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Gorensteinness, homological invariants and Gorenstein derived categories

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Abstract Relations between Gorenstein derived categories, Gorenstein defect categories and Gorenstein stable categories are established. Using these, the Gorensteinness of an algebra A and invariants with respect to recollements of the bounded Gorenstein derived category $D^b_{gp}(A\text{-mod})$ of A are investigated. Specifically, the Gorensteinness of A is characterized in terms of recollements of $D^b_{gp}(A\text{-mod})$ and Gorenstein derived equivalences. It is also shown that Cohen-Macaulay-finiteness is invariant with respect to the recollements of $D^b_{gp}(A\text{-mod})$.

Keywords Gorenstein-projective modules, CM-finite algebras, virtually Gorenstein algebras, Gorenstein derived categories, Gorenstein defect categories, Gorenstein stable categories

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1 Introduction

The Gorensteinness of an algebra is of interest in the representation theory of algebras, in Gorenstein homological algebra, and in the theory of singularity categories (see, e.g., [4, 6, 11, 16, 21, 26, 27]). How to characterize the Gorenstein property is a basic problem.

An algebra has many invariants, for example, the finiteness of global dimension, the finitistic dimension, and Cohen-Macaulay (CM)-finiteness (see, e.g., [7, 22, 30, 33]). How to describe and compare these homological invariants is a major topic of interest.

One can approach these questions by derived (and related) categories, as well as comparisons of derived categories. Derived categories, introduced by Verdier [32], have been playing an increasingly important role in various areas of mathematics, including representation theory, algebraic geometry, and mathematical physics. There are two ways to compare derived categories. One way is by derived equivalences. For example, Happel [20] has shown the finiteness of global dimension of an algebra is invariant under derived equivalences. Another way is by recollements, which have been introduced by Beilinson et al. [9]. A recollement of a derived category by another two derived categories is a diagram of six functors between these categories, generalising Grothendieck's six functors. Suppose that A, B and C are three finite dimensional algebras over a field. If $D^b(A\text{-mod})$ admits a recollement with respect to the bounded derived categories $D^b(B\text{-mod})$ and $D^b(C\text{-mod})$ of B and C, Wiedemann [33] has shown that A has finite global dimension if and only if so have B and C, and Happel [22] has shown that A has finite finitistic dimension if and only if so have B and C.

For Gorenstein homological algebra we refer to [3, 14, 15, 17, 24]. Gao and Zhang [19] defined the corresponding version of the derived category in Gorenstein homological algebra. Gao and Zhang [19] introduced the notions of Gorenstein derived category and Gorenstein derived equivalence which are needed in this context. Following [19], the bounded Gorenstein derived category $D^b_{gp}(A\text{-mod})$ of an algebra A is defined as the Verdier quotient of the bounded homotopy category $K^b(A\text{-mod})$ with respect to the triangulated subcategory $K^b_{\text{gpac}}(A\text{-mod})$ of A-$\mathcal{G}$proj-acyclic complexes. Later, a necessary and sufficient criterion was given by Gao [18] for the existence of recollements of Gorenstein derived categories. Based on these work, two questions arise.

(1) Can we characterize the Gorensteinness of an algebra in terms of the corresponding Gorenstein derived category?

(2) Which invariants can be compared along recollements of Gorenstein derived categories?

In this paper, we will provide answers to these questions. Our answers to Question (1) are the combination of Corollary 3.2, and Theorems 3.3 and 3.5(2). We state them as Theorem A.

Theorem A. *Let A be an artin algebra. Consider the following statements*:

(1) *A is Gorenstein*;

(2) *A is virtually Gorenstein, and there exist Gorensein algebras B and C and a recollement*

$$D^b_{gp}(B\text{-mod}) \underset{\overset{\longleftarrow}{i^!}}{\overset{\overset{i^*}{\longleftarrow}}{\xrightarrow{i_*}}} D^b_{gp}(A\text{-mod}) \underset{\overset{\longleftarrow}{j_*}}{\overset{\overset{j_!}{\longleftarrow}}{\xrightarrow{j^*}}} D^b_{gp}(C\text{-mod}).$$

($2'$) *A is virtually Gorenstein, and for arbitrary virtually Gorensein algebras B and C, if there exists the following recollement*:

$$D^b_{gp}(B\text{-mod}) \underset{\overset{\longleftarrow}{i^!}}{\overset{\overset{i^*}{\longleftarrow}}{\xrightarrow{i_*}}} D^b_{gp}(A\text{-mod}) \underset{\overset{\longleftarrow}{j_*}}{\overset{\overset{j_!}{\longleftarrow}}{\xrightarrow{j^*}}} D^b_{gp}(C\text{-mod}),$$

then B and C are Gorenstein;

(3) *There is a triangle-equivalence $D^b_{gp}(A\text{-mod}) \cong K^b(A\text{-}\mathcal{G}\text{proj})$*;

(4) *A is Gorenstein derived equivalent to an algebra B, which is Gorenstein.*

We have the following relations between these statements:

(i) $(1) \Leftrightarrow (2) \Leftrightarrow (2')$.

