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Indefinite stochastic linear-quadratic optimal control problems with random jumps and related stochastic Riccati equations

Na Li[1], Zhen Wu[2] & Zhiyong Yu[2,*]

[1]*School of Statistics, Shandong University of Finance and Economics, Jinan 250014, China;*
[2]*School of Mathematics, Shandong University, Jinan 250100, China*

Email: naibor@163.com, wuzhen@sdu.edu.cn, yuzhiyong@sdu.edu.cn

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Abstract We discuss the stochastic linear-quadratic (LQ) optimal control problem with Poisson processes under the indefinite case. Based on the wellposedness of the LQ problem, the main idea is expressed by the definition of relax compensator that extends the stochastic Hamiltonian system and stochastic Riccati equation with Poisson processes (SREP) from the positive definite case to the indefinite case. We mainly study the existence and uniqueness of the solution for the stochastic Hamiltonian system and obtain the optimal control with open-loop form. Then, we further investigate the existence and uniqueness of the solution for SREP in some special case and obtain the optimal control in close-loop form.

Keywords stochastic linear-quadratic problem, Hamiltonian system, Riccati equation, Poisson process, indefinite case

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1 Introduction

The linear-quadratic (LQ) control problem is an important tool in research areas, such as stochastic signal analysis, mathematical finance, control theory and so on. In reality, the LQ problems describe a large number of models in the optimal control problems. Furthermore, there exist many nonlinear control problems that can be approximated by the LQ problem. In the early literature on LQ problems, the control weight costs were supposed to be positive definite when the systems are deterministic or stochastic. In recent years, there has been tremendous interest in developing stochastic linear systems, in which the control not only affects the drift component but also the diffusion component. For this kind of stochastic LQ problems, there exists an essential difference between the deterministic case and the stochastic case. Peng [7] and Chen et al. [2] pointed out that the positive definite condition could be relaxed in stochastic cases. In [2], Chen et al. studied the following stochastic linear system:

$$\begin{cases} dx(t) = (A(t)x(t) + B(t)v(t))dt + (C(t)x(t) + D(t)v(t))dW(t), \\ x_0 = a, \end{cases}$$

* Corresponding author

and the cost functional

$$\mathcal{J}(v(\cdot)) = \frac{1}{2}\mathrm{E}\int_0^T [\langle L_t x_t, x_t\rangle + \langle M_t v_t, v_t\rangle]dt + \frac{1}{2}\mathrm{E}\langle S x_T, x_T\rangle.$$

The conclusion is that if $D(t) \neq 0$, then $D(t)$ can supply a compensation in some sense, so that the control weight cost might be singular or indefinite. The essential reason for this relaxation depends on the solvability of the following deterministic Riccati equation on the interval $[0,T]$:

$$\begin{cases} \dot{P}(t) + P(t)A(t) + A(t)^{\mathrm{T}}P(t) + C^{\mathrm{T}}(t)P(t)C(t) + (P(t)B(t) + C^{\mathrm{T}}(t)P(t)D(t)) \\ \quad \times (M(t) + D^{\mathrm{T}}(t)P(t)D(t))^{-1}(B^{\mathrm{T}}(t)P(t) + D^{\mathrm{T}}(t)P(t)C(t)) + L(t) = 0, \\ P(T) = S. \end{cases} \tag{1.1}$$

If there exists a solution of (1.1), then $D^{\mathrm{T}}PD$ needs to be positive enough such that

$$\tilde{M}(t) \equiv M(t) + D^{\mathrm{T}}(t)P(t)D(t) > 0$$

(instead of $M > 0$). So, when $M < 0$ (but not too negative), $D^{\mathrm{T}}PD$ plays a role of the compensation for M. However, in [2], they could only give the existence and uniqueness for the Riccati equation (1.1) in some special cases where $C(t) = 0$ and all the coefficients are deterministic.

Since then, many researchers became interested in this kind of indefinite stochastic LQ problems, such as Rami et al. [11,12], Qian and Zhou [10], Yu [15], Huang and Yu [4] and so on. Rami et al. [11,12] studied a series of indefinite LQ problems with infinite time horizon by *linear matrix inequality* (LMI) method, which included continuous time models and discrete time models and the coefficients in the system were constants or deterministic functions. In 2013, Yu [15] introduced the *equivalent cost functional* method to deal with indefinite stochastic LQ problems. This method is effective to solve problems with random coefficients. Moreover, he also dealt with the corresponding stochastic Hamiltonian systems when the coefficients did not satisfy the classical monotonicity conditions (see [3,9]). In the following work, Huang and Yu [4] studied the stochastic Riccati equations (SREs) related to indefinite LQ problems with random coefficients, and they gave the existence and uniqueness of SREs and obtained feedback forms of optimal controls.

In this paper, we study the stochastic linear system with Poisson processes as follows:

$$\begin{cases} dx_t = (A_t x_t + B_t v_t)dt + \displaystyle\sum_{i=1}^m (C_t^i x_t + D_t^i v_t)dW_t^i + \sum_{j=1}^n \int_{\mathcal{E}} [E_t^j(e)x_{t-} + F_t^j(e)v_t]\widetilde{N}^j(dt,de), \\ x_0 = a, \end{cases}$$

and minimize the following cost functional:

$$\mathcal{J}(v(\cdot)) = \frac{1}{2}\mathrm{E}\int_0^T [\langle L_t x_t, x_t\rangle + 2\langle R_t v_t, x_t\rangle + \langle M_t v_t, v_t\rangle]dt + \frac{1}{2}\mathrm{E}\langle S x_T, x_T\rangle.$$

Based on the equivalent cost functional method in [15], we introduce a new concept: the *relax compensator*. As we represented above, the efficiency of the method in [2] depended crucially on the solvability of the corresponding Riccati equation. However, it is difficult to obtain the solvability of the stochastic Riccati equations with Poisson processes (SREPs). For example, in [6], Meng only could solve some special cases for SREPs. So the method in [2] is not easy to apply generally. While, the advantage of relax compensators is that it allows us to overcome the difficulty caused by the lack of the solvability of the corresponding Riccati equations. On the other hand, when the coefficients are deterministic, the solution of the Riccati equation (if it exists) could be regarded as a special relax compensator (see Remark 5.6). Moreover, the corresponding stochastic Hamiltonian system with jumps and some special SREP under the indefinite case are solved.

The rest of this paper is organized as follows. In Section 2, we give some notation and formulate the problem. In Section 3, we study the wellposedness of the LQ problem by relax compensators. Section 4 is devoted to studying the stochastic Hamiltonian system under the indefinite case, and we obtain the open-loop form of the optimal control. In Section 5, we focus on the SREP under the indefinite condition and obtain a feedback form of the optimal control in some special cases. Section 6 is the conclusion of this paper.

2 Notation and problem formulation

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with the usual norm $|\cdot|$ and the usual inner product $\langle\cdot,\cdot\rangle$. Let $\mathbb{R}^{m\times n}$ be the collection of $m\times n$ matrices with the inner product $\langle A,B\rangle=\mathrm{tr}\{AB^{\mathrm{T}}\}$, for any $A,B\in\mathbb{R}^{m\times n}$, where T denotes the transpose of matrices.

Let $T>0$ be a constant and $[0,T]$ denote the finite time span. Let $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ be a complete filtered probability space. The filtration $\mathbb{F}=\{\mathcal{F}_t;0\leqslant t\leqslant T\}$ is generated by two mutually independent stochastic processes. One is an m-dimensional Brownian motion $\{W_t:=(W_t^1,\dots,W_t^m)^{\mathrm{T}},\ 0\leqslant t\leqslant T\}$, and the other is an n-dimensional Poisson random measure $\{N(\cdot,\cdot)=(N^1(\cdot,\cdot),\dots,N^n(\cdot,\cdot))^{\mathrm{T}}\}$ defined on $\mathbb{R}_+\times\mathcal{E}$, where $\mathcal{E}=\mathbb{R}^d\setminus\{0\}$. The compensator of N is $\overline{N}(dt,de)=\pi(de)dt$, which makes $\{\widetilde{N}((0,t]\times A)=(N-\overline{N})((0,t]\times A)\}_{t\geqslant0}$ a martingale for any A belonging to the Borel field $\mathcal{B}(\mathcal{E})$ with $\pi(A)<\infty$. Here, π is a given σ-finite measure on the measurable space $(\mathcal{E},\mathcal{B}(\mathcal{E}))$ satisfying that

$$\int_{\mathcal{E}}(1\wedge|e|^2)\pi(de)<\infty,$$

which is called a Lévy measure. $\mathcal{F}_t$ is defined by

$$\sigma\{W_s:0\leqslant s\leqslant t\}\vee\sigma\bigg\{\int_{\mathcal{E}}\widetilde{N}(s,de):0\leqslant s\leqslant t\bigg\}\vee\mathcal{N},\quad 0\leqslant t\leqslant T,$$

where $\mathcal{N}$ denotes the totality of $\mathbb{P}$-null sets, and $\mathcal{F}=\mathcal{F}_T$.

