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An H^m-conforming spectral element method on multi-dimensional domain and its application to transmission eigenvalues

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Abstract We develop an H^m-conforming ($m \geqslant 1$) spectral element method on multi-dimensional domain associated with the partition into multi-dimensional rectangles. We construct a set of basis functions on the interval $[-1, 1]$ that are made up of the generalized Jacobi polynomials (GJPs) and the nodal basis functions. So the basis functions on multi-dimensional rectangles consist of the tensorial product of the basis functions on the interval $[-1, 1]$. Then we construct the spectral element interpolation operator and prove the associated interpolation error estimates. Finally, we apply the H^2-conforming spectral element method to the Helmholtz transmission eigenvalues that is a hot problem in the field of engineering and mathematics.

Keywords spectral element method, multi-dimensional domain, interpolation error estimates, transmission eigenvalues

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1 Introduction

Spectral method is an efficient method in scientific and engineering computations which can provide superior accuracy for the solution of partial differential equations in fluid dynamics [6, 17]. But spectral method lacks the domain flexibility. So the spectral element method is developed to overcome this defect. Up to now the spectral element method have attracted more and more scholars' attention. Guo and Jia [10] studied the quadrilateral spectral method and extended it to H^1-conforming spectral element method for polygons. Shen et al. [18] provided an H^1-conforming spectral element method by constructing directly the modal basis functions on the triangle and Samson et al. [16] built a new H^1-conforming spectral element method using the basis on the triangle by the rectangle-triangle mapping. Yu and Guo [23] developed an H^2-conforming spectral element method with rectangular partition in two dimensions.

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The orthogonal Jacobi polynomials $\widehat{J}_i^{\alpha,\beta}(\widehat{x})$ ($\widehat{x} \in I := [-1,1]$, $\alpha, \beta > -1$) weighted with

$$w^{\alpha,\beta}(\widehat{x}) := (1-\widehat{x})^{\alpha}(1+\widehat{x})^{\beta}$$

are usually adopted to construct the modal basis functions in spectral method and spectral element method. Shen et al. [17] extended these polynomials to the generalized case, namely the GJPs for $\alpha, \beta \in \mathbb{R}$. It is worth indicating an important property of GJPs that they together with their first few order derivatives vanish at the endpoints ± 1. So Shen et al. used them as a set of basis functions in $H_0^m(I)$ ($m \geqslant 1$) as well as apply them to the general order PDEs. Using the GJPs one can easily construct the basis functions in H_0^m for spectral method on multi-dimensional rectangle. Canuto et al. mentioned in the book [6, Subsection 8.5] a type of modal boundary-adapted basis functions on $[-1,1]$. They consist of the modal basis functions, viewed as a compact combination of Legendre polynomials, and the nodal basis functions (without derivatives) at ± 1 so that one can easily establish H^1-conforming spectral element approximation. Note that these modal basis functions are no other than the GJPs $\widehat{J}_i^{-1,-1}(\widehat{x})$. Using the similar way Yu and Guo [23] developed an H^2-conforming spectral element method with rectangular partition coupled with an error analysis. As the high dimensional problems are issues of common concern in scientific computing, this motivates us to extend this case to H^m-conforming spectral elements on multi-dimensional domain.

In this paper, we aim to develop an H^m-conforming spectral element method on multi-dimensional domain which is the same as the one in [23] for the case $m=2$ in two dimensions, using a proof method different from that of [23]. We construct a set of basis functions on the interval $[-1,1]$ that are made up of the GJPs and the nodal basis functions, the former of which can be regarded as bubble functions. So the basis functions on multi-dimensional rectangular element consist of the tensorial product of the basis functions on the interval $[-1,1]$ by an affine mapping. Then we construct the spectral element interpolation operator and prove the associated interpolation error estimates. Finally, we shall apply the H^2-conforming spectral element method presented in this paper to the Helmholtz transmission eigenvalue problem that is a quadratic eigenvalue problem arising in inverse scattering theory for an inhomogeneous medium [4,8,14]. In recent years, the numerical methods of the transmission eigenvalue problem are hot topics in the field of engineering and computational mathematics (see [1,5,9,12,13,20,21,24]). Among them, An and Shen [1] studied the spectral methods on the rectangle. But to our knowledge the above works do not involve spectral element method with d-dimensional rectangular partition ($d=2,3$). In this paper, we adopt the H^2-conforming method built in [22] to construct a spectral element approximation for transmission eigenvalues. Our theoretical analysis and numerical results show that the H^2-conforming spectral element method can obtain the transmission eigenvalues of high accuracy numerically.

2 An H^m-conforming spectral element method

In this section, we shall discuss an H^m-conforming spectral element method on d-dimensional domain D ($d \geqslant 1$). We associate D with a sequence of rectangular partitions $\{\pi_h\}_{h>0}$ into elements κ whose edges are parallel to axis. First of all, we consider the construction of the basis functions on one-dimensional standard interval $I=[-1,1]$ containing nodal and modal basis functions. Before presentation, we use the notation $P_N(K)$ to denote the polynomial space of degree less than or equal to N in each variable on K.

First of all, we define $2m$ nodal basis functions $\widehat{\phi}_j(\widehat{x})$ ($j=0,\ldots,2m-1$) for the polynomial space $P_{2m-1}(I)$ satisfying

$$\partial_{\widehat{x}}^j \widehat{\phi}_i(-1) = \partial_{\widehat{x}}^j \widehat{\phi}_{i+m}(1) = \delta_{i,j} \quad \text{and} \quad \partial_{\widehat{x}}^j \widehat{\phi}_i(1) = \partial_{\widehat{x}}^j \widehat{\phi}_{i+m}(-1) = 0, \quad i,j=0,\ldots,m-1.$$

The introduction of the nodal basis functions guarantees the H^m-conformity of spectral element space across the adjacent elements. Then we would like to increase the degree of polynomial space from $2m-1$ to N. For this purpose, we shall construct the modal basis functions on I which are actually the polynomial bubble functions on I. Let $\widehat{J}_j^{\alpha,\beta}(\widehat{x})$ be the Jacobi polynomials which are orthogonal with respect to the weight function $\widehat{\omega}^{\alpha,\beta}(\widehat{x}) = (1-\widehat{x})^\alpha(1+\widehat{x})^\beta$ $(\alpha,\beta > -1)$ on I,

$$\int_{-1}^{1} \widehat{J}_i^{\alpha,\beta}(\widehat{x})\widehat{J}_j^{\alpha,\beta}(\widehat{x})\widehat{\omega}^{\alpha,\beta}(\widehat{x}) = \gamma_j^{\alpha,\beta}\delta_{i,j}.$$

where

$$\gamma_j^{\alpha,\beta} = \frac{2^{\alpha+\beta+1}\Gamma(j+\alpha+1)\Gamma(j+\beta+1)}{(2j+\alpha+\beta+1)j!\Gamma(j+\alpha+\beta+1)}.$$

The GJPs are defined by

$$\widehat{J}_j^{\alpha,\beta}(\widehat{x}) = \begin{cases} (1-\widehat{x})^{-\alpha}(1+\widehat{x})^{-\beta}\widehat{J}_{j-j_0}^{-\alpha,-\beta}(\widehat{x}), & \alpha,\beta \leqslant -1, \\ (1-\widehat{x})^{-\alpha}\widehat{J}_{j-j_0}^{-\alpha,\beta}(\widehat{x}), & \alpha \leqslant -1, \quad \beta > -1, \\ (1+\widehat{x})^{-\beta}\widehat{J}_{j-j_0}^{\alpha,-\beta}(\widehat{x}), & \alpha > -1, \quad \beta \leqslant -1, \end{cases}$$

where $j \geqslant j_0$ with $j_0 = -(\alpha+\beta), -\alpha$ and $-\beta$ for the above three cases, respectively.