(ii) *If A-$\mathcal{G}$proj is contravariantly finite in A-mod, then* $(1) \Leftrightarrow (3)$.

(iii) *If A is CM-finite, then* $(1) \Leftrightarrow (4)$.

Our answer to Question (2) is Theorem 3.5(1). We state it as Theorem B.

Theorem B. *Let A, B and C be artin algebras. Assume that the bounded Gorenstein derived category $D^b_{gp}(A\text{-mod})$ admits a recollement with respect to $D^b_{gp}(B\text{-mod})$ and $D^b_{gp}(C\text{-mod})$. If A, B and C are virtually Gorenstein, then A is CM-finite if and only if so are B and C.*

This paper is structured as follows. In Section 2, we recall some concepts to be used. In Section 3, we characterize the Gorensteinness of an algebra A in two ways: recollements of $D^b_{gp}(A\text{-mod})$ and Gorenstein derived equivalences. In Section 4, we show the Karoubianness of Gorenstein defect categories, and establish relations between Gorenstein defect categories and Gorenstein stable categories. In Section 5, we make a conclusion.

Let us end this introduction by mentioning that in private communication with Javad Asadollahi, he pointed out that he and his collaborators also had proofs for Theorem 3.5 in this paper. Their proofs were obtained independently, and also are different from the proofs given in this paper (see [2]). The author thanks him for letting us know their proofs.

2 Preliminaries

In this section, we fix notation and recall the main concepts to be used.

Let A be an artin algebra. Denote by A-Mod (resp. A-mod) the category of left A-modules (resp. the category of finitely-generated left A-modules), and A-Proj (resp. A-proj) the full subcategory of projective A-modules (resp. the full subcategory of finitely-generated projective A-modules). Let D^b (A-mod) be the bounded derived category of A. Following [29], the singularity category $D_{sg}(A)$ of A is the Verdier quotient category (see [32, Chapter II, Subsection 2.1.8]) of D^b(A-mod) with respect to the subcategory formed by bounded complexes of projective modules.

An A-module M is said to be Gorenstein-projective in A-Mod (resp. A-mod), if there is an exact sequence $P^\bullet = \cdots \to P^{-1} \to P^0 \xrightarrow{d^0} P^1 \to P^2 \to \cdots$ in A-Proj (resp. A-proj) with $\mathrm{Hom_A(P^\bullet, Q)}$ exact for any A-module Q in A-Proj (resp. A-proj), such that $M \cong \ker d^0$ (see [17]). Denote by A-$\mathcal{GP}$ (resp. A-$\mathcal{G}$proj) the full subcategory of Gorenstein-projective modules in A-Mod (resp. A-mod), and similarly denote by A-$\mathcal{GI}$ the full subcategory of Gorenstein-injective modules in A-Mod.

A proper Gorenstein-projective resolution of A-module M in A-mod is an exact sequence $E^\bullet = \cdots \to G_1 \to G_0 \to M \to 0$ such that all $G_i \in A$-$\mathcal{G}$proj, and that $\mathrm{Hom}_A(G, E^\bullet)$ stays exact for each $G \in A$-$\mathcal{G}$proj.

A complex $C^\bullet$ of finitely-generated A-modules is A-$\mathcal{G}$proj-acyclic, if $\mathrm{Hom}_A(G, C^\bullet)$ is acyclic for any $G \in A$-$\mathcal{G}$proj. It is also called proper exact for example in [3]. A chain map $f^\bullet : X^\bullet \to Y^\bullet$ is an A-$\mathcal{G}$proj-quasi-isomorphism, if $\mathrm{Hom}_A(G, f^\bullet)$ is a quasi-isomorphism for any $G \in A$-$\mathcal{G}$proj, i.e., there are isomorphisms of abelian groups

$$\mathrm{H}^n \mathrm{Hom}_A(G, f^\bullet) : \mathrm{H}^n \mathrm{Hom}_A(G, X^\bullet) \cong \mathrm{H}^n \mathrm{Hom}_A(G, Y^\bullet), \quad \forall\, n \in \mathbb{Z}, \quad \forall\, G \in A\text{-}\mathcal{G}\mathrm{proj}.$$