Moreover, we denote the sets of matrices as follows:

- $\mathbb{S}^d$: the space of all $d\times d$ symmetric matrices.
- $\mathbb{S}^d_+$: the subset of $\mathbb{S}^d$ consisting of all positive semi-definite matrices.
- $\hat{\mathbb{S}}^d_+$: the subset of $\mathbb{S}^d$ consisting of all positive definite matrices.

For any Euclidean space $\mathbb{R}^d$, we introduce the following notation:

- $L^2_{\mathbb{F}}(0,T;\mathbb{R}^d)=\{g:[0,T]\times\Omega\to\mathbb{R}^d\,|\,g(\cdot)$ is an $\mathbb{R}^d$-valued $\mathbb{F}$-adapted stochastic process such that $\|g\|^2_{L^2_{\mathbb{F}}}=\mathbb{E}\int_0^T|g(t)|^2dt<\infty\}$.
- $L^\infty_{\mathbb{F}}(0,T;\mathbb{R}^d)=\{g:[0,T]\times\Omega\to\mathbb{R}^d\,|\,g(\cdot)$ is an $\mathbb{R}^d$-valued $\mathbb{F}$-adapted bounded stochastic process$\}$.
- $L^\infty(\mathcal{F}_T;\mathbb{R}^d)=\{\varsigma\,|\,\varsigma$ is an $\mathbb{R}^d$-valued $\mathcal{F}_T$-measurable bounded random variable$\}$.
- $\mathcal{C}^2_{\mathbb{F}}(0,T;\mathbb{R}^d)=\{g:[0,T]\times\Omega\to\mathbb{R}^d\,|\,g(\cdot)$ is a càdlàg process in $L^2_{\mathbb{F}}(0,T;\mathbb{R}^d)$ such that $\|g\|^2_{\mathcal{C}^2_{\mathbb{F}}}=\mathbb{E}[\sup_{0\leqslant t\leqslant T}|g(t)|^2]<\infty\}$.
- $M^2_{\mathbb{F}}(0,T;\mathbb{R}^d)=\{r:[0,T]\times\mathcal{E}\times\Omega\to\mathbb{R}^d\,|\,r(\cdot,\cdot)$ is an $\mathbb{R}^d$-valued $\mathbb{F}$-adapted process such that $\|r\|^2_{M^2_{\mathbb{F}}}=\mathbb{E}\int_0^T\int_{\mathcal{E}}|r(t,e)|^2\pi(de)dt<\infty\}$.

In this paper, we consider the following controlled linear stochastic differential equation with Poisson processes (SDEP) as follows:

$$\begin{cases}dx_t=(A_tx_t+B_tv_t)dt+\displaystyle\sum_{i=1}^m(C_t^ix_t+D_t^iv_t)dW_t^i+\sum_{j=1}^n\int_{\mathcal{E}}[E_t^j(e)x_{t-}+F_t^j(e)v_t]\widetilde{N}^j(dt,de),\\ x_0=a,\end{cases}\tag{2.1}$$

where $a\in\mathbb{R}^d$, $A(\cdot),C^i(\cdot)\in L^\infty_{\mathbb{F}}(0,T;\mathbb{R}^{d\times d})$, $B(\cdot),D^i(\cdot)\in L^\infty_{\mathbb{F}}(0,T;\mathbb{R}^{d\times k})$ $(i=1,2,\dots,m)$, $E^j(\cdot,\cdot)\in M^\infty_{\mathbb{F}}(0,T;\mathbb{R}^{d\times d})$ and $F^j(\cdot,\cdot)\in M^\infty_{\mathbb{F}}(0,T;\mathbb{R}^{d\times k})$ $(j=1,2,\dots,n)$. The admissible control set is given by $\mathcal{V}_{\mathrm{ad}}=L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)$, and each element $v(\cdot)$ of $\mathcal{V}_{\mathrm{ad}}$ is called an admissible control. We also consider the following cost functional:

$$\mathcal{J}(v(\cdot))=\frac{1}{2}\mathrm{E}\int_0^T[\langle L_tx_t,x_t\rangle+2\langle R_tv_t,x_t\rangle+\langle M_tv_t,v_t\rangle]dt+\frac{1}{2}\mathrm{E}\langle Sx_T,x_T\rangle,\tag{2.2}$$

where $L(\cdot)\in L_{\mathbb{F}}^{\infty}(0,T;\mathbb{S}^d)$, $R(\cdot)\in L_{\mathbb{F}}^{\infty}(0,T;\mathbb{R}^{d\times k})$, $M(\cdot)\in L_{\mathbb{F}}^{\infty}(0,T;\mathbb{S}^k)$ and $S\in L^{\infty}(\mathcal{F}_T;\mathbb{S}^d)$.

Problem (LQ SOC). The problem is to look for an admissible control $u(\cdot)\in\mathcal{V}_{\rm ad}$ satisfying

$$\mathcal{J}(u(\cdot))=\inf_{v(\cdot)\in\mathcal{V}_{\rm ad}}\mathcal{J}(v(\cdot)). \tag{2.3}$$

Such an admissible control $u(\cdot)$ is called an optimal control, and $x(\cdot)=x^u(\cdot)$ is called the corresponding optimal trajectory.

We point out that any definiteness of the coefficients is not assumed in the above. Next, we categorize the Problem (LQ SOC) into two classes: the positive definite case and the indefinite case.

Assumption 2.1 (Positive definite condition)**.**

$$\begin{bmatrix} L(\cdot) & R(\cdot)\\ R^{\rm T}(\cdot) & M(\cdot)\end{bmatrix}\in L_{\mathbb{F}}^{\infty}(0,T;\mathbb{S}_+^{d+k}),\quad S\in L^{\infty}(\mathcal{F}_T;\mathbb{S}_+^d)$$

and $M(\cdot)\in L_{\mathbb{F}}^{\infty}(0,T;\hat{\mathbb{S}}_+^k)$. *Moreover, the inverse matrix* $M^{-1}(\cdot)$ *is also bounded.*

If the positive definite condition is satisfied, we call Problem (LQ SOC) positive definite. Otherwise, it is called indefinite. The aim of this paper is to study the indefinite case of Problem (LQ SOC) by finding a new component to relax the positive definite condition, which is inspired by the equivalent cost functional method introduced in [4, 15]. This method is more general than that used in [2, 12]. It is still effective even though the solution of the corresponding Riccati equation does not exist.

At the end of this section, we introduce the following lemma which is useful in the sequel.

Lemma 2.2 (Schur's lemma, see [1])**.** *Let matrices* $L=L^{\rm T}$, $M=M^{\rm T}$ *and* R *be given with appropriate dimensions. The following conditions are equivalent*:

(i) $L-RM^{-1}R^{\rm T}\geqslant 0$, $M>0$;

(ii) $\left[\begin{smallmatrix} L & R\\ R^{\rm T} & M\end{smallmatrix}\right]\geqslant 0$, $M>0$.

3 Wellposedness of Problem (LQ SOC)

In this section, we discuss the wellposedness for Problem (LQ SOC) with indefinite control weight cost. By virtue of the wellposedness, we extend the LQ problem from the positive definite case to the indefinite case.

Firstly, we give the definition of wellposedness.

Definition 3.1. If the value $V=\inf_{v(\cdot)\in\mathcal{V}_{\rm ad}}\mathcal{J}(v(\cdot))>-\infty$, then the Problem (LQ SOC) is called well-posed.

