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A boundary Schwarz lemma on the classical domain of type $\mathcal{I}$

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Abstract Let $\mathcal{R}_\mathcal{I}(m,n)$ be the classical domain of type $\mathcal{I}$ in $\mathbb{C}^{m\times n}$ with $1 \leqslant m \leqslant n$. We obtain the optimal estimates of the eigenvalues of the Fréchet derivative $Df(\mathring{Z})$ at a smooth boundary fixed point $\mathring{Z}$ of $\mathcal{R}_\mathcal{I}(m,n)$ for a holomorphic self-mapping f of $\mathcal{R}_\mathcal{I}(m,n)$. We provide a necessary and sufficient condition such that the boundary points of $\mathcal{R}_\mathcal{I}(m,n)$ are smooth, and give some properties of the smooth boundary points of $\mathcal{R}_\mathcal{I}(m,n)$. Our results extend the classical Schwarz lemma at the boundary of the unit disk Δ to $\mathcal{R}_\mathcal{I}(m,n)$, which may be applied to get some optimal estimates in several complex variables.

Keywords holomorphic mapping, Schwarz lemma at the boundary, the classical domain of type $\mathcal{I}$

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1 Introduction

The Schwarz lemma is one of the most influential results in the classical complex analysis, which is widely applied in many branches of mathematical research such as geometric function theory, hyperbolic geometry, complex dynamical systems and theory of quasi-conformal mappings. We refer to [1, 4, 5, 11] for a more complete insight on the Schwarz lemma.

Cartan first studied the Schwarz lemma in several complex variables in [3]. In fact, this is the well-known rigidity theorem of Cartan [3].

Theorem 1.1 (See [3])**.** *Suppose that Ω is a bounded domain in $\mathbb{C}^n$ and $f : \Omega \to \Omega$ is a holomorphic mapping such that $f(z) = z + o(\|z - z_0\|)$ as $z \to z_0$ for some $z_0 \in \Omega$. Then $f(z) \equiv z$.*

Look discussed Schwarz lemma of several complex variables in [17], which extends the result of Cartan [3]. This result is stated as follows.

Theorem 1.2 (See [17])**.** *Let Ω be a bounded domain in $\mathbb{C}^n$, and let f be a holomorphic self-mapping of Ω which fixes a point $p \in \Omega$. Then*

(1) *the eigenvalues of $J_f(p)$ all have modulus not exceeding* 1;
(2) $|\det J_f(p)| \leqslant 1$;
(3) *if $|\det J_f(p)| = 1$, then f is a biholomorphism of Ω.*

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Naturally, higher-dimensional Schwarz lemma at the boundary has attracted the attention of many mathematicians as well. From Theorem 1.1, Burns and Krantz [2] first considered a new rigidity problem of holomorphic mappings at the boundary points. They obtained a rigidity theorem for holomorphic mappings on the bounded strongly pseudoconvex domain in $\mathbb{C}^n$. Huang strengthened the Burns-Krantz result for holomorphic mappings with an interior fixed point in [9]. See [8, 10] for more on these matters. Here is one of typical results in these papers.

Theorem 1.3 (See [9]). *Let $\Omega \subset\subset \mathbb{C}^n$ $(n > 1)$ be a simply connected pseudoconvex domain with C^∞ boundary. Suppose that $p \in \partial\Omega$ is a strongly pseudoconvex point. If $f : \Omega \to \Omega$ is a holomorphic mapping such that $f(z_0) = z_0$ for some $z_0 \in \Omega$ and $f(z) = z + o(\|z - p\|^2)$ as $z \to p$, then $f(z) \equiv z$.*

It is natural to consider that what are the multidimensional generalizations of the boundary Schwarz lemma corresponding to Theorem 1.2. It is this problem which motivated our study in this paper. Recently, we first established the Schwarz lemma at the boundary of the unit ball in $\mathbb{C}^n$, and gave some applications in the geometric function theory of several complex variables in [16]. We discussed the same problem on the unit polydisk and the strongly pseudoconvex domain, respectively in [15, 18]. Our main purpose here is to study the boundary Schwarz lemma on the classical domain of type $\mathcal{I}$. We also study the properties of the smooth boundary points of the classical domain of type $\mathcal{I}$.

The rest of the article is organized as follows. In Section 2, we give some characterizations of the smooth boundary points of $\mathcal{R}_{\mathcal{I}}(m,n)$, which will be used in the subsequent section. In Section 3, we establish the Schwarz lemma at the boundary on the classical domain of type $\mathcal{I}$, i.e., we obtain the optimal estimates of the eigenvalues of the Fréchet derivative of holomorphic self-mappings of $\mathcal{R}_{\mathcal{I}}(m,n)$ at a smooth boundary fixed point of $\mathcal{R}_{\mathcal{I}}(m,n)$. In Section 4, we present some conclusions of the paper.

2 Preliminaries

2.1 Notation

Let $\mathbb{C}^{m\times n}$ be the set of all $m \times n$ complex matrices with $1 \leqslant m \leqslant n$. For any $Z, W \in \mathbb{C}^{m\times n}$, the inner product and the corresponding norm are given by

$$\langle Z, W\rangle = \sum_{i=1}^{m}\sum_{j=1}^{n} z_{ij}\overline{w}_{ij}, \quad \|Z\| = \langle Z, Z\rangle^{\frac{1}{2}},$$

where $Z = (z_{ij})_{m\times n}$ and $W = (w_{ij})_{m\times n}$. It is well known that as real vectors in $\mathbb{R}^{2mn}$, Z and W are orthogonal if and only if $\Re\langle Z, W\rangle = 0$. Throughout this paper, denote by Z' and $\overline{Z}$, respectively, the transpose and the complex conjugate of Z.

Let A be a square matrix of order m and let B be a square matrix of order n. Then for each $Z, W \in \mathbb{C}^{m\times n}$,

$$\begin{aligned}\langle AZ, W\rangle &= \sum_{k=1}^{m}\sum_{l=1}^{n}(AZ)_{kl}\overline{w}_{kl} = \sum_{k=1}^{m}\sum_{l=1}^{n}\bigg(\sum_{i=1}^{m} a_{ki}z_{il}\bigg)\overline{w}_{kl}\\ &= \sum_{i=1}^{m}\sum_{l=1}^{n} z_{il}\overline{\bigg(\sum_{k=1}^{m}\overline{a}_{ki}w_{kl}\bigg)} = \sum_{i=1}^{m}\sum_{l=1}^{n} z_{il}\overline{(\overline{A}'W)_{il}} = \langle Z, \overline{A}'W\rangle,\end{aligned}$$

where $A = (a_{ij})_{m\times m}$. Similarly, we have

$$\langle ZB, W\rangle = \langle Z, W\overline{B}'\rangle, \quad \langle Z, AW\rangle = \langle \overline{A}'Z, W\rangle, \quad \langle Z, WB\rangle = \langle Z\overline{B}', W\rangle.$$

Moreover, if U is a unitary square matrix of order m and V is a unitary square matrix of order n, then

$$\langle UZV, UWV\rangle = \langle Z, W\rangle.$$

The classical domain of type $\mathcal{I}$, denoted by $\mathcal{R}_{\mathcal{I}}(m,n)$, is defined as

$$\mathcal{R}_{\mathcal{I}}(m,n) = \{Z \in \mathbb{C}^{m\times n} : I_m - Z\overline{Z}' > 0\},$$

where I_m is the unit square matrix of order m, and the inequality sign means that the left-hand side is positive definite. It is easy to check that $\mathcal{R}_\mathcal{I}(m,n)$ is a bounded convex circular domain in $\mathbb{C}^{m\times n}$.

Let $\partial\mathcal{R}_\mathcal{I}(m,n)$ be the boundary of $\mathcal{R}_\mathcal{I}(m,n)$, and write $\mathbb{C}^{1\times m}$ as $\mathbb{C}^m$. Let $B^m\subset\mathbb{C}^m$ be the open unit ball under the Euclidean metric. The Minkowski functional $\rho(Z)$ of $\mathcal{R}_\mathcal{I}(m,n)$ is defined by

$$\rho(Z)=\max\{\|\alpha Z\|:\alpha\in\partial B^m\},\quad Z\in\mathbb{C}^{m\times n},$$

where ∂B^m is the boundary of B^m. By [7], we know that $\rho(Z)$ is a Banach norm of $\mathbb{C}^{m\times n}$, $(\rho(Z))^2$ is the largest eigenvalue of $Z\overline{Z}'$, and

$$\mathcal{R}_\mathcal{I}(m,n)=\{Z\in\mathbb{C}^{m\times n}:\rho(Z)<1\}.$$

In particular, $\mathcal{R}_\mathcal{I}(1,n)$ is just the open unit ball B^n for which the Minkowski functional is $\rho(Z)=\|Z\|$. For the unitary square matrix U of order m and the unitary square matrix V of order n, it is easy to see that

$$Z\in\mathcal{R}_\mathcal{I}(m,n)\Leftrightarrow UZV\in\mathcal{R}_\mathcal{I}(m,n)\quad\text{and}\quad Z\in\partial\mathcal{R}_\mathcal{I}(m,n)\Leftrightarrow UZV\in\partial\mathcal{R}_\mathcal{I}(m,n).$$

Since $Z\in\mathcal{R}_\mathcal{I}(m,n)$ means that the elements in the principal diagonal of $I_m-Z\overline{Z}'$ are positive, we have $|z_{ij}|<1$ for $i=1,\dots,m$ and $j=1,\dots,n$. We also obtain $\rho(UZV)=\rho(Z)$ for each $Z\in\mathbb{C}^{m\times n}$.

Assume that $f:\mathcal{R}_\mathcal{I}(m,n)\to\mathbb{C}^{m\times n}$ is a holomorphic mapping. The Fréchet derivative of f at $a\in\mathcal{R}_\mathcal{I}(m,n)$ is defined as

$$(Df(a)(W))_{ij}=\sum_{s=1}^{m}\sum_{t=1}^{n}\frac{\partial f_{ij}}{\partial z_{st}}(a)w_{st},\quad W\in\mathbb{C}^{m\times n}.$$

It is easy to see that $Df(a)$ is a linear transformation from $\mathbb{C}^{m\times n}$ to $\mathbb{C}^{m\times n}$ and $df(Z)\,|_{Z=a}=Df(a)(dZ)$. The adjoint transformation of $Df(a)$ is denoted by $D^*f(a)$, i.e.,

$$\langle D^*f(a)(Z),W\rangle=\langle Z,Df(a)(W)\rangle,\quad Z\in\mathbb{C}^{m\times n},\quad W\in\mathbb{C}^{m\times n}.$$

Obviously, $D^*f(a)$ is also a linear transformation from $\mathbb{C}^{m\times n}$ to $\mathbb{C}^{m\times n}$. Moreover,

$$(D^*f(a)(Z))_{ij}=\sum_{s=1}^{m}\sum_{t=1}^{n}\frac{\partial\overline{f}_{st}}{\partial\overline{z}_{ij}}(a)z_{st}.$$

In fact, let $e_{ij}\in\mathbb{C}^{m\times n}$ be a matrix which has 1 at i-th row and j-th column, and 0's elsewhere. Then

$$(D^*f(a)(Z))_{ij}=\langle D^*f(a)(Z),e_{ij}\rangle=\langle Z,Df(a)(e_{ij})\rangle=\left\langle Z,\frac{\partial f}{\partial z_{ij}}(a)\right\rangle=\sum_{s=1}^{m}\sum_{t=1}^{n}\frac{\partial\overline{f}_{st}}{\partial\overline{z}_{ij}}(a)z_{st}.$$

It is clear that λ is an eigenvalue of $Df(a)$ if and only if $\overline{\lambda}$ is an eigenvalue of $D^*f(a)$.