Here, we fix $\alpha = \beta = -m$ then the GJPs $\{\widehat{J}_j^{-m,-m}(\widehat{x})\}_{j\geqslant 2m}$ satisfy

$$\int_{-1}^{1} \widehat{J}_i^{-m,-m}(\widehat{x})\widehat{J}_j^{-m,-m}(\widehat{x})\widehat{\omega}^{-m,-m}(\widehat{x})d\widehat{x} = \gamma_{j-2m}^{m,m}\delta_{i,j}.$$

An attractive property of the GJPs is that

$$\partial_{\widehat{x}}^j \widehat{J}_i^{-m,-m}(\pm 1) = 0, \quad j = 0,1,\ldots,m-1, \quad i \geqslant 2m.$$

In addition, the GJPs can be represented as a compact combination of Legendre polynomials (see [17, Lemma 6.1] and Remark 2.5), which is convenient for computation. So we adopt them to set the bubble functions on I,

$$\widehat{\phi}_j(\widehat{x}) = \widehat{J}_j^{-m,-m}(\widehat{x}), \quad j = 2m, 2m+1, \ldots, N. \tag{2.1}$$

It is known that $\{\widehat{\phi}_j\}_{j=2m}^N$ are a set of basis functions of $P_N^0(I) \subset H_0^m(I)$ (see [17]). Hence, $\{\widehat{\phi}_j\}_{j=0}^N$ constitutes a set of basis functions of $P_N(I)$. Next, we consider the case of an arbitrary interval $[a,b]$. For this purpose, we define

$$\phi_j(x) = \left(\frac{b-a}{2}\right)^j \widehat{\phi}_j(\widehat{x}) \quad \text{and} \quad \phi_{j+m}(x) = \left(\frac{b-a}{2}\right)^j \widehat{\phi}_{j+m}(\widehat{x}) \quad \text{for} \quad 0 \leqslant j \leqslant m-1, \tag{2.2}$$

$$\phi_j(x) = \widehat{\phi}_j(\widehat{x}) \quad \text{for} \quad 2m \leqslant j \leqslant N \tag{2.3}$$

in terms with the linear transformation

$$x = \frac{b-a}{2}\widehat{x} + \frac{b+a}{2}.$$

Then $\{\phi_j\}_{j=0}^N$ constitutes a set of basis functions of $P_N([a,b])$.

Now we consider the basis functions on the arbitrary element

$$\kappa := [a_1,b_1] \times \cdots \times [a_d,b_d] \subset \mathbb{R}^d.$$

A natural choice of the basis functions on κ is the tensor product of one-dimensional basis functions. We define the linear transformation

$$x_i = \frac{b_i - a_i}{2}\widehat{x}_i + \frac{b_i + a_i}{2}, \quad i = 1,\ldots,d,$$

where $\widehat{\boldsymbol{x}} := (\widehat{x}_1,\ldots,\widehat{x}_d)^{\mathrm{T}}$ and $\boldsymbol{x} := (x_1,\ldots,x_d)^{\mathrm{T}}$ are the vectors defined on I^d and κ, respectively. Based on the previous discussion for one dimension, one can use

$$\left\{\prod_{i=1}^d \phi_{j_i}(x_i)\right\}_{j_1,\ldots,j_d=0}^N \subset P_N(\kappa)$$

as a set of basis functions on the element κ. For reading convenience, we classify these basis functions on κ as follows:

(A) Nodal basis functions: $\prod_{i=1}^{d}\phi_{j_i}(x_i), j_1,\dots,j_d=0,\dots,2m-1$.

(B) q-face basis functions ($1\leqslant q\leqslant d-1$): For any rearranged sequence $\{i_l\}_{l=1}^{d}$ of $\{i\}_{i=1}^{d}$, define $\prod_{i=1}^{d}\phi_{j_i}(x_i)$, $j_{i_1},\dots,j_{i_{d-q}}=0,\dots,2m-1, j_{i_{d-q+1}},\dots,j_{i_d}\geqslant 2m$.

(C) Element bubble basis functions: $\prod_{i=1}^{d}\phi_{j_i}(x_i), j_1,\dots,j_d\geqslant 2m$.

One can easily verify the H^m-conformity for basis functions between the adjacent elements κ_1 and κ_2. We consider only the case that κ_1 and κ_2 share the common $(d-1)$-face

$$\partial\kappa_1\cap\partial\kappa_2=a\times[a_2,b_2]\times\cdots\times[a_d,b_d].$$

For $s=0,\dots,m-1$, we have

$$\begin{aligned}&\partial_{x_1}^{s}\phi_{j_1}\,|_{\kappa_1}(a)=\partial_{x_1}^{s}\phi_{j_1}\,|_{\kappa_2}(a),\quad j_1=0,\dots,2m-1,\\&\partial_{x_1}^{s}\phi_{j_1}\,|_{\kappa_1}(a)=\partial_{x_1}^{s}\phi_{j_1}\,|_{\kappa_2}(a)=0,\quad j_1\geqslant 2m.\end{aligned}$$

For $s=0,\dots,m-1$ and $i=2,\dots,d$, we have with $x_i\in[a_i,b_i]$,

$$\begin{aligned}&\partial_{x_i}^{s}\phi_{j_i}\,|_{\kappa_1}(x_i)=\partial_{x_i}^{s}\phi_{j_i}\,|_{\kappa_2}(x_i)=\left(\frac{2}{b_i-a_i}\right)^{s-j_i}\partial_{\widehat{x}_i}^{s}\widehat{\phi}_{j_i}(\widehat{x}_i),\quad j_i=0,\dots,m-1,\\&\partial_{x_i}^{s}\phi_{j_i}\,|_{\kappa_1}(x_i)=\partial_{x_i}^{s}\phi_{j_i}\,|_{\kappa_2}(x_i)=\left(\frac{2}{b_i-a_i}\right)^{s-j_i+m}\partial_{\widehat{x}_i}^{s}\widehat{\phi}_{j_i}(\widehat{x}_i),\quad m\leqslant j_i\leqslant 2m-1,\\&\partial_{x_i}^{s}\phi_{j_i}\,|_{\kappa_1}(x_i)=\partial_{x_i}^{s}\phi_{j_i}\,|_{\kappa_2}(x_i)=\left(\frac{2}{b_i-a_i}\right)^{s}\partial_{\widehat{x}_i}^{s}\widehat{J}_{j_i}^{-m,-m}(\widehat{x}_i),\quad j_i\geqslant 2m.\end{aligned}$$

Therefore, the basis functions $\prod_{i=1}^{d}\phi_{j_i}(x_i)$ ($0\leqslant j_1,\dots,j_d\leqslant N$) together with their derivatives of order less than or equal to $m-1$ are equal on $\partial\kappa_1\cap\partial\kappa_2$.

In what follows, we mainly introduce some interpolation operators that will be used in the sequent argument. For reading convenience, we use the symbols $\widehat{v}\,|_{\widehat{x}_i}$ and $v\,|_{x_i}$ to denote the restriction of the multi-variable functional $\widehat{v}$ to the variable $\widehat{x}_i\in I$ and the restriction of the multi-variable functional v to $x_i\in[a_i,b_i]$, respectively.

Introduce the interpolation operator $\widehat{\Pi}_i^1$ for $\widehat{v}=\widehat{v}(\widehat{\boldsymbol{x}})\in C^{(m-1)d}(I^d)$ corresponding to the variable $\widehat{x}_i\in I$ ($1\leqslant i\leqslant d$):

$$(\widehat{\Pi}_i^1\widehat{v})(\widehat{\boldsymbol{x}})=\sum_{j=0}^{m-1}((\partial_{\widehat{x}_i}^{j}\widehat{v})\,|_{\widehat{x}_i=-1}\widehat{\phi}_j(\widehat{x}_i)+(\partial_{\widehat{x}_i}^{j}\widehat{v})\,|_{\widehat{x}_i=1}\widehat{\phi}_{j+m}(\widehat{x}_i)),$$

and the orthogonal projector $\widehat{\Pi}_i^2$ for $\widehat{v}:=\widehat{v}(\widehat{\boldsymbol{x}})\in C^{md}(I^d)$ satisfying $\widehat{v}\,|_{\widehat{x}_i}\in H_0^m(I)$ so that

$$\widehat{\Pi}_i^2\widehat{v}\,|_{\widehat{x}_i}\in P_N^0(I):=P_N(I)\cap H_0^m(I):\int_{-1}^{1}\partial_{\widehat{x}_i}^{m}(\widehat{\Pi}_i^2\widehat{v}(\widehat{\boldsymbol{x}})-\widehat{v}(\widehat{\boldsymbol{x}}))\partial_{\widehat{x}_i}^{m}\widehat{v}_N(\widehat{x}_i)d\widehat{x}_i=0,\quad\forall\,\widehat{v}_N(\widehat{x}_i)\in P_N^0(I).$$

We can infer

$$\widehat{\Pi}_i^2\widehat{v}(\widehat{\boldsymbol{x}})=\sum_{j=1}^{N+1-2m}\widehat{\varphi}_j(\widehat{x}_i)\int_{-1}^{1}\partial_{\widehat{x}_i}^{m}\widehat{v}(\widehat{\boldsymbol{x}})\partial_{\widehat{x}_i}^{m}\widehat{\varphi}_j(\widehat{x}_i)d\widehat{x}_i,$$

where $\{\widehat{\varphi}_j\}_{j=1}^{N+1-2m}$ constitutes a set of orthonormal basis functions in $H_0^m(I)$.