Secondly, we denote the following set of jump-diffusion processes:

$$\Upsilon:=\Bigg\{K\in L_{\mathbb{F}}^{\infty}(0,T;\mathbb{S}^d)\;\Bigg|\;K_t=K_0+\int_0^t\Theta_s ds+\sum_{i=1}^m\int_0^t\Phi_s^i dW_s^i+\sum_{j=1}^n\int_0^t\int_{\mathcal{E}}\Psi_s^j(e)\widetilde{N}^j(ds,de)$$
$$\text{for all } t\in[0,T], \text{where } \Theta,\Phi^i,\Psi^j\in L_{\mathbb{F}}^{\infty}(0,T;\mathbb{S}^d), i=1,\ldots,m, \text{ and } j=1,\ldots,n\Bigg\}$$

and introduce the following notation:

$$\begin{aligned}\mathcal{L}(K)&=L+\Theta+KA+A^{\rm T}K+\sum_{i=1}^m[\Phi^iC^i+(C^i)^{\rm T}\Phi^i+(C^i)^{\rm T}KC^i]+\sum_{j=1}^n\int_{\mathcal{E}}[\Phi^j(e)E^j(e)\\&\quad+(E^j(e))^{\rm T}\Psi^j(e)+(E^j(e))^{\rm T}(K+\Psi^j(e))E^j(e)]\pi^j(de),\\ \mathcal{R}(K)&=R+KB+\sum_{i=1}^m[\Phi^iD^i+(C^i)^{\rm T}KD^i]+\sum_{j=1}^n\int_{\mathcal{E}}[\Psi^j(e)F^j(e)+(E^j(e))^{\rm T}(K\\&\quad+\Psi^j(e))F^j(e)]\pi^j(de),\end{aligned}$$

$$\mathcal{M}(K) = M + \sum_{i=1}^{m}(D^i)^{\mathrm{T}}KD^i + \sum_{j=1}^{n}\int_{\mathcal{E}}(F^j(e))^{\mathrm{T}}(K+\Psi^j(e))F^j(e)\pi^j(de),$$
$$\mathcal{S}(K) = S - K_T.$$

We give the definition of relax compensator as follows.

Definition 3.2. If $K \in \Upsilon$ such that $(\mathcal{L}(K), \mathcal{R}(K), \mathcal{M}(K), \mathcal{S}(K))$ satisfies Assumption 2.1, then we call K a relax compensator.

Theorem 3.3. *If there exists a relax compensator K, then Problem* (SOC LQ) *is well-posed.*

Proof. For any $K \in \Upsilon$, applying Itô's formula to $\langle K_t x_t, x_t\rangle$ on the interval $[0,T]$ and taking expectation, we get

$$\begin{aligned}
&\mathrm{E}[\langle K_T x_T, x_T\rangle] - \langle K_0 a, a\rangle \\
&= \mathrm{E}\int_0^T \bigg[\bigg\langle\bigg\{K_tA_t + A_t^{\mathrm{T}}K_t + \sum_{i=1}^{m}[\Phi_t^iC_t^i + (C_t^i)^{\mathrm{T}}\Phi_t^i + (C_t^i)^{\mathrm{T}}K_tC_t^i] + \sum_{j=1}^{n}\int_{\mathcal{E}}[\Phi_t^j(e)E_t^j(e) \\
&\quad + (E_t^j(e))^{\mathrm{T}}\Psi_t^j(e) + (E_t^j(e))^{\mathrm{T}}(K_t+\Psi_t^j(e))E_t^j(e)]\pi^j(de) + \Theta_t\bigg\}x_t, x_t\bigg\rangle + 2\bigg\langle\bigg\{K_tB_t \\
&\quad + \sum_{i=1}^{m}[\Phi_t^iD_t^i + (C_t^i)^{\mathrm{T}}K_tD_t^i] + \sum_{j=1}^{n}\int_{\mathcal{E}}[\Psi_t^j(e)F_t^j(e) + (E_t^j(e))^{\mathrm{T}}(K_t+\Psi_t^j(e))F_t^j(e)]\pi^j(de)\bigg\}v_t, x_t\bigg\rangle \\
&\quad + \bigg\langle\bigg\{\sum_{i=1}^{m}(D_t^i)^{\mathrm{T}}K_tD_t^i + \sum_{j=1}^{n}\int_{\mathcal{E}}[(F_t^j(e))^{\mathrm{T}}(K_t+\Psi_t^j(e))F_t^j(e)]\pi^j(de)\bigg\}v_t, v_t\bigg\rangle\bigg]dt.
\end{aligned}$$

By the definition of $\mathcal{L}(K)$, $\mathcal{R}(K)$, $\mathcal{M}(k)$ and $\mathcal{S}(K)$,

$$\begin{aligned}
0 &= \mathrm{E}\int_0^T[\langle(\mathcal{L}_t(K)-L_t)x_t, x_t\rangle + 2\langle(\mathcal{R}_t(K)-R_t)v_t, x_t\rangle + \langle(\mathcal{M}_t(K)-M_t)v_t, v_t\rangle]dt \\
&\quad + \mathrm{E}[\langle(\mathcal{S}(K)-S)x_T, x_T\rangle] + \langle K_0a, a\rangle. \qquad (3.1)
\end{aligned}$$

Combining (3.1) with the definition of $\mathcal{J}(v(\cdot))$, we have

$$\begin{aligned}
\mathcal{J}(v(\cdot)) &= \frac{1}{2}\mathrm{E}\int_0^T[\langle\mathcal{L}_t(K)x_t, x_t\rangle + 2\langle\mathcal{R}_t(K)v_t, x_t\rangle + \langle\mathcal{M}_t(K)v_t, v_t\rangle]dt \\
&\quad + \frac{1}{2}\mathrm{E}\langle\mathcal{S}(K)x_T, x_T\rangle + \frac{1}{2}\langle K_0a, a\rangle \\
&= \frac{1}{2}\mathrm{E}\int_0^T \begin{bmatrix} x_t \\ v_t \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathcal{L}_t(K) & \mathcal{R}_t(K) \\ \mathcal{R}_t^{\mathrm{T}}(K) & \mathcal{M}_t(K) \end{bmatrix} \begin{bmatrix} x_t \\ v_t \end{bmatrix} dt + \frac{1}{2}\mathrm{E}\langle\mathcal{S}(K)x_T, x_T\rangle + \frac{1}{2}\langle K_0a, a\rangle.
\end{aligned}$$

According to Assumption 2.1,

$$\begin{bmatrix} \mathcal{L}_t(K) & \mathcal{R}_t(K) \\ \mathcal{R}_t^{\mathrm{T}}(K) & \mathcal{M}_t(K) \end{bmatrix} \in L_{\mathbb{F}}^{\infty}(0,T;\mathcal{S}_+^{n+m})$$

and

$$\mathcal{S}(K) \in L^{\infty}(\mathcal{F}_T;\mathcal{S}_+^{n+m}).$$

Hence,

$$\mathcal{J}(v(\cdot)) \geqslant \frac{1}{2}\langle K_0a, a\rangle > -\infty.$$

We proved that Problem (LQ SOC) is well-posed. □

Remark 3.4. Theorem 3.3 reveals some essence of the indefinite LQ problem. In [2], Chen et al. studied the LQ problems by solving the associated Riccati equation directly. In general, the solvability of Riccati equation is very difficult, and they only obtained the solvability result in some particular cases. In this paper, the controlled system is generalized to a jump-diffusion model (see (2.1)), and the solvability of the corresponding Riccati equation is more challenging. However, the viewpoint of Theorem 3.3 is to look for some relax compensator instead of to solve the Riccati equation.

4 Stochastic Hamiltonian system with Poisson process under the indefinite case

In this section, we focus on researching the stochastic Hamiltonian system with Poisson processes. Firstly, we study the stochastic Hamiltonian system with Poisson processes under the positive definite case. Then, by relax compensators, we extend the problem from the positive definite case to the indefinite case.