2.2 Smooth boundary points of $\mathcal{R}_\mathcal{I}(m,n)$

For $\mathring{Z}\in\mathbb{C}^{m\times n}$, by [13] we know that $\mathring{Z}$ has the following polar decomposition, i.e.,

$$\mathring{Z}=U\begin{pmatrix} r_1 & 0 & \cdots & 0 & 0 & \cdots & 0\\ 0 & r_2 & \cdots & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & r_m & 0 & \cdots & 0\end{pmatrix}V,$$

where $r_1\geqslant r_2\geqslant\cdots\geqslant r_m\geqslant0$, U is a unitary square matrix of order m and V is a unitary square matrix of order n. We give the properties of the smooth boundary points of $\mathcal{R}_\mathcal{I}(m,n)$.

Proposition 2.1. *Suppose that $\mathring{Z} \in \mathbb{C}^{m\times n}$ has the polar decomposition above. Then $\mathring{Z}$ is a smooth boundary point of $\mathcal{R}_{\mathcal{I}}(m,n)$ if and only if $1 = r_1 > r_2 \geqslant \cdots \geqslant r_m \geqslant 0$. Furthermore, $\rho(Z)$ is holomorphic about Z and $\overline{Z}$ near $\mathring{Z}$, and the gradient of ρ at $\mathring{Z}$, i.e.,*

$$\nabla\rho(\mathring{Z}) = U \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix} V$$

is the unit outward normal vector to $\partial\mathcal{R}_{\mathcal{I}}(m,n)$ at $\mathring{Z}$ with $\langle \mathring{Z}, \nabla\rho(\mathring{Z})\rangle = 1$.

Proof. It is easy to verify that $\mathring{Z} \in \partial\mathcal{R}_{\mathcal{I}}(m,n)$ if and only if $r_1 = 1$. Suppose that $1 = r_1 > r_2 \geqslant \cdots \geqslant r_m \geqslant 0$. Assume that the characteristic polynomial of $Z\overline{Z}'$ is

$$\Phi(x,Z) = \det(xI_m - Z\overline{Z}') = x^m - tr(Z\overline{Z}')x^{m-1} + \cdots + (-1)^m \det(Z\overline{Z}').$$

Then $(\rho(Z))^2$ is a root of $\Phi(x,Z)$ near $\mathring{Z}$. Notice that

$$\Phi(1,\mathring{Z}) = 0, \quad \frac{\partial\Phi}{\partial x}(1,\mathring{Z}) = [(x-1)(x-r_2^2)\cdots(x-r_m^2)]'\,|_{x=1} = \prod_{k=2}^{m}(1-r_k^2) > 0. \tag{2.1}$$

It follows from the implicit function existence theorem that $(\rho(Z))^2$ is holomorphic about Z and $\overline{Z}$ near $\mathring{Z}$, and satisfies $(\rho(\mathring{Z}))^2 = 1$. Hence, $\rho(Z)$ is also holomorphic about Z and $\overline{Z}$ near $\mathring{Z}$. Now, we compute $\nabla\rho(\mathring{Z})$. Since $\Phi((\rho(Z))^2, Z) = \det((\rho(Z))^2 I_m - Z\overline{Z}') \equiv 0$ near $\mathring{Z}$, we have

$$\frac{\partial\Phi}{\partial x}(1,\mathring{Z})2\rho(\mathring{Z})\frac{\partial\rho}{\partial\overline{z}_{ij}}(\mathring{Z}) + \frac{\partial\Phi}{\partial\overline{z}_{ij}}(1,\mathring{Z}) = 0.$$

This, together with (2.1), implies

$$2\prod_{k=2}^{m}(1-r_k^2)\frac{\partial\rho}{\partial\overline{z}_{ij}}(\mathring{Z}) + \frac{\partial\Phi}{\partial\overline{z}_{ij}}(1,\mathring{Z}) = 0. \tag{2.2}$$

On the other hand,

$$\Phi(x,Z)\,|_{(x,Z)=(1,\mathring{Z})} = \det(xI_m - Z\overline{Z}')\,|_{(x,Z)=(1,\mathring{Z})} = \det(I_m - \overline{U}'Z\overline{Z}'U)\,|_{Z=\mathring{Z}},$$

$$I_m - \overline{U}'\mathring{Z}\overline{\mathring{Z}}'U = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1-r_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1-r_m^2 \end{pmatrix}$$

and the algebraic cofactor of the element at s-th row and t-th column for $\det(I_m - \overline{U}'\mathring{Z}\overline{\mathring{Z}}'U)$ is

$$J_{st} = \begin{cases} \prod_{k=2}^{m}(1-r_k^2), & (s,t) = (1,1), \\ 0, & (s,t) \neq (1,1). \end{cases}$$

Thus, we obtain

$$\begin{aligned} \frac{\partial\Phi}{\partial\overline{z}_{ij}}(1,\mathring{Z}) &= -\sum_{s,t=1}^{m}\frac{\partial}{\partial\overline{z}_{ij}}(\overline{U}'Z\overline{Z}'U)_{st}\,|_{Z=\mathring{Z}}J_{st} \\ &= -\prod_{k=2}^{m}(1-r_k^2)\frac{\partial}{\partial\overline{z}_{ij}}(\overline{U}'Z\overline{Z}'U)_{11}\,|_{Z=\mathring{Z}} \end{aligned}$$

$$
\begin{aligned}
&= -\prod_{k=2}^{m}(1-r_k^2)\frac{\partial}{\partial \overline{z}_{ij}}\Big(\sum_{l=1}^{m}\sum_{s=1}^{n}\sum_{t=1}^{m}\overline{u}_{l1}z_{ls}\overline{z}_{ts}u_{t1}\Big)\Big|_{Z=\mathring{Z}}\\
&= -\prod_{k=2}^{m}(1-r_k^2)\sum_{l=1}^{m}\overline{u}_{l1}\mathring{z}_{lj}u_{i1}\\
&= -\prod_{k=2}^{m}(1-r_k^2)\sum_{l=1}^{m}\overline{u}_{l1}u_{i1}\Big(\sum_{s=1}^{m}u_{ls}r_s v_{sj}\Big)\\
&= -\prod_{k=2}^{m}(1-r_k^2)u_{i1}v_{1j}\\
&= -\prod_{k=2}^{m}(1-r_k^2)\left(U\begin{pmatrix}1 & 0 & \cdots & 0 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0\end{pmatrix}V\right)_{ij},
\end{aligned}
$$

where $U=(u_{ij})_{m\times m}$, $V=(v_{ij})_{n\times n}$ and $\mathring{Z}=(\mathring{z}_{ij})_{m\times n}$. It follows from this and (2.2) that

$$
(\nabla\rho(\mathring{Z}))_{ij}=2\frac{\partial\rho}{\partial\overline{z}_{ij}}(\mathring{Z})=\left(U\begin{pmatrix}1 & 0 & \cdots & 0 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0\end{pmatrix}V\right)_{ij}.
$$

That means

$$
\nabla\rho(\mathring{Z})=U\begin{pmatrix}1 & 0 & \cdots & 0 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0\end{pmatrix}V\neq 0.
$$

Hence, $\partial\mathcal{R}_{\mathcal{I}}(m,n)$ is smooth near $\mathring{Z}$. Moreover, since $\langle UZV,UWV\rangle=\langle Z,W\rangle$ for any $Z,W\in\mathbb{C}^{m\times n}$, we get $\langle\mathring{Z},\nabla\rho(\mathring{Z})\rangle=1$.

Conversely, suppose that $\mathring{Z}$ is a smooth boundary point of $\mathcal{R}_{\mathcal{I}}(m,n)$. Assume

$$
1=r_1=r_2\geqslant\cdots\geqslant r_m\geqslant 0.
$$

Then any two nonzero outward normal vectors to $\partial\mathcal{R}_{\mathcal{I}}(m,n)$ at $\mathring{Z}$ have the same direction. Now, we consider the following two different $(2mn-1)$-dimensional real affine spaces through $\mathring{Z}$ in $\mathbb{C}^{m\times n}$:

$$
\Sigma_1=\{\mathring{Z}+U\alpha V:\alpha\in\mathbb{C}^{m\times n},\ \Re\alpha_{11}=0\}\quad\text{and}\quad\Sigma_2=\{\mathring{Z}+U\alpha V:\alpha\in\mathbb{C}^{m\times n},\ \Re\alpha_{22}=0\}.
$$

To simplify our notation, set

$$
T_1=\begin{pmatrix}1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0\\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0\\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0\end{pmatrix}\quad\text{and}\quad T_2=\begin{pmatrix}0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0\\ 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0\\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0\end{pmatrix}.
$$

For any $\mathring{Z}+U\alpha V\in\Sigma_1$, we have $\Re\langle U\alpha V,UT_1V\rangle=\Re\alpha_{11}=0$. This shows that UT_1V is a normal vector to Σ_1 at $\mathring{Z}$ (see Figure 1).

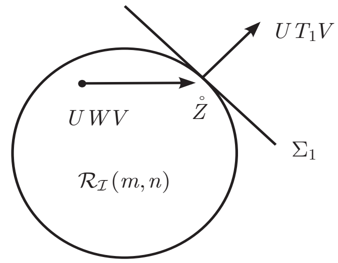

Figure 1 The affine tangent space and normal vector to $\partial\mathcal{R}_{\mathcal{I}}(m,n)$ at $\mathring{Z}$

Similarly, UT_2V is also a normal vector to Σ_2 at $\mathring{Z}$. On the other hand, for each $UWV \in \mathcal{R}_{\mathcal{I}}(m,n)$ we obtain

$$\begin{aligned}\Re\langle \mathring{Z} - UWV, UT_1V\rangle &= \Re\left\langle U\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0\\ 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0\\ 0 & 0 & r_3 & \cdots & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & r_m & 0 & \cdots & 0\end{pmatrix}V - UWV, UT_1V\right\rangle\\ &= 1 - \Re w_{11} > 0.\end{aligned}$$

This implies that $\mathcal{R}_{\mathcal{I}}(m,n)$ is located on one side of Σ_1, i.e., Σ_1 is an affine tangent space to $\partial\mathcal{R}_{\mathcal{I}}(m,n)$ at $\mathring{Z}$. Similar to the above proof, we know that Σ_2 is an affine tangent space to $\partial\mathcal{R}_{\mathcal{I}}(m,n)$ at $\mathring{Z}$ as well. Because $\mathring{Z}$ is a smooth boundary point of $\mathcal{R}_{\mathcal{I}}(m,n)$, this contradicts with $\Sigma_1 \neq \Sigma_2$. Therefore, we have $1 = r_1 > r_2 \geqslant \cdots \geqslant r_m \geqslant 0$. The proof is complete. □

2.3 Some lemmas

Let Δ be the open unit disk in the complex plane $\mathbb{C}$. The following lemma is the classical Schwarz lemma at the boundary.