Define $(\Pi_i^1v)(\boldsymbol{x})=(\widehat{\Pi}_i^1\widehat{v})(\widehat{\boldsymbol{x}})$ and $(\Pi_i^2v)(\boldsymbol{x})=(\widehat{\Pi}_i^2\widehat{v})(\widehat{\boldsymbol{x}})$ with $v(\boldsymbol{x}):=\widehat{v}(\widehat{\boldsymbol{x}})$. Then it is obvious that Π_i^2 is an orthogonal projector satisfying

$$\begin{aligned}&\Pi_i^2v\,|_{x_i}\in P_N^0([a_i,b_i]):=P_N([a_i,b_i])\cap H_0^m([a_i,b_i]),\\&\int_{a_i}^{b_i}\partial_{x_i}^{m}(\Pi_i^2v(\boldsymbol{x})-v(\boldsymbol{x}))\partial_{x_i}^{m}v_N(x_i)dx_i=0,\quad\forall\,v_N(x_i)\in P_N^0([a_i,b_i]),\end{aligned}$$

and

$$(\Pi_i^1 v)(\widehat{\boldsymbol{x}}) = \sum_{j=0}^{m-1} ((\partial_{x_i}^j v)\,|_{x_i=-1}\phi_j(x_i) + (\partial_{x_i}^j v)\,|_{x_i=1}\phi_{j+m}(x_i)).$$

Define the interpolation $\widehat{I}_i$ for $\widehat{v} := \widehat{v}(\widehat{\boldsymbol{x}}) \in C^{md}(I^d)$ such that $(\widehat{I}_i\widehat{v})(\widehat{\boldsymbol{x}}) = \widehat{v}(\widehat{\boldsymbol{x}})$ $(i = 1,\dots,d)$ for any $\widehat{v}\,|_{\widehat{x}_i} \in P_N(I)$ satisfying

$$(\widehat{I}_i\widehat{v})(\widehat{\boldsymbol{x}}) = (\widehat{\Pi}_i^1\widehat{v} + \widehat{\Pi}_i^2 \circ (\boldsymbol{I} - \widehat{\Pi}_i^1)\widehat{v})(\widehat{\boldsymbol{x}}),$$

where $\boldsymbol{I}$ is the identity operator.

Define $v(\boldsymbol{x}) = \widehat{v}(\widehat{\boldsymbol{x}})$ and the interpolation I_i for $v := v(\boldsymbol{x}) \in C^{md}(\kappa)$ such that $(I_iv)(\boldsymbol{x}) = v(\boldsymbol{x})$, $\forall v\,|_{x_i} \in P_N([a_i,b_i])$ and

$$(I_iv)(\boldsymbol{x}) = (\Pi_i^1 v + \Pi_i^2 \circ (\boldsymbol{I} - \Pi_i^1)v)(\boldsymbol{x}).$$

It is obvious that $(I_iv)(\boldsymbol{x}) = (\widehat{I}_i\widehat{v})(\widehat{\boldsymbol{x}})$.

Let $H^s(K)$ be the standard Sobolev space with norm $\|\cdot\|_{s,K}$ for a given $K \subseteq D$ and we shall omit the subscript K if $K = D$. Hereafter in this paper, we use the symbol $x \lesssim y$ to mean $x \leqslant Cy$ for a constant C that is independent of the mesh size and the degree of piecewise polynomial space and may be different at different occurrences. Now we start with the interpolation error estimates in one dimension.

Lemma 2.1. *Assume $\widehat{v}(\widehat{\boldsymbol{x}}) \in C^t(I^d)$ $(m \leqslant t \leqslant N+1)$. Then there holds for $0 \leqslant s \leqslant m$,*

$$\|\widehat{I}_i\widehat{v} - \widehat{v}\|_{s,I} \lesssim (1/N)^{t-s}\|\widehat{v}\|_{t,I}.$$

Here, $\|\widehat{v}\|_{t,I}$ denotes the norm of $\widehat{v}$ with respect to the variable $\widehat{x}_i$.

Proof. From [17, Theorem 6.1], we know if $\widehat{f}(\widehat{x}_i) \in H_0^m(I) \cap H^t(I)$ $(t \geqslant m)$ then there holds for $0 \leqslant s \leqslant m$,

$$\|\widehat{\Pi}_i^2\widehat{f} - \widehat{f}\|_{s,I} \lesssim (1/N)^{t-s}\|\widehat{f}\|_{t,I}.$$

Note that $(\widehat{v} - \widehat{\Pi}_i^1\widehat{v})\,|_{\widehat{x}_i} \in H_0^m(I)$ and $\widehat{\Pi}_i^1(\widehat{v} - \widehat{\Pi}_i^1\widehat{v}) = 0$. Then

$$\begin{aligned}\|\widehat{I}_i\widehat{v} - \widehat{v}\|_{s,I} &= \|\widehat{I}_i(\widehat{v} - \widehat{\Pi}_i^1\widehat{v}) - (\widehat{v} - \widehat{\Pi}_i^1\widehat{v})\|_{s,I}\\ &= \|\widehat{\Pi}_i^2 \circ (\boldsymbol{I} - \widehat{\Pi}_i^1)(\widehat{v} - \widehat{\Pi}_i^1\widehat{v}) - (\widehat{v} - \widehat{\Pi}_i^1\widehat{v})\|_{s,I}\\ &= \|\widehat{\Pi}_i^2(\widehat{v} - \widehat{\Pi}_i^1\widehat{v}) - \widehat{v} - \widehat{\Pi}_i^1\widehat{v}\|_{s,I}\\ &\lesssim (1/N)^{t-s}\|\widehat{v} - \widehat{\Pi}_i^1\widehat{v}\|_{t,I}\\ &\lesssim (1/N)^{t-s}\|\widehat{v}\|_{t,I}.\end{aligned}$$

This concludes the proof. □

We define the multi-dimensional interpolation operator $C^{md}(I^d) \to P_N(I^d)$ on I^d as

$$(\widehat{\boldsymbol{I}}_N\widehat{v})(\widehat{\boldsymbol{x}}) = (\widehat{I}_1 \circ \cdots \circ \widehat{I}_d\widehat{v})(\widehat{\boldsymbol{x}}). \tag{2.4}$$

One can easily verify that $(\widehat{\boldsymbol{I}}_N\widehat{v})(\widehat{\boldsymbol{x}}) = \widehat{v}(\widehat{\boldsymbol{x}})$ holds for any $\widehat{v}(\widehat{\boldsymbol{x}}) \in P_N(I^d)$.

