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Least energy solutions of nonlinear Schrödinger equations involving the fractional Laplacian and potential wells

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Abstract We are concerned with the existence of least energy solutions of nonlinear Schrödinger equations involving the fractional Laplacian

$$(-\Delta)^s u(x) + \lambda V(x)u(x) = u(x)^{p-1}, \quad u(x) \geqslant 0, \quad x \in \mathbb{R}^N,$$

for sufficiently large λ, $2 < p < \frac{2N}{N-2s}$ for $N \geqslant 2$. $V(x)$ is a real continuous function on $\mathbb{R}^N$. Using variational methods we prove the existence of least energy solution $u_\lambda(x)$ which localizes near the potential well int $V^{-1}(0)$ for λ large. Moreover, if the zero sets int $V^{-1}(0)$ of $V(x)$ include more than one isolated component, then $u_\lambda(x)$ will be trapped around all the isolated components. However, in Laplacian case $s = 1$, when the parameter λ is large, the corresponding least energy solution will be trapped around only one isolated component and become arbitrarily small in other components of int $V^{-1}(0)$. This is the essential difference with the Laplacian problems since the operator $(-\Delta)^s$ is nonlocal.

Keywords nonlinear Schrödinger equation, least energy solution, fractional Laplacian, variational methods

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1 Introduction and main results

We are concerned with the following nonlinear Schrödinger equations involving the fractional Laplacian:

$$\begin{cases} (-\Delta)^s u(x) + \lambda V(x)u(x) = u(x)^{p-1}, & x \in \mathbb{R}^N, \\ u(x) \geqslant 0, \quad u(x) \in H^s(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $2 < p < 2_s^\sharp := \frac{2N}{N-2s}$ for $N \geqslant 2$, $V(x)$ is the potential, which is a real valued function on $\mathbb{R}^N$.

In recent years, much attention has been devoted to the study of the fractional Laplacian. The fractional powers of the Laplacian, which are called fractional Laplacian and correspond to Lévy stable processes, appear in anomalous diffusion phenomena in physics, biology as well as other areas. They occur in flame propagation, chemical reaction in liquids and population dynamics. Lévy diffusion processes have discontinuous sample paths and heavy tails, while Brownian motion has continuous sample paths and

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exponential decaying tails. These processes have been applied to American options in mathematical finance for modeling the jump processes of the financial derivatives such as futures, forwards, options, and swaps (see [1] and references therein). Moreover, they play important roles in the study of the quasi-geostrophic equations in geophysical fluid dynamics.

There are many results which are concerned with the problems involving the fractional Laplacian. Firstly, we refer the readers to the work by Caffarelli and Silvestre [7], in which a new formulation of the fractional Laplacian through Dirichlet-Neumann maps was introduced. By this formulation, they transferred the nonlocal problem to a local problem defined in a higher half space. After their pioneering work, there are many investigations to the fractional Laplacian problem by using variational methods. For example, using variational methods, Cabré and Tan [5] established the existence of positive solutions for fractional problems in a bounded domain with power-type nonlinearities in the subcritical case. We also refer the work by Dávila et al. [10], where they considered the following fractional problem:

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u - u^q = 0, \quad u > 0, \quad u \in H^{2s}(\mathbb{R}^N), \tag{1.2}$$

where $0 < s < 1$, $1 < q < \frac{N+2s}{N-2s}$, $V(x)$ is a sufficiently smooth potential with $\inf_{\mathbb{R}^N} V(x) > 0$ and $\varepsilon > 0$ is a small parameter. Via a Lyapunov-Schmidt variational reduction, they proved the existence of multiple spike solutions which concentrate as ε small at separate places in the case of stable critical points and the existence of multiple spikes which concentrate as ε small at the same points.

For the following related fractional Schrödinger equation:

$$(-\Delta)^s u + V(x)u = f(x,u), \quad x \in \mathbb{R}^N, \tag{1.3}$$

with $0 < s < 1$ and $V : \mathbb{R}^N \to \mathbb{R}$ being the potential function, there are also many investigations. See also Barrios et al. [2], Bona and Li [3], de Bouard and Saut [11], Cabré and Sire [4], Caffarelli et al. [6], Cheng [8], Choi et al. [9], Dipierro et al. [12], Felmer et al. [13], Frank and Lenzmann [14], Maris [17], Silvestre [19], Sire and Valdinoci [20], Tan [21, 22], Yan et al. [24] and references therein.

We also refer the readers to the paper by Jin et al. [15], where the authors considered the following fractional Laplacian equations with lower order terms:

$$(-\Delta)^s u = au + b, \quad x \in B_1, \tag{1.4}$$

where $a, b \in C^\alpha(B_1)$ with $0 < \alpha \notin \mathbb{N}$ and $2s + \alpha$ is not an integer. They proved some priori estimates results for the solutions of the above equation (1.4), such as the local Schauder estimates for non-negative solutions. We also refer the work by Tan and Xiong [23], where they established a Harnack inequality in the case of $u \in C^2(B_1) \cap C(\overline{B_1})$.

In our previous paper (see Niu and Tang [18]), we considered the existence of least energy solutions for nonlinear Schrödinger equations (1.1) in the case of $s = 1/2$, namely the following problem:

$$\begin{cases} (-\Delta)^{1/2}u(x) + \lambda V(x)u(x) = u(x)^{p-1}, & x \in \mathbb{R}^N, \\ u(x) \geqslant 0, \quad u(x) \in H^{1/2}(\mathbb{R}^N), \end{cases} \tag{1.5}$$

where $2 < p < \frac{2N}{N-1}$ for $N \geqslant 2$, $V(x)$ is the potential, which is a real valued function on $\mathbb{R}^N$. Using variational methods we proved the existence of the least energy solution $u_\lambda(x)$ which localizes near the potential well $\mathrm{int}(V^{-1}(0))$ for λ large.

In the present paper, as a continuation of our previous paper, we consider the fractional problem for more general cases $0 < s < 1$. Our main assumptions are the following:

(V_1) $V(x) \in C(\mathbb{R}^N, \mathbb{R})$ such that $V(x) \geqslant 0$, $\Omega := \mathrm{int}\ V^{-1}(0)$ is non-empty with smooth boundary and $\bar{\Omega} = V^{-1}(0)$;

(V_2) There exists $M_0 > 0$ such that

$$\mu(\{x \in \mathbb{R}^N : V(x) \leqslant M_0\}) < \infty,$$

where μ denotes the Lebesgue measure on $\mathbb{R}^N$.

Before stating our main results, we firstly give some notation and remarks.

To treat the nonlocal problem (1.1), we will study a corresponding extension problem in one more dimensional space, which allows us to investigate (1.1) by studying a local problem via classical nonlinear variational methods.

The homogeneous fractional Sobolev space $D^s(\mathbb{R}^N)$ $(0 < s < 1)$ is given by

$$D^s(\mathbb{R}^N) := \left\{ u \in L^{\frac{2N}{N-2s}}(\mathbb{R}^N) : \|u\|_{D^s(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi \right)^{1/2} < \infty \right\},$$

where $\hat{u}$ denotes the Fourier transform of u.