Lemma 2.2 (See [5]). *Let $f : \Delta \to \Delta$ be a holomorphic function. If f is holomorphic at $z = 1$ with $f(0) = 0$ and $f(1) = 1$, then $f'(1) \geqslant 1$. Moreover, the inequality is sharp.*

If the condition $f(0) = 0$ is removed, then one has the following estimate instead:

$$f'(1) \geqslant \frac{|1-\overline{f(0)}|^2}{1-|f(0)|^2} > 0 \tag{2.3}$$

by applying Lemma 2.2 to $g(z) = \frac{1-\overline{f(0)}}{1-f(0)} \frac{f(z)-f(0)}{1-\overline{f(0)}f(z)}$.

Lemma 2.3 (See [13]). *Let*

$$a = A\begin{pmatrix} l_1 & 0 & \cdots & 0 & 0 & \cdots & 0\\ 0 & l_2 & \cdots & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & l_m & 0 & \cdots & 0\end{pmatrix} B \in \mathcal{R}_{\mathcal{I}}(m,n).$$

Write

$$Q = A\begin{pmatrix} \frac{1}{\sqrt{1-l_1^2}} & & 0\\ & \ddots & \\ 0 & & \frac{1}{\sqrt{1-l_m^2}}\end{pmatrix}\overline{A}', \quad R = \overline{B}'\begin{pmatrix} \frac{1}{\sqrt{1-l_1^2}} & & & 0\\ & \ddots & & \\ & & \frac{1}{\sqrt{1-l_m^2}} & \\ 0 & & & I_{n-m}\end{pmatrix}B.$$

Here, $1 > l_1 \geqslant \cdots \geqslant l_m \geqslant 0$, A is a unitary square matrix of order m and B is a unitary square matrix of order n. Then for any $Z \in \overline{\mathcal{R}_{\mathcal{I}}(m,n)}$,

$$\varphi_a(Z) = Q^{-1}(I_m - Z\overline{a}')^{-1}(a - Z)R$$

is a holomorphic automorphism of $\mathcal{R}_{\mathcal{I}}(m,n)$, which interchanges 0 and a. Moreover, φ_a is biholomorphic in a neighborhood of $\overline{\mathcal{R}_{\mathcal{I}}(m,n)}$, and

$$\varphi_a^{-1} = \varphi_a, \quad d\varphi_a(Z)\,|_{Z=a} = -QdZR, \quad d\varphi_a(Z)\,|_{Z=0} = -Q^{-1}dZR^{-1}.$$

From now on, we always denote by $F(Z,\xi)$ the infinitesimal form of Carathéodory metric or Kobayashi metric on $\mathcal{R}_{\mathcal{I}}(m,n)$, where $Z \in \mathcal{R}_{\mathcal{I}}(m,n)$ and $\xi \in \mathbb{C}^{m\times n}$ (see [12] for details).

Corollary 2.4. *Let $\rho(Z)$ be the Minkowski functional of $\mathcal{R}_{\mathcal{I}}(m,n)$. Under the notation of Lemma 2.3, for any $\xi \in \mathbb{C}^{m\times n}$,*

$$F(a,\xi) = \rho(Q\xi R).$$

Proof. Since $\varphi_a(Z)$ is a holomorphic automorphism of $\mathcal{R}_{\mathcal{I}}(m,n)$ and $F(Z,\xi)$ is a biholomorphically invariant metric on $\mathcal{R}_{\mathcal{I}}(m,n)$, we get

$$F(a,\xi) = F(0, D\varphi_a(a)(\xi)).$$

This, together with $D\varphi_a(a)(dZ) = d\varphi_a(Z)\,|_{Z=a}$ and Lemma 2.3, yields

$$F(a,\xi) = F(0, D\varphi_a(a)(\xi)) = F(0, -Q\xi R) = F(0, Q\xi R).$$

Thus, by [6, Lemma 3.2], we have $F(a,\xi) = F(0,Q\xi R) = \rho(Q\xi R)$. The proof is complete. □

Lemma 2.5. *Let $\mathring{Z}$ be a smooth boundary point of $\mathcal{R}_{\mathcal{I}}(m,n)$. Then*

$$|\langle W, \nabla\rho(\mathring{Z})\rangle| \leqslant \rho(W)$$

for any $W \in \mathbb{C}^{m\times n}$.

Proof. Without loss of generality, we may assume $W \neq 0$. Then $\frac{W}{\rho(W)} \in \partial\mathcal{R}_{\mathcal{I}}(m,n)$. Notice that $\mathcal{R}_{\mathcal{I}}(m,n)$ is a bounded convex circular domain. Then for each $\theta \in \mathbb{R}$, we obtain

$$\Re\left\langle \mathring{Z} - \mathrm{e}^{\mathrm{i}\theta}\frac{W}{\rho(W)}, \nabla\rho(\mathring{Z})\right\rangle \geqslant 0.$$

It follows from this and Proposition 2.1 that

$$\Re\frac{\mathrm{e}^{\mathrm{i}\theta}}{\rho(W)}\langle W, \nabla\rho(\mathring{Z})\rangle \leqslant \Re\langle \mathring{Z}, \nabla\rho(\mathring{Z})\rangle = 1.$$

This implies $|\langle W, \nabla\rho(\mathring{Z})\rangle| \leqslant \rho(W)$. The proof is complete. □

Lemma 2.6 (See [14])**.** *Let $f : \mathcal{R}_{\mathcal{I}}(m,n) \to \mathcal{R}_{\mathcal{I}}(m,n)$ be a holomorphic mapping and let $f(0) = 0$. Then for any $Z \in \mathcal{R}_{\mathcal{I}}(m,n)$,*

$$\rho(f(Z)) \leqslant \rho(Z).$$

3 Main result

Now, we establish the Schwarz lemma at the smooth boundary points for holomorphic self-mappings of $\mathcal{R}_{\mathcal{I}}(m,n)$.

Let

$$\mathring{Z} = U\begin{pmatrix} r_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_m & 0 & \cdots & 0 \end{pmatrix} V$$

be a smooth boundary point of $\mathcal{R}_{\mathcal{I}}(m,n)$, where U is a unitary square matrix of order m, V is a unitary square matrix of order n and $1=r_1>r_2\geqslant\cdots\geqslant r_m\geqslant 0$. Then Proposition 2.1 implies that the tangent space $T_{\mathring{Z}}(\partial\mathcal{R}_{\mathcal{I}}(m,n))$ to $\partial\mathcal{R}_{\mathcal{I}}(m,n)$ at $\mathring{Z}$ is

$$T_{\mathring{Z}}(\partial\mathcal{R}_{\mathcal{I}}(m,n))=\{\beta\in\mathbb{C}^{m\times n}:\Re\langle\beta,\nabla\rho(\mathring{Z})\rangle=0\}=\{U\alpha V:\alpha\in\mathbb{C}^{m\times n},\Re\alpha_{11}=0\},$$

and the holomorphic tangent space $T^{1,0}_{\mathring{Z}}(\partial\mathcal{R}_{\mathcal{I}}(m,n))$ to $\partial\mathcal{R}_{\mathcal{I}}(m,n)$ at $\mathring{Z}$ is

$$T^{1,0}_{\mathring{Z}}(\partial\mathcal{R}_{\mathcal{I}}(m,n))=\{\beta\in\mathbb{C}^{m\times n}:\langle\beta,\nabla\rho(\mathring{Z})\rangle=0\}=\{U\alpha V:\alpha\in\mathbb{C}^{m\times n},\alpha_{11}=0\}.$$

Suppose that $f:\mathcal{R}_{\mathcal{I}}(m,n)\to\mathcal{R}_{\mathcal{I}}(m,n)$ is a holomorphic mapping and f is holomorphic at $\mathring{Z}$ with $f(\mathring{Z})=\mathring{Z}$. Then it is easy to check that the $(mn-1)$-dimensional space $T^{1,0}_{\mathring{Z}}(\partial\mathcal{R}_{\mathcal{I}}(m,n))\subset\mathbb{C}^{m\times n}$ is an invariant subspace of $Df(\mathring{Z})$.

Theorem 3.1. *Let $f:\mathcal{R}_{\mathcal{I}}(m,n)\to\mathcal{R}_{\mathcal{I}}(m,n)$ be a holomorphic mapping with $f(0)=a$, and let*

$$\mathring{Z}=U\begin{pmatrix} r_1 & 0 & \cdots & 0 & 0 & \cdots & 0\\ 0 & r_2 & \cdots & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & r_m & 0 & \cdots & 0\end{pmatrix}V$$

be a smooth boundary point of $\mathcal{R}_{\mathcal{I}}(m,n)$, where $1=r_1>r_2\geqslant\cdots\geqslant r_m\geqslant 0$, U is a unitary square matrix of order m and V is a unitary square matrix of order n. If f is holomorphic at $\mathring{Z}$ and $f(\mathring{Z})=\mathring{Z}$, then all the eigenvalues λ,μ_i $(i=1,\dots,m+n-2)$ and ν_i $(i=1,\dots,(m-1)(n-1))$ of $Df(\mathring{Z})$ have the following properties:

(1) *The unit outward normal vector $\nabla\rho(\mathring{Z})$ to $\partial\mathcal{R}_{\mathcal{I}}(m,n)$ at $\mathring{Z}$ is an eigenvector of $D^*f(\mathring{Z})$ and the corresponding eigenvalue is a real number λ that we just mentioned above, i.e.,*

$$D^*f(\mathring{Z})(\nabla\rho(\mathring{Z}))=\lambda\nabla\rho(\mathring{Z}).$$

(2) $\lambda\geqslant\frac{1-\rho(a)}{1+\rho(a)}>0$, *and if $m=1$ then* $\lambda\geqslant\frac{|1-\mathring{Z}\overline{a}'|^2}{1-\|a\|^2}\geqslant\frac{1-\|a\|}{1+\|a\|}>0$.

(3) $T^{1,0}_{\mathring{Z}}(\partial\mathcal{R}_{\mathcal{I}}(m,n))=M\oplus N$, *where*

$$N=\{U\alpha V:\alpha\in\mathbb{C}^{m\times n},\alpha_{11}=0,(\alpha_{21},\dots,\alpha_{m1})'=0,(\alpha_{12},\dots,\alpha_{1n})=0\}$$

is an $(m-1)(n-1)$-dimensional invariant subspace of $Df(\mathring{Z})$, and M is an $(m+n-2)$-dimensional invariant subspace of $Df(\mathring{Z})$. Moreover, the eigenvalues μ_i of $Df(\mathring{Z})$, which is a linear transformation on M, satisfy

$$|\mu_i|\leqslant\sqrt{\lambda},\quad i=1,\dots,m+n-2;$$

and the eigenvalues ν_i of $Df(\mathring{Z})$, which is a linear transformation on N, satisfy

$$|\nu_i|\leqslant 1,\quad i=1,\dots,(m-1)(n-1).$$

(4) $|\det Df(\mathring{Z})|\leqslant\lambda^{\frac{m+n}{2}}$, $|\mathrm{tr}Df(\mathring{Z})|\leqslant\lambda+\sqrt{\lambda}(m+n-2)+(m-1)(n-1)$. *Then $m=1$ shows* $|\det Df(\mathring{Z})|\leqslant\lambda^{\frac{n+1}{2}}$, $|\mathrm{tr}Df(\mathring{Z})|\leqslant\lambda+\sqrt{\lambda}(n-1)$.