Lemma 2.2. *For any $\widehat{v} \in C^t(I^d)$ with $md \leqslant t \leqslant N+1$, there holds*

$$\|\widehat{\boldsymbol{I}}_N\widehat{v} - \widehat{v}\|_{s,I^d} \lesssim (1/N)^{t-s}\|\widehat{v}\|_{t,I^d}, \quad 0 \leqslant s \leqslant m.$$

Proof. Let d nonnegative integers $\alpha_1,\dots,\alpha_d$ satisfy $\sum_{i=1}^d \alpha_i = s$. We obtain from Lemma 2.1 that

$$\|\partial_{x_1}^{\alpha_1}\cdots\partial_{x_d}^{\alpha_d}(\widehat{I}_1\widehat{v} - \widehat{v})\|_{0,I^d} \lesssim (1/N)^{t-s}\|\widehat{v}\|_{t,I^d},$$

and

$$\begin{aligned}&\|\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\cdots\partial_{x_d}^{\alpha_d}(\widehat{I}_1 - \boldsymbol{I})(\widehat{I}_2 \circ \cdots \circ \widehat{I}_d\widehat{v} - \widehat{v})\|_{s,I^d}\\ &\quad\lesssim (1/N)^{m-\alpha_1}\|\partial_{x_1}^{m}\partial_{x_2}^{\alpha_2}\cdots\partial_{x_d}^{\alpha_d}(\widehat{I}_2 \circ \cdots \circ \widehat{I}_d\widehat{v} - \widehat{v})\|_{0,I^d},\end{aligned}$$

where $\boldsymbol{I}$ is the identity operator. Hence by the triangular inequality and (2.4), we have

$$\begin{aligned}
&\|\partial_{x_1}^{\alpha_1}\cdots\partial_{x_d}^{\alpha_d}(\widehat{\boldsymbol{I}}_N\widehat{v}-\widehat{v})\|_{0,I^d}\\
&\quad\lesssim \|\partial_{x_1}^{\alpha_1}\cdots\partial_{x_d}^{\alpha_d}(\widehat{I}_1\widehat{v}-\widehat{v})\|_{0,I^d}+\|\partial_{x_1}^{\alpha_1}\cdots\partial_{x_d}^{\alpha_d}(\widehat{I}_2\circ\cdots\circ\widehat{I}_d\widehat{v}-\widehat{v})\|_{0,I^d}\\
&\qquad+\|\partial_{x_1}^{\alpha_1}\cdots\partial_{x_d}^{\alpha_d}(\widehat{I}_1-\boldsymbol{I})(\widehat{I}_2\circ\cdots\circ\widehat{I}_d\widehat{v}-\widehat{v})\|_{0,I^d}\\
&\quad\lesssim (1/N)^{t-s}\|\widehat{v}\|_{t,I^d}+\|\partial_{x_1}^{\alpha_1}\cdots\partial_{x_d}^{\alpha_d}(\widehat{I}_2\circ\cdots\circ\widehat{I}_d\widehat{v}-\widehat{v})\|_{0,I^d}\\
&\qquad+(1/N)^{m-\alpha_1}\|\partial_{x_1}^{m}\partial_{x_2}^{\alpha_2}\cdots\partial_{x_d}^{\alpha_d}(\widehat{I}_2\circ\cdots\circ\widehat{I}_d\widehat{v}-\widehat{v})\|_{0,I^d}\\
&\quad\lesssim (1/N)^{t-s}\|\widehat{v}\|_{t,I^d}+\|\partial_{x_1}^{\alpha_1}\cdots\partial_{x_d}^{\alpha_d}(\widehat{I}_3\circ\cdots\circ\widehat{I}_d\widehat{v}-\widehat{v})\|_{0,I^d}\\
&\qquad+(1/N)^{m-\alpha_2}\|\partial_{x_1}^{\alpha_1}\partial_{2}^{m}\cdots\partial_{x_d}^{\alpha_d}(\widehat{I}_3\circ\cdots\circ\widehat{I}_d\widehat{v}-\widehat{v})\|_{0,I^d}\\
&\qquad+(1/N)^{m-\alpha_1}\|\partial_{x_1}^{m}\partial_{x_2}^{\alpha_2}\cdots\partial_{x_d}^{\alpha_d}(\widehat{I}_3\circ\cdots\circ\widehat{I}_d\widehat{v}-\widehat{v})\|_{0,I^d}\\
&\qquad+(1/N)^{(m-\alpha_1)+(m-\alpha_2)}\|\partial_{x_1}^{m}\partial_{x_2}^{m}\cdots\partial_{x_d}^{\alpha_d}(\widehat{I}_3\circ\cdots\circ\widehat{I}_d\widehat{v}-\widehat{v})\|_{0,I^d}.
\end{aligned}$$

Repeating the above argument, we get

$$\begin{aligned}
&\|\partial_{x_1}^{\alpha_1}\cdots\partial_{x_d}^{\alpha_d}(\widehat{\boldsymbol{I}}_N\widehat{v}-\widehat{v})\|_{0,I^d}\\
&\quad\lesssim (1/N)^{t-s}\|\widehat{v}\|_{t,I^d}+\|\partial_{x_1}^{\alpha_1}\cdots\partial_{x_d}^{\alpha_d}(\widehat{I}_d\widehat{v}-\widehat{v})\|_{0,I^d}\\
&\qquad+\sum_{k=1}^{d-1}\sum_{i_1<\cdots<i_k\leqslant d-1}\left(\frac{1}{N}\right)^{(m-\alpha_{i_1})+\cdots+(m-\alpha_{i_k})}\|\partial_{x_{i_1}}^{m}\cdots\partial_{x_{i_k}}^{m}\partial_{x_{i_{k+1}}}^{\alpha_{i_{k+1}}}\cdots\partial_{x_{i_{d-1}}}^{\alpha_{i_{d-1}}}\partial_{x_d}^{\alpha_d}(\widehat{I}_d\widehat{v}-\widehat{v})\|_{0,I^d}\\
&\quad\lesssim (1/N)^{t-s}\|\widehat{v}\|_{t,I^d}.
\end{aligned}$$

This ends this proof. □

Likewise the multi-dimensional interpolation $\boldsymbol{I}_N^{\kappa}: C^{md}(\kappa)\to P_N(\kappa)$ on κ is defined as

$$(\boldsymbol{I}_N^{\kappa}v)(\boldsymbol{x})=(I_1\circ\cdots\circ I_d v)(\boldsymbol{x}).$$

In order to guarantee the H^m-conformity for the interpolations $\boldsymbol{I}_N^{\kappa_1}$ and $\boldsymbol{I}_N^{\kappa_2}$ between the adjacent elements κ_1 and κ_2, we need to verify for any $v\in C^{md}(\overline{D})$ and $|\alpha|\leqslant m-1$ there holds

$$\partial_{x_1}^{\alpha_1}\cdots\partial_{x_d}^{\alpha_d}\boldsymbol{I}_N^{\kappa_1}v(\boldsymbol{x})=\partial_{x_1}^{\alpha_1}\cdots\partial_{x_d}^{\alpha_d}\boldsymbol{I}_N^{\kappa_2}v(\boldsymbol{x}),\quad\forall\,\boldsymbol{x}\in\partial\kappa_1\cap\partial\kappa_2$$

and $\partial\kappa_1\cap\partial\kappa_2\neq\emptyset$. As before we only consider the case $\partial\kappa_1\cap\partial\kappa_2=a\times[a_2,b_2]\times\cdots\times[a_d,b_d]$. Due to the H_0^m-orthogonality of $\widehat{\Pi}_i^2$, we have for $i=1,\dots,d$,

$$v_0=(1-\Pi_i^1)v:\int_{a_i}^{b_i}\partial_{x_i}^m(\Pi_i^2\,|_{\kappa_1}v_0(\boldsymbol{x})-\Pi_i^2\,|_{\kappa_2}v_0(\boldsymbol{x}))\partial_{x_i}^m v_N(x_i)=0,\quad\forall\,v_N\in P_N^0([a_i,b_i]).$$

Then obviously $\partial_{x_i}^j\Pi_i^2\,|_{\kappa_1}v_0(\boldsymbol{x})=\partial_{x_i}^j\Pi_i^2\,|_{\kappa_2}v_0(\boldsymbol{x}),\forall\,\boldsymbol{x}\in\partial\kappa_1\cap\partial\kappa_2$, for $0\leqslant j\leqslant m,1\leqslant i\leqslant d$. Also, it is obvious that $\partial_{x_i}^j\Pi_i^1\,|_{\kappa_1}v(\boldsymbol{x})=\partial_{x_i}^j\Pi_i^1\,|_{\kappa_2}v(\boldsymbol{x}),\forall\,\boldsymbol{x}\in\partial\kappa_1\cap\partial\kappa_2$, for $0\leqslant j\leqslant m-1,1\leqslant i\leqslant d$. Hence, $\partial_{x_i}^j I_i\,|_{\kappa_1}v(\boldsymbol{x})=\partial_{x_i}^j I_i\,|_{\kappa_2}v(\boldsymbol{x}),\forall\,\boldsymbol{x}\in\partial\kappa_1\cap\partial\kappa_2$, for $0\leqslant j\leqslant m-1,1\leqslant i\leqslant d$. Then for $0\leqslant j\leqslant m-1,1\leqslant i\leqslant d$, there holds $\partial_{x_i}^j\boldsymbol{I}_N^{\kappa_1}v(\boldsymbol{x})=\partial_{x_i}^j\boldsymbol{I}_N^{\kappa_2}v(\boldsymbol{x}),\forall\,\boldsymbol{x}\in\partial\kappa_1\cap\partial\kappa_2$. According to the definition of $\boldsymbol{I}_N^{\kappa}$, repeating the above argument we yield the assertion.