Denote by $K^-(A$-$\mathcal{G}$proj) the upper bounded homotopy category of A-$\mathcal{G}$proj, and by $K^{-,gpb}(A$-$\mathcal{G}$proj) the full subcategory,

$$K^{-,gpb}(A\text{-}\mathcal{G}\mathrm{proj}) := \{G^\bullet \in K^-(A\text{-}\mathcal{G}\mathrm{proj}) \mid \exists \text{ a positive integer } N \text{ such that } H^{-n}\mathrm{Hom}_A(E, G^\bullet) = 0, \forall\, n > N, \forall\, E \in A\text{-}\mathcal{G}\mathrm{proj}\}.$$

Note from the proof of [19, Theorem 3.6] that if A-$\mathcal{G}$proj is contravariantly finite in A-mod, then there is a triangle-equivalence $D^b_{gp}(A\text{-mod}) \cong K^{-,gpb}(A\text{-}\mathcal{G}\mathrm{proj})$.

We say that two artin algebras A and B are Gorenstein derived equivalent, if there is a triangle-equivalence $D^b_{gp}(A\text{-mod}) \cong D^b_{gp}(B\text{-mod})$.

Recall from [7,12] that an artin algebra A is Cohen-Macaulay finite (simply, CM-finite) if there are only finitely many isomorphism classes of finitely-generated indecomposable Gorenstein-projective A-modules. Suppose $G_1, \ldots, G_n$ are all the pairwise non-isomorphic indecomposable finitely-generated Gorenstein-projective A-modules. Throughout the paper, we set

$$G_A := \bigoplus_{1 \leqslant i \leqslant n} G_i.$$

Recall from [21] that an artin algebra A is Gorenstein if $\mathrm{inj.dim}_A A < \infty$ and $\mathrm{inj.dim} A_A < \infty$. Recall from [7, 10] that an artin algebra A is called virtually Gorenstein if $A\text{-}\mathcal{GP}^\perp = {}^\perp A\text{-}\mathcal{GI}$. Note that a Gorenstein algebra is virtually Gorenstein, but, the converse does not hold in general.

Let A be an artin algebra. Denote by A-$\mathcal{G}$proj(M, N) the subgroup of $\mathrm{Hom}_A(M, N)$ of A-maps from M to N which factors through the finitely-generated Gorenstein-projective modules, and A-mod/A-$\mathcal{G}$proj the quotient category of A-mod modulo A-$\mathcal{G}$proj, i.e., the objects are same as those of A-mod, and the morphism space from M to N is the quotient group $\mathrm{Hom}_A(M, N)/A$-$\mathcal{G}$proj(M, N). In the following, we call A-mod/A-$\mathcal{G}$proj the Gorenstein stable category of A.

Recall from [5] that A-mod/A-$\mathcal{G}$proj carries a left triangulated structure and the stabilization of A-mod/A-$\mathcal{G}$proj is a pair $(S, S(A$-mod $/A$-$\mathcal{G}$proj$))$, where $S(A$-mod/A-$\mathcal{G}$proj$)$ is a triangulated category and S : A-mod/A-$\mathcal{G}$proj $\to S(A$-mod $/A$-$\mathcal{G}$proj$)$ is an exact functor, such that for any exact functor F : A-mod/A-$\mathcal{G}$ proj $\to \mathcal{C}$ to a triangulated category $\mathcal{C}$, there exists a unique triangle functor F^* : $S(A$-mod/A-$\mathcal{G}$proj$) \to \mathcal{C}$ such that $F^*S = F$. For the construction of $S(A$-mod/A-$\mathcal{G}$proj$)$, see [5, 23].

Let $\mathcal{C}'$, $\mathcal{C}$ and $\mathcal{C}''$ be triangulated categories. Recall from [9] that a recollement of $\mathcal{C}$ relative to $\mathcal{C}'$ and $\mathcal{C}''$ is a diagram of triangle functors

$$\mathcal{C}' \begin{array}{c} \xleftarrow{\ i^*\ } \\ \xrightarrow{\ i_*\ } \\ \xleftarrow{\ i^!\ } \end{array} \mathcal{C} \begin{array}{c} \xleftarrow{\ j_!\ } \\ \xrightarrow{\ j^*\ } \\ \xleftarrow{\ j_*\ } \end{array} \mathcal{C}''$$

such that

(R1) $(i^*, i_*), (i_*, i^!), (j_!, j^*)$ and (j^*, j_*) are adjoint pairs;

(R2) $i_*, j_!$ and j_* are fully faithful;

(R3) $j^* i_* = 0$ (and hence $i^* j_! = 0$ and $i^! j_* = 0$);

(R4) for each $X \in \mathcal{C}$ there are distinguished triangles

$$j_! j^* X \xrightarrow{\epsilon_X} X \xrightarrow{\eta_X} i_* i^* X \to (j_! j^* X)[1], \quad i_* i^! X \xrightarrow{\omega_X} X \xrightarrow{\zeta_X} j_* j^* X \to (i_* i^! X)[1],$$

where ϵ_X is the counit of $(j_!, j^*)$, η_X is the unit of (i^*, i_*), ω_X is the counit of $(i_*, i^!)$, and ζ_X is the unit of (j^*, j_*).