The inequalities in (2)–(4) *are sharp.*

Proof. (1) For any $\beta\in T_{\mathring{Z}}(\partial\mathcal{R}_{\mathcal{I}}(m,n))$, we have $Df(\mathring{Z})(\beta)\in T_{\mathring{Z}}(\partial\mathcal{R}_{\mathcal{I}}(m,n))$. Then

$$\Re\langle\beta,D^*f(\mathring{Z})(\nabla\rho(\mathring{Z}))\rangle=\Re\langle Df(\mathring{Z})(\beta),\nabla\rho(\mathring{Z})\rangle=0.$$

It follows that there exists $\lambda\in\mathbb{R}$ such that

$$D^*f(\mathring{Z})(\nabla\rho(\mathring{Z}))=\lambda\nabla\rho(\mathring{Z}).$$

This means that λ is an eigenvalue of $D^*f(\mathring{Z})$, and $\nabla\rho(\mathring{Z})$ is an eigenvector of $D^*f(\mathring{Z})$ with respect to λ. Since $\lambda\in\mathbb{R}$, we know that λ is also an eigenvalue of $Df(\mathring{Z})$. The proof of (1) is complete.

(2) We divide the proof of (2) into two cases.

Case 1. $f(0)=a=0$. For any $t\in(0,1)$, by Lemma 2.6 we get

$$\rho(f(t\mathring{Z}))\leqslant\rho(t\mathring{Z})=t.$$

This, together with Lemma 2.5, implies

$$\Re\langle f(t\mathring{Z}),\nabla\rho(\mathring{Z})\rangle\leqslant\rho(f(t\mathring{Z}))\leqslant t. \tag{3.1}$$

By Proposition 2.1, $\langle\mathring{Z},\nabla\rho(\mathring{Z})\rangle=1$. Hence, combine $f(t\mathring{Z})=\mathring{Z}-(1-t)Df(\mathring{Z})(\mathring{Z})+O(|t-1|^2)(t\to1^-)$ and (3.1) to obtain

$$1-(1-t)\Re\langle Df(\mathring{Z})(\mathring{Z}),\nabla\rho(\mathring{Z})\rangle+O(|t-1|^2)\leqslant t,$$

which gives

$$\Re\langle\mathring{Z},D^*f(\mathring{Z})(\nabla\rho(\mathring{Z}))\rangle+O(|t-1|)\geqslant1. \tag{3.2}$$

Since $D^*f(\mathring{Z})(\nabla\rho(\mathring{Z}))=\lambda\nabla\rho(\mathring{Z})$ and $\langle\mathring{Z},\nabla\rho(\mathring{Z})\rangle=1$, (3.2) yields

$$\lambda+O(|t-1|)\geqslant1.$$

Taking $t\to1^-$, we get $\lambda\geqslant1$.

Case 2. $f(0)=a\neq0$. Suppose that

$$a=A\begin{pmatrix} l_1 & 0 & \cdots & 0 & 0 & \cdots & 0\\ 0 & l_2 & \cdots & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & l_m & 0 & \cdots & 0\end{pmatrix}B\in\mathcal{R}_{\mathcal{I}}(m,n)$$

is a polar decomposition of a. Then by Lemma 2.3, $g=\varphi_a\circ f:\mathcal{R}_{\mathcal{I}}(m,n)\to\mathcal{R}_{\mathcal{I}}(m,n)$ is a holomorphic mapping, and g is holomorphic at $\mathring{Z}$ with $g(0)=0$. Moreover,

$$\mathring{W}=g(\mathring{Z})=\varphi_a(\mathring{Z})=Q^{-1}(I_m-\mathring{Z}\bar{a}')^{-1}(a-\mathring{Z})R$$

is also a smooth boundary point of $\mathcal{R}_{\mathcal{I}}(m,n)$. Since $D\varphi_a(\mathring{Z})(\beta)\in T_{\mathring{W}}(\partial\mathcal{R}_{\mathcal{I}}(m,n))$ for any $\beta\in T_{\mathring{Z}}(\partial\mathcal{R}_{\mathcal{I}}(m,n))$, we get

$$\Re\langle D\varphi_a(\mathring{Z})(\beta),\nabla\rho(\mathring{W})\rangle=0,\quad \Re\langle\beta,D^*\varphi_a(\mathring{Z})(\nabla\rho(\mathring{W}))\rangle=0.$$

It follows that there is $\mu\in\mathbb{R}$ such that

$$D^*\varphi_a(\mathring{Z})(\nabla\rho(\mathring{W}))=\mu\nabla\rho(\mathring{Z}). \tag{3.3}$$

Set

$$h_1(\zeta)=\langle\varphi_a(\zeta\mathring{Z}),\nabla\rho(\mathring{W})\rangle,\quad \zeta\in\Delta.$$

Then $h_1:\Delta\to\Delta$ is a holomorphic function, and h_1 is holomorphic at $\zeta=1$ with $h_1(1)=\langle\mathring{W},\nabla\rho(\mathring{W})\rangle=1$. This, together with (2.3) and (3.3), implies

$$\mu=\langle\mathring{Z},\mu\nabla\rho(\mathring{Z})\rangle=\langle\mathring{Z},D^*\varphi_a(\mathring{Z})(\nabla\rho(\mathring{W}))\rangle=\langle D\varphi_a(\mathring{Z})(\mathring{Z}),\nabla\rho(\mathring{W})\rangle=h_1'(1)>0.$$

Take

$$h_2(\zeta)=\langle g(\zeta\mathring{Z}),\nabla\rho(\mathring{W})\rangle,\quad \zeta\in\Delta.$$

Then $h_2 : \Delta \to \Delta$ is a holomorphic function, and h_2 is holomorphic at $\zeta = 1$ with $h_2(0) = 0$ and $h_2(1) = 1$. Thus, by Lemma 2.2, (3.3) and (1) we obtain

$$\begin{aligned}1 \leqslant h_2'(1) &= \langle Dg(\mathring{Z})(\mathring{Z}), \nabla\rho(\mathring{W})\rangle \\ &= \langle D\varphi_a(\mathring{Z})(Df(\mathring{Z})(\mathring{Z})), \nabla\rho(\mathring{W})\rangle = \langle Df(\mathring{Z})(\mathring{Z}), D^*\varphi_a(\mathring{Z})(\nabla\rho(\mathring{W}))\rangle \\ &= \mu\langle Df(\mathring{Z})(\mathring{Z}), \nabla\rho(\mathring{Z})\rangle = \mu\langle \mathring{Z}, D^*f(\mathring{Z})(\nabla\rho(\mathring{Z}))\rangle = \lambda\mu.\end{aligned}$$

This shows $\lambda \geqslant \frac{1}{\mu}$. Now, we estimate $\mu = \langle D\varphi_a(\mathring{Z})(\mathring{Z}), \nabla\rho(\mathring{W})\rangle$. It is easy to check that for each $X \in \mathbb{C}^{m\times k}$, $Y \in \mathbb{C}^{k\times l}$ and $Z \in \mathbb{C}^{l\times n}$, the maximum eigenvalue of $(XYZ)\overline{(XYZ)}'\leqslant$ (the maximum eigenvalue of $X\overline{X}'$) $\times$ (the maximum eigenvalue of $(YZ)\overline{(YZ)}'$) $\leqslant$ (the maximum eigenvalue of $X\overline{X}'$) $\times$ (the maximum eigenvalue of $Y\overline{Y}'$) $\times$ (the maximum eigenvalue of $Z\overline{Z}'$). This gives

$$\rho_{m\times n}(XYZ) \leqslant \rho_{m\times k}(X)\rho_{k\times l}(Y)\rho_{l\times n}(Z).$$

Here, $\rho_{m\times n}$, $\rho_{m\times k}$, $\rho_{k\times l}$ and $\rho_{l\times n}$ are the corresponding matrix norms. Notice that

$$\begin{aligned}&D\varphi_a(\mathring{Z})(\mathring{Z}) \\ &\quad= Q^{-1}(I_m - \mathring{Z}\overline{a}')^{-1}\mathring{Z}\overline{a}'(I_m - \mathring{Z}\overline{a}')^{-1}(a - \mathring{Z})R - Q^{-1}(I_m - \mathring{Z}\overline{a}')^{-1}\mathring{Z}R \\ &\quad= Q^{-1}(I_m - \mathring{Z}\overline{a}')^{-1}\mathring{Z}\overline{a}'Q\mathring{W} + \mathring{W} - Q^{-1}(I_m - \mathring{Z}\overline{a}')^{-1}aR \\ &\quad= Q^{-1}(I_m - \mathring{Z}\overline{a}')^{-1}(\mathring{Z}\overline{a}' - I_m)Q\mathring{W} + Q^{-1}(I_m - \mathring{Z}\overline{a}')^{-1}Q\mathring{W} + \mathring{W} - Q^{-1}(I_m - \mathring{Z}\overline{a}')^{-1}Qa \\ &\quad= Q^{-1}(I_m - \mathring{Z}\overline{a}')^{-1}Q(\mathring{W} - a)\end{aligned}$$

and

$$\begin{aligned}&Q^{-1}(I_m - \mathring{Z}\overline{a}')^{-1}(a - \mathring{Z})R\overline{a}' - I_m = \mathring{W}\overline{a}' - I_m, \\ &Q^{-1}(I_m - \mathring{Z}\overline{a}')^{-1}[(a - \mathring{Z})R\overline{a}' - (I_m - \mathring{Z}\overline{a}')Q] = \mathring{W}\overline{a}' - I_m, \\ &Q^{-1}(I_m - \mathring{Z}\overline{a}')^{-1}(Q - aR\overline{a}') = I_m - \mathring{W}\overline{a}', \\ &Q^{-1}(I_m - \mathring{Z}\overline{a}')^{-1}Q^{-1} = I_m - \mathring{W}\overline{a}', \\ &Q^{-1}(I_m - \mathring{Z}\overline{a}')^{-1} = (I_m - \mathring{W}\overline{a}')Q.\end{aligned}$$

Then we obtain

$$D\varphi_a(\mathring{Z})(\mathring{Z}) = (I_m - \mathring{W}\overline{a}')Q^2(\mathring{W} - a).$$

This, together with Lemma 2.5, yields

$$\begin{aligned}\mu &= \langle D\varphi_a(\mathring{Z})(\mathring{Z}), \nabla\rho(\mathring{W})\rangle \leqslant \rho(D\varphi_a(\mathring{Z})(\mathring{Z})) \\ &\leqslant [\rho_{m\times m}(I_m) + \rho_{m\times n}(\mathring{W})\rho_{n\times m}(\overline{a}')][\rho_{m\times m}(Q)]^2[\rho_{m\times n}(\mathring{W}) + \rho_{m\times n}(a)] \\ &= [1 + \rho(a)]^2[1 - (\rho(a))^2]^{-1} = \frac{1 + \rho(a)}{1 - \rho(a)}.\end{aligned}$$

It follows that

$$\lambda \geqslant \frac{1}{\mu} \geqslant \frac{1 - \rho(a)}{1 + \rho(a)}.$$

In particular, if $m = 1$ then

$$\mu = \langle D\varphi_a(\mathring{Z})(\mathring{Z}), \mathring{W}\rangle = (1 - \mathring{W}\overline{a}')Q^2(1 - a\overline{\mathring{W}}') = |1 - \mathring{W}\overline{a}'|^2Q^2 = \frac{1}{Q^2|1 - \mathring{Z}\overline{a}'|^2} = \frac{1 - \|a\|^2}{|1 - \mathring{Z}\overline{a}'|^2}.$$

This implies

$$\lambda \geqslant \frac{1}{\mu} = \frac{|1 - \mathring{Z}\overline{a}'|^2}{1 - \|a\|^2} \geqslant \frac{(1 - \|a\|)^2}{1 - \|a\|^2} = \frac{1 - \|a\|}{1 + \|a\|}.$$

The proof of (2) is complete.