Note that $D^s(\mathbb{R}^N)$ is a Hilbert space equipped with an inner product

$$(u, v)_{D^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |\xi|^{2s} \hat{u}(\xi)\hat{v}(\xi) d\xi.$$

We also define a fractional Laplacian operator on the whole space,

$$(-\Delta)^s : D^s(\mathbb{R}^N) \to D^{-s}(\mathbb{R}^N)$$

by

$$\langle (-\Delta)^s u, v\rangle_{D^{-s}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |\xi|^{2s} \hat{u}(\xi)\hat{v}(\xi) d\xi,$$

where $D^{-s}(\mathbb{R}^N)$ is the dual of $D^s(\mathbb{R}^N)$.

Then, one can easily check that if $u \in D^{2s}(\mathbb{R}^N)$, we have $(-\Delta)^s u \in L^2(\mathbb{R}^N)$ such that

$$(-\Delta)^s u = \mathcal{F}^{-1}(|\xi|^{2s} \hat{u}(\xi)),$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform.

We see that for u, $v \in D^s(\mathbb{R}^N)$,

$$(u, v)_{D^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} (-\Delta)^{s/2} u \cdot (-\Delta)^{s/2} v dx$$

and assuming additionally $u \in D^{2s}(\mathbb{R}^N)$, $v \in L^2(\mathbb{R}^N)$, we can apply integration by parts and get

$$\int_{\mathbb{R}^N} (-\Delta)^{s/2} u \cdot (-\Delta)^{s/2} v dx = \int_{\mathbb{R}^N} (-\Delta)^s u \cdot v dx.$$

Finally, the notation $H^s(\mathbb{R}^N)$ denotes the standard fractional Sobolev space defined as

$$H^s(\mathbb{R}^N) := D^s(\mathbb{R}^N) \cap L^2(\mathbb{R}^N),$$

with the norm

$$\|u\|_{H^s(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} |u(x)|^2 dx + \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi \right)^{1/2}.$$

Similarly, it holds by taking the trace that

$$D^s(\mathbb{R}^N) = \left\{ u = \mathrm{tr}|_{\mathbb{R}^N \times \{0\}} U : \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U|^2 dx dt < \infty \right\}$$

and

$$\|U(\cdot, 0)\|_{D^s(\mathbb{R}^N)} \leqslant C \left(\int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U|^2 dx dt \right)^{1/2}$$

for some $C > 0$ independent of

$$U \in \left\{ U \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^{N+1}_+) : \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U|^2 dx dt < \infty \right\}.$$

Now we introduce the concept of s-harmonic extension of a function $u \in D^s(\mathbb{R}^N)$, which provides a way to represent the fractional Laplacian operators as a form of Dirichlet-to-Neumann map.

By works of Caffarelli and Silvestre [7] (for $\mathbb{R}^N$), it is known that there is one unique function

$$U \in H(t^{1-2s}, \mathbb{R}^{N+1}_+) := \left\{ U : \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U|^2 dx dt < \infty, \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |U|^2 dx dt < \infty \right\},$$

which satisfies the equation

$$\begin{cases} \operatorname{div}(t^{1-2s} \nabla U) = 0, & \text{in } \mathbb{R}^{N+1}_+, \\ U(x,0) = u, & \text{for } x \in \mathbb{R}^N, \end{cases} \tag{1.6}$$

respectively in the distributional sense. Moreover, if u is compactly supported and smooth, then the following limits

$$\partial^s_\nu U(x,0) := -C_s^{-1} \left(\lim_{t \to 0^+} t^{1-2s} \frac{\partial U}{\partial t}(x,t) \right) \quad \text{with} \quad C_s := \frac{2^{1-2s}\Gamma(1-s)}{\Gamma(s)}$$

are well defined and one must have

$$(-\Delta)^s u = \partial^s_\nu U(x,0).$$

We call U the s-harmonic extension of u.

Let

$$E := \left\{ U \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^{N+1}_+) : \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U|^2 dx dt < \infty, U(\cdot, 0) \in L^2(\mathbb{R}^N) \right\}.$$

Then E is the Hilbert space under the inner product

$$(U,W)_E = \int_{\mathbb{R}^{N+1}_+} t^{1-2s} \nabla U \cdot \nabla W \, dx dt + \int_{\mathbb{R}^N} U(x,0) W(x,0) dx,$$

and the norm induced by the inner product $(\cdot,\cdot)$ is

$$\|U\|_E = \left(\int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U|^2 dx dt + \int_{\mathbb{R}^N} U(x,0)^2 dx \right)^{1/2}.$$

Indeed for every $U(x,t) \in E$, we denote $U(x,0)$ to be the trace of $U(x,t)$ on $\mathbb{R}^N$ and we take

$$\mathrm{tr}_{\mathbb{R}^N} E := \{ U(x,0) : U(x,t) \in E \}.$$

Then by the definition of E, we have

$$\mathrm{tr}_{\mathbb{R}^N} E = H^s(\mathbb{R}^N). \tag{1.7}$$

We take

$$E_\lambda := \left\{ U \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^{N+1}_+) : \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U|^2 dx dt < \infty, \int_{\mathbb{R}^N} \lambda V(x) U(x,0)^2 dx < \infty \right\},$$

then E_λ is the Hilbert space under the inner product

$$(U,W)_\lambda = \int_{\mathbb{R}^{N+1}_+} t^{1-2s} \nabla U \cdot \nabla W \, dx dt + \int_{\mathbb{R}^N} \lambda V(x) U(x,0) W(x,0) dx,$$

and the norm induced by the inner product $(\cdot,\cdot)_\lambda$ is

$$\|U\|_\lambda = \left(\int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U|^2 dx dt + \int_{\mathbb{R}^N} \lambda V(x) U(x,0)^2 dx \right)^{1/2}.$$

We can study (1.1) by variational methods for a local problem. More precisely, we will study the following boundary value problem in a half space:

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla U)=0, & \text{in } \mathbb{R}^{N+1}_+=\mathbb{R}^N\times(0,\infty),\\ \partial^s_\nu U(\cdot,0)=U^{p-1}-\lambda V(x)U, & \text{on } \partial\mathbb{R}^{N+1}_+=\mathbb{R}^N\times\{0\},\\ U\geqslant 0, & \text{in } \mathbb{R}^{N+1}_+=\mathbb{R}^N\times(0,\infty), \end{cases} \tag{1.8}$$

where

$$\partial^s_\nu U(x,0):=-C_s^{-1}\left(\lim_{t\to0^+}t^{1-2s}\frac{\partial U}{\partial t}(x,t)\right)$$

with

$$C_s:=\frac{2^{1-2s}\Gamma(1-s)}{\Gamma(s)}.$$

If U satisfies (1.8), then the trace u on $\mathbb{R}^N\times\{0\}$ of the function U will be a solution of (1.1). By studying (1.8), we establish the results for (1.1).

The energy functional associated with (1.8) is defined by

$$\begin{aligned} J_\lambda(U):=&\frac12\int_{\mathbb{R}^{N+1}_+}t^{1-2s}|\nabla U|^2dxdt+\frac{\lambda}{2}\int_{\mathbb{R}^N}V(x)U(x,0)^2dx\\ &-\frac1p\int_{\mathbb{R}^N}U^+(x,0)^pdx \quad \text{for} \quad U\in E_\lambda, \end{aligned} \tag{1.9}$$

where U^+ denotes the positive part of U for every function U. In other words, $U^+=\max\{U,0\}$.