Now we introduce a backward stochastic differential equation with Poisson processes (BSDEP) which is called the adjoint equation to (2.1):

$$\begin{cases} -dq_t = \left(A_t^{\mathrm{T}} q_t + \sum_{i=1}^m (C_t^i)^{\mathrm{T}} r_t^i + \sum_{j=1}^n \int_{\mathcal{E}} (E_t^j(e))^{\mathrm{T}} \theta_t^j(e)\pi(de) + L_t x_t^v + R_t v_t\right)dt \\ \qquad\quad - \sum_{i=1}^m r_t^i dW_t^i - \sum_{j=1}^n \int_{\mathcal{E}} \theta_t^j(e)\widetilde{N}^j(dt, de), \\ q_T = Sx_T, \end{cases} \tag{4.1}$$

where $q(\cdot), r(\cdot)$ and $\theta(\cdot,\cdot)$ are called the adjoint processes. Noting that from the classical BSDEP theory with jumps, (4.1) has a unique solution $(q(\cdot), r(\cdot), \theta(\cdot,\cdot))$ for any given admissible pair $(x^v(\cdot), v(\cdot))$. For simplicity, we denote $(x(\cdot), q(\cdot), r(\cdot), \theta(\cdot,\cdot)) = (x^u(\cdot), q^u(\cdot), r^u(\cdot), \theta^u(\cdot,\cdot))$. We link Problem (LQ SOC) to the following stochastic Hamiltonian system:

$$\begin{cases} 0 = M_t u_t + R_t^{\mathrm{T}} x_t + B_t^{\mathrm{T}} q_t + \sum_{i=1}^m (D_t^i)^{\mathrm{T}} r_t^i + \sum_{j=1}^n \int_{\mathcal{E}} (F_t^j(e))^{\mathrm{T}} \theta_t^j(e)\pi^j(de), \\ dx_t = (A_t x_t + B_t u_t)dt + \sum_{i=1}^m (C_t^i x_t + D_t^i u_t)dW_t^i + \sum_{j=1}^n \int_{\mathcal{E}} [E_t^j(e)x_{t-} + F_t^j(e)u_t]\widetilde{N}^j(dt, de), \\ -dq_t = \left(A_t^{\mathrm{T}} q_t + \sum_{i=1}^m (C_t^i)^{\mathrm{T}} r_t^i + \sum_{j=1}^n \int_{\mathcal{E}} (E_t^j(e))^{\mathrm{T}} \theta_t^j(e)\pi^j(de) + L_t x_t + R_t u_t\right)dt \\ \qquad\quad - \sum_{i=1}^m r_t^i dW_t^i - \sum_{j=1}^n \int_{\mathcal{E}} \theta_t^j(e)\widetilde{N}^j(dt, de), \\ x_0 = a, \quad q_T = Sx_T. \end{cases} \tag{4.2}$$

Theorem 4.1. *Under Assumption* 2.1*, the Hamiltonian system* (4.2) *admits a unique solution*

$$(x(\cdot), u(\cdot), q(\cdot), r(\cdot), \theta(\cdot,\cdot)) \in L_{\mathbb{F}}^2(0,T;\mathbb{R}^d) \times \mathcal{V}_{\mathrm{ad}} \times L_{\mathbb{F}}^2(0,T;\mathbb{R}^d) \times L_{\mathbb{F}}^2(0,T;\mathbb{R}^d) \times M_{\mathbb{F}}^2(0,T;\mathbb{R}^d),$$

in which $(x(\cdot), u(\cdot))$ *is the unique optimal pair of Problem* (LQ SOC).

Proof. Under Assumption 2.1, we can solve $u(\cdot)$ directly from the first equation of the Hamiltonian system (4.2):

$$u_t = -M_t^{-1}\left[R_t^{\mathrm{T}} x_t + B_t^{\mathrm{T}} q_t + \sum_{i=1}^m (D_t^i)^{\mathrm{T}} r_t^i + \sum_{j=1}^n \int_{\mathcal{E}} (F_t^j(e))^{\mathrm{T}} \theta_t^j(e)\pi^j(de)\right]. \tag{4.3}$$

Then the problem is reduced to solve the following fully coupled forward-backward stochastic differential equation with Poisson processes (FBSDEP):

$$\begin{cases} dx_t = \bigg[(A_t - B_tM_t^{-1}R_t^{\mathrm{T}})x_t - B_tM_t^{-1}B_t^{\mathrm{T}}q_t - \displaystyle\sum_{i=1}^m B_tM_t^{-1}(D_t^i)^{\mathrm{T}}r_t^i \\ \quad - \displaystyle\sum_{j=1}^n \int_{\mathcal{E}} B_tM_t^{-1}(F_t^j(e))^{\mathrm{T}}\theta_t^j(e)\pi^j(de)\bigg]dt + \sum_{i=1}^m \bigg[(C_t^i - D_t^iM_t^{-1}R_t^{\mathrm{T}})x_t \\ \quad - D_t^iM_t^{-1}B_t^{\mathrm{T}}q_t - \displaystyle\sum_{k=1}^m D_t^iM_t^{-1}(D_t^k)^{\mathrm{T}}r_t^k - \sum_{j=1}^n \int_{\mathcal{E}} D_t^iM_t^{-1}(F_t^j(e))^{\mathrm{T}}\theta_t^j(e)\pi^j(de)\bigg]dW_t^i \\ \quad + \displaystyle\sum_{j=1}^n \int_{\mathcal{E}} \bigg[(E_t^j(e) - F_t^j(e)M_t^{-1}R_t^{\mathrm{T}})x_{t-} - F_t^j(e)M_t^{-1}B_t^{\mathrm{T}}q_t - \sum_{i=1}^m F_t^j(e)M_t^{-1}(D_t^i)^{\mathrm{T}}r_t^i \\ \quad - \displaystyle\sum_{k=1}^n \int_{\mathcal{E}} F_t^j(e)M_t^{-1}(F_t^k(e))^{\mathrm{T}}\theta_t^k(e)\pi^j(de)\bigg]\widetilde{N}^j(dt,de), \\ -dq_t = \bigg[(A_t^{\mathrm{T}} - R_tM_t^{-1}B_t^{\mathrm{T}})q_t + \displaystyle\sum_{i=1}^m((C_t^i)^{\mathrm{T}} - R_tM_t^{-1}(D_t^i)^{\mathrm{T}})r_t^i + \sum_{j=1}^n \int_{\mathcal{E}}((E_t^j(e))^{\mathrm{T}} \\ \quad - R_tM_t^{-1}(F_t^j(e))^{\mathrm{T}})\theta_t^j(e)\pi^j(de) + (L_t - R_tM_t^{-1}R_t^{\mathrm{T}})x_t\bigg]dt - \displaystyle\sum_{i=1}^m r_t^i dW_t^i \\ \quad - \displaystyle\sum_{j=1}^n \int_{\mathcal{E}} \theta_t^j(e)\widetilde{N}^j(dt,de), \\ x_0 = a, \quad q_T = Sx_T. \end{cases} \tag{4.4}$$

The above linear FBSDEP (4.4) admits a unique solution, since it is a special case of the relevant arguments of Li and Yu [5]. So (4.2) admits a unique solution $(x(\cdot), u(\cdot), q(\cdot), r(\cdot), \theta(\cdot,\cdot))$.

Now, we prove that $(x(\cdot), u(\cdot))$ is an optimal pair of Problem (LQ SOC). For any admissible control $v(\cdot)$, the corresponding state is denoted by $x^v(\cdot)$. We analyze the difference between $\mathcal{J}(u(\cdot))$ and $\mathcal{J}(v(\cdot))$. Under Assumption 2.1, $\mathcal{J}(v(\cdot))$ is convex, so

$$\begin{aligned} \mathcal{J}(u(\cdot)) - \mathcal{J}(v(\cdot)) &= \frac{1}{2}\mathrm{E}\int_0^T [\langle L_tx_t, x_t\rangle + 2\langle R_tu_t, x_t\rangle + \langle M_tu_t, u_t\rangle]dt + \frac{1}{2}\mathrm{E}\langle Sx_T, x_T\rangle \\ &\quad - \frac{1}{2}\mathrm{E}\int_0^T [\langle L_tx_t^v, x_t^v\rangle + 2\langle R_tv_t, x_t^v\rangle + \langle M_tv_t, v_t\rangle]dt - \frac{1}{2}\mathrm{E}\langle Sx_T^v, x_T^v\rangle \\ &\leqslant \mathrm{E}\int_0^T [\langle L_tx_t, x_t - x_t^v\rangle + \langle R_tu_t, x_t - x_t^v\rangle + \langle R_t^{\mathrm{T}}x_t, u_t - v_t\rangle + \langle M_tu_t, u_t - v_t\rangle]dt \\ &\quad + \mathrm{E}[\langle Sx_T, x_T - x_T^v\rangle]. \end{aligned} \tag{4.5}$$

Applying Itô's formula to $\langle q(\cdot), x(\cdot) - x^v(\cdot)\rangle$ on the interval $[0,T]$, we get the right-hand side of the above inequality (4.5) is zero. So $\mathcal{J}(u(\cdot)) - \mathcal{J}(v(\cdot)) \leqslant 0$, which implies $u(\cdot)$ is an optimal control.