(3) It is clear that $T^{1,0}_{\mathring{Z}}(\partial\mathcal{R}_{\mathcal{I}}(m,n)) = \{U\alpha V : \alpha \in \mathbb{C}^{m\times n}, \alpha_{11} = 0\}$ is an invariant subspace of $Df(\mathring{Z})$, i.e., $(\overline{U}'Df(\mathring{Z})(\beta)\overline{V}')_{11} = 0$ for any $\beta \in T^{1,0}_{\mathring{Z}}(\partial\mathcal{R}_{\mathcal{I}}(m,n))$. Now, we claim that $N = \{U\alpha V : \alpha \in \mathbb{C}^{m\times n}, \alpha_{11} = 0, (\alpha_{21},\dots,\alpha_{m1})' = 0, (\alpha_{12},\dots,\alpha_{1n}) = 0\}$ is an invariant subspace of $Df(\mathring{Z})$. We only need to prove that for each

$$\beta = U\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix} V \in N,$$

if we set $\varepsilon = \overline{U}'Df(\mathring{Z})(\beta)\overline{V}' \in \mathbb{C}^{m\times n}$, then $\varepsilon_{11} = 0$, $(\varepsilon_{21},\dots,\varepsilon_{m1})' = 0$ and $(\varepsilon_{12},\dots,\varepsilon_{1n}) = 0$.

Since $Df(\mathring{Z})(\beta) \in T^{1,0}_{\mathring{Z}}(\partial\mathcal{R}_{\mathcal{I}}(m,n))$, we know $\varepsilon_{11} = 0$. For $t \in (0,1)$, the polar decompositions of $t\mathring{Z}$ and $f(t\mathring{Z})$ are

$$t\mathring{Z} = U\begin{pmatrix} t & 0 & \cdots & 0 & 0 \cdots 0 \\ 0 & tr_2 & \cdots & 0 & 0 \cdots 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \ddots \vdots \\ 0 & 0 & \cdots & tr_m & 0 \cdots 0 \end{pmatrix} V$$

and

$$f(t\mathring{Z}) = U(t)\begin{pmatrix} r_1(t) & 0 & \cdots & 0 & 0 \cdots 0 \\ 0 & r_2(t) & \cdots & 0 & 0 \cdots 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \ddots \vdots \\ 0 & 0 & \cdots & r_m(t) & 0 \cdots 0 \end{pmatrix} V(t),$$

respectively, where $1 > r_1(t) \geqslant r_2(t) \geqslant \cdots \geqslant r_m(t) \geqslant 0$, $U(t)$ is a unitary square matrix of order m and $V(t)$ is a unitary square matrix of order n. Then by Lemma 2.3, corresponding to $a = t\mathring{Z}$ and $a = f(t\mathring{Z})$, we take

$$Q = U\begin{pmatrix} \frac{1}{\sqrt{1-t^2}} & & & 0 \\ & \frac{1}{\sqrt{1-t^2r_2^2}} & & \\ & & \ddots & \\ 0 & & & \frac{1}{\sqrt{1-t^2r_m^2}} \end{pmatrix}\overline{U}', \quad R = \overline{V}'\begin{pmatrix} \frac{1}{\sqrt{1-t^2}} & & & & 0 \\ & \frac{1}{\sqrt{1-t^2r_2^2}} & & & \\ & & \ddots & & \\ & & & \frac{1}{\sqrt{1-t^2r_m^2}} & \\ 0 & & & & I_{n-m} \end{pmatrix} V$$

and

$$Q(t) = U(t)\begin{pmatrix} \frac{1}{\sqrt{1-r_1^2(t)}} & & & 0 \\ & \frac{1}{\sqrt{1-r_2^2(t)}} & & \\ & & \ddots & \\ 0 & & & \frac{1}{\sqrt{1-r_m^2(t)}} \end{pmatrix}\overline{U(t)}',$$

$$R(t) = \overline{V(t)}'\begin{pmatrix} \frac{1}{\sqrt{1-r_1^2(t)}} & & & & 0 \\ & \frac{1}{\sqrt{1-r_2^2(t)}} & & & \\ & & \ddots & & \\ & & & \frac{1}{\sqrt{1-r_m^2(t)}} & \\ 0 & & & & I_{n-m} \end{pmatrix} V(t).$$

Because $\lim_{t\to 1^-} f(t\mathring{Z}) = \mathring{Z}$, we have

$$\lim_{t\to 1^-} r_1(t) = 1, \quad \lim_{t\to 1^-} r_2(t) = r_2, \quad \ldots, \quad \lim_{t\to 1^-} r_m(t) = r_m.$$

In addition, we also get

$$U(t) = U + O(|t-1|), \quad V(t) = V + O(|t-1|) \quad \text{and} \quad Df(t\mathring{Z})(\beta) = Df(\mathring{Z})(\beta) + O(|t-1|)$$

as $t \to 1^-$. Moreover, it follows from $f(t\mathring{Z}) = \mathring{Z} - (1-t)Df(\mathring{Z})(\mathring{Z}) + O(|t-1|^2)$ that

$$\begin{aligned} r_1(t) &= \rho(f(t\mathring{Z})) \\ &= 1 - (1-t)2\Re\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{\partial\rho}{\partial z_{ij}}(\mathring{Z})Df_{ij}(\mathring{Z})(\mathring{Z}) + O(|t-1|^2) \\ &= 1 - (1-t)\Re\langle Df(\mathring{Z})(\mathring{Z}), \nabla\rho(\mathring{Z})\rangle + O(|t-1|^2) \\ &= 1 - (1-t)\Re\langle \mathring{Z}, D^*f(\mathring{Z})(\nabla\rho(\mathring{Z}))\rangle + O(|t-1|^2) \\ &= 1 - \lambda(1-t) + O(|t-1|^2) \end{aligned}$$

as $t \to 1^-$. This implies

$$\sqrt{1-r_1^2(t)} = \sqrt{1-[1-\lambda(1-t)+O(|t-1|^2)]^2} = \sqrt{2\lambda(1-t)+O(|t-1|^2)} \tag{3.4}$$

as $t \to 1^-$. By Corollary 2.4, we have

$$\begin{aligned} &F(t\mathring{Z},\beta) \\ &= \rho\left[U\begin{pmatrix} \frac{1}{\sqrt{1-t^2}} & & & 0 \\ & \frac{1}{\sqrt{1-t^2r_2^2}} & & \\ & & \ddots & \\ 0 & & & \frac{1}{\sqrt{1-t^2r_m^2}} \end{pmatrix}\overline{U}'\beta\overline{V}'\begin{pmatrix} \frac{1}{\sqrt{1-t^2}} & & & & 0 \\ & \frac{1}{\sqrt{1-t^2r_2^2}} & & & \\ & & \ddots & & \\ & & & \frac{1}{\sqrt{1-t^2r_m^2}} & \\ 0 & & & & I_{n-m} \end{pmatrix}V\right] \\ &= \rho\left[\begin{pmatrix} 1 & & & 0 \\ & \frac{1}{\sqrt{1-t^2r_2^2}} & & \\ & & \ddots & \\ 0 & & & \frac{1}{\sqrt{1-t^2r_m^2}} \end{pmatrix}\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix}\begin{pmatrix} 1 & & & & 0 \\ & \frac{1}{\sqrt{1-t^2r_2^2}} & & & \\ & & \ddots & & \\ & & & \frac{1}{\sqrt{1-t^2r_m^2}} & \\ 0 & & & & I_{n-m} \end{pmatrix}\right], \end{aligned}$$

which gives $\lim_{t\to 1^-}\sqrt{1-t^2}F(t\mathring{Z},\beta) = 0$. Similarly, we obtain

$$\begin{aligned} &F[f(t\mathring{Z}), Df(t\mathring{Z})(\beta)] \\ &= \rho\left[U(t)\begin{pmatrix} \frac{1}{\sqrt{1-r_1^2(t)}} & & & 0 \\ & \frac{1}{\sqrt{1-r_2^2(t)}} & & \\ & & \ddots & \\ 0 & & & \frac{1}{\sqrt{1-r_m^2(t)}} \end{pmatrix}\overline{U(t)}'Df(t\mathring{Z})(\beta)\overline{V(t)}'\right. \\ &\quad \left.\times\begin{pmatrix} \frac{1}{\sqrt{1-r_1^2(t)}} & & & & 0 \\ & \frac{1}{\sqrt{1-r_2^2(t)}} & & & \\ & & \ddots & & \\ & & & \frac{1}{\sqrt{1-r_m^2(t)}} & \\ 0 & & & & I_{n-m} \end{pmatrix}V(t)\right]. \end{aligned}$$

Notice that

$$\overline{U(t)}'Df(t\mathring{Z})(\beta)\overline{V(t)}' = \overline{U}'Df(\mathring{Z})(\beta)\overline{V}' + O(|t-1|) = \varepsilon + O(|t-1|)$$

as $t \to 1^-$. This, together with (3.4), implies

$$\begin{aligned}
&\lim_{t\to1^-}\sqrt{1-r_1^2(t)}F[f(t\mathring{Z}), Df(t\mathring{Z})(\beta)]\\
&= \lim_{t\to1^-}\sqrt{1-r_1^2(t)}\rho\left[\begin{pmatrix} 1 & & & 0\\ & \frac{1}{\sqrt{1-r_2^2(t)}} & & \\ & & \ddots & \\ 0 & & & \frac{1}{\sqrt{1-r_m^2(t)}}\end{pmatrix}\right.\\
&\quad\times\begin{pmatrix} \frac{O(|t-1|)}{1-r_1^2(t)} & \frac{\varepsilon_{12}+O(|t-1|)}{\sqrt{1-r_1^2(t)}} & \cdots & \frac{\varepsilon_{1n}+O(|t-1|)}{\sqrt{1-r_1^2(t)}}\\ \frac{\varepsilon_{21}+O(|t-1|)}{\sqrt{1-r_1^2(t)}} & \varepsilon_{22}+O(|t-1|) & \cdots & \varepsilon_{2n}+O(|t-1|)\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\varepsilon_{m1}+O(|t-1|)}{\sqrt{1-r_1^2(t)}} & \varepsilon_{m2}+O(|t-1|) & \cdots & \varepsilon_{mn}+O(|t-1|)\end{pmatrix}\left.\begin{pmatrix} 1 & & & & 0\\ & \frac{1}{\sqrt{1-r_2^2(t)}} & & & \\ & & \ddots & & \\ & & & \frac{1}{\sqrt{1-r_m^2(t)}} & \\ 0 & & & & I_{n-m}\end{pmatrix}\right]\\
&= \rho\left[\begin{pmatrix} 0 & \frac{\varepsilon_{12}}{\sqrt{1-r_2^2}} & \cdots & \frac{\varepsilon_{1m}}{\sqrt{1-r_m^2}} & \varepsilon_{1(m+1)} & \cdots & \varepsilon_{1n}\\ \frac{\varepsilon_{21}}{\sqrt{1-r_2^2}} & 0 & \cdots & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ \frac{\varepsilon_{m1}}{\sqrt{1-r_m^2}} & 0 & \cdots & 0 & 0 & \cdots & 0\end{pmatrix}\right].
\end{aligned}$$