We define the Nehari manifold

$$\mathcal{M}_\lambda:=\left\{U\in E_\lambda\setminus\{0\}:\int_{\mathbb{R}^{N+1}_+}t^{1-2s}|\nabla U|^2dxdt+\lambda\int_{\mathbb{R}^N}V(x)U(x,0)^2dx=\int_{\mathbb{R}^N}U^+(x,0)^pdx\right\}$$

and let

$$c_\lambda:=\inf\{J_\lambda(U):U\in\mathcal{M}_\lambda\}$$

be the infimum of J_λ on the Nehari manifold $\mathcal{M}_\lambda$.

For λ large, the following problem:

$$\begin{cases} (-\Delta)^su(x)=u(x)^{p-1}, & x\in\Omega,\\ u(x)\geqslant0, & x\in\Omega,\\ u(x)=0, & x\in\mathbb{R}^N\setminus\Omega, \end{cases} \tag{1.10}$$

is some kind of limit problem of (1.1). We shall prove that there exists a least energy solution of (1.1) converging, for $\lambda\to\infty$, to a least energy solution of (1.10).

Similarly, to consider (1.10), we will study the following boundary value problem in a half space:

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla U)=0, & \text{in } \mathbb{R}^{N+1}_+=\mathbb{R}^N\times(0,\infty),\\ U=0, & \text{on } \mathbb{R}^N\setminus\Omega\times\{0\},\\ \partial^s_\nu U(\cdot,0)=U^{p-1}, & \text{on } \Omega\times\{0\},\\ U\geqslant0, & \text{in } \mathbb{R}^{N+1}_+=\mathbb{R}^N\times(0,\infty), \end{cases} \tag{1.11}$$

where

$$\partial^s_\nu U(x,0):=-C_s^{-1}\left(\lim_{t\to0^+}t^{1-2s}\frac{\partial U}{\partial t}(x,t)\right)$$

with

$$C_s:=\frac{2^{1-2s}\Gamma(1-s)}{\Gamma(s)}.$$

If U satisfies (1.11), then the trace u on $\mathbb{R}^N \times \{0\}$ of the function U will be a solution of (1.10).

To consider (1.11), we define a subspace E_0 of E as follows:

$$E_0 := \{U(x,t) \in E : U(x,0) = 0 \text{ in } \mathbb{R}^N \setminus \Omega\}. \tag{1.12}$$

Similarly, we also have

$$\mathrm{tr}_\Omega E_0 = H^s(\Omega).$$

The energy functional associated with (1.11) is defined by

$$\Phi(U) = \frac{1}{2}\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla U|^2 dxdt - \frac{1}{p}\int_\Omega U^+(x,0)^p dx \quad \text{for} \quad U \in E_0.$$

Comparing with the Nehari manifold $\mathcal{M}_\lambda$, we define the Nehari manifold

$$\mathcal{N} := \left\{U \in E_0 \setminus \{0\} : \int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla U|^2 dxdt = \int_\Omega U^+(x,0)^p dx\right\}$$

and let

$$c(\Omega) := \inf\{\Phi(U) : U \in \mathcal{N}\}$$

be the infimum of Φ on the Nehari manifold $\mathcal{N}$.

Remark 1.1. We say that a function $u_\lambda(x) = U_\lambda(x,0)$ is a least energy solution of (1.1) if c_λ is achieved by some $U_\lambda \in \mathcal{M}_\lambda$ which is a critical point of J_λ. Similarly, we say that a function $u(x) = U(x,0)$ is a least energy solution of (1.10) if $c(\Omega)$ is achieved by some $U \in \mathcal{N}$ which is a critical point of Φ.

Now we give our main results which are the following:

Theorem 1.2. *Suppose* (V_1) *and* (V_2) *hold. Then for* λ *large,* (1.1) *has a least energy solution* $u_\lambda(x) = U_\lambda(x,0)$. *Furthermore, for any sequence* $\lambda_n \to \infty$ $(\lambda_n \to \infty$ *as* $n \to \infty)$, $\{u_{\lambda_n}(x)\}$ *has a subsequence such that* u_{λ_n} *converges in* $H^s(\mathbb{R}^N)$ *along the subsequence to a least energy solution* u *of* (1.10).

As in the case of the least energy solution of (1.1), any solution of (1.1) converges, for $\lambda \to \infty$, to the solution of (1.10). More precisely, we have the following result.

Theorem 1.3. *Suppose* (V_1) *and* (V_2) *hold. Let* $u_n = U_n(\cdot,0), n \in \mathbb{N}$ *be a sequence of solutions of* (1.1) *with* λ *being replaced by* λ_n $(\lambda_n \to \infty$ *as* $n \to \infty)$ *such that*

$$\limsup_{n\to\infty} J_\lambda(U_n) < \infty.$$

Then $u_n(x) = U_n(x,0)$ *converges strongly along a subsequence in* $H^s(\mathbb{R}^N)$ *to a solution* u *of* (1.10).

Our paper is organized as follows: In Section 2, we give a compactness result, Section 3 is devoted to the "limit" problem and Section 4 contains the proofs of the main results.

We will use the same C to denote various generic positive constant, and we will use $o(1)$ to denote quantities that tend to 0 as λ (or n) tends to ∞.

2 Compactness result

The main result in this section is the following compactness result.

Proposition 2.1. *Suppose* (V_1) *and* (V_2) *hold. Then for any* $C_0 > 0$, *there exists* $\Lambda_0 > 0$ *such that* J_λ *satisfies the* $(PS)_c$*-condition for all* $\lambda \geqslant \Lambda_0$ *and* $c \leqslant C_0$.

The proof consists of a series of lemmas which occupy the rest of this section.

Lemma 2.2. *Let* $\lambda_0 > 0$ *be a fixed constant. Then for* $\lambda \geqslant \lambda_0 > 0$, $V(x)$ *satisfies* (V_1) *and* (V_2), E_λ *is continuously embedded in* E.

Proof. From the definition of E and E_λ, to show the lemma, we only need to prove the following estimate:

$$\int_{\mathbb{R}^N} U(x,0)^2dx \leqslant C\bigg(\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla U|^2dxdt + \lambda\int_{\mathbb{R}^N} V(x)U(x,0)^2dx\bigg). \tag{2.1}$$

Let us denote

$$D := \{x\in\mathbb{R}^N : V(x)\leqslant M_0\}$$

and

$$D^{\delta_0} := \{x\in\mathbb{R}^N : \mathrm{dist}(x,D)\leqslant \delta_0\}.$$

Take $\zeta\in C^\infty(\mathbb{R}^N)$, $0\leqslant\zeta\leqslant 1$ and for the above small fixed δ_0,

$$\zeta(x)=\begin{cases}1, & x\in D,\\ 0, & x\notin D^{\delta_0},\end{cases}\qquad |\nabla\zeta|\leqslant C/\delta_0. \tag{2.2}$$