In the end, we introduce the spectral element space

$$\begin{aligned}
S^{N,h}=\{v:v\,|_{\kappa}\in P_N(\kappa),\ \forall\,\kappa\in\pi_h\ \text{and}\ \partial^{\alpha}v\ (0\leqslant|\alpha|\leqslant m-1)\\
\text{are continuous accross}\ \partial\kappa_1\cap\partial\kappa_2\ \text{for}\ \kappa_1,\kappa_2\in\pi_{\rm h}\ \text{and}\ \partial\kappa_1\cap\partial\kappa_2\neq\emptyset\}.
\end{aligned}$$

We define the spectral element interpolation operator $\boldsymbol{I}_{N,h}: C^{md}(\overline{D})\to S^{N,h}$ as $\boldsymbol{I}_{N,h}\,|_{\kappa}=\boldsymbol{I}_N^{\kappa}$ for any $\kappa\in\pi_h$. One can easily verify that $\boldsymbol{I}_{N,h}v=v$ and $(\boldsymbol{I}_{N,h}v)(\boldsymbol{x})=(\widehat{\boldsymbol{I}}_N\widehat{v})(\widehat{\boldsymbol{x}})$ for any $v\in P_N(\kappa)$ and any $\boldsymbol{x}\in\kappa$.

Note that if $(m+\frac{1}{2})d<t\leqslant N+1$ then $H^t(I^d)\hookrightarrow C^{md}(I^d)$ and $\widehat{\boldsymbol{I}}_N: H^t(I^d)\to H^m(I^d)$ is bounded, then Lemma 2.2 is also true for any v in $H^t(I^d)$ due to the density of $C^{\infty}(I^d)$ in $H^t(I^d)$. Using the scaling argument, we can easily derive the interpolation error estimate on the element κ and the entire domain D.

Lemma 2.3. *For any $v \in H^t(\kappa)$ with $(m+\frac{1}{2})d < t \leqslant N+1$, there holds*

$$\|\boldsymbol{I}_{N,h}v - v\|_{s,\kappa} \lesssim (h/N)^{t-s}\|v\|_{t,\kappa}, \quad 0 \leqslant s \leqslant m.$$

Theorem 2.4. *For any $v \in H^t(D)$ with $(m+\frac{1}{2})d < t \leqslant N+1$,*

$$\|\boldsymbol{I}_{N,h}v - v\|_{s,D} \lesssim (h/N)^{t-s}\|v\|_{t,D}, \quad 0 \leqslant s \leqslant m.$$

Remark 2.5. We consider the special case of $m = 2$. The nodal basis functions with respect to the endpoint -1 are respectively,

$$\widehat{\phi}_0(\widehat{x}) = \frac{(\widehat{x}-1)^2(\widehat{x}+2)}{4}, \quad \widehat{\phi}_1(\widehat{x}) = \frac{(\widehat{x}-1)^2(\widehat{x}+1)}{4}. \tag{2.5}$$

Meanwhile, the nodal basis functions with respect to the endpoint 1 are respectively,

$$\widehat{\phi}_2(\widehat{x}) = -\frac{(\widehat{x}+1)^2(\widehat{x}-2)}{4}, \quad \widehat{\phi}_3(\widehat{x}) = \frac{(\widehat{x}+1)^2(\widehat{x}-1)}{4}. \tag{2.6}$$

One can easily verify that $\widehat{\phi}_0(-1) = 1$, $\widehat{\phi}'_1(-1) = 1$, $\widehat{\phi}_2(1) = 1$ and $\widehat{\phi}'_3(1) = 1$.

Legendre polynomials and Chebyshev polynomials are two most popular Jacobi polynomials. Now we adopt Legendre polynomials $\{\widehat{L}_j\}_{j=0}^N$ or Chebyshev polynomials $\{\widehat{T}_j\}_{j=0}^N$ to give the bubble basis functions on I. One may set

$$\widehat{\phi}_j(\widehat{x}) = (2j-1)\widehat{L}_{j-4}(\widehat{x}) - 2(2j-3)\widehat{L}_{j-2}(\widehat{x}) + (2j-5)\widehat{L}_j(\widehat{x}), \quad j = 4,5,\ldots,N, \tag{2.7}$$

since

$$\widehat{\phi}_j(\widehat{x}) = \frac{(2j-1)(2j-3)(2j-5)}{4(j-2)(j-3)}\widehat{J}_j^{-2,-2}(\widehat{x});$$

another choice is $\widehat{\phi}_j(\widehat{x}) = (j-1)\widehat{T}_{j-4}(\widehat{x}) - 2(j-2)\widehat{T}_{j-2}(\widehat{x}) + (j-3)\widehat{T}_j(\widehat{x})$, $j = 4,5,\ldots,N$.

Remark 2.6. One may set different polynomial degrees for each element. Figure 1 shows three elements κ_1–κ_3 and the basis functions of $P_4(\kappa_1)$ on κ_1. If one wants to decrease the polynomial degrees to 3 on κ_1 and κ_2, one should delete the basis functions $\phi_4(x_1)\phi_4(x_2)$ on both κ_1 and κ_2, $\phi_2(x_1)\phi_4(x_2)$ and $\phi_3(x_1)\phi_4(x_2)$ on the common edge of κ_1 and κ_2.

Remark 2.7. The H^m-conforming spectral elements can deal with the problem with mixed boundary condition on multi-dimensional domain due to adopting the nodal basis functions at the endpoint ± 1. Though we restrict our attention to the spectral method on rectangular domain, it can be extended to spectral method on non-rectangular domains like the way as in [10]. Likewise the mesh adopted by the spectral element method can be improved for approximating general domains better.

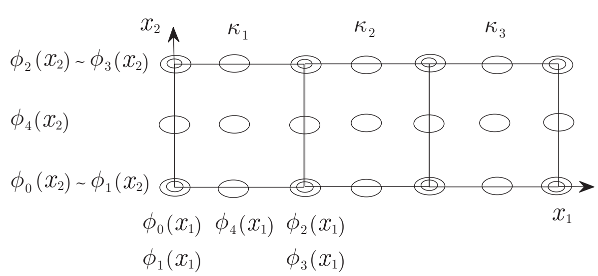

Figure 1 The elements κ_1–κ_3 and the basis functions on κ_1

3 $\boldsymbol{H}^2$-conforming spectral elements for transmission eigenvalues

In this section, we aim to apply the H^2-conforming spectral elements in the last section to the transmission eigenvalue problem.

Consider the Helmholtz transmission eigenvalue problem: Find $k \in \mathbb{C}$, $\omega, \sigma \in L^2(D)$, $\omega - \sigma \in H^2(D)$ such that

$$\Delta\omega + k^2 n(x)\omega = 0 \quad \text{in} \quad D, \tag{3.1}$$

$$\Delta\sigma + k^2\sigma = 0 \quad \text{in} \quad D, \tag{3.2}$$

$$\omega - \sigma = 0 \quad \text{on} \quad \partial D, \tag{3.3}$$

$$\frac{\partial\omega}{\partial\nu} - \frac{\partial\sigma}{\partial\nu} = 0 \quad \text{on} \quad \partial D, \tag{3.4}$$

where $D \subset \mathbb{R}^d$ $(d = 2, 3)$ is an open bounded simply connected inhomogeneous medium, ν is the unit outward normal to ∂D.