By the contraction property of the Kobayashi metric, we have $F[f(t\mathring{Z}), Df(t\mathring{Z})(\beta)] \leqslant F(t\mathring{Z},\beta)$. It follows from this and (3.4) that

$$\begin{aligned}
&\rho\left[\begin{pmatrix} 0 & \frac{\varepsilon_{12}}{\sqrt{1-r_2^2}} & \cdots & \frac{\varepsilon_{1m}}{\sqrt{1-r_m^2}} & \varepsilon_{1(m+1)} & \cdots & \varepsilon_{1n}\\ \frac{\varepsilon_{21}}{\sqrt{1-r_2^2}} & 0 & \cdots & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ \frac{\varepsilon_{m1}}{\sqrt{1-r_m^2}} & 0 & \cdots & 0 & 0 & \cdots & 0\end{pmatrix}\right]\\
&= \lim_{t\to1^-}\sqrt{1-r_1^2(t)}F[f(t\mathring{Z}), Df(t\mathring{Z})(\beta)]\\
&\leqslant \lim_{t\to1^-}\frac{\sqrt{1-r_1^2(t)}}{\sqrt{1-t^2}}\sqrt{1-t^2}F(t\mathring{Z},\beta)\\
&= \sqrt{\lambda}\lim_{t\to1^-}\sqrt{1-t^2}F(t\mathring{Z},\beta) = 0.
\end{aligned}$$

That means

$$(\varepsilon_{21},\ldots,\varepsilon_{m1})' = 0, \quad (\varepsilon_{12},\ldots,\varepsilon_{1n}) = 0.$$

This shows that N is an $(m-1)(n-1)$-dimensional invariant subspace of $Df(\mathring{Z})$. Hence, there exists an $(m+n-2)$-dimensional invariant subspace M of $Df(\mathring{Z})$ such that

$$T^{1,0}_{\mathring{Z}}(\partial\mathcal{R}_{\mathcal{I}}(m,n)) = M \oplus N.$$

Since $M \cap N = \{0\}$, we have $(\alpha_{21},\ldots,\alpha_{m1})' \neq 0$ or $(\alpha_{12},\ldots,\alpha_{1n}) \neq 0$ for any $\beta = U\alpha V \in M\setminus\{0\}$.

For each eigenvalue μ_i of $Df(\mathring{Z})$ on M, suppose that $\beta^{(i)} = U\alpha^{(i)}V \in M\setminus\{0\}$ is a nonzero eigenvector with respect to μ_i. Here, $\alpha^{(i)}_{11} = 0$, $(\alpha^{(i)}_{21},\ldots,\alpha^{(i)}_{m1})' \neq 0$ or $(\alpha^{(i)}_{12},\ldots,\alpha^{(i)}_{1n}) \neq 0$, $\overline{U}'Df(\mathring{Z})(\beta^{(i)})\overline{V}' =$

$\mu_i\alpha^{(i)}(i=1,\dots,m+n-2)$. By Corollary 2.4, we get

$$\begin{aligned}
&F(t\mathring{Z},\beta^{(i)})\\
&=\rho\left[U\begin{pmatrix}\frac{1}{\sqrt{1-t^2}} & & & 0\\ & \frac{1}{\sqrt{1-t^2r_2^2}} & & \\ & & \ddots & \\ 0 & & & \frac{1}{\sqrt{1-t^2r_m^2}}\end{pmatrix}\overline{U}'\beta^{(i)}\overline{V}'\begin{pmatrix}\frac{1}{\sqrt{1-t^2}} & & & & 0\\ & \frac{1}{\sqrt{1-t^2r_2^2}} & & & \\ & & \ddots & & \\ & & & \frac{1}{\sqrt{1-t^2r_m^2}} & \\ 0 & & & & I_{n-m}\end{pmatrix}V\right]\\
&=\rho\left[\begin{pmatrix}1 & & & 0\\ & \frac{1}{\sqrt{1-t^2r_2^2}} & & \\ & & \ddots & \\ 0 & & & \frac{1}{\sqrt{1-t^2r_m^2}}\end{pmatrix}\begin{pmatrix}0 & \frac{\alpha_{12}^{(i)}}{\sqrt{1-t^2}} & \cdots & \frac{\alpha_{1n}^{(i)}}{\sqrt{1-t^2}}\\ \frac{\alpha_{21}^{(i)}}{\sqrt{1-t^2}} & \alpha_{22}^{(i)} & \cdots & \alpha_{2n}^{(i)}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\alpha_{m1}^{(i)}}{\sqrt{1-t^2}} & \alpha_{m2}^{(i)} & \cdots & \alpha_{mn}^{(i)}\end{pmatrix}\right.\\
&\quad\left.\times\begin{pmatrix}1 & & & & 0\\ & \frac{1}{\sqrt{1-t^2r_2^2}} & & & \\ & & \ddots & & \\ & & & \frac{1}{\sqrt{1-t^2r_m^2}} & \\ 0 & & & & I_{n-m}\end{pmatrix}\right].
\end{aligned}$$

Thus, we obtain

$$\lim_{t\to1^-}\sqrt{1-t^2}F(t\mathring{Z},\beta^{(i)})=\rho\left[\begin{pmatrix}0 & \frac{\alpha_{12}^{(i)}}{\sqrt{1-r_2^2}} & \cdots & \frac{\alpha_{1m}^{(i)}}{\sqrt{1-r_m^2}} & \alpha_{1(m+1)}^{(i)} & \cdots & \alpha_{1n}^{(i)}\\ \frac{\alpha_{21}^{(i)}}{\sqrt{1-r_2^2}} & 0 & \cdots & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ \frac{\alpha_{m1}^{(i)}}{\sqrt{1-r_m^2}} & 0 & \cdots & 0 & 0 & \cdots & 0\end{pmatrix}\right]\neq0.$$

On the other hand,

$$\begin{aligned}
&F[f(t\mathring{Z}),Df(t\mathring{Z})(\beta^{(i)})]\\
&=\rho\left[U(t)\begin{pmatrix}\frac{1}{\sqrt{1-r_1^2(t)}} & & & 0\\ & \frac{1}{\sqrt{1-r_2^2(t)}} & & \\ & & \ddots & \\ 0 & & & \frac{1}{\sqrt{1-r_m^2(t)}}\end{pmatrix}\overline{U(t)}'Df(t\mathring{Z})(\beta^{(i)})\overline{V(t)}'\right.\\
&\quad\left.\times\begin{pmatrix}\frac{1}{\sqrt{1-r_1^2(t)}} & & & & 0\\ & \frac{1}{\sqrt{1-r_2^2(t)}} & & & \\ & & \ddots & & \\ & & & \frac{1}{\sqrt{1-r_m^2(t)}} & \\ 0 & & & & I_{n-m}\end{pmatrix}V(t)\right].
\end{aligned}$$

Notice that $\overline{U(t)}'Df(t\mathring{Z})(\beta^{(i)})\overline{V(t)}'=\mu_i\alpha^{(i)}+O(|t-1|)(t\to1^-)$ and $\alpha_{11}^{(i)}=0$. Then

$$\lim_{t\to1^-}\sqrt{1-r_1^2(t)}F[f(t\mathring{Z}),Df(t\mathring{Z})(\beta^{(i)})]$$

$$
\begin{aligned}
&= \lim_{t\to 1^-}\sqrt{1-r_1^2(t)}\rho\left[\begin{pmatrix}1 & & & 0\\ & \frac{1}{\sqrt{1-r_2^2(t)}} & & \\ & & \ddots & \\ 0 & & & \frac{1}{\sqrt{1-r_m^2(t)}}\end{pmatrix}\right.\\
&\quad\times\begin{pmatrix}\frac{O(|t-1|)}{1-r_1^2(t)} & \frac{\mu_i\alpha_{12}^{(i)}+O(|t-1|)}{\sqrt{1-r_1^2(t)}} & \cdots & \frac{\mu_i\alpha_{1n}^{(i)}+O(|t-1|)}{\sqrt{1-r_1^2(t)}}\\ \frac{\mu_i\alpha_{21}^{(i)}+O(|t-1|)}{\sqrt{1-r_1^2(t)}} & \mu_i\alpha_{22}^{(i)}+O(|t-1|) & \cdots & \mu_i\alpha_{2n}^{(i)}+O(|t-1|)\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\mu_i\alpha_{m1}^{(i)}+O(|t-1|)}{\sqrt{1-r_1^2(t)}} & \mu_i\alpha_{m2}^{(i)}+O(|t-1|) & \cdots & \mu_i\alpha_{mn}^{(i)}+O(|t-1|)\end{pmatrix}\left.\begin{pmatrix}1 & & & & 0\\ & \frac{1}{\sqrt{1-r_2^2(t)}} & & & \\ & & \ddots & & \\ & & & \frac{1}{\sqrt{1-r_m^2(t)}} & \\ 0 & & & & I_{n-m}\end{pmatrix}\right]\\
&= |\mu_i|\rho\left[\begin{pmatrix}0 & \frac{\alpha_{12}^{(i)}}{\sqrt{1-r_2^2}} & \cdots & \frac{\alpha_{1m}^{(i)}}{\sqrt{1-r_m^2}} & \alpha_{1(m+1)}^{(i)} & \cdots & \alpha_{1n}^{(i)}\\ \frac{\alpha_{21}^{(i)}}{\sqrt{1-r_2^2}} & 0 & \cdots & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ \frac{\alpha_{m1}^{(i)}}{\sqrt{1-r_m^2}} & 0 & \cdots & 0 & 0 & \cdots & 0\end{pmatrix}\right] = |\mu_i|\lim_{t\to 1^-}\sqrt{1-t^2}F(t\mathring{Z},\beta^{(i)}).
\end{aligned}
$$

It follows from this and (3.4) that

$$
1 \geqslant \lim_{t\to 1^-}\frac{F[f(t\mathring{Z}),Df(t\mathring{Z})(\beta^{(i)})]}{F(t\mathring{Z},\beta^{(i)})} = \lim_{t\to 1^-}\frac{\sqrt{1-t^2}}{\sqrt{1-r_1^2(t)}}\frac{\sqrt{1-r_1^2(t)}}{\sqrt{1-t^2}}\frac{F[f(t\mathring{Z}),Df(t\mathring{Z})(\beta^{(i)})]}{F(t\mathring{Z},\beta^{(i)})} = \frac{|\mu_i|}{\sqrt{\lambda}}.
$$