Let $(\bar{x}(\cdot), \bar{u}(\cdot))$ be an optimal pair of Problem (LQ SOC). From some standard variational calculus and dual representation considerations, which is a straightforward consequence of the stochastic maximum principle in [14], there exists $(\bar{q}(\cdot), \bar{r}(\cdot), \bar{\theta}(\cdot,\cdot)) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^d) \times L^2_{\mathbb{F}}(0,T;\mathbb{R}^d) \times M^2_{\mathbb{F}}(0,T;\mathbb{R}^d)$ such that $(\bar{x}(\cdot), \bar{u}(\cdot), \bar{q}(\cdot), \bar{r}(\cdot), \bar{\theta}(\cdot,\cdot))$ satisfies the Hamiltonian system (4.2). From the uniqueness of (4.2), we obtain the uniqueness of the optimal control. □

The above discussions are about the positive definite case. Now, we turn our attention to the indefinite case. Inspired by the idea of relax compensators, for any $K \in \Upsilon$, we consider a new adjoint equation instead of the original one:

$$\begin{cases} -dq_t^K = \bigg(A_t^{\mathrm{T}}q_t^K + \displaystyle\sum_{i=1}^m (C_t^i)^{\mathrm{T}}r_t^{K,i} + \sum_{j=1}^n \int_{\mathcal{E}} (E_t^j(e))^{\mathrm{T}}\theta_t^{K,j}(e)\pi^j(de) + \mathcal{L}_t(K)x_t + \mathcal{R}_t(K)u_t\bigg)dt \\ \qquad - \displaystyle\sum_{i=1}^m r_t^{K,i}dW_t^i - \sum_{j=1}^n \int_{\mathcal{E}} \theta_t^{K,j}(e)\widetilde{N}^j(dt,de), \\ q_T^K = \mathcal{S}(K)x_T. \end{cases} \tag{4.6}$$

Comparing (4.6) with (4.1), we take $(\mathcal{L}(K),\mathcal{R}(K),\mathcal{M}(K),\mathcal{S}(K))$ instead of (L,R,M,S). Then, we study the following new stochastic Hamiltonian system with $K\in\Upsilon$:

$$\begin{cases} 0=\mathcal{M}_t(K)u_t^K+\mathcal{L}_t^{\mathrm{T}}(K)x_t^K+B_t^{\mathrm{T}}q_t^K+\displaystyle\sum_{i=1}^m(D_t^i)^{\mathrm{T}}r_t^{K,i}+\sum_{j=1}^n\int_{\mathcal{E}}(F_t^j(e))^{\mathrm{T}}\theta_t^{K,j}(e)\pi(de),\\ dx_t^K=(A_tx_t^K+B_tu_t^K)dt+\displaystyle\sum_{i=1}^m(C_t^ix_t^K+D_t^iu_t^K)dW_t^i\\ \qquad\quad+\displaystyle\sum_{j=1}^n\int_{\mathcal{E}}[E_t^j(e)x_{t-}^K+F_t^j(e)u_t^K]\widetilde{N}^j(dt,de),\\ -dq_t^K=\left(A_t^{\mathrm{T}}q_t^K+\displaystyle\sum_{i=1}^m(C_t^i)^{\mathrm{T}}r_t^{K,i}+\sum_{j=1}^n\int_{\mathcal{E}}(E_t^j(e))^{\mathrm{T}}\theta_t^{K,j}(e)\pi^j(de)+\mathcal{L}_t(K)x_t+\mathcal{R}_t(K)u_t\right)dt\\ \qquad\quad-\displaystyle\sum_{i=1}^m r_t^{K,i}dW_t^i-\sum_{j=1}^n\int_{\mathcal{E}}\theta_t^{K,j}(e)\widetilde{N}^j(dt,de),\\ x_0=a,\quad q_T^K=\mathcal{S}(K)x_T. \end{cases}\tag{4.7}$$

Based on Theorem 4.1, we obtain the following theorem.

Theorem 4.2. *If there exists a relax compensator $\tilde{K}$, then for any $K\in\Upsilon$, the stochastic Hamiltonian system* (4.7) *admits a unique solution $(x^K(\cdot),u^K(\cdot),q^K(\cdot),r^K(\cdot),\theta^K(\cdot,\cdot))\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^d)\times\mathcal{V}_{\mathrm{ad}}\times L^2_{\mathbb{F}}(0,T;\mathbb{R}^d)\times L^2_{\mathbb{F}}(0,T;\mathbb{R}^d)\times M^2_{\mathbb{F}}(0,T;\mathbb{R}^d)$. Moreover, $(x^K(\cdot),u^K(\cdot))=(x^{\tilde{K}}(\cdot),u^{\tilde{K}}(\cdot))$ is the unique optimal pair of Problem* (LQ SOC) *corresponding to $\mathcal{J}(\cdot)$.*

Proof. Firstly, we prove the existence and uniqueness of the Hamiltonian systems (4.7) with K and $\tilde{K}$ are equivalent. If $(x^K(\cdot),u^K(\cdot),q^K(\cdot),r^K(\cdot),\theta^K(\cdot,\cdot))$ is a solution of Hamiltonian system (4.7) with K, then

$$\begin{aligned} &x_t=x_t^K,\quad u_t=u_t^K,\\ &q_t=q_t^K+K_tx_t,\\ &r_t^i=r_t^{K,i}+(\Phi_t^i+K_tC_t^i)x_t+K_tD_t^iu_t,\quad i=1,\dots,m,\\ &\theta_t^j(e)=\theta_t^{K,j}(e)+[\Psi_t^j(e)+(K_t+\Psi_t^j(e))E_t^j(e)]x_t+[K_t+\Psi_t^j(e)]F_t^j(e)u_t,\quad j=1,\dots,n, \end{aligned}\tag{4.8}$$

solves the Hamiltonian system (4.2). Because (4.8) is invertible, the existence and uniqueness of the Hamiltonian systems (4.7) with $\tilde{K}$ and K are equivalent.

If $\tilde{K}$ is a relax compensator, then $(\mathcal{L}(\tilde{K}),\mathcal{R}(\tilde{K}),\mathcal{M}(\tilde{K}),\mathcal{S}(\tilde{K}))$ satisfies Assumption 2.1. By Theorem 4.1, the Hamiltonian system (4.7) with $\tilde{K}$ admits a unique solution, so the Hamiltonian system (4.7) with K admits a unique solution either. Particularly, $(x(\cdot),u(\cdot))$ is the unique optimal pair for Problem (LQ SOC). □

Remark 4.3. Inspired by the idea of relax compensators, in Theorem 4.2, we use $(\mathcal{L}(\tilde{K}),\mathcal{R}(\tilde{K}),\mathcal{M}(\tilde{K}),\mathcal{S}(\tilde{K}))$ instead of (L,R,M,S), which relaxes the condition on solvability of the stochastic Hamiltonian system (4.2). Then we can solve more cases of the stochastic Hamiltonian system. In other words, the existence of the relax compensator gives a new condition for the FBSDEP (4.4) in some special case which does not satisfy the monotonicity conditions (see [3,9]).

Example 4.4. Let processes $x(\cdot)$ and $v(\cdot)$ be 1-dimensional. We consider the LQ problem as follows: minimize the cost functional $\mathcal{J}(v(\cdot))=\frac{1}{2}\mathrm{E}\int_0^T[\alpha_tx_t^2+\beta_tv_t^2]dt-\frac{1}{2}\mathbb{E}[(\int_0^T\alpha_tdt)x_T^2]$ subject to

$$\begin{cases} dx_t=\displaystyle\int_{\mathcal{E}}u_t\widetilde{N}(dt,de),\\ x_0=a, \end{cases}$$

where $\alpha(\cdot)$ and $\beta(\cdot)$ are $\mathbb{F}$-adapted bounded real-valued processes. In the positive definite LQ problems, $L_t=\alpha_t$ and $S=-\int_0^T\alpha_tdt$ are required to be non-negative, and $M_t=\beta_t$ is required to be positive, then α must be zero. As we discussed above, if we find a relax compensator, even if $\alpha \neq 0$, this LQ problem can be well-posed too. We select $K_t = \int_0^t \alpha_s ds$, then, $\mathcal{L}(K) = \mathcal{R}(K) = \mathcal{S}(K) = 0$ and $\mathcal{M}(K) = \beta_t + \pi(\mathcal{E}) \int_0^t \alpha(s)ds$. If there exists a constant $k > 0$ such that $\beta_t + \pi(\mathcal{E}) \int_0^t \alpha(s)ds > k$, $t \in [0,T]$, then $(\mathcal{L}(K), \mathcal{R}(K), \mathcal{M}(K), \mathcal{S}(K))$ satisfies Assumption 2.1. Moreover, the corresponding Hamiltonian system has the following form:

$$\begin{cases} 0 = \beta_t u_t + \displaystyle\int_{\mathcal{E}} \theta_t(e)\pi(de), \\ dx_t = \displaystyle\int_{\mathcal{E}} u_t \widetilde{N}(dt, de), \\ -dq_t = \alpha_t x_t dt - r_t dW_t - \displaystyle\int_{\mathcal{E}} \theta_t(e)\widetilde{N}(dt, de), \\ x_0 = a, \quad q_T = x_T \displaystyle\int_0^T \alpha_t dt. \end{cases} \tag{4.9}$$

Furthermore, the unique solution of the Hamiltonian system (4.9) is

$$(x_t, u_t, q_t, r_t, \theta_t(\cdot)) = \left(a, 0, a\int_0^t \alpha_s ds, 0, 0\right), \quad t \in [0,T],$$

which also means that the unique optimal pair is $(x_t, u_t) = (a, 0)$.