Thus for any function $U\in E_\lambda$, we obtain that

$$\begin{aligned}\int_{\mathbb{R}^N}(1-\zeta^2)U(x,0)^2dx &= \int_{\mathbb{R}^N\setminus D}(1-\zeta^2)U(x,0)^2dx + \int_D(1-\zeta^2)U(x,0)^2dx\\ &\leqslant \frac{1}{\lambda_0M_0}\lambda\int_{\mathbb{R}^N}V(x)U(x,0)^2dx \end{aligned}\tag{2.3}$$

and

$$\begin{aligned}\int_{\mathbb{R}^N}\zeta^2U(x,0)^2dx &= \int_{D^{\delta_0}}\zeta^2U(x,0)^2dx\\ &\leqslant \mu(D^{\delta_0})^{1-\frac{2}{2^\sharp_s}}\bigg(\int_{D^{\delta_0}}|U(x,0)|^{2^\sharp_s}dx\bigg)^{\frac{2}{2^\sharp_s}}\\ &\leqslant \mu(D^{\delta_0})^{1-\frac{2}{2^\sharp_s}}\bigg(\int_{\mathbb{R}^N}|U(x,0)|^{2^\sharp_s}dx\bigg)^{\frac{2}{2^\sharp_s}}\\ &\leqslant \frac{\mathcal{S}_{N,s}}{\sqrt{C_s}}\mu(D^{\delta_0})^{1-\frac{2}{2^\sharp_s}}\int_{\mathbb{R}^{N+1}_+}t^{1-2s}|\nabla U|^2dxdt,\end{aligned}\tag{2.4}$$

where we have used Assumption (V_2) and the well-known Sobolev trace inequality which states for $U\in D^1(t^{1-2s},\mathbb{R}^{N+1}_+)$,

$$\bigg(\int_{\mathbb{R}^N}|U(x,0)|^{2^\sharp_s}dx\bigg)^{1/2^\sharp_s}\leqslant\frac{\mathcal{S}_{N,s}}{\sqrt{C_s}}\bigg(\int_{\mathbb{R}^{N+1}_+}t^{1-2s}|\nabla U|^2dxdt\bigg)^{1/2},$$

where

$$\mathcal{S}_{N,s}=2^{-2s}\pi^{-s}\frac{\Gamma(\frac{N-2s}{2})}{\Gamma(\frac{N+2s}{2})}\bigg[\frac{\Gamma(N)}{\Gamma(N/2)}\bigg]^{2s/N}$$

and $C_s := \frac{2^{1-2s}\Gamma(1-s)}{\Gamma(s)}$. Therefore, we indeed have proved (2.1) by adding two inequalities (2.3) and (2.4) together. Thus the proof of the lemma is completed. □

The following Lemma shows that 0 is an isolated critical point of J_λ.

Lemma 2.3. *Let K_λ denote the set of critical points of J_λ and $\lambda\geqslant\lambda_0>0$. Then there exists $\sigma>0$ independent of λ such that $\|U\|_\lambda\geqslant\sigma$ for all $U\in K_\lambda\setminus\{0\}$.*

Proof. By Lemma 2.2, for any $U\in K_\lambda\setminus\{0\}$,

$$\begin{aligned}0=J'_\lambda(U)\cdot U &= \int_{\mathbb{R}^{N+1}_+}t^{1-2s}|\nabla U|^2dxdt+\lambda\int_{\mathbb{R}^N}V(x)U(x,0)^2dx-\int_{\mathbb{R}^N}U^+(x,0)^pdx\\ &\geqslant \|U\|^2_\lambda - C\|U\|^p_E\\ &\geqslant \|U\|^2_\lambda - C\|U\|^p_\lambda,\end{aligned}$$

where $C>0$ is independent of $\lambda\geqslant 0$. Thus we see that there exists $\sigma>0$ such that $\|U\|_\lambda\geqslant\sigma$. □

Lemma 2.4. *There exists $c_0 > 0$ independent of $\lambda \geqslant \lambda_0 > 0$ such that if $\{U_n\}$ is a $(PS)_c$-sequence of J_λ, then*

$$\limsup_{n\to\infty} \|U_n\|_\lambda^2 \leqslant \frac{2p}{p-2}c \tag{2.5}$$

and either $c \geqslant c_0$ or $c = 0$.

Proof. First we prove that any $(PS)_c$-sequence must be bounded; in fact,

$$\begin{aligned} c &= \limsup_{n\to\infty} \left(J_\lambda(U_n) - \frac{1}{p} J'_\lambda(U_n)U_n \right) \\ &\geqslant \limsup_{n\to\infty} \left(\frac{1}{2} - \frac{1}{p} \right) \left(\int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U_n|^2 dxdt + \lambda \int_{\mathbb{R}^N} V(x) U_n(x,0)^2 dx \right) \\ &= \frac{p-2}{2p} \limsup_{n\to\infty} \|U_n\|_\lambda^2, \end{aligned}$$

which proves (2.5).

On the other hand, there is a constant $C > 0$ independent of $\lambda \geqslant \lambda_0 \geqslant 0$ such that

$$\begin{aligned} J'_\lambda(U) \cdot U &= \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U|^2 dxdt + \lambda \int_{\mathbb{R}^N} V(x) U(x,0)^2 dx - \int_{\mathbb{R}^N} U^+(x,0)^p dx \\ &\geqslant \|U\|_\lambda^2 - C\|U\|_\lambda^p. \end{aligned}$$

Thus there exists $\sigma_1 > 0$ independent of λ such that

$$\frac{1}{4}\|U\|_\lambda^2 \leqslant J'_\lambda(U) \cdot U \quad \text{for} \quad \|U\|_\lambda < \sigma_1. \tag{2.6}$$

Now, if $c < \frac{\sigma_1^2(p-2)}{2p}$ and $\{U_n\}$ is a $(PS)_c$-sequence of J_λ, then

$$\limsup_{n\to\infty} \|U_n\|_\lambda^2 \leqslant \frac{2cp}{p-2} < \sigma_1^2.$$

Hence, $\|U_n\|_\lambda < \sigma_1$ for n large. Then by (2.6),

$$\frac{1}{4}\|U_n\|_\lambda^2 \leqslant J'_\lambda(U_n) \cdot U_n = o(1)\|U_n\|_\lambda,$$

which implies $\|U_n\|_\lambda \to 0$ as $n \to \infty$. Therefore $J_\lambda(U_n) \to 0$, i.e., $c = 0$. Thus $c_0 = \frac{\sigma_1^2(p-2)}{2p}$ is as required. □

Lemma 2.5. *There exists $\delta_0 > 0$ such that any $(PS)_c$-sequence $\{U_n\}$ of J_λ with $\lambda \geqslant 0$ and $c > 0$ satisfies*

$$\liminf_{n\to\infty} \|U_n^+(\cdot,0)\|_{L^p(\mathbb{R}^N)}^p \geqslant \delta_0 c. \tag{2.7}$$

Proof. From the proof of Lemma 2.4 we know that $\{U_n\}$ is bounded and hence

$$\begin{aligned} c &= \lim_{n\to\infty} \left(J_\lambda(U_n) - \frac{1}{2} J'_\lambda(U_n) \cdot U_n \right) \\ &= \left(\frac{1}{2} - \frac{1}{p} \right) \lim_{n\to\infty} \int_{\mathbb{R}^N} U_n^+(x,0)^p dx \\ &= \frac{(p-2)}{2p} \lim_{n\to\infty} \|U_n^+(\cdot,0)\|_{L^p(\mathbb{R}^N)}^p, \end{aligned}$$

which implies (2.7) with $\delta_0 = \frac{2p}{p-2}$. □

Lemma 2.6. *Let C_1 be fixed. Then for any $\varepsilon > 0$ there exists $\Lambda_\varepsilon > 0$ and $R_\varepsilon > 0$ such that if $\{U_n\}$ is a $(PS)_c$-sequence of J_λ with $\lambda \geqslant \Lambda_\varepsilon$, $c \leqslant C_1$, then*

$$\limsup_{n\to\infty} \int_{B^c_{R_\varepsilon}} U_n^+(x,0)^p dx \leqslant \varepsilon, \tag{2.8}$$

where $B^c_{R_\varepsilon} = \{x \in \mathbb{R}^N : |x| \geqslant R_\varepsilon\}$.

