \mathbb{Z}$ and any open set $U \ni (z_m, 0)$. There is $N > 0$ such that $(z_m, y) \in U$ for every $y \leqslant 4^{-N}$. Fix any $d > 0$ and take $n > \max\{d, N, |m|\}$. Now if we take any $j = 2^n, \ldots, 2^n + d < 2^{n+1} - 1$, then

$$p_j = (z_m, 4^{-n} - j2^{-n-1}4^{-n-1}) \in Y \cap U.$$

By the definition of f, for $j = 0, \ldots, d-1$ we have $f^{2n+1}(p_j) = p_{j+1}$. In other words,

$$p_d \in U \cap f^{-2n-1}(U) \cap \cdots \cap f^{-(2n+1)d}(U) \neq \emptyset.$$

Indeed $(z_m, 0) \in \Omega^{(\infty)}(f)$. But

$$\Omega^{(\infty)}(f|_{\Omega^{(\infty)}(f)}) = \Omega(f|_{\Omega^{(\infty)}(f)}) = \{(0,0), (0,2)\}.$$

It follows that the van der Waerden center can be a proper subset of $\Omega^{(\infty)}(f)$.

Remark 6.8. It is shown in [31] that if α is a countable ordinal, then there exists a dynamical system (X,T) with $\mathrm{depth}(X,T) = \alpha$. We define the *van der Waerden depth* of (X,T) as

$$\mathrm{depth}^{(\infty)}(X,T) = \min\{\alpha\colon \Omega^{(\infty)}_{\alpha+1}(X,T) = \Omega^{(\infty)}_{\alpha}(X,T)\}.$$

We conjectured that the van der Waerden depth is a countable ordinal and for every countable ordinal number α there exists a dynamical system (X,T) such that $\mathrm{depth}^{(\infty)}(X,T) = \alpha$[2).]

Acknowledgements This work was supported by the National Science Centre (Grant No. DEC-2012/07/E/ST1/00185), National Natural Science Foundation of China (Grant Nos. 11401362, 11471125, 11326135, 11371339 and 11431012), Shantou University Scientific Research Foundation for Talents (Grant No. NTF12021) and the Project of LQ1602 IT4Innovations Excellence in Science. The authors thank Jie Li for the careful reading and helpful comments. A substantial part of this paper was written when the authors attended the activity "Dynamics and Numbers", June–July 2014, held at the Max Planck Institute für Mathematik (MPIM) in Bonn, Germany. Some part of the work was continued when the authors attended a conference held at the Wuhan Institute of Physics and Mathematics, China, and the satellite conference of 2014 ICM at Chungnam National University, South Korea. The authors are grateful to the organizers for their hospitality.

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