Example 4.5. Let processes $x(\cdot)$ and $v(\cdot)$ be 1-dimensional. We consider an LQ problem as follows: minimize $\mathcal{J}(v(\cdot)) = \frac{1}{2}\mathrm{E}\int_0^T [-\gamma|v_t|^2]dt + \frac{1}{2}\mathbb{E}[Sx_T^2]$ subject to

$$\begin{cases} dx_t = \displaystyle\sum_{j=1}^n \int_{\mathcal{E}} [E_t^j(e)x_{t-} + u_t]\widetilde{N}^j(dt, de), \\ x_0 = a, \end{cases}$$

where $\gamma \geqslant 0$. For $M_t = -\gamma I$ is non-positive, obviously, the standard positive definite condition is not satisfied. Now, we try to find some relax compensator. If there exist two constants l_1 and l_2 such that $\gamma < l_1 < l_2 < \infty$ and $S \geqslant l_2 I$, we could find a relax compensator $k(t) \in C^1(0,T;\mathbb{R})$ such that $k(T) \in [l_1, l_2]$, $k(t) \geqslant l_1$ for all $t \in [0,T]$, meanwhile, $k'(t)I - \frac{\gamma k(t)}{k(t)-\gamma}\sum_{j=1}^n \int_{\mathcal{E}} E_t^j(e)(E_t^j(e))^{\mathrm{T}}\pi(de) \geqslant 0$, and then this LQ problem is well-posed. Moreover, the LQ problem has a unique optimal pair $(x(\cdot), u(\cdot)) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^d) \times \mathcal{V}_{\mathrm{ad}}$, and the corresponding stochastic Hamiltonian system is as follows:

$$\begin{cases} 0 = \displaystyle\sum_{j=1}^n \int_{\mathcal{E}} \theta_t^j(e)\pi^j(de) - \gamma u_t, \\ dx_t = \left[\displaystyle\sum_{j=1}^n \int_{\mathcal{E}} E_t^j(e)x_t + u_t\right]\widetilde{N}^j(dt, de), \\ -dq_t = \displaystyle\sum_{j=1}^n \int_{\mathcal{E}} (E_t^j(e))^{\mathrm{T}}\theta_t(e)\pi^j(de)dt - \sum_{j=1}^n \int_{\mathcal{E}} \theta_t^j(e)\widetilde{N}^j(dt, de), \\ x_0 = a, \quad q_T = Sx_T. \end{cases}$$

In summary, if there exists a relax compensator, then Problem (LQ SOC) is unique solvable. Moreover, in this case, there exists a unique solution to the corresponding stochastic Hamiltonian system (4.2), and the unique optimal control of Problem (LQ SOC) is given by (4.3), which is in an "open-loop" form. On the further step, we go to study the "closed-loop" form for the optimal control of Problem (LQ SOC) by the related stochastic Riccati equation.

5 Stochastic Riccati equation under the indefinite case

In this section, we study the relationship between the stochastic Riccati equation and the LQ problem under the indefinite case. Moreover, we shall give the existence and uniqueness result of Riccati equation in some special case. We introduce the following stochastic Riccati equations with Poisson processes (SREP):

$$\begin{cases}-dP_t = G(A_t, B_t, C_t, D_t, E_t, F_t; L_t, R_t, M_t; P_t, \Lambda_t, \Gamma_t)dt \\ \qquad\qquad - \sum_{i=1}^{m} \Lambda_t^i dW_t^i - \sum_{j=1}^{n} \int_{\mathcal{E}} \Gamma_t^j(e) \widetilde{N}^j(dt, de), \\ P_T = S,\end{cases} \tag{5.1}$$

where

$$\begin{cases}G(A, B, C, D, E, F; L, R, M; P, \Lambda, \Gamma) = \tilde{L}(A, C, E; L; P, \Lambda, \Gamma) - \tilde{R}(B, C, D, E, F; R; P, \Lambda, \Gamma) \\ \qquad\qquad \times \tilde{M}^{-1}(D, F; R; P, \Gamma)\tilde{R}^{\mathrm{T}}(B, C, D, E, F; R; P, \Lambda, \Gamma), \\ \tilde{L}(A, C, E; L; P, \Lambda, \Gamma) = L + PA + A^{\mathrm{T}}P + \sum_{i=1}^{m}[\Lambda^i C^i + (C^i)^{\mathrm{T}}\Lambda^i + (C^i)^{\mathrm{T}} PC^i] \\ \qquad\qquad + \sum_{j=1}^{n} \int_{\mathcal{E}} [\Gamma^j(e)E^j(e) + (E^j(e))^{\mathrm{T}}\Gamma^j(e) \\ \qquad\qquad + (E^j(e))^{\mathrm{T}}(P + \Gamma^j(e))E^j(e)]\pi^j(de), \\ \tilde{R}(B, C, D, E, F; R; P, \Lambda, \Gamma) = R + PB + \sum_{i=1}^{m}[\Lambda^i D^i + (C^i)^{\mathrm{T}} PD^i] + \sum_{j=1}^{n} \int_{\mathcal{E}} [\Gamma^j(e)F^j(e) \\ \qquad\qquad + (E^j(e))^{\mathrm{T}}(P + \Gamma^j(e))F^j(e)]\pi^j(de), \\ \tilde{M}(D, F; M; P, \Gamma) = M + \sum_{i=1}^{m}(D^i)^{\mathrm{T}} PD^i + \sum_{j=1}^{n} \int_{\mathcal{E}} (F^j(e))^{\mathrm{T}}(P + \Gamma^j(e))F^j(e)\pi^j(de).\end{cases} \tag{5.2}$$

The above Riccati equation is a BSDE with jumps. Obviously, the generator $G(A_t, B_t, C_t, D_t, E_t, F_t; L_t, R_t, M_t; P, \Lambda, \Gamma)$ is nonlinear in P, Λ and Γ.

Firstly, we give the connection of SREP (5.1) to the stochastic Hamiltonian system (4.2) and to the stochastic LQ problem.