3 Gorenstein derived categories

In this section, we characterize the Gorensteinness of an algebra A in two ways: recollements of $D^b_{gp}(A\text{-mod})$ and Gorenstein derived equivalences.

Let A be an artin algebra. Denote by $\underline{A\text{-}\mathcal{G}\text{proj}}$ the sable category of A-$\mathcal{G}$proj. Buchweitz's Theorem [13, Subsection 4.4.1] has shown that there is an embedding $F : \underline{A\text{-}\mathcal{G}\text{proj}} \to D_{sg}(A)$ given by $F(G) = G$, where the second G is the corresponding stalk complex at degree 0, and that if A is Gorenstein, then F is a triangle-equivalence. The converse is also true (see [5, Theorem 6.9]). In general, to measure how far a ring is from being Gorenstein, Bergh et al. [11] defined the Gorenstein defect category $D^b_{\text{def}}(A) := D_{sg}(A)/\text{Im}F$, and also they have shown that A is Gorenstein if and only if $D^b_{\text{def}}(A) = 0$.

To recognize Gorenstein rings via Gorenstein derived categories, we compare a Gorenstein defect category with a Gorenstein derived category, and try to construct a precise relation. We start with the following lemma.

Lemma 3.1. *Let A be an artin algebra such that A-$\mathcal{G}$proj is contravariantly finite in A-mod. Then there is a triangle-equivalence $D^b_{\text{def}}(A) \cong D^b_{gp}(A\text{-mod})/K^b(A\text{-}\mathcal{G}\text{proj})$.*

Proof. By [25, Final Remark], we have a triangle-equivalence $D^b_{\text{def}}(A) \cong K^{-gpb}(A\text{-}\mathcal{G}\text{proj})/K^b(A\text{-}\mathcal{G}\text{proj})$. Since A-$\mathcal{G}$proj is contravariantly finite in A-mod, it follows from the proof of [19, Theorem 3.6(ii)] that $D^b_{gp}(A\text{-mod}) \cong K^{-gpb}(A\text{-}\mathcal{G}\text{proj})$. This completes the proof. □

Now we test the Gorensteinness of A by the structure of $D^b_{gp}(A\text{-mod})$.

Corollary 3.2. *Let A be an artin algebra such that A-$\mathcal{G}$proj is contravariantly finite in A-mod. Then A is Gorenstein if and only if there is a triangle-equivalence $D^b_{gp}(A\text{-mod}) \cong K^b(A\text{-}\mathcal{G}\text{proj})$.*

Proof. By [19, Corollary 3.8], we only need to verify the sufficiency. Since $D^b_{gp}(A\text{-mod}) \cong K^b(A\text{-}\mathcal{G}\text{proj})$, it follows that $D^b_{gp}(A\text{-mod})/K^b(A\text{-}\mathcal{G}\text{proj}) = 0$. By Lemma 3.1, we get that $D^b_{\text{def}}(A) = 0$. [11, Theorem 2.8] implies A is Gorenstein. □

Naturally there are two ways to compare Gorenstein derived categories. One way is by Gorenstein derived equivalence, the other is by recollements of Gorenstein derived categories. We first show the Gorensteinness is invariant under Gorenstein derived equivalences.

Theorem 3.3. *Let A and B be two artin algebras such that A-$\mathcal{G}$proj and B-$\mathcal{G}$proj are contravariantly finite respectively. If A and B are Gorenstein derived equivalent, then A is Gorenstein if and only if B is Gorenstein.*