This implies

$$
|\mu_i| \leqslant \sqrt{\lambda}, \quad i = 1,\dots,m+n-2.
$$

For any eigenvalue ν_i of $Df(\mathring{Z})$ on N, suppose that $\beta^{(i)} = U\alpha^{(i)}V \in N\setminus\{0\}$ is a nonzero eigenvector with respect to ν_i. Here, $\alpha_{11}^{(i)} = 0$, $(\alpha_{21}^{(i)},\dots,\alpha_{m1}^{(i)})' = 0$, $(\alpha_{12}^{(i)},\dots,\alpha_{1n}^{(i)}) = 0$,

$$
\begin{pmatrix}\alpha_{22}^{(i)} & \cdots & \alpha_{2n}^{(i)}\\ \vdots & \ddots & \vdots\\ \alpha_{m2}^{(i)} & \cdots & \alpha_{mn}^{(i)}\end{pmatrix} \neq 0
$$

and $\overline{U}'Df(\mathring{Z})(\beta^{(i)})\overline{V}' = \nu_i\alpha^{(i)}$ for $i = 1,\dots,(m-1)(n-1)$. Then by Corollary 2.4, we have

$$
\begin{aligned}
&F(t\mathring{Z},\beta^{(i)})\\
&\quad= \rho\left[U\begin{pmatrix}\frac{1}{\sqrt{1-t^2}} & & & 0\\ & \frac{1}{\sqrt{1-t^2r_2^2}} & & \\ & & \ddots & \\ 0 & & & \frac{1}{\sqrt{1-t^2r_m^2}}\end{pmatrix}\overline{U}'\beta^{(i)}\overline{V}'\begin{pmatrix}\frac{1}{\sqrt{1-t^2}} & & & & 0\\ & \frac{1}{\sqrt{1-t^2r_2^2}} & & & \\ & & \ddots & & \\ & & & \frac{1}{\sqrt{1-t^2r_m^2}} & \\ 0 & & & & I_{n-m}\end{pmatrix}V\right]\\
&\quad= \rho\left[\begin{pmatrix}1 & & & 0\\ & \frac{1}{\sqrt{1-t^2r_2^2}} & & \\ & & \ddots & \\ 0 & & & \frac{1}{\sqrt{1-t^2r_m^2}}\end{pmatrix}\begin{pmatrix}0 & 0 & \cdots & 0\\ 0 & \alpha_{22}^{(i)} & \cdots & \alpha_{2n}^{(i)}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & \alpha_{m2}^{(i)} & \cdots & \alpha_{mn}^{(i)}\end{pmatrix}\begin{pmatrix}1 & & & & 0\\ & \frac{1}{\sqrt{1-t^2r_2^2}} & & & \\ & & \ddots & & \\ & & & \frac{1}{\sqrt{1-t^2r_m^2}} & \\ 0 & & & & I_{n-m}\end{pmatrix}\right].
\end{aligned}
$$

Hence, we get

$$\lim_{t\to 1^-} F(t\mathring{Z},\beta^{(i)}) = \rho\left[\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \frac{\alpha_{22}^{(i)}}{1-r_2^2} & \cdots & \frac{\alpha_{2m}^{(i)}}{\sqrt{1-r_2^2}\sqrt{1-r_m^2}} & \frac{\alpha_{2(m+1)}^{(i)}}{\sqrt{1-r_2^2}} & \cdots & \frac{\alpha_{2n}^{(i)}}{\sqrt{1-r_2^2}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{\alpha_{m2}^{(i)}}{\sqrt{1-r_m^2}\sqrt{1-r_2^2}} & \cdots & \frac{\alpha_{mm}^{(i)}}{1-r_m^2} & \frac{\alpha_{m(m+1)}^{(i)}}{\sqrt{1-r_m^2}} & \cdots & \frac{\alpha_{mn}^{(i)}}{\sqrt{1-r_m^2}} \end{pmatrix}\right] \neq 0.$$

On the other hand,

$$\begin{aligned} &F[f(t\mathring{Z}), Df(t\mathring{Z})(\beta^{(i)})] \\ &= \rho\left[U(t)\begin{pmatrix} \frac{1}{\sqrt{1-r_1^2(t)}} & & & 0 \\ & \frac{1}{\sqrt{1-r_2^2(t)}} & & \\ & & \ddots & \\ 0 & & & \frac{1}{\sqrt{1-r_m^2(t)}} \end{pmatrix}\overline{U(t)}' Df(t\mathring{Z})(\beta^{(i)})\overline{V(t)}'\right. \\ &\quad \left.\times \begin{pmatrix} \frac{1}{\sqrt{1-r_1^2(t)}} & & & & 0 \\ & \frac{1}{\sqrt{1-r_2^2(t)}} & & & \\ & & \ddots & & \\ & & & \frac{1}{\sqrt{1-r_m^2(t)}} & \\ 0 & & & & I_{n-m} \end{pmatrix} V(t)\right]. \end{aligned}$$

Notice that $\overline{U(t)}' Df(t\mathring{Z})(\beta^{(i)})\overline{V(t)}' = \nu_i\alpha^{(i)} + O(|t-1|)$ and $r_1(t) = 1-\lambda(1-t)+O(|t-1|^2)$ as $t\to 1^-$. Then

$$\begin{aligned} &\lim_{t\to 1^-} F[f(t\mathring{Z}), Df(t\mathring{Z})(\beta^{(i)})] \\ &= \lim_{t\to 1^-}\rho\left[\begin{pmatrix} 1 & & & 0 \\ & \frac{1}{\sqrt{1-r_2^2(t)}} & & \\ & & \ddots & \\ 0 & & & \frac{1}{\sqrt{1-r_m^2(t)}} \end{pmatrix}\right. \\ &\quad \left.\times\begin{pmatrix} \frac{O(|t-1|)}{1-r_1^2(t)} & \frac{O(|t-1|)}{\sqrt{1-r_1^2(t)}} & \cdots & \frac{O(|t-1|)}{\sqrt{1-r_1^2(t)}} \\ \frac{O(|t-1|)}{\sqrt{1-r_1^2(t)}} & \nu_i\alpha_{22}^{(i)}+O(|t-1|) & \cdots & \nu_i\alpha_{2n}^{(i)}+O(|t-1|) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{O(|t-1|)}{\sqrt{1-r_1^2(t)}} & \nu_i\alpha_{m2}^{(i)}+O(|t-1|) & \cdots & \nu_i\alpha_{mn}^{(i)}+O(|t-1|) \end{pmatrix}\begin{pmatrix} 1 & & & & 0 \\ & \frac{1}{\sqrt{1-r_2^2(t)}} & & & \\ & & \ddots & & \\ & & & \frac{1}{\sqrt{1-r_m^2(t)}} & \\ 0 & & & & I_{n-m} \end{pmatrix}\right] \\ &= \rho\left[\begin{pmatrix} b & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \frac{\nu_i\alpha_{22}^{(i)}}{1-r_2^2} & \cdots & \frac{\nu_i\alpha_{2m}^{(i)}}{\sqrt{1-r_2^2}\sqrt{1-r_m^2}} & \frac{\nu_i\alpha_{2(m+1)}^{(i)}}{\sqrt{1-r_2^2}} & \cdots & \frac{\nu_i\alpha_{2n}^{(i)}}{\sqrt{1-r_2^2}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{\nu_i\alpha_{m2}^{(i)}}{\sqrt{1-r_m^2}\sqrt{1-r_2^2}} & \cdots & \frac{\nu_i\alpha_{mm}^{(i)}}{1-r_m^2} & \frac{\nu_i\alpha_{m(m+1)}^{(i)}}{\sqrt{1-r_m^2}} & \cdots & \frac{\nu_i\alpha_{mn}^{(i)}}{\sqrt{1-r_m^2}} \end{pmatrix}\right] \geqslant |\nu_i|\lim_{t\to 1^-} F(t\mathring{Z},\beta^{(i)}), \end{aligned}$$

where $b=\lim_{t\to 1^-}\frac{(\overline{U(t)}'Df(t\mathring{Z})(\beta^{(i)})\overline{V(t)}')_{11}}{1-r_1^2(t)}$. It follows that

$$1\geqslant \lim_{t\to 1^-}\frac{F[f(t\mathring{Z}),Df(t\mathring{Z})(\beta^{(i)})]}{F(t\mathring{Z},\beta^{(i)})}\geqslant |\nu_i|.$$

This shows

$$|\nu_i|\leqslant 1,\quad i=1,\dots,(m-1)(n-1).$$

The proof of (3) is complete.

(4) Since $T^{1,0}_{\mathring{Z}}(\partial\mathcal{R}_{\mathcal{I}}(m,n))=\{U\alpha V:\alpha\in\mathbb{C}^{m\times n},\alpha_{11}=0\}=M\oplus N$ is an $(mn-1)$-dimensional invariant subspace of $Df(\mathring{Z})$, we know that there is a one-dimensional invariant subspace L of $Df(\mathring{Z})$ such that

$$\mathbb{C}^{m\times n}=L\oplus M\oplus N.$$

It follows from $L\cap T^{1,0}_{\mathring{Z}}(\partial\mathcal{R}_{\mathcal{I}}(m,n))=\{0\}$ that for any $\beta=U\alpha V\in L\setminus\{0\}$ we have $\alpha_{11}\neq 0$. Now, we claim that λ is just the eigenvalue of $Df(\mathring{Z})$ on L. Assume that $\widetilde{\lambda}$ is an eigenvalue of $Df(\mathring{Z})$ on L, and $\beta=U\alpha V\in L\setminus\{0\}$ is a nonzero eigenvector of $Df(\mathring{Z})$ with respect to $\widetilde{\lambda}$. Then by Proposition 2.1, we obtain

$$\langle Df(\mathring{Z})(\beta),\nabla\rho(\mathring{Z})\rangle=\widetilde{\lambda}\langle\beta,\nabla\rho(\mathring{Z})\rangle=\widetilde{\lambda}\alpha_{11}.$$

On the other hand,

$$\langle Df(\mathring{Z})(\beta),\nabla\rho(\mathring{Z})\rangle=\langle\beta,D^*f(\mathring{Z})(\nabla\rho(\mathring{Z}))\rangle=\lambda\langle\beta,\nabla\rho(\mathring{Z})\rangle=\lambda\alpha_{11}.$$

This, together with $\alpha_{11}\neq 0$, yields $\widetilde{\lambda}=\lambda$. Therefore, λ, μ_i $(i=1,\dots,m+n-2)$ and ν_i $(i=1,\dots,(m-1)(n-1))$ are all the eigenvalues of $Df(\mathring{Z})$ on $\mathbb{C}^{m\times n}$. This implies

$$|\det Df(\mathring{Z})|\leqslant\lambda^{\frac{m+n}{2}},\quad |\mathrm{tr}Df(\mathring{Z})|\leqslant\lambda+\sqrt{\lambda}(m+n-2)+(m-1)(n-1).$$

The proof of (4) is complete. □

Remark 3.2. From the view of geometry, N is an invariant subspace of $Df(\mathring{Z})$ perhaps because the Levi form of ρ at $\mathring{Z}$ is positive semi-definite and not positive definite on N. We get the same conclusions of $|\mu_i|\leqslant\sqrt{\lambda}$ $(i=1,\dots,m+n-2)$ with [15, Theorem 3.1] perhaps because the Levi form of ρ at $\mathring{Z}$ is positive definite on M.