Theorem 5.1. *Let $(x(\cdot), u(\cdot), q(\cdot), r(\cdot), \theta(\cdot,\cdot))$ be the solution of the stochastic Hamiltonian system* (4.2). *If SREP* (5.1) *admits a solution $(P(\cdot), \Lambda(\cdot), \Gamma(\cdot,\cdot))$ such that P is uniformly bounded, and $\Lambda(\cdot)$ and $\Gamma(\cdot,\cdot)$ are square integrable. Then we have*

$$\begin{aligned} &q_t = P_t x_t, \\ &r_t^i = (\Lambda_t^i + P_t C_t^i)x_t + P_t D_t^i u_t, \quad i = 1, \ldots, m, \\ &\theta_t^j(e) = [\Gamma_t^j(e) + (P_t + \Gamma_t^i(e))E_t^j(e)]x_t + [P_t + \Gamma_t^j(e)]F_t^j(e)u_t, \quad j = 1, \ldots, n. \end{aligned} \tag{5.3}$$

Moreover, assume that Problem (LQ SOC) *admits an optimal control. If $\tilde{M}(D, F; R; P, \Gamma)$ is invertible and the inverse matrix process is bounded, then the optimal control u has the following feedback form*:

$$u_t = -\tilde{M}^{-1}(D_t, F_t; M_t; P_t, \Gamma_t)\tilde{R}^{\mathrm{T}}(B_t, C_t, D_t, E_t, F_t; R_t; P_t, \Lambda_t, \Gamma_t)x_t. \tag{5.4}$$

Proof. Applying Itô's formula to $P(\cdot)x(\cdot)$ and comparing it with $q(\cdot)$, we obtain the first conclusion (5.3). By the maximum principle in [14], the optimal control must have the form (4.3). Substituting (5.3) into (4.3), we get the feedback form (5.4). □

It is clear that the solvability of SREP plays a crucial role for the feedback form of the optimal control. However, even under the positive definite condition, there are two difficulties to overcome. One is that the generator $G(A_t, B_t, C_t, D_t, E_t, F_t; L_t, R_t, M_t; P, \Lambda, \Gamma)$ is nonlinear in P, Λ and Γ, which means that we cannot guarantee the existence and uniqueness by the classical theory of BSDEs. The other one is that $\tilde{M}(D_t, F_t; M_t; P_t, \Gamma_t)$ includes two unknown elements $P(\cdot)$ and $\Gamma(\cdot,\cdot)$ as we show in (5.2), so we cannot guarantee that whether $\tilde{M}(D_t, F_t; M_t; P_t, \Gamma_t)$ is positive definite or not. In [13], under the positive definite condition, Tang gave a complete result of the existence and uniqueness of the stochastic Riccati equation only driven by Brownian motions. While, in our paper, the Riccati equation (5.1) is driven not only by Brownian motions but also by Poisson processes. The method introduced by [13] cannot be used directly because it essentially depends on the continuity of the state process $x(\cdot)$, and this point is not satisfied in our paper. By virtue of the method of Peng [8], also under the positive condition, Meng [6] obtained a solvability result of SREP (5.1) in some special case.

Now, we give the following two assumptions adopted in [6].

Assumption 5.2. *Let $n=1$. Suppose there exists a number $m_1\in\mathbb{N}$ with $1\leqslant m_1\leqslant m$ such that the coefficients satisfy*

$$\begin{aligned}
&C=(C^1,\dots,C^m)=(C^1,\dots,C^{m_1},C^{m_1+1},\dots,C^m),\\
&D=(D^1,\dots,D^m)=(O,\dots,O,D^{m_1+1},\dots,D^m),\\
&F=O,
\end{aligned}$$

where O is the zero matrix.

Furthermore, denote $\mathbb{F}^*=\{\mathcal{F}_t^*,t\geqslant 0\}$ to be the natural filtration which is generated by the Brownian motion $(W_t^1,\dots,W_t^{m_1})^{\mathrm{T}}$ and the Poisson random martingale measure $\widetilde{N}(dt,de)=(\widetilde{N}^1(dt,de))$ augmented by all $\mathbb{P}$-null sets.

Assumption 5.3. *Assume that A_t, B_t, C_t, D_t, E_t, L_t, R_t and M_t are $\mathbb{F}^*$-adapted matrix-valued processes, and the random matrix S is $\mathcal{F}_T^*$-measurable.*

Under Assumptions 5.2 and 5.3, the controlled system (2.1) is reduced as follows:

$$\begin{cases}
dx_t=(A_tx_t+B_tv_t)dt+\displaystyle\sum_{i=1}^{m_1}C_t^ix_tdW_t^i+\sum_{i=m_1+1}^{m}(C_t^ix_t+D_t^iv_t)dW_t^i+\int_{\mathcal{E}}E_t(e)x_{t-}\widetilde{N}(dt,de),\\
x_0=a.
\end{cases}\tag{5.5}$$

Due to the restriction on the measurability of coefficients (see Assumption 5.3), some later components of Λ in SREP (5.1) vanish, i.e., $\Lambda_t^{m_1+1}=\cdots=\Lambda_t^m=0$. For more details, we refer to [6,8]. Then the Riccati equation (5.1) is reduced to

$$\begin{cases}
-dP_t=G(A_t,B_t,C_t,D_t,E_t,O;L_t,R_t,M_t;P_t,\Lambda_t,\Gamma_t)dt-\displaystyle\sum_{i=1}^{m_1}\Lambda_t^idW_t^i-\int_{\mathcal{E}}\Gamma_t(e)\widetilde{N}(dt,de),\\
P_T=S,
\end{cases}\tag{5.6}$$

where

$$\begin{cases}
G(A,B,C,D,E,O;L,R,M;P,\Lambda,\Gamma)=\tilde{L}(A,C,E;L;P,\Lambda,\Gamma)-\tilde{R}(B,C,D;R;P)\\
\qquad\qquad\times\tilde{M}^{-1}(D;M;P)\tilde{R}^{\mathrm{T}}(B,C,D;R;P),\\
\tilde{L}(A,C,E;L;P,\Lambda,\Gamma)=L+PA+A^{\mathrm{T}}P+\displaystyle\sum_{i=1}^{m_1}[\Lambda^iC^i+(C^i)^{\mathrm{T}}\Lambda^i+(C^i)^{\mathrm{T}}PC^i]\\
\qquad\qquad+\displaystyle\sum_{i=m_1+1}^{m}(C^i)^{\mathrm{T}}PC^i+\int_{\mathcal{E}}[\Gamma(e)E(e)+(E(e))^{\mathrm{T}}\Gamma(e)\\
\qquad\qquad+(E(e))^{\mathrm{T}}(P+\Gamma(e))E(e)]\pi(de),\\
\tilde{R}(B,C,D;R;P)=R+PB+\displaystyle\sum_{i=m_1+1}^{m}(C^i)^{\mathrm{T}}PD^i,\\
\tilde{M}(D;M;P)=M+\displaystyle\sum_{i=m_1+1}^{m}(D^i)^{\mathrm{T}}PD^i.
\end{cases}$$

We notice that, in the cost functional studied in [6], there exists no cross items involved. In order to apply the solvability result on SREP obtained in [6], we would like to link Problem (LQ SOC) in which a cross item $\langle R_t v_t, x_t\rangle$ is involved with a new one without cross items by an invertible linear transformation:

$$\bar{v}_t = v_t + M_t^{-1} R_t^{\mathrm{T}} x_t. \tag{5.7}$$

Denote

$$\begin{aligned} &\bar{L}_t \triangleq L_t - R_t M_t^{-1} R_t^{\mathrm{T}}, \quad \bar{A}_t \triangleq A_t - B_t M_t^{-1} R_t^{\mathrm{T}}, \\ &\bar{C}_t^i \triangleq C_t^i - D_t^i M_t^{-1} R_t^{\mathrm{T}}, \quad \bar{E}_t(e) \triangleq E_t(e). \end{aligned} \tag{5.8}$$

Then (5.5) is rewritten as

$$\begin{cases} dx_t = (\bar{A}_t x_t + B_t \bar{v}_t)dt + \displaystyle\sum_{i=1}^{m_1} \bar{C}_t^i x_t dW_t^i + \sum_{i=m_1+1}^{m} (\bar{C}_t^i x_t + D_t^i \bar{v}_t) dW_t^i + \int_{\mathcal{E}} \bar{E}_t(e) x_{t-} \widetilde{N}(dt, de), \\ x_0 = a, \end{cases} \tag{5.9}$$

and the cost functional is rewritten as the following form without cross-item:

$$\mathcal{J}(\bar{v}(\cdot)) = \frac{1}{2}\mathrm{E}\int_0^T [\langle \bar{L}_t x_t, x_t\rangle + \langle M_t \bar{v}_t, \bar{v}_t\rangle] dt + \frac{1}{2}\mathrm{E}\langle S x_T, x_T\rangle. \tag{5.10}$$

Moreover, corresponding to the new problems (5.9) and (5.10), we have the following Riccati equation:

$$\begin{cases} -dP_t = G(\bar{A}_t, B_t, \bar{C}_t, D_t, \bar{E}_t, O; \bar{L}, O, M_t; P_t, \Lambda_t, \Gamma_t)dt - \displaystyle\sum_{i=1}^{m_1} \Lambda_t^i dW_t^i - \int_{\mathcal{E}} \Gamma_t(e) \widetilde{N}(dt, de), \\ P_T = S. \end{cases} \tag{5.11}$$

By a straightforward calculation, it is verified that

$$G(\bar{A}, B, \bar{C}, D, \bar{E}, O; \bar{L}, O, M; P, \Lambda, \Gamma) = G(A, B, C, D, E, O; L, R, M; P, \Lambda, \Gamma),$$

i.e., the SREP (5.6) and the SREP (5.11) are the same.