Proof. Suppose that $F : D^b_{gp}(A\text{-mod}) \to D^b_{gp}(B\text{-mod})$ is the given triangle-equivalence and F^{-1} is its inverse. Since F is fully faithful, it follows that there is an isomorphism $\text{Hom}_{D^b_{gp}(A\text{-mod})}(X^\bullet, F^{-1}(Y^\bullet)) \cong \text{Hom}_{D^b_{gp}(B\text{-mod})}(F(X^\bullet), Y^\bullet)$ for any $X^\bullet \in D^b_{gp}(A\text{-mod})$ and $Y^\bullet \in D^b_{gp}(B\text{-mod})$. This means that (F, F^{-1})

is an adjoint pair. Thus we get from [19, Lemma 4.3] that $F : D^b_{gp}(A\text{-mod}) \to D^b_{gp}(B\text{-mod})$ restricts to $K^b(A\text{-}\mathcal{G}\text{proj})$, and $F : K^b(A\text{-}\mathcal{G}\text{proj}) \to K^b(B\text{-}\mathcal{G}\text{proj})$ is fully faithful. It follows that there is a triangle-equivalence $K^b(A\text{-}\mathcal{G}\text{proj}) \cong K^b(B\text{-}\mathcal{G}\text{proj})$. Thus by Lemma 3.1 we get a triangle-equivalence $D^b_{\text{def}}(A) \cong D^b_{\text{def}}(B)$. This implies that A is Gorenstein if and only if B is Gorenstein. □

Next, we will compare the invariants along recollements of Gorenstein derived categories. We show the Gorensteinness and CM-finiteness are invariant with respect to such recollements. We need the following lemma, which is inspired by [1].

Lemma 3.4. *Let A and B be two artin algebras and $F : K^{-,gpb}(A\text{-}\mathcal{G}\text{proj}) \to K^{-,gpb}(B\text{-}\mathcal{G}\text{proj})$ be a triangle functor. The following two conditions are equivalent*:

(1) *F restricts to $K^b(A\text{-}\mathcal{G}\text{proj})$*;

(2) *$F(G) \in K^b(B\text{-}\mathcal{G}\text{proj})$ for any $G \in A\text{-}\mathcal{G}\text{proj}$.*

Theorem 3.5. *Let A, B and C be virtually Gorenstein algebras. Assume that $D^b_{gp}(A\text{-mod})$ admits the following recollement*:

$$D^b_{gp}(B\text{-mod}) \underset{\overset{\longleftarrow}{i^!}}{\overset{\overset{i^*}{\longleftarrow}}{\xrightarrow{i_*}}} D^b_{gp}(A\text{-mod}) \underset{\overset{\longleftarrow}{j_*}}{\overset{\overset{j_!}{\longleftarrow}}{\xrightarrow{j^*}}} D^b_{gp}(C\text{-mod}).$$

Then

(1) *A is CM-finite if and only if B and C are so*;

(2) *A is Gorenstein if and only if B and C are so.*

Proof. Since A is virtually Gorenstein, it follows from [8] that $A\text{-}\mathcal{G}\text{proj}$ is contravariantly finite in A-mod. This implies that $D^b_{gp}(A) \cong K^{-,gpb}(A\text{-}\mathcal{G}\text{proj})$ by the proof of [19, Theorem 3.6(ii)]. Similar to B and C. Let $Y^\bullet$ be an object in $D^b_{gp}(B\text{-mod})$. Since (i^*, i_*) is an adjoint pair, there is an isomorphism $\mathrm{Hom}_{D^b_{gp}(B\text{-mod})}(i^*(X^\bullet), Y^\bullet) \cong \mathrm{Hom}_{D^b_{gp}(A\text{-mod})}(X^\bullet, i_*(Y^\bullet))$ for any $X^\bullet \in D^b_{gp}(A\text{-mod})$. Thus we get from [19, Lemma 4.3] that $i^* : D^b_{gp}(A\text{-mod}) \to D^b_{gp}(B\text{-mod})$ restricts to $K^b(A\text{-}\mathcal{G}\text{proj})$. Similarly, j_* restricts to $K^b(A\text{-}\mathcal{G}\text{proj})$, $j_!$ restricts to $K^b(C\text{-}\mathcal{G}\text{proj})$ and i_* restricts to $K^b(B\text{-}\mathcal{G}\text{proj})$.

If A is CM-finite, then we get from Lemma 3.4 and the above argument that $i^*(G_A) \in K^b(B\text{-}\mathcal{G}\text{proj})$. Furthermore, using the isomorphism $i^*i_* \cong \mathrm{Id}_{D^b_{gp}(B\text{-mod})}$, $i^*(G_A)$ is a generator of $D^b_{gp}(B\text{-mod})$. This implies that B is CM-finite. For the converse, by above arguments and Lemma 3.4 we get that $i_*(G_B)$ and $j_!(G_C)$ are in $K^b(A\text{-}\mathcal{G}\text{proj})$. Let $X^\bullet$ be any object in $D^b_{gp}(A\text{-mod})$. Then there is a distinguished triangle $j_!j^*(X^\bullet) \to X^\bullet \to i_*i^*(X^\bullet) \to j_!j^*(X^\bullet)[1]$. This means that $i_*(G_B)$ and $j_!(G_C)$ generate $D^b_{gp}(A\text{-mod})$. It follows that A is CM-finite.