Proof. For $R > 0$, we set

$$A(R) := \{x \in \mathbb{R}^N : |x| > R, V(x) \geqslant M_0\}$$

and

$$B(R) := \{x \in \mathbb{R}^N : |x| > R, V(x) < M_0\}.$$

Then by Lemma 2.4, we can obtain that

$$\begin{aligned}\int_{A(R)} U_n(x,0)^2 dx &\leqslant \frac{1}{\lambda M_0}\int_{\mathbb{R}^N} \lambda V(x) U_n(x,0)^2 dx \\ &\leqslant \frac{1}{\lambda M_0}\bigg(\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla U_n|^2 dxdt + \int_{\mathbb{R}^N}\lambda V(x)U_n(x,0)^2dx\bigg) \\ &\leqslant \frac{1}{\lambda M_0}\bigg(\frac{2p}{p-2}C_1 + o(1)\bigg).\end{aligned} \tag{2.9}$$

Using Hölder's inequality and (2.5), we obtain that for $1 < q < N/(N-2s)$,

$$\begin{aligned}\int_{B(R)} U_n(x,0)^2 dx &\leqslant \bigg(\int_{\mathbb{R}^N} U_n(x,0)^{2q} dx\bigg)^{1/q} \mu(B(R))^{1/q'} \\ &\leqslant C\|U_n\|^2_\lambda \cdot \mu(B(R))^{1/q'} \\ &\leqslant C\frac{2p}{p-2}C_1 \cdot \mu(B(R))^{1/q'},\end{aligned} \tag{2.10}$$

where $C = C(N,q)$ is a positive constant and q' is such that $1/q + 1/q' = 1$, $\mu(B(R))$ denotes the Lebesgue measure of $B(R)$. Setting $\theta = \frac{N}{s}\frac{p-2}{2p}$, the interpolation inequality and the Sobolev trace inequality yield

$$\begin{aligned}\int_{B^c_R} U_n^+(x,0)^p dx &\leqslant \bigg(\int_{B^c_R} U_n(x,0)^2 dx\bigg)^{\frac{p(1-\theta)}{2}} \cdot \bigg(\int_{B^c_R} |U_n(x,0)|^{2^\sharp_s} dx\bigg)^{\frac{p\theta}{2^\sharp_s}} \\ &\leqslant \bigg(\int_{B^c_R} U_n(x,0)^2 dx\bigg)^{\frac{p(1-\theta)}{2}} \bigg(\int_{\mathbb{R}^N} |U_n(x,0)|^{2^\sharp_s} dx\bigg)^{\frac{p\theta}{2^\sharp_s}} \\ &\leqslant C\bigg(\int_{B^c_R} U_n(x,0)^2 dx\bigg)^{\frac{p(1-\theta)}{2}} \bigg(\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla U_n|^2 dxdt\bigg)^{\frac{p\theta}{2}} \\ &\leqslant C\bigg(\int_{A(R)} U_n(x,0)^2 dx + \int_{B(R)} U_n(x,0)^2 dx\bigg)^{\frac{p(1-\theta)}{2}} \|U_n\|^{p\theta}_\lambda.\end{aligned}$$

From (2.9), the first summand on the right can be made arbitrarily small if λ is large. On the other hand, from (2.10), the second summand on the right will be arbitrarily small if R is large since $\mu(B(R)) \to 0$ as $R \to \infty$ by Assumption (V_2). This completes the proof. □

The following lemma is well known and we only give the result without proof.

Lemma 2.7 (Brézis-Lieb lemma). *Let $\{u_n\} \subset L^p(\mathbb{R}^N)$ and $1 \leqslant p < \infty$. If*

(a) *$\{u_n\}$ is bounded in $L^p(\mathbb{R}^N)$,*

(b) *$u_n \to u$ almost everywhere on $\mathbb{R}^N$, then*

$$\lim_{n\to\infty}(|u_n|^p_p - |u_n - u|^p_p) = |u|^p_p. \tag{2.11}$$

Lemma 2.8. *Let $\lambda \geqslant \lambda_0 > 0$ be fixed and let $\{U_n\}$ be a $(PS)_c$-sequence of J_λ. Then up to a subsequence, $U_n \rightharpoonup U$ in E_λ with U being a weak solution of (1.8). Moreover, $U_n^1 = U_n - U$ is a $(PS)_{c'}$-sequence with $c' = c - J_\lambda(U)$.*

Proof. Firstly, by Lemma 2.4 we know that $\{U_n\}$ is bounded in E_λ. Then up to a subsequence $U_n \rightharpoonup U$ in E_λ as $n \to \infty$. We recall (1.7) and obtain that

$$U_n(\cdot,0) \rightharpoonup U(\cdot,0) \quad \text{in} \quad H^s(\mathbb{R}^N), \tag{2.12}$$
$$U_n(\cdot,0) \rightharpoonup U(\cdot,0) \quad \text{in} \quad L^p(\mathbb{R}^N) \quad \text{for any} \quad 2 \leqslant p < 2_s^\sharp, \tag{2.13}$$
$$U_n(\cdot,0) \to U(\cdot,0) \quad \text{in} \quad L^p_{\mathrm{loc}}(\mathbb{R}^N) \quad \text{for any} \quad 2 \leqslant p < 2_s^\sharp, \tag{2.14}$$
$$U_n(\cdot,0) \to U(\cdot,0) \quad \text{almost everywhere on} \quad \mathbb{R}^N, \tag{2.15}$$

where $2_s^\sharp = \frac{2N}{N-2s}$ is the critical Sobolev exponent. Thus for any $W \in E_\lambda$ we have

$$\begin{aligned} J'_\lambda(U_n)\cdot W = {} & \int_{\mathbb{R}^{N+1}_+} t^{1-2s}\nabla U_n \nabla W + \lambda\int_{\mathbb{R}^N} V(x)U_n(x,0)W(x,0)dx \\ & - \int_{\mathbb{R}^N} U_n^+(x,0)^{p-1}W(x,0)dx \\ \to {} & \int_{\mathbb{R}^{N+1}_+} t^{1-2s}\nabla U\nabla W + \lambda\int_{\mathbb{R}^N} V(x)U(x,0)W(x,0)dx \\ & - \int_{\mathbb{R}^N} U^+(x,0)^{p-1}W(x,0)dx \\ = {} & J'_\lambda(U)\cdot W. \end{aligned}$$

Therefore,

$$\langle J'_\lambda(U), W\rangle = \lim_{n\to\infty}\langle J'_\lambda(U_n), W\rangle = 0, \tag{2.16}$$

which indicates that U is a critical point of J_λ.