In [6, Theorem 5.3], under the positive definite condition, Meng proved that the SREP (5.11) (SREP (5.6), equivalently) admits a unique solution $(P(\cdot), \Lambda(\cdot), \Gamma(\cdot,\cdot))$ in the space $L_{\mathbb{F}}^{\infty}(0,T;\mathbb{S}^d) \times L_{\mathbb{F}}^2(0,T;(\mathbb{S}^d)^{m_1}) \times M_{\mathbb{F}}^2(0,T;\mathbb{S}^d)$. Moreover, $P(\cdot)$ is non-negative. Combining with Theorem 5.1, we obtain a relatively complete result for Problem (LQ SOC) under the positive definite condition, which is summed up in the following theorem.

Theorem 5.4. *Let Assumptions* 2.1, 5.2 *and* 5.3 *hold. The SREP* (5.6) *admits a unique solution* $(P(\cdot), \Lambda(\cdot), \Gamma(\cdot,\cdot)) \in L_{\mathbb{F}}^{\infty}(0,T;\mathbb{S}^d) \times L_{\mathbb{F}}^2(0,T;(\mathbb{S}^d)^{m_1}) \times M_{\mathbb{F}}^2(0,T;\mathbb{S}^d)$*. Moreover,* $P(\cdot)$ *is non-negative for a.e. a.s.* $(t,\omega) \in [0,T]\times\Omega$*. In addition, the LQ problems* (2.2) *and* (5.5) *admit a unique optimal pair* $(x(\cdot), u(\cdot))$ *determined by*

$$\begin{cases} u_t = -\left[M_t + \displaystyle\sum_{i=m_1+1}^{m} (D_t^i)^{\mathrm{T}} P_t D_t^i\right]^{-1} \left[R_t + P_t B_t + \sum_{i=m_1+1}^{m} (C_t^i)^{\mathrm{T}} P_t D_t^i\right]^{\mathrm{T}} x_t, \\ dx_t = (A_t x_t + B_t u_t)dt + \displaystyle\sum_{i=1}^{m_1} C_t^i x_t dW_t^i + \sum_{i=m_1+1}^{m} (C_t^i x_t + D_t^i u_t) dW_t^i + \int_{\mathcal{E}} E_t(e) x_{t-} \widetilde{N}(dt, de), \\ x_0 = a. \end{cases} \tag{5.12}$$

Now, we are in the position to give the corresponding result for the indefinite case with the help of relax compensators.

Theorem 5.5. *Let Assumptions* 5.2 *and* 5.3 *hold. If there exists a relax compensator* $K \in \Upsilon$*, then the SREP* (5.6) *admits a unique solution* $(P(\cdot), \Lambda(\cdot), \Gamma(\cdot,\cdot)) \in L_{\mathbb{F}}^{\infty}(0,T;\mathbb{S}_+^d) \times L_{\mathbb{F}}^2(0,T;(\mathbb{S}^d)^{m_1}) \times M_{\mathbb{F}}^2(0,T;\mathbb{S}^d)$*. Moreover,*

$$P(\cdot) \geqslant K(\cdot), \quad \textit{for a.e. a.s.} \quad (t,\omega) \in [0,T]\times\Omega. \tag{5.13}$$

In addition, the LQ problems (2.2) *and* (5.5) *admit a unique optimal pair* $(x(\cdot), u(\cdot))$ *determined by*

$$\begin{cases} u_t = -\left[M_t + \sum_{i=m_1+1}^{m} (D_t^i)^{\mathrm{T}} P_t D_t^i\right]^{-1} \left[R_t + P_t B_t + \sum_{i=m_1+1}^{m} (C_t^i)^{\mathrm{T}} P_t D_t^i\right]^{\mathrm{T}} x_t, \\ dx_t = (A_t x_t + B_t v_t)dt + \sum_{i=1}^{m_1} C_t^i x_t dW_t^i + \sum_{i=m_1+1}^{m} (C_t^i x_t + D_t^i v_t) dW_t^i + \int_{\mathcal{E}} E_t(e) x_{t-} \widetilde{N}(dt, de), \\ x_0 = a. \end{cases} \tag{5.14}$$

Proof. We first prove the equivalence between the existence and uniqueness of SREP (5.6) and that of the following SREP associated with the relax compensator K:

$$\begin{cases} -dP_t^K = G(A_t, B_t, C_t, D_t, E_t, O; \mathcal{L}_t(K), \mathcal{R}_t(K), \mathcal{M}_t(K); P_t^K, \Lambda_t^K, \Gamma_t^K)dt - \sum_{i=1}^{m_1} \Lambda_t^{K,i} dW_t^i \\ \qquad\qquad - \int_{\mathcal{E}} \Gamma_t^K(e) \widetilde{N}(dt, de), \\ P_T^K = \mathcal{S}(K). \end{cases} \tag{5.15}$$

In fact, it is verified that the solution $(P(\cdot), \Lambda(\cdot), \Gamma(\cdot,\cdot))$ of SREP (5.6) and $(P^K(\cdot), \Lambda^K(\cdot), \Gamma^K(\cdot,\cdot))$ of SREP (5.15) have the following relationship:

$$P_t = P_t^K + K_t, \quad \Lambda_t = \Lambda_t^K + \Phi_t, \quad \Gamma_t = \Gamma_t^K + \Psi_t, \quad t \in [0, T].$$

Since $K \in \Upsilon$ is a relax compensator, $(\mathcal{L}(K), \mathcal{R}(K), \mathcal{M}(K), \mathcal{S}(K))$ satisfies Assumption 2.1. By Theorem 5.4, the SREP (5.15) admits a unique solution $(P^K(\cdot), \Lambda^K(\cdot), \Gamma^K(\cdot,\cdot))$ and $P^K(\cdot)$ is uniformly bounded and non-negative for a.e. a.s. $(t,\omega) \in [0,T] \times \Omega$. Then, equivalently, the SREP (5.6) has the unique solution $(P(\cdot), \Lambda(\cdot), \Gamma(\cdot,\cdot))$. Moreover, from $P_t^K = P_t - K_t \geqslant 0$, we get

$$P_t \geqslant K_t. \tag{5.16}$$

Furthermore, by a direct calculation, we have

$$M + \sum_{i=m_1+1}^{m} (D^i)^{\mathrm{T}} P D^i = \mathcal{M}(K) + \sum_{i=m_1+1}^{m} (D^i)^{\mathrm{T}} P^K D^i. \tag{5.17}$$

Therefore, $M + \sum_{i=m_1+1}^{m} (D^i)^{\mathrm{T}} P D^i$ is invertible and the inverse matrix is bounded. Theorem 5.1 works again to complete the proof. □

Remark 5.6. If all of the coefficients in (5.5) are deterministic, then the Riccati equation (5.6) is reduced to an ordinary differential equation with the vanishing Λ and Γ. If the corresponding deterministic Riccati equation admits a solution P such that $\tilde{M}(D, F; M; P, O)$ is positive and the inverse matrix $\tilde{M}^{-1}$ is bounded, then $\mathcal{S}(P) = 0$, $\mathcal{M}(P) = \tilde{M}(D, F; M; P, O)$, $\mathcal{R}(P) = \tilde{R}(B, C, D, E, F; R; P, O, O)$ and $\mathcal{L}(P) = \mathcal{R}(P)(\mathcal{M}(P))^{-1}(\mathcal{R}(P))^{\mathrm{T}}$ which satisfy Assumption 2.1. So, in this case, we could regard the solution of Riccati equation as a special relax compensator.

6 Conclusion

In this paper, we discuss the LQ problem of the stochastic system with jumps under the indefinite case. The relax compensators are the major elements introduced by the wellposedness of the LQ problem, which extend the condition from the positive definite case to the indefinite case for the corresponding stochastic Hamiltonian system and Riccati equation with jumps. Firstly, we construct the LQ problem with jumps under the indefinite condition by relax compensators. Secondly, we study the corresponding Hamiltonian system under the positive definite condition, and then we extend to the indefinite condition. Moreover, we also give a new case of solvability for FBSDEP. Thirdly, we give the existence and uniqueness of stochastic Riccati equation with jumps under the indefinite condition in some special case. The relax compensators play a crucial role to deal with all of the problems under the indefinite case in this paper.

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