If A is Gorenstein, then by [19, Corollary 3.8] we have that $D^b_{gp}(A) \cong K^b(A\text{-}\mathcal{G}\text{proj})$. Since $i^*(G) \in K^b(B\text{-}\mathcal{G}\text{proj})$ for any $G \in A\text{-}\mathcal{G}\text{proj}$ by Lemma 3.4, we get that every finitely-generated B-module M admits a proper Gorenstein-projective resolution of finite length. This means $D^b_{gp}(B) \cong K^b(B\text{-}\mathcal{G}\text{proj})$. Hence by Corollary 3.2 we get that B is Gorenstein. For the converse, suppose B and C are Gorenstein. Note that there is a distinguished triangle $j_!j^*(M) \to M \to i_*i^*(M) \to j_!j^*(M)[1]$ for any object $M \in A$-mod. Since $j_!j^*(M) \in K^b(A\text{-}\mathcal{G}\text{proj})$ and $i_*i^*(M) \in K^b(A\text{-}\mathcal{G}\text{proj})$, we obtain that $D^b_{gp}(A) \cong K^b(A\text{-}\mathcal{G}\text{proj})$. This implies that A is Gorenstein. □

4 Gorenstein defect categories

Gorenstein defect categories are used as a crucial tool in the previous section. In this section, we will show the Karoubianness of Gorenstein defect categories, and establish relations between Gorenstein defect categories and Gorenstein stable categories.

We first determine the dimension of Gorenstein defect categories for a simple class of algebras, where the dimension is in the sense of Rouquier [31].

Example 4.1. Let A be a representation-finite artin algebra. Then $\dim D^b_{\text{def}}(A) \leqslant 1$.

Proof. Since A is representation-finite, it follows from [28] that $\dim D^b(A) \leqslant 1$. By [31, Lemma 3.4] we get that $\dim D^b_{\rm def}(A) \leqslant \dim D_{sg}(A) \leqslant \dim D^b(A)$. Hence $\dim D^b_{\rm def}(A) \leqslant 1$. □

Now we study the Karoubianness of the Gorenstein defect category of an algebra. We need some preparation.

Suppose A is CM-finite. We take a right A-$\mathcal{G}$proj-approximation of M for any $M \in A\text{-mod}/A\text{-}\mathcal{G}\text{proj}$, and denote its kernel by M^1. Then we have a functor $\overline{\Omega} : A\text{-mod}/A\text{-}\mathcal{G}\text{proj} \to A\text{-mod}/A\text{-}\mathcal{G}\text{proj}$, view $\overline{\Omega}(M) := M^1$. Denote by $\overline{\Omega}^n$ the n-th composition functor of $\overline{\Omega}$ for any positive integer $n \geqslant 2$. Then we have the following lemma.

Lemma 4.2. *Let $X^\bullet$ be a complex in $D^b_{gp}(A\text{-mod})/K^b(A\text{-}\mathcal{G}\text{proj})$ and $n_0 > 0$. Then for any n large enough, there exists a module M in $\overline{\Omega}^{n_0}(A\text{-mod})$ such that $X^\bullet \cong Q(M)[n]$, where $Q : D^b_{gp}(A\text{-mod}) \to D^b_{gp}(A\text{-mod})/K^b(A\text{-}\mathcal{G}\text{proj})$ is the Verdier quotient functor (see [32, Chapter II, Subsection 2.1.8]).*

Proof. Since A is CM-finite, it follows from [8] that every object X in A-mod has a proper Gorenstein-projective resolution $G^\bullet_X \in K^-(A\text{-}\mathcal{G}\text{proj})$. By induction on the length of the complex $X^\bullet$, we obtain that there is an A-$\mathcal{G}$proj-quasi-isomorphism $G^\bullet \to X^\bullet$ with $G^\bullet \in K^-(A\text{-}\mathcal{G}\text{proj})$. This implies that $H^n\text{Hom}_A(E, G^\bullet) \cong H^n\text{Hom}_A(E, X^\bullet)$ for all $n \in \mathbb{Z}$ and $E \in A\text{-}\mathcal{G}\text{proj}$. So by $X^\bullet$ is a bounded complex, we can take $n \geqslant n_0$ such that $H^i\text{Hom}_A(E, X^\bullet) = 0$ for all $i \leqslant n_0 - n$ and $E \in A\text{-}\mathcal{G}\text{proj}$. Consider the truncation $\tau^{\geqslant -n}G^\bullet = \cdots \to 0 \to M \to G^{1-n} \to G^{2-n} \to \cdots$ of $G^\bullet$, which is A-$\mathcal{G}$proj-quasi-isomorphic to $G^\bullet$. Then the cone of the obvious chain map $\tau^{\geqslant -n}G^\bullet \to M[n]$ is in $K^b(A\text{-}\mathcal{G}\text{proj})$, which becomes an isomorphism in $D^b_{gp}(A\text{-mod})/K^b(A\text{-}\mathcal{G}\text{proj})$. This shows that $X^\bullet \cong Q(M)[n]$ and M lies in $\overline{\Omega}^{n_0}(A\text{-mod})$. □