The eigenvalue problem (3.1)–(3.4) can be stated as the classical weak formulation below (see [4,9,15]): Find $k^2 \in \mathbb{C}$, $k^2 \neq 0$, nonzero $u \in H_0^2(D)$ such that

$$\left(\frac{1}{n(x)-1}(\Delta u + k^2 u), \Delta v + \overline{k}^2 n(x) v\right)_0 = 0, \quad \forall\, v \in H_0^2(D), \tag{3.5}$$

where $(\cdot,\cdot)_0$ is the inner product of $L^2(D)$. As usual, we define $\lambda = k^2$ as the transmission eigenvalue in this paper. We suppose that the index of refraction $n \in L^\infty(D)$ is a real valued function such that $n-1$ is strictly positive (strictly negative) almost everywhere in D.

Define Hilbert space $\boldsymbol{H} = H_0^2(D) \times L^2(D)$ and define $\boldsymbol{H}^s(K) = H^s(K) \times H^{s-2}(K)$ with norm $\|(u,w)\|_{s,K} = \|u\|_{s,K} + \|w\|_{s-2,K}$ for a given $K \subseteq D$. We write $\boldsymbol{H}^1 := \boldsymbol{H}^1(D)$ for simplicity.

Although (3.5) is a quadratic eigenvalue problem, it can be linearized by introducing some variables. Using the linearized way in [22], we introduce $w = \lambda u$, then (3.5) is equivalent to the following linear weak formulation: Find $(\lambda, u, w) \in \mathbb{C} \times H_0^2(D) \times L^2(D)$ such that

$$\begin{aligned}\left(\frac{1}{n-1}\Delta u, \Delta v\right)_0 &= -\lambda\left(\frac{1}{n-1}u, \Delta v\right)_0 \\ &\quad - \lambda\left(\Delta u, \frac{n}{n-1}v\right)_0 - \lambda\left(\frac{n}{n-1}v\right)_0, \quad \forall\, v \in H_0^2(D),\end{aligned} \tag{3.6}$$

$$(w,z)_0 = \lambda(u,z)_0, \quad \forall\, z \in L^2(D). \tag{3.7}$$

We introduce the following sesquilinear forms:

$$A((u,w),(v,z)) = \left(\frac{1}{n-1}\Delta u, \Delta v\right)_0 + (w,z)_0,$$

$$B((u,w),(v,z)) = -\left(\frac{1}{n-1}u, \Delta v\right)_0 - \left(\Delta u, \frac{n}{n-1}v\right)_0 - \left(\frac{n}{n-1}w, v\right)_0 + (u,z)_0.$$

Then (3.5) can be rewritten as the following problem: Find $\lambda \in \mathbb{C}$, nonzero $(u,w) \in \boldsymbol{H}$ such that

$$A((u,w),(v,z)) = \lambda B((u,w),(v,z)), \quad \forall\, (v,z) \in \boldsymbol{H}. \tag{3.8}$$

Let norm $\|\cdot\|_A$ be induced by inner product $A(\cdot,\cdot)$. Then it is clear $\|\cdot\|_A$ is equivalent to $\|\cdot\|_{2,D}$ in $\boldsymbol{H}$.

When $n \in W^{1,\infty}(D)$ for any given $(f,g) \in \boldsymbol{H}^1$, $B((f,g),(v,z))$ is a continuous linear form on $\boldsymbol{H}^1$:

$$B((f,g),(v,z)) \lesssim \|(f,g)\|_{1,D}\|(v,z)\|_{1,D}, \quad \forall\, (v,z) \in \boldsymbol{H}^1. \tag{3.9}$$

Consider the dual problem of (3.8): Find $\lambda^* \in \mathbb{C}$, nonzero $(u^*, w^*) \in \boldsymbol{H}$ such that

$$A((v,z),(u^*,w^*)) = \overline{\lambda^*} B((v,z),(u^*,w^*)), \quad \forall\, (v,z) \in \boldsymbol{H}. \tag{3.10}$$

In order to discretize the space $\boldsymbol{H}$, we need finite element spaces to discretize $H_0^2(D)$ and $L^2(D)$, respectively. Since $H_0^2(D) \subset L^2(D)$ here we can construct the spectral element space $S_0^{N,h} := S^{N,h} \cap H_0^2(D)$ such that $\boldsymbol{H}_{N,h} := S_0^{N,h} \times S_0^{N,h} \subset H_0^2(D) \times L^2(D)$.

The conforming spectral element approximation of (3.8) is given by the following: Find $\lambda_{N,h} \in \mathbb{C}$, nonzero $(u_{N,h}, w_{N,h}) \in \boldsymbol{H}_{N,h}$ such that

$$A((u_{N,h}, w_{N,h}), (v, z)) = \lambda_{N,h} B((u_{N,h}, w_{N,h}), (v, z)), \quad \forall\, (v, z) \in \boldsymbol{H}_{N,h}. \tag{3.11}$$

According to Theorem 2.4 and the operator interpolation theory, we know the following error estimates hold for spectral element space. For any $\psi \in H_0^2(D) \cap H^{2+r}(D)$ $(0 \leqslant r \leqslant N-1, N+1 > (m+\frac{1}{2})d)$, there holds

$$\inf_{v \in S^{N,h}} \|\psi - v\|_s \lesssim (h/N)^{2+r-s} \|\psi\|_{2+r}, \quad s = 0, 1, 2.$$

To give the error of eigenfunction $(u_{N,h}, w_{N,h})$ in the norm $\|\cdot\|_{1,D}$ we need the following regularity assumption (see [3]):

R(D): For any $\varrho \in H^{-1}(D)$, there exists $\psi \in H^{2+r_1}(D)$ satisfying

$$\Delta\left(\frac{1}{n-1}\Delta\psi\right) = \varrho \quad \text{in} \quad D,$$
$$\psi = \frac{\partial \psi}{\partial \nu} = 0 \quad \text{on} \quad \partial D,$$

and

$$\|\psi\|_{2+r_1} \leqslant C_p \|\varrho\|_{-1}, \tag{3.12}$$

where $r_1 \in (0, 1]$, C_p denotes the prior constant dependent on the equation and D but independent of the right-hand side ϱ of the equation.

In this paper, let λ be an eigenvalue of (3.8) with the ascent α. Let $M(\lambda)$ and $M(\lambda_{N,h})$ be the space spanned by all generalized eigenfunctions corresponding respectively to the eigenvalues λ and $\lambda_{N,h}$. As for the dual problem (3.10), the definitions of $M^*(\lambda^*)$ are made similarly to $M(\lambda)$.

In what follows, to characterize the approximation of the finite element space $\boldsymbol{H}_{N,h}$ to $M(\lambda)$ and $M^*(\lambda^*)$, we introduce the following quantities:

$$\delta_{N,h}(\lambda) = \sup_{\substack{(v,z) \in M(\lambda) \\ \|(v,z)\|_{2,D}=1}} \inf_{(v_h, z_h) \in \boldsymbol{H}_{N,h}} \|(v, z) - (v_{N,h}, z_{N,h})\|_{2,D},$$
$$\delta^*_{N,h}(\lambda^*) = \sup_{\substack{(v,z) \in M^*(\lambda^*) \\ \|(v,z)\|_{2,D}=1}} \inf_{(v_{N,h}, z_{N,h}) \in \boldsymbol{H}_{N,h}} \|(v, z) - (v_{N,h}, z_{N,h})\|_{2,D}.$$

Note that if $M(\lambda), M^*(\lambda^*) \subset \boldsymbol{H}^t(D)$ with $2 \leqslant t \leqslant N+1, N+1 > (m+\frac{1}{2})d$, then

$$\delta_{N,h}(\lambda) \lesssim (h/N)^{t-2} \quad \text{and} \quad \delta^*_{N,h}(\lambda^*) \lesssim (h/N)^{t-2}. \tag{3.13}$$

Using the spectral approximation theory in [2, 7], [22] established the a priori error estimates for the conforming finite element version of (3.11). According to [22], the following theorem is valid for the conforming spectral elements, as well as for the spectral method as a special case.