We consider a new sequence $U_n^1 := U_n - U$. We will prove that as $n \to \infty$,

$$J_\lambda(U_n^1) \to c - J_\lambda(U) \tag{2.17}$$

and

$$J'_\lambda(U_n^1) \to 0. \tag{2.18}$$

To show (2.17), we observe that

$$\begin{aligned} J_\lambda(U_n^1) = {} & \frac{1}{2}\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla U_n^1|^2 dxdt + \frac{1}{2}\int_{\mathbb{R}^N} \lambda V(x)U_n^1(x,0)^2 dx - \frac{1}{p}\int_{\mathbb{R}^N} U_n^{1+}(x,0)^p dx \\ = {} & \frac{1}{2}\int_{\mathbb{R}^{N+1}_+} t^{1-2s}(|\nabla U_n|^2 + |\nabla U|^2 - 2\nabla U_n\nabla U)dxdt \\ & + \frac{1}{2}\int_{\mathbb{R}^N} \lambda V(x)(U_n(x,0)^2 + U(x,0)^2 - 2U_n(x,0)U(x,0))dx - \frac{1}{p}\int_{\mathbb{R}^N} U_n^{1+}(x,0)^p dx \\ = {} & J_\lambda(U_n) - J_\lambda(U) - (U_n^1, U)_\lambda - \frac{1}{p}\int_{\mathbb{R}^N} U_n^{1+}(x,0)^p dx + \frac{1}{p}\int_{\mathbb{R}^N} U_n^+(x,0)^p dx \\ & - \frac{1}{p}\int_{\mathbb{R}^N} U^+(x,0)^p dx. \end{aligned} \tag{2.19}$$

From Lemma 2.7,

$$\int_{\mathbb{R}^N} U_n^+(x,0)^p dx - \int_{\mathbb{R}^N} U^+(x,0)^p dx - \int_{\mathbb{R}^N} U_n^{1+}(x,0)^p dx \to 0$$

as $n \to \infty$. On the other hand, we know that $(U_n^1, U)_\lambda \to 0$, as $n \to \infty$. Thus from (2.19) we indeed have obtained (2.17). Now we come to show (2.18). Since $L^p(\mathbb{R}^N) \subseteq \mathrm{tr}E_\lambda$ is self-conjugate, up to a subsequence, we may also assume that $U_n(\cdot, 0) \rightharpoonup U(\cdot, 0)$ in $L^p(\mathbb{R}^N)$. From (2.16) we have for any $W \in E_\lambda$,

$$\langle J'_\lambda(U_n^1), W\rangle = \langle J'_\lambda(U_n), W\rangle - \int_{\mathbb{R}^N} (U_n^{1+})^{p-1} W(x,0)dx + \int_{\mathbb{R}^N} (U_n^+)^{p-1} W(x,0)dx - \int_{\mathbb{R}^N} (U^+)^{p-1} W(x,0)dx. \tag{2.20}$$

Since $J'_\lambda(U_n) \to 0$ and $U_n(\cdot, 0) \rightharpoonup U(\cdot, 0)$ in $L^p(\mathbb{R}^N, \mathbb{R})$, we have

$$\lim_{n\to\infty} \sup_{\|W\|_\lambda \leqslant 1} \int_{\mathbb{R}^N} ((U_n^{1+})^{p-1}(x,0)W(x,0) - (U_n^+)^{p-1}W(x,0) + (U^+)^{p-1}W(x,0))dx = 0.$$

Thus we have

$$\lim_{n\to\infty} \sup_{\|W\|_\lambda \leqslant 1} |\langle J'_\lambda(U_n^1), W\rangle| = 0,$$

which implies (2.18) and this completes the proof. □

Now we come to prove Proposition 2.1.

Proof of Proposition 2.1. We choose $0 < \varepsilon < \delta_0 c_0/2$, where $c_0 > 0$ and $\delta_0 > 0$ are same constants defined in Lemmas 2.4 and 2.5, respectively. Then for the given constant $C_0 > 0$, we choose $\Lambda_\varepsilon > 0$ and $R_\varepsilon > 0$ as in Lemma 2.6. Thus we claim that $\Lambda_0 := \Lambda_\varepsilon$ is the constant as required in Proposition 2.1.

Take $\{U_n\}$ to be a $(PS)_c$-sequence of J_λ with $\lambda \geqslant \Lambda_0$ and $c \leqslant C_0$. As in Lemma 2.8, we may assume that $U_n \rightharpoonup U$ in E_λ and $U_n^1 = U_n - U$ is a $(PS)_{c'}$-sequence of J_λ with $c' = c - J_\lambda(U)$. If $c' > 0$, then $c' \geqslant c_0$ by Lemma 2.4. As a consequence of Lemma 2.5,

$$\liminf_{n\to\infty} \|U_n^{1+}(\cdot, 0)\|^p_{L^p(\mathbb{R}^N)} \geqslant \delta_0 c' \geqslant \delta_0 c_0.$$

On the other hand, Lemma 2.6 implies

$$\limsup_{n\to\infty} \int_{B^c_{R_\varepsilon}} U_n^{1+}(x,0)^p \leqslant \varepsilon < \frac{\delta_0 c_0}{2}.$$

This implies $U_n^1 \rightharpoonup U^1$ in E_λ with $U^1 \neq 0$, which leads to a contradiction. Therefore $c' = 0$; hence $U_n^1 \to 0$ in E_λ by Lemma 2.4. This completes the proof of Proposition 2.1. □

Recalling the definition of c_λ in Section 1 and applying Proposition 2.1 to the functional J_λ, we obtain the following corollary.

Corollary 2.9. *For any $p \in (2, 2_s^\sharp)$, there exists $\Lambda_0 > 0$ such that c_λ is achieved for all $\lambda \geqslant \Lambda_0$ at some $U_\lambda \in E_\lambda$ which is a solution of* (1.8).

3 Limit problem

Let us recall that the following problem is the "limit" problem of (1.8):

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla U) = 0, & \text{in } \mathbb{R}^{N+1}_+ = \mathbb{R}^N \times (0,\infty), \\ U = 0, & \text{on } \mathbb{R}^N \setminus \Omega \times \{0\}, \\ \partial^s_\nu U(\cdot, 0) = U^{p-1}, & \text{on } \Omega \times \{0\}, \\ U \geqslant 0, & \text{in } \mathbb{R}^{N+1}_+ = \mathbb{R}^N \times (0,\infty), \end{cases} \tag{3.1}$$

and the corresponding functional of (3.1) is defined by

$$\Phi(U) = \frac{1}{2}\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla U|^2 dxdt - \frac{1}{p}\int_\Omega U^+(x,0)^p dx \quad \text{for} \quad U \in E_0,$$

where E_0 is defined as in Section 1. Again as defined in Section 1, the following energy:

$$c(\Omega) := \inf\{\Phi(U) : U \in \mathcal{N}\}$$

is the infimum of Φ on the Nehari manifold $\mathcal{N}$. We will see that $c(\Omega)$ is achieved by a least energy $U \in \mathcal{N}$. To show that, we firstly give an imbedding lemma which is standard.

Lemma 3.1. *Let $2 < p < 2_s^\sharp := \frac{2N}{N-2s}$ for $N \geqslant 2$. Then $\mathrm{tr}_\Omega E_0$ is compactly embedded in $L^p(\Omega)$.*

Proof. Note that $\mathrm{tr}_\Omega E_0 \subset H^s(\Omega)$ and the fact that the embedding $H^s(\Omega) \hookrightarrow L^p(\Omega)$ is compact for $2 < p < \frac{2N}{N-2s}$ for $N \geqslant 2$ immediately implies Lemma 3.1.

By a standard argument applying the above compactness result Lemma 3.1, we can obtain the following existence lemma.

Lemma 3.2. *The infimum $c(\Omega)$ is achieved by a function $U \in \mathcal{N}$ which is a least energy solution of* (3.1).