Lemma 4.3. *Let $0 \to M \to G^{1-n} \to \cdots \to G^0 \to N \to 0$ be an A-$\mathcal{G}$proj-acyclic complex with each G^i Gorenstein-projective. Then we have an isomorphism $Q(N) \cong Q(M)[n]$ in $D^b_{gp}(A\text{-mod})/K^b(A\text{-}\mathcal{G}\text{proj})$. In particular, for an A-module M, we have a natural isomorphism $Q(\overline{\Omega}^n(M)) \cong Q(M)[-n]$.*

Proof. The stalk complex N is A-$\mathcal{G}$proj-quasi-isomorphic to $Z^\bullet := \cdots \to 0 \to M \to G^{1-n} \to \cdots \to G^0 \to 0$. This follows that $N \cong Z^\bullet$ in $D^b_{gp}(A\text{-mod})$ and there is a triangle $M[n-1] \to G^\bullet \to Z^\bullet \to M[n]$, where $G^\bullet := G^{1-n} \to \cdots \to G^0$. Thus we have a morphism $N \to M[n]$ in $D^b_{gp}(A\text{-mod})$, whose cone is in $K^b(A\text{-}\mathcal{G}\text{proj})$. Then this morphism becomes an isomorphism in $D^b_{gp}(A\text{-mod})/K^b(A\text{-}\mathcal{G}\text{proj})$.

We consider the composite functor $Q' : A\text{-mod} \hookrightarrow D^b_{gp}(A\text{-mod}) \xrightarrow{Q} D^b_{gp}(A\text{-mod})/\ K^b(A\text{-}\mathcal{G}\text{proj})$. It vanishes on A-$\mathcal{G}$proj, so it induces uniquely a functor $A\text{-mod}/A\text{-}\mathcal{G}\text{proj} \to D^b_{gp}(A\text{-mod})/K^b(A\text{-}\mathcal{G}\text{proj})$, which still denote by Q'. Then for any modules M and N in $A\text{-mod}/A\text{-}\mathcal{G}\text{proj}$, the functor Q' induces a natural map by Lemma 4.3,

$$\Phi^0 : \text{Hom}_A(M,N)/A\text{-}\mathcal{G}\text{proj}(M,N) \to \text{Hom}_{D^b_{gp}(A\text{-mod})/K^b(A\text{-}\mathcal{G}\text{proj})}(Q(M), Q(N)).$$

We represent the isomorphism $Q(M) \cong Q(\overline{\Omega}^n(M)[n])$ by θ_M. Then the above map induces a map

$$\Phi^n : \text{Hom}_A(\overline{\Omega}^n(M), \overline{\Omega}^n(N))/A\text{-}\mathcal{G}\text{proj}(M,N) \to \text{Hom}_{D^b_{gp}(A\text{-mod})/K^b(A\text{-}\mathcal{G}\text{proj})}(Q(M), Q(N))$$

given by $\Phi^n(f) = (\theta_N)^{-1} \circ (\Phi^0(f)[n]) \circ \theta_M$.

Consider the chain of maps $\text{Hom}_A(\overline{\Omega}^n(M), \overline{\Omega}^n(N))/A\text{-}\mathcal{G}\text{proj}(M,N) \to \text{Hom}_A(\overline{\Omega}^{n+1}(M),\ \overline{\Omega}^{n+1}(N))$ $/A\text{-}\mathcal{G}\text{proj}(M,N)$ induced by $\overline{\Omega}$. Then we have an induced map

$$\Phi : \varinjlim \text{Hom}_A((\overline{\Omega}^n(M), \overline{\Omega}^n(N))/A\text{-}\mathcal{G}\text{proj}(M,N) \to \text{Hom}(Q(M), Q(N)).$$

The proof is complete. □

Lemma 4.4. *Let A be a CM-finite algebra, and M, N be in $A\text{-mod}/A\text{-}\mathcal{G}\text{proj}$. Then the map Φ is an isomorphism.*

Proof. We refer to [5, Theorem 3.8] for a detailed proof. □

Recall that an additive category $\mathcal{A}$ is Karoubian (i.e., idempotent split) provided that each idempotent $e: X \to X$ splits, i.e., it admits a factorization $X \xrightarrow{u} Y \xrightarrow{v} X$ with $u \circ v = \mathrm{Id}_Y$. In particular, for an artin algebra A, the quotient category A-mod/A-$\mathcal{G}$proj is Karoubian.