Remark 3.3. From the proof of Theorem 3.1, it is clear that we need only to assume that the mapping f is C^1 up to the boundary of $\mathcal{R}_{\mathcal{I}}(m,n)$ near $\mathring{Z}$.

Remark 3.4. When $m=1,n=1,f(0)=0$ and $\mathcal{R}_{\mathcal{I}}(1,1)=\Delta$, Theorem 3.1 is just Lemma 2.2. When $m=1$ and $\mathcal{R}_{\mathcal{I}}(1,n)=B^n$, Theorem 3.1 is just [16, Theorem 3.1].

Finally, we give the following example to show that the inequalities in (2)–(4) of Theorem 3.1 are sharp.

Example 3.5. Let

$$a=\begin{pmatrix}\varepsilon & 0 & \cdots & 0 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0\end{pmatrix}\in\mathcal{R}_{\mathcal{I}}(m,n)$$

and $0<\varepsilon<1$. Write $e_{ij}\in\mathbb{C}^{m\times n}$ as a matrix, which has 1 at i-th row and j-th column, and 0's elsewhere. By Lemma 2.3, take

$$Q=\begin{pmatrix}\frac{1}{\sqrt{1-\varepsilon^2}} & 0\\ 0 & I_{m-1}\end{pmatrix},\quad R=\begin{pmatrix}\frac{1}{\sqrt{1-\varepsilon^2}} & 0\\ 0 & I_{n-1}\end{pmatrix}.$$

Suppose that

$$\mathring{Z} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_m & 0 & \cdots & 0 \end{pmatrix}$$

is a smooth boundary point of $\mathcal{R}_{\mathcal{I}}(m,n)$ and

$$f(Z) = -\varphi_{-a}(Z) = Q^{-1}(I_m + Z\overline{a}')^{-1}(a+Z)R,$$

where $1 > r_2 \geqslant \cdots \geqslant r_m \geqslant 0$. Then $f : \mathcal{R}_{\mathcal{I}}(m,n) \to \mathcal{R}_{\mathcal{I}}(m,n)$ is a holomorphic mapping with $f(0) = a$, and f is holomorphic at $\mathring{Z}$. Moreover, f has the following properties.

(1) $f(\mathring{Z}) = \mathring{Z}$.

(2) For any $\beta \in \mathbb{C}^{m\times n}$,

$$Df(\mathring{Z})(\beta) = \begin{pmatrix} \sqrt{\frac{1-\rho(a)}{1+\rho(a)}} & 0 \\ 0 & I_{m-1} \end{pmatrix} \beta \begin{pmatrix} \sqrt{\frac{1-\rho(a)}{1+\rho(a)}} & 0 \\ 0 & I_{n-1} \end{pmatrix}.$$

(3) $Df(\mathring{Z})(e_{11}) = \frac{1-\rho(a)}{1+\rho(a)} e_{11}$. This shows that one of eigenvalues of $Df(\mathring{Z})$ is $\frac{1-\rho(a)}{1+\rho(a)}$.

(4) $Df(\mathring{Z})(e_{1j}) = \sqrt{\frac{1-\rho(a)}{1+\rho(a)}} e_{1j}$ $(j = 2,\dots,n)$ and $Df(\mathring{Z})(e_{i1}) = \sqrt{\frac{1-\rho(a)}{1+\rho(a)}} e_{i1}$ $(i = 2,\dots,m)$. This shows that the $m+n-2$ eigenvalues of $Df(\mathring{Z})$ are all $\sqrt{\frac{1-\rho(a)}{1+\rho(a)}}$.

(5) $Df(\mathring{Z})(e_{ij}) = e_{ij}$ $(i = 2,\dots,m; j = 2,\dots,n)$. This shows that the $(m-1)(n-1)$ eigenvalues of $Df(\mathring{Z})$ are all 1.

Proof. By Lemma 2.3, it is clear that $f : \mathcal{R}_{\mathcal{I}}(m,n) \to \mathcal{R}_{\mathcal{I}}(m,n)$ is a holomorphic mapping with $f(0) = a$, and f is holomorphic at $\mathring{Z}$.

(1) It is obvious that $\rho(a) = \varepsilon$. Since

$$\mathring{Z}\overline{a}' = \begin{pmatrix} \varepsilon & 0 \\ 0 & 0 \end{pmatrix}, \quad I_m + \mathring{Z}\overline{a}' = \begin{pmatrix} 1+\varepsilon & 0 \\ 0 & I_{m-1} \end{pmatrix} \quad \text{and} \quad a + \mathring{Z} = \begin{pmatrix} 1+\varepsilon & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_m & 0 & \cdots & 0 \end{pmatrix},$$

we have

$$\begin{aligned} f(\mathring{Z}) &= Q^{-1}(I_m + \mathring{Z}\overline{a}')^{-1}(a+\mathring{Z})R \\ &= \begin{pmatrix} \sqrt{1-\varepsilon^2} & 0 \\ 0 & I_{m-1} \end{pmatrix} \begin{pmatrix} \frac{1}{1+\varepsilon} & 0 \\ 0 & I_{m-1} \end{pmatrix} \begin{pmatrix} 1+\varepsilon & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_m & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{1-\varepsilon^2}} & 0 \\ 0 & I_{n-1} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_m & 0 & \cdots & 0 \end{pmatrix} \\ &= \mathring{Z}. \end{aligned}$$

(2) By a straightforward calculation, we get

$$Df(\mathring{Z})(\beta) = Q^{-1}(I_m + \mathring{Z}\overline{a}')^{-1}\beta R - Q^{-1}(I_m + \mathring{Z}\overline{a}')^{-1}\beta\overline{a}'(I_m + \mathring{Z}\overline{a}')^{-1}(a+\mathring{Z})R$$

$$= Q^{-1}(I_m + \mathring{Z}\overline{a}')^{-1}\beta R - Q^{-1}(I_m + \mathring{Z}\overline{a}')^{-1}\beta\overline{a}'Qf(\mathring{Z})$$
$$= \begin{pmatrix} \sqrt{\frac{1-\varepsilon}{1+\varepsilon}} & 0 \\ 0 & I_{m-1} \end{pmatrix} \beta(R - \overline{a}'Q\mathring{Z})$$
$$= \begin{pmatrix} \sqrt{\frac{1-\varepsilon}{1+\varepsilon}} & 0 \\ 0 & I_{m-1} \end{pmatrix} \beta \left[\begin{pmatrix} \frac{1}{\sqrt{1-\varepsilon^2}} & 0 \\ 0 & I_{n-1} \end{pmatrix} - \begin{pmatrix} \frac{\varepsilon}{\sqrt{1-\varepsilon^2}} & 0 \\ 0 & 0 \end{pmatrix} \right]$$
$$= \begin{pmatrix} \sqrt{\frac{1-\varepsilon}{1+\varepsilon}} & 0 \\ 0 & I_{m-1} \end{pmatrix} \beta \begin{pmatrix} \sqrt{\frac{1-\varepsilon}{1+\varepsilon}} & 0 \\ 0 & I_{n-1} \end{pmatrix}$$
$$= \begin{pmatrix} \sqrt{\frac{1-\rho(a)}{1+\rho(a)}} & 0 \\ 0 & I_{m-1} \end{pmatrix} \beta \begin{pmatrix} \sqrt{\frac{1-\rho(a)}{1+\rho(a)}} & 0 \\ 0 & I_{n-1} \end{pmatrix}. \tag{3.5}$$

(3)–(5) Replacing β with e_{ij} $(i = 1, \dots, m; j = 1, \dots, n)$ in (3.5), we can obtain

$$Df(\mathring{Z})(e_{11}) = \frac{1-\rho(a)}{1+\rho(a)} e_{11},$$
$$Df(\mathring{Z})(e_{1j}) = \sqrt{\frac{1-\rho(a)}{1+\rho(a)}} e_{1j}, \quad j = 2, \dots, n,$$
$$Df(\mathring{Z})(e_{i1}) = \sqrt{\frac{1-\rho(a)}{1+\rho(a)}} e_{i1}, \quad i = 2, \dots, m,$$
$$Df(\mathring{Z})(e_{ij}) = e_{ij}, \quad i = 2, \dots, m, \quad j = 2, \dots, n$$

at once. The proof is complete. □

4 Conclusions

In this paper, we considered the Schwarz lemma at the smooth boundary points of $\mathcal{R}_\mathcal{I}(m,n)$. There are some interesting problems that deserve further investigation such as the Schwarz lemma at the non-smooth boundary points and the boundary Schwarz lemma on the other classical domains. In addition, we can apply it to obtain some new results in the geometric function theory of several complex variables.

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References

1 Abate M. Iteration Theory of Holomorphic Maps on Taut Manifolds. Rende: Mediterranean Press, 1989
2 Burns D M, Krantz S G. Rigidity of holomorphic mappings and a new Schwarz lemma at the boundary. J Amer Math Soc, 1994, 7: 661–676
3 Cartan H. Les fonction de deux variables complexs et le problème de la reprèsentation analytique. J Math Pures Appl, 1931, 96: 1–114
4 Elin M, Jacobzon F, Levenshtein M, et al. The Schwarz lemma: Rigidity and dynamics. In: Harmonic and Complex Analysis and Its Applications. New York: Springer, 2014, 135–230
5 Garnett J B. Bounded Analytic Functions. New York: Academic Press, 1981
6 Gong S, Liu T S. Distortion therorems for biholomorphic convex mappings on bounded convex circular domains. Chin Ann Math Ser B, 1999, 20: 297–304
7 Hua L K. Harmonic Analysis of Functions of Several Complex Variables in the Classical Domains. Providence: Amer Math Soc, 1963
8 Huang X J. Some applications of Bell's theorem to weakly pseudoconvex domains. Pacific J Math, 1993, 158: 305–315
9 Huang X J. A preservation principle of extremal mappings near a strongly pseudoconvex point and its applications. Illinois J Math, 1994, 38: 283–302

10 Huang X J. A boundary rigidity problem for holomorphic mappings on some weakly pseudoconvex domains. Canad J Math, 1995, 47: 405–420

11 Kim K T, Lee H. Schwarz's Lemma from a Differential Geometric Viewpoint. Bangalore: IISc Press, 2011

12 Krantz S G. Function Theory of Several Complex Variables, 2nd ed. Providence: Amer Math Soc, 2001

13 Liu T S. The growth theorems, covering theorems and distortion theorems for biholomorphic mappings on classical domains. Doctoral Dissertion. Hefei: University of Science and Technology of China, 1989

14 Liu T S, Ren G B. Decomposition theorem of normalized biholomorphic convex mapping. J Reine Angew Math, 1998, 496: 1–13

15 Liu T S, Tang X M. Schwarz lemma at the boundary of strongly pseudoconvex domain in $\mathbb{C}^n$. Math Ann, 2016, 366: 655–666

16 Liu T S, Wang J F, Tang X M. Schwarz lemma at the boundary of the unit ball in $\mathbb{C}^n$ and its applications. J Geom Anal, 2015, 25: 1890–1914

17 Look K H. Schwarz lemma and analytic invariants. Scientia Sinica, 1958, 7: 453–504

18 Tang X M, Liu T S, Lu J. Schwarz lemma at the boundary of the unit polydisk in $\mathbb{C}^n$. Sci China Math, 2015, 58: 1639–1652