Theorem 3.1. *Suppose $n \in L^\infty(D)$, and $M(\lambda), M^*(\lambda^*) \subset \boldsymbol{H}^t(D)$ with $2 \leqslant t \leqslant N+1$, $N+1 > (m+\frac{1}{2})d$. Let $\lambda_{N,h}$ be an eigenvalue of* (3.11) *that converges to λ. Let $(u_{N,h}, w_{N,h}) \in M(\lambda_{N,h})$ and $\|(u_{N,h}, w_{N,h})\|_A = 1$. Then there exists $(u, w) \in M(\lambda)$ such that*

$$\|(u_{N,h}, w_{N,h}) - (u, w)\|_{2,D} \lesssim (h/N)^{\frac{t-2}{\alpha}}, \tag{3.14}$$
$$|\lambda - \lambda_{N,h}| \lesssim (h/N)^{\frac{2(t-2)}{\alpha}}; \tag{3.15}$$

*further suppose $n \in W^{1,\infty}(D)$ and **R(D)** is valid then*

$$\|(u_{N,h}, w_{N,h}) - (u, w)\|_{1,D} \lesssim (h/N)^{\frac{r_1}{\alpha} + \frac{t-2}{\alpha}}. \tag{3.16}$$

Proof. From [22, Theorems 3.3 and 3.5] (see also [11, Lemma 2.1]) we know if $(u_{N,h}, w_{N,h}) \in M(\lambda_{N,h})$ and $\|(u_{N,h}, w_{N,h})\|_A = 1$ then there exists $(u, w) \in M(\lambda)$ such that

$$\|(u_{N,h}, w_{N,h}) - (u, w)\|_{2,D} \lesssim \delta_{N,h}(\lambda)^{1/\alpha},$$
$$|\lambda - \lambda_{N,h}| \lesssim (\delta_{N,h}(\lambda)\delta^*_{N,h}(\lambda^*))^{1/\alpha};$$

Table 1 Numerical eigenvalues obtained by SM on $(-0.5, 0.5)^2$

n	N	dof	$k_{1,h}$	$k_{2,h}$,$k_{3,h}$	$k_{4,h}$	$k_{5,h}$
16	15	288	1.87959117836	2.4442361007	2.86643909864	3.14011071773664
16	20	578	1.87959117345	2.4442360999	2.86643911078	3.14011071380238
16	25	968	1.87959117325	2.4442360993	2.86643910989	3.14011071380235
16	30	1,458	1.87959117313	2.4442360992	2.86643910981	3.14011071380234
n	N	dof	$k_{1,h}$	$k_{2,h}$	$k_{3,h}$	$k_{5,h}, k_{6,h}$
f_1	15	288	2.8221893622	3.5386966987	3.5389915453	4.4965518722
						±0.8714816081i
f_1	20	578	2.8221893415	3.5386966965	3.5389915430	4.4965519547
						±0.8714817833i
f_1	25	968	2.8221893411	3.5386966953	3.5389915418	4.4965519545
						±0.8714817812i
f_1	30	1,458	2.8221893409	3.5386966952	3.5389915416	4.4965519545
						±0.8714817805i
n	N	dof	$k_{1,h}$,$k_{2,h}$	$k_{3,h}$	$k_{4,h}$	$k_{5,h}$
f_2	15	288	4.3184549937	4.5885145655	4.6472932515	4.95760056967
			±0.6549618762i			
f_2	20	578	4.3184553572	4.5885144838	4.6472932378	4.95759998805
			±0.6549618008i			
f_2	25	968	4.3184553557	4.5885144805	4.6472932351	4.95759999028
			±0.6549617996i			
f_2	30	1,458	4.3184553554	4.5885144801	4.6472932348	4.95759999016
			±0.6549617990i			

Table 2 Numerical eigenvalues obtained by SEM on $(-1, 1)^2\backslash([0, 1) \times (-1, 0])$

n	h	N	dof	$k_{1,h}$	$k_{2,h}$	$k_{3,h}$	$k_{4,h}$
16	$\sqrt{2}$	15	960	1.47854	1.569782	1.705721	1.78312049
16	$\sqrt{2}$	20	1,870	1.47742	1.569746	1.705408	1.78311760
16	$\sqrt{2}$	25	3,080	1.47691	1.569735	1.705269	1.78311674
16	$\sqrt{2}$	30	4,590	1.47665	1.569730	1.705195	1.78311641
16	$\frac{\sqrt{2}}{2}$	15	4,264	1.47722	1.569741	1.705355	1.78311725
16	$\frac{\sqrt{2}}{4}$	15	17,928	1.47663	1.569730	1.705189	1.78311639
16	$\frac{\sqrt{2}}{8}$	15	73,480	1.47635	1.569727	1.705111	1.78311614
n	h	N	dof	$k_{1,h}$	$k_{2,h}$	$k_{3,h}$	$k_{5,h}, k_{6,h}$
f_1	$\sqrt{2}$	15	960	2.30499	2.395810	2.64178134	2.92613
							±0.56694i
f_1	$\sqrt{2}$	30	4,590	2.30277	2.395702	2.64177929	2.92466
							±0.56511i
f_1	$\frac{\sqrt{2}}{2}$	15	4,264	2.30343	2.395724	2.64177970	2.92510
							±0.56566i
f_1	$\frac{\sqrt{2}}{4}$	15	17,928	2.30274	2.395701	2.64177927	2.92464
							±0.56509i
f_1	$\frac{\sqrt{2}}{8}$	15	73,480	2.30241	2.395694	2.64177916	2.92442
							±0.56482i

Table 3 Numerical eigenvalues obtained by SM on $(0,1)^3$

n	N	dof	$k_{1,h}$	$k_{2,h},k_{3,h},k_{4,h}$	$k_{5,h},k_{6,h},k_{7,h}$	$k_{8,h},k_{9,h}$
16	5	8	2.094055156	2.664272514	3.0661457744	3.406897998
16	10	343	2.067227464	2.584867751	2.9870636216	3.246721378
16	15	7,304	2.067227678	2.584856761	2.9870431376	3.246569769
16	20	4,913	2.067227671	2.584856755	2.9870431377	3.246569737
n	N	dof	$k_{1,h}$	$k_{2,h}$	$k_{3,h}$	$k_{4,h}$
f_1	5	8	3.098469114	3.865559775	3.8745648277	3.877187844
f_1	10	343	3.025670231	3.722083630	3.7247284048	3.724785357
f_1	15	7,304	3.025670590	3.722061785	3.7247087240	3.724765624
f_1	20	4,913	3.025670572	3.722061777	3.7247087161	3.724765616

Table 4 Numerical eigenvalues obtained by SEM on $((-1,1)^2\backslash(-1,0]^2)\times(0,1)$

n	h	N	dof	$k_{1,h}$	$k_{2,h}$	$k_{3,h}$	$k_{4,h}$
16	$\sqrt{3}$	4	14	1.85647	1.90219	1.96071	2.07418
16	$\frac{\sqrt{3}}{2}$	4	512	1.82025	1.87691	1.93480	2.04285
16	$\frac{\sqrt{3}}{4}$	4	6,800	1.80961	1.87089	1.92939	2.03857
16	$\sqrt{3}$	7	512	1.81154	1.87122	1.92966	2.03879
16	$\sqrt{3}$	8	950	1.80956	1.87078	1.92929	2.03848
16	$\sqrt{3}$	9	1,584	1.80841	1.87069	1.92925	2.03845
16	$\sqrt{3}$	10	2,450	1.80760	1.87064	1.92922	2.03843
n	h	N	dof	$k_{1,h}$	$k_{2,h}$	$k_{3,h}$	$k_{4,h}$
f_1	$\sqrt{3}$	4	14	2.69988	2.73831	2.92444	3.07209
f_1	$\frac{\sqrt{3}}{2}$	4	512	2.65535	2.66530	2.85574	2.98575
f_1	$\frac{\sqrt{3}}{4}$	4	6,800	2.64344	2.64820	2.84249	2.97109
f_1	$\sqrt{3}$	7	512	2.64455	2.65045	2.84339	2.97245
f_1	$\sqrt{3}$	8	950	2.64325	2.64810	2.84230	2.97092
f_1	$\sqrt{3}$	9	1,584	2.64269	2.64702	2.84203	2.97042
f_1	$\sqrt{3}$	10	2,450	2.64215	2.64641	2.84186	2.97005