Theorem 4.5. *The Gorenstein defect category $D^b_{\mathrm{def}}(A)$ of a CM-finite algebra A is Karoubian.*

Proof. We claim that $D^b_{gp}(A\text{-mod})/K^b(A\text{-}\mathcal{G}\mathrm{proj})$ is Karoubian. By Lemma 4.3 and [19, Proposition 2.9] it suffices to show that for each module M in A-mod/A-$\mathcal{G}$proj, an idempotent $e: Q(M) \to Q(M)$ splits. Lemma 4.4 implies that for a large n, there is an idempotent $e^n: \overline{\Omega}^n(M) \to \overline{\Omega}^n(M)$ in A-mod/A-$\mathcal{G}$proj which is mapped by Φ to e. Note that the idempotent e^n splits. Then the idempotent e splits. By Lemma 3.1 we have a triangle-equivalence

$$D^b_{\mathrm{def}}(A) \cong D^b_{gp}(A\text{-mod})/K^b(A\text{-}\mathcal{G}\mathrm{proj}).$$

Hence $D^b_{\mathrm{def}}(A)$ is Karoubian. □

Next, we will show the Gorenstein defect category of an algebra A is triangular equivalent to the stabilization of the Gorenstein stable category of A.

Lemma 4.6. *Let A be an artin algebra such that A-$\mathcal{G}$proj is contravariantly finite in A-mod. Then there is a triangle-equivalence $S(A\text{-mod}/A\text{-}\mathcal{G}\mathrm{proj}) \cong D^b_{\mathrm{def}}(A)$.*

Proof. Since A-$\mathcal{G}$proj is contravariantly finite in A-mod, it follows from [5, Theorem 3.8] that there is a triangle-equivalence $S(A\text{-mod}/A\text{-}\mathcal{G}\mathrm{proj}) \cong K^{-,gpb}(A\text{-}\mathcal{G}\mathrm{proj})/K^b(A\text{-}\mathcal{G}\mathrm{proj})$. By [25, Final Remark] we get a triangle-equivalence $S(A\text{-mod}/A\text{-}\mathcal{G}\mathrm{proj}) \cong D^b_{\mathrm{def}}(A)$. □

As an application we show the equivalences of Gorenstein stable categories can induce the equivalences of Gorenstein defect categories for two CM-finite algebras. For convenience, we introduce two definitions.

Definition 4.7. Two artin algebras A and B are said to be Gorenstein stably equivalent if their Gorenstein stable categories A-mod/A-$\mathcal{G}$proj and B-mod/B-$\mathcal{G}$proj are equivalent as left triangulated categories, where the left triangulated structure is in the sense of Beligiannis [5].

Definition 4.8. Two artin algebras A and B are said to be Gorenstein defect equivalent if there is a triangle-equivalence $D^b_{\mathrm{def}}(A) \cong D^b_{\mathrm{def}}(B)$.

Theorem 4.9. *Let A and B be two artin algebras such that A-$\mathcal{G}$proj and B-$\mathcal{G}$proj are contravariantly finite respectively. If A and B are Gorenstein stably equivalent, then A and B are Gorenstein defect equivalent.*

Proof. Since A and B are Gorenstein stably equivalent, there is a triangle-equivalence by [5, Corollary 3.3], $S(A\text{-mod}/A\text{-}\mathcal{G}\mathrm{proj}) \cong S(B\text{-mod}/B\text{-}\mathcal{G}\mathrm{proj})$. By Lemma 4.6 we have triangle-equivalences

$$S(A\text{-mod}/A\text{-}\mathcal{G}\mathrm{proj}) \cong D^b_{\mathrm{def}}(A) \quad \text{and} \quad S(B\text{-mod}/B\text{-}\mathcal{G}\mathrm{proj}) \cong D^b_{\mathrm{def}}(B).$$

Hence we get a triangle-equivalence $D^b_{\mathrm{def}}(A) \cong D^b_{\mathrm{def}}(B)$. □

5 Conclusion

In this paper, we construct the relations between Gorenstein derived categories, Gorenstein defect categories and Gorenstein stable categories. On the basis of these, we characterize the Gorensteinness of an algebra A in terms of recollements of $D^b_{gp}(A\text{-mod})$ and Gorenstein derived equivalences, and also show that CM-finiteness is invariant with respect to recollements of the bounded Gorenstein derived category $D^b_{gp}(A\text{-mod})$ of A.

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