Proof. Indeed, by Ekeland variational principle, there is a P.S. sequence $U_n \in E_0$ such that

$$\Phi(U_n) \to c(\Omega) \quad \text{and} \quad \Phi'(U_n) \to 0.$$

Then we have

$$\begin{aligned} c(\Omega) + o(1)\|U_n\| &\geqslant \Phi(U_n) - \frac{1}{p}\Phi'(U_n) \cdot U_n \\ &= \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\mathbb{R}_+^{N+1}} t^{1-2s} |\nabla U_n|^2 dxdt \end{aligned}$$

and

$$\begin{aligned} c(\Omega) + o(1)\|U_n\| &\geqslant \Phi(U_n) - \frac{1}{2}\Phi'(U_n) \cdot U_n \\ &= \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\Omega} U_n^+(x,0)^p dx. \end{aligned}$$

It is easy to see that $\{U_n\}$ is bounded in E_0. Then, we can easily find a subsequence of $\{U_n\}$ (we still denote U_n) such that $U_n \rightharpoonup U \in E_0$. Thus by Lemma 3.1, there is a subsequence of $\{U_n\}$ (we denote it as itself) such that $U_n(\cdot,0) \to U(\cdot,0)$ in $L^p(\Omega)$. Obviously, we have

$$\begin{aligned} c(\Omega) &\leqslant \frac{1}{2} \int_{\mathbb{R}_+^{N+1}} t^{1-2s} |\nabla U|^2 dxdt - \frac{1}{p} \int_\Omega U^+(x,0)^p dx \\ &\leqslant \lim_{n\to\infty} \frac{1}{2} \int_{\mathbb{R}_+^{N+1}} t^{1-2s} |\nabla U_n|^2 dxdt - \frac{1}{p} \int_\Omega U_n^+(x,0)^p dx = c(\Omega). \end{aligned}$$

Then U achieves $c(\Omega)$ which is a least energy solution of (3.1). □

Remark 3.3. When the zero set $\Omega = \mathrm{int} V^{-1}(0)$ has more than one isolated component, for example, $\Omega = \Omega_1 \cup \Omega_2$ with $\overline{\Omega}_1 \cap \overline{\Omega}_2 = \emptyset$. Suppose $U \in \mathcal{N}$ is the least energy solution of (3.1) with $U(x,0) = 0$ in Ω_1 and $U(x,0) \gneqq 0$ in Ω_2. Then we have $(-\Delta)^s U(x,0) = \int_{\mathbb{R}^N} \frac{U(x,0)-U(y,0)}{|x-y|^{N+2s}} dy < 0$ in Ω_1. However, on the other hand $(-\Delta)^s U(x,0) = U(x,0)^{p-1} = 0$ for $x \in \Omega_1$. This contradiction shows that the least energy solution $U(x,y)$ of (3.1) satisfies $U(x,0) \gneqq 0$ both in Ω_1 and in Ω_2. The phenomenon is totally different from the local operator Laplacian since in Laplacian case, $u = 0$ in Ω immediately indicates that $\Delta u = 0$ in Ω for any domain Ω. For the fractional Laplacian case, it is not the case.

4 Proofs of the main results

In this section we will give the proofs of our main results. To begin with, we firstly give an asymptotic behavior for c_λ as λ is large. More precisely, we have the following lemma.

Lemma 4.1. *$c_\lambda \to c(\Omega)$ as $\lambda \to \infty$.*

Proof. It is easy to see that $c_\lambda \leqslant c(\Omega)$ for all $\lambda \geqslant 0$. By the definition of c_λ, it is not difficult to check that c_λ is monotone increasing with respect to $\lambda > 0$.

Now, assume on the contrary that for a sequence $\{\lambda_n\}$ with $\lambda_n \to \infty$ as $n \to \infty$,

$$\lim_{n\to\infty} c_{\lambda_n} = k < c(\Omega).$$

First of all, Lemma 2.4 implies $k > 0$ and by Corollary 2.9, for n large enough, there exists a sequence $U_n \in \mathcal{M}_{\lambda_n}$ which is a solution of (1.8) with λ being replaced by λ_n such that $J_{\lambda_n}(U_n) = c_{\lambda_n}$. Similar to the proof of Lemma 2.4, it is easy to verify that $\{U_n\}$ is bounded in E. Thus up to a subsequence, we may assume that $U_n \rightharpoonup U$ in E and

$$U_n(x,0) \to U(x,0) \quad \text{in} \quad L^\theta_{\text{loc}}(\mathbb{R}^N) \quad \text{for} \quad 2 \leqslant \theta < 2^\sharp_s. \tag{4.1}$$

We firstly claim that $U(\cdot,0)|_{\Omega^c} = 0$ and hence $U \in E_0$, where $\Omega^c = \{x \in \mathbb{R}^N : x \notin \Omega\}$. In fact, if $U(\cdot,0)|_{\Omega^c} \neq 0$, then there exists a compact subset $F \subset \Omega^c$ with $\text{dist}(F,\Omega) > 0$ such that $U(\cdot,0)|_F \neq 0$. Then by (4.1),

$$\int_F U_n(x,0)^2 dx \to \int_F U(x,0)^2 dx > 0.$$

However, since $V(x) \geqslant \varepsilon_0 > 0$ for all $x \in F$ and for some $\varepsilon_0 > 0$, it follows that

$$J_{\lambda_n}(U_n) \geqslant \frac{p-2}{2p}\lambda_n \int_F V(x) U_n(x,0)^2 dx \geqslant \frac{p-2}{2p}\lambda_n \varepsilon_0 \int_F U_n(x,0)^2 dx \to \infty \quad \text{as} \quad n \to \infty,$$

which leads to a contradiction.

Next, we show that $U_n(\cdot,0) \to U(\cdot,0)$ in $L^p(\mathbb{R}^N)$ for $2 < p < 2^\sharp_s$. Indeed, if not, then by the concentration compactness lemma of Lions [16], there exist $\delta > 0, \rho > 0$ and $x_n \in \mathbb{R}^N$ with $|x_n| \to \infty$ such that

$$\liminf_{n\to\infty} \int_{B_\rho(x_n)} |U_n(x,0) - U(x,0)|^2 dx \geqslant \delta > 0.$$

Then we have

$$\begin{aligned}
J_{\lambda_n}(U_n) &= \frac{p-2}{2p} \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U_n|^2 dx dt + \frac{p-2}{2p} \int_{\mathbb{R}^N} \lambda_n V(x) U_n(x,0)^2 dx \\
&\geqslant \frac{p-2}{2p}\lambda_n \int_{B_\rho(x_n)\cap\{x:V(x)\geqslant M_0\}} V(x) U_n(x,0)^2 dx \\
&= \frac{p-2}{2p}\lambda_n \int_{B_\rho(x_n)\cap\{x:V(x)\geqslant M_0\}} V(x) |U_n(x,0) - U(x,0)|^2 dx \\
&\geqslant \frac{p-2}{2p}\lambda_n \bigg(M_0 \int_{B_\rho(x_n)} |U_n(x,0) - U(x,0)|^2 dx - M_0 \int_{B_\rho(x_n)\cap\{x:V(x)\leqslant M_0\}} U_n(x,0)^2 dx \bigg) \\
&\geqslant \frac{p-2}{2p}\lambda_n \bigg(M_0 \int_{B_\rho(x_n)} |U_n(x,0) - U(x,0)|^2 dx - o(1) \bigg) \\
&\to \infty \quad \text{as} \quad n \to \infty.
\end{aligned}$$

For the last inequality we have used Hölder's inequality and the fact

$$\mu(B_\rho(x_n) \cap \{x : V(x) \leqslant M_0\}) \to 0 \quad \text{as} \quad n \to \infty.$$