Table 5 Numerical eigenvalues obtained by SEM on $(-1,1)^3\backslash(-1,0)^3$

n	h	N	dof	$k_{1,h}$	$k_{2,h},k_{3,h}$	$k_{4,h}$,$k_{5,h}$	$k_{6,h}$
16	$\sqrt{3}$	4	74	1.4993	1.5552	1.6676	1.6857
16	$\frac{\sqrt{3}}{2}$	4	1,568	1.4443	1.5199	1.6443	1.6515
16	$\frac{\sqrt{3}}{4}$	4	17,840	1.4277	1.5097	1.6392	1.6420
16	$\frac{\sqrt{3}}{4}$	5	45,808	1.4225	1.5069	1.6388	1.6395
16	$\sqrt{3}$	6	774	1.4402	1.5161	1.6405	1.6477
16	$\sqrt{3}$	7	1,568	1.4325	1.5121	1.6397	1.6442
16	$\sqrt{3}$	8	2,770	1.4279	1.5097	1.6392	1.6421
16	$\sqrt{3}$	9	4,464	1.4248	1.5081	1.6389	1.6406
n	h	N	dof	$k_{1,h}$	$k_{2,h}$	$k_{3,h}$	$k_{4,h}$
f_1	$\sqrt{3}$	4	74	2.2003	2.2307	2.2957	2.3602
f_1	$\frac{\sqrt{3}}{2}$	4	1,568	2.1191	2.1675	2.2282	2.3227
f_1	$\frac{\sqrt{3}}{4}$	4	17,840	2.0953	2.1523	2.2105	2.3152
f_1	$\frac{\sqrt{3}}{4}$	5	45,808	2.0880	2.1480	2.2066	2.3143
f_1	$\sqrt{3}$	6	774	2.1126	2.1625	2.2197	2.3178
f_1	$\sqrt{3}$	7	1,568	2.1020	2.1561	2.2141	2.3161
f_1	$\sqrt{3}$	8	2,770	2.0955	2.1523	2.2105	2.3151
f_1	$\sqrt{3}$	9	4,464	2.0913	2.1498	2.2083	2.3146

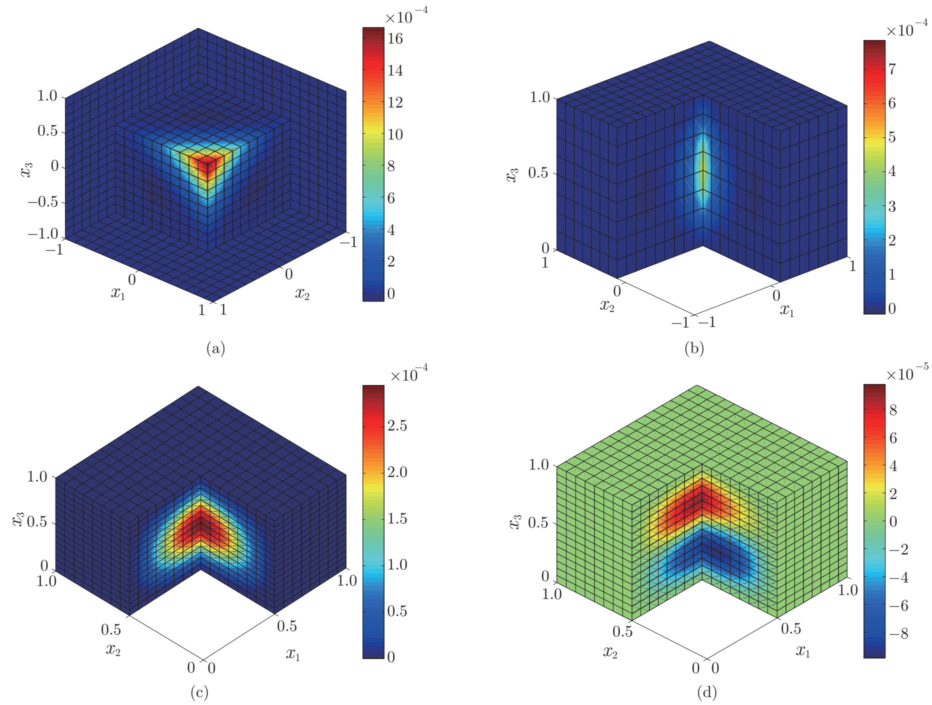

Figure 2 The 1st eigenfunction for $n = 16$ on $(-1,1)^3\backslash(-1,0)^3$ (a), on $((-1,1)^2\backslash(-1,0]^2)\times(0,1)$ (b) and on $(-1,1)^3$ (c); the 2nd eigenfunction for $n = 16$ on $(-1,1)^3$ (d)

furthermore, if $n \in W^{1,\infty}(D)$ and $\boldsymbol{R(D)}$ is valid then

$$\|(u_{N,h}, w_{N,h}) - (u,w)\|_{1,D} \lesssim (h/N)^{r_1/\alpha}\delta_{N,h}(\lambda)^{1/\alpha}.$$

Substituting (3.13) into the above estimates yields the desired results (3.14)–(3.16). □

4 Numerical experiment

In this section, we will report some numerical experiments for solving the transmission eigenvalue problem (3.8) by the H^2 conforming spectral element method (SEM) on non-rectangular domain or by the spectral method (SM) on rectangular domain. In computation, the basis functions for both SM and SEM are adopted according to (A)–(C), where the functions $\phi_j,\ j = 0,1,\ldots,N$ are defined by (2.2)–(2.7) (here, we use $\widehat{\phi}_j,\ j = 4,5,\ldots,N$ defined by (2.7) instead of (2.1)). Notice that the spectral scheme in [1] is based on the iterative method in [19], which is different from the one in this paper. An obvious feature of the method in [1] is using an estimated eigenvalue to initialize the iterative procedure. We consider the case when the medium D is the unit square $(-0.5,0.5)^2$ or the L-shaped domain $(-1,1)^2\backslash([0,1)\times(-1,0])$ in two dimensions and when D is the unit cube $(0,1)^3$, the thick L-shaped domain $((-1,1)^2\backslash(-1,0]^2)\times(0,1)$ or the Fichera corner domain $(-1,1)^3\backslash(-1,0)^3$ in three dimensions; we set the homogeneous index of refraction $n = 16$, inhomogeneous $n = f_1(x), f_2(x)$ with $f_1(x) = 8 + x_1 - x_2$ and $f_2(x) = 4 + \mathrm{e}^{x_1+x_2}$. We use uniform rectangular refinement to obtain a sequence of partitions of D. Accordingly some numerical eigenvalues on these domains are listed in Tables 1–5. We also depict the cross section of some eigenfunctions on the thick L-shaped domain and the Fichera corner domain with $n = 16$ (see Figure 2).

We use Matlab 2012a to solve (3.11) by the sparse solver eigs on a Lenovo G480 PC with 4G memory.

For reading convenience, we denote by $k_{j,h} = \sqrt{\lambda_{j,h}}$ the numerical eigenvalue obtained on the space $\boldsymbol{H}_h$, which approximates the j-th eigenvalue $\sqrt{\lambda_j}$.

Tables 1 and 3 show that the numerical eigenvalues obtained by SM on both the unit square and the unit cube with different n own superior accuracy; specifically, they achieve about eight-digit accuracy with $N = 15$. Whereas the numerical eigenvalues obtained by SEM on the L-shaped domain, the thick

L-shaped domain and the Fichera corner domain do not have such accuracy (see Tables 2, 4 and 5). This is due to the fact that the eigenfunctions on the unit square and the unit cube are often smooth whereas those on the L-shaped domain, the thick L-shaped domain and the Fichera corner domain have the singularities towards the reentrant corner (see Figure 2).

5 Conclusions and further work

In this paper, we study the H^m-conforming rectangular spectral element methods on arbitrarily dimensional domain, which is a basic and significant work. We also apply it to solve the Helmholtz transmission eigenvalue problem. A more significant and challenging work is to extend the rectangular spectral elements to quadrilateral spectral elements on arbitrarily dimensional domain even for $m = 2$. We are not able to complete it so far.

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