This contradiction implies $U_n(\cdot,0) \to U(\cdot,0)$ in $L^p(\mathbb{R}^N)$. By this strong convergence, one can easily check that $U \geqslant 0$ is a solution of the following problem:

$$\begin{cases}
\text{div}(t^{1-2s}\nabla U) = 0, & \text{in} \quad \mathbb{R}^{N+1}_+ = \mathbb{R}^N \times (0,\infty), \\
\partial^s_\nu U(\cdot,0) = U^{p-1}(x,0), & \text{on} \quad \Omega, \\
U = 0, & \text{on} \quad \mathbb{R}^N \setminus \Omega.
\end{cases} \tag{4.2}$$

Furthermore, again from the strong convergence $U_n^+(\cdot,0)\to U^+(\cdot,0)$ in $L^p(\mathbb{R}^N)$, we have

$$\begin{aligned}k&=\lim_{n\to\infty}c_{\lambda_n}=\lim_{n\to\infty}J_{\lambda_n}(U_n)\\&=\lim_{n\to\infty}J_{\lambda_n}(U_n)-\frac{1}{2}J'_{\lambda_n}(U_n)\cdot U_n\\&=\lim_{n\to\infty}\left(\frac{1}{2}-\frac{1}{p}\right)\int_{\mathbb{R}^N}U_n^+(x,0)^pdx\\&=\left(\frac{1}{2}-\frac{1}{p}\right)\int_{\Omega}U^+(x,0)^pdx.\end{aligned}$$

Namely we obtain that

$$\Phi(U)=\left(\frac{1}{2}-\frac{1}{p}\right)\int_{\Omega}U^+(x,0)^pdx=k,$$

which implies $U\in\mathcal{N}$ and $k\geqslant c(\Omega)$. This also leads to a contradiction. Thus we proved that $\lim_{\lambda\to\infty}c_\lambda=c(\Omega)$. Thus the proof of this lemma is completed. □

Now we are ready to give the proofs of Theorems 1.2 and 1.3.

Proof of Theorem 1.2. It suffices to prove that any sequence of $U_n\in E_{\lambda_n}$ with $U_n\in\mathcal{M}_{\lambda_n}$, $J_{\lambda_n}(U_n)=c_{\lambda_n}$ ($\lambda_n\to\infty$ as $n\to\infty$) converges in E along a subsequence to a least energy solution of (1.11). As in the proof of Lemma 2.4, we can obtain that such a sequence U_n must be bounded in E. Thus without loss of generality, we may assume that $U_n\rightharpoonup U$ in E and $U_n(\cdot,0)\to U(\cdot,0)$ in $L^\theta_{\rm loc}(\mathbb{R}^N)$ for $2<\theta<2^\sharp_s$.

To complete the proof, it is sufficient to prove that $U_n\to U$ strongly in E and $U\in\mathcal{N}$ is a least energy solution of (1.11) such that $\Phi(U)=c(\Omega)$. Firstly, as in the proof of Lemma 4.1, we can prove that $U\geqslant0$ is a solution of the following problem:

$$\begin{cases}\operatorname{div}(t^{1-2s}\nabla U)=0, & \text{in}\quad \mathbb{R}^{N+1}_+=\mathbb{R}^N\times(0,\infty),\\ \partial^s_\nu U(\cdot,0)=U^{p-1}(x,0), & \text{on}\quad \Omega,\\ U=0, & \text{on}\quad \mathbb{R}^N\setminus\Omega,\end{cases}\tag{4.3}$$

and $U_n^+(x,0)\to U^+(x,0)$ strongly in $L^p(\mathbb{R}^N)$.

Now we claim that

$$\lambda_n\int_{\mathbb{R}^N}V(x)U_n(x,0)^2dx\to0\quad\text{and}\quad\int_{\mathbb{R}^{N+1}_+}t^{1-2s}|\nabla U_n|^2dxdt\to\int_{\mathbb{R}^{N+1}_+}t^{1-2s}|\nabla U|^2dxdt.$$

Indeed, if either

$$\limsup_{n\to\infty}\lambda_n\int_{\mathbb{R}^N}V(x)U_n(x,0)^2dx>0$$

or

$$\limsup_{n\to\infty}\int_{\mathbb{R}^{N+1}_+}t^{1-2s}|\nabla U_n|^2dxdt>\int_{\mathbb{R}^{N+1}_+}t^{1-2s}|\nabla U|^2dxdt.$$

Then we get that

$$\int_{\mathbb{R}^{N+1}_+}t^{1-2s}|\nabla U|^2dxdy<\int_{\Omega}U^+(x,0)^pdx,$$

thus there is $\alpha\in(0,1)$ such that $\alpha U\in\mathcal{N}$ and

$$\begin{aligned}c(\Omega)&\leqslant\Phi(\alpha U)\\&=\frac{p-2}{2p}\int_{\mathbb{R}^{N+1}_+}t^{1-2s}|\nabla\alpha U|^2dxdt\\&<\frac{p-2}{2p}\int_{\mathbb{R}^{N+1}_+}t^{1-2s}|\nabla U|^2dxdt\\&\leqslant\lim_{n\to\infty}\frac{p-2}{2p}\left(\int_{\mathbb{R}^{N+1}_+}t^{1-2s}|\nabla U_n|^2dxdt+\int_{\mathbb{R}^N}\lambda_nV(x)U_n(x,0)^2dx\right)\end{aligned}$$

$$= \lim_{n\to\infty} J_{\lambda_n}(U_n)$$
$$= c(\Omega),$$

which leads to a contradiction. Thus we complete the proof of Theorem 1.2. □

Proof of Theorem 1.3. Suppose $\{u_n = U_n(\cdot,0)\} \in H^s(\mathbb{R}^N)$ is a solution of (1.1) with λ being replaced by λ_n ($\lambda_n \to \infty$ as $n \to \infty$). It is easy to see that such a sequence U_n must be bounded in E. We may assume that $U_n \rightharpoonup U$ in E and $U_n(\cdot,0) \to U(\cdot,0)$ in $L^\theta_{\rm loc}(\mathbb{R}^N)$ for $2 < \theta < 2^\sharp_s$. As in the proof of Lemma 4.1, we can prove that $U(\cdot,0)|_{\Omega^c} = 0$ and $U \in E_0$ is a solution of (1.11). Moreover, $U_n(\cdot,0) \to U(\cdot,0)$ in $L^\theta(\mathbb{R}^N)$ for $2 < \theta < 2^\sharp_s$. As in the proof of Theorem 1.2, it suffices to show $U_n \to U$ in E. We observe that

$$\begin{aligned}
&\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla(U_n - U)|^2 dxdt + \int_{\mathbb{R}^N} \lambda_n V(x)|U_n(x,0) - U(x,0)|^2 dx \\
&= \int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla U_n|^2 dxdt + \int_{\mathbb{R}^N} \lambda_n V(x) U_n(x,0)^2 dx - \int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla U|^2 dxdt \\
&\quad - \int_{\mathbb{R}^N} \lambda_n V(x) U(x,0)^2 dx + o(1) \\
&= \int_{\mathbb{R}^N} U_n^+(x,0)^p dx - \int_\Omega U^+(x,0)^p dx + o(1) \\
&= o(1).
\end{aligned}$$

Here we used the fact that U_n and U lie on the Nehari manifold $\mathcal{M}_{\lambda_n}$ and $\mathcal{N}$, respectively. This completes the proof of Theorem 1.3. □

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