SCIENCE CHINA
Mathematics

• ARTICLES •

December 2015 Vol. 58 No. 12: 2683–2690
doi: 10.1007/s11425-014-4958-0

Twice Q-polynomial distance-regular graphs of diameter 4

MA JianMin[1] & KOOLEN Jack H.[2,*]

[1] *College of Mathematics & Information Science and Hebei Key Lab of Computational Mathematics & Applications, Hebei Normal University, Shijiazhuang* 050016*, China;*
[2] *School of Mathematical Sciences, University of Science and Technology of China, Hefei* 230026*, China*

Email: jianminma@yahoo.com, koolen@ustc.edu.cn

Received July 28, 2014; accepted November 17, 2014; published online December 5, 2014

Abstract It is known that a distance-regular graph with valency k at least three admits at most two Q-polynomial structures. We show that all distance-regular graphs with diameter four and valency at least three admitting two Q-polynomial structures are either dual bipartite or almost dual bipartite. By the work of Dickie (1995) this implies that any distance-regular graph with diameter d at least four and valency at least three admitting two Q-polynomial structures is, provided it is not a Hadamard graph, either the cube $H(d,2)$ with d even, the half cube $1/2H(2d+1,2)$, the folded cube $\tilde{H}(2d+1,2)$, or the dual polar graph on $[^2A_{2d-1}(q)]$ with $q \geqslant 2$ a prime power.

Keywords distance-regular graph, P- or Q-polynomial structure, tight

MSC(2010) 05E30

Citation: Ma J M, Koolen J H. Q-polynomial distance-regular graphs of diameter 4. Sci China Math, 2015, 58: 2683–2690, doi: 10.1007/s11425-014-4958-0

1 Introduction

Bannai and Ito [1] proposed the research program to classify the Q-polynomial distance-regular graphs with large enough diameter in the early 1980s. While there has been tremendous progress, it seems still far from reaching this goal [5]. Meanwhile, they studied distance-regular graphs with more than one Q-polynomial or P-polynomial structures, showed that such distance-regular graphs with diameter at least 34 have at most two Q-polynomial or P-polynomial structures if they are not ordinary polygons. They further determined the parameters for these graphs. See [1, Subsection 3.7]. In 1995, Dickie [6] (in collaboration with Paul Terwilliger) extended the work of Bannai and Ito and obtained the following classification for the distance-regular graphs with two Q-polynomial structures.

Theorem 1.1 (See [6, Theorem 8.1.2]). *Let Γ be a distance regular graph with diameter $d \geqslant 5$ and valency $k \geqslant 3$. Then Γ has two Q-polynomial structures if and only if Γ is one of the following*:

(i) *the cube $H(d,2)$ with d even*;
(ii) *the half cube $1/2H(2d+1,2)$*;
(iii) *the folded cube $\tilde{H}(2d+1,2)$*;
(iv) *the dual polar graph on $[^2A_{2d-1}(q)]$, where $q \geqslant 2$ is a prime power.*

In this case, Γ has at most two Q-polynomial structures.

*Corresponding author

For brevity we call a distance-regular graph (or an association scheme) with exactly two Q-polynomial structures *twice Q-polynomial*. In a recent survey paper, van Dam et al. [5] raised the question to classify the distance-regular graphs with two Q-polynomial structures and diameter three or four. We show that Theorem 1.2 can be extended to include the diameter four case. The following result is key to doing so.

Theorem 1.2. *Let Γ denote a twice Q-polynomial distance-regular graph of diameter four and valency at least three. Then one of the Q-polynomial structures has $a_1^* = a_2^* = a_3^* = 0$, i.e., this structure is either dual bipartite or almost dual bipartite.*

Theorem 1.2 shows the converse of [6, Lemma 8.2.1] (see Lemma 2.4 below) does not hold. We note that this lemma played a key role in the proof of Theorem 1.1; see [6]. The proof of Theorem 1.2 involves a very delicate argument to determine the local structure information about Γ; see Subsection 3.1.2. As a consequence of Theorems 1.1, 1.2 and the classification of distance-regular graphs of diameter at least four that are either dual bipartite or almost dual bipartite [6], we obtain the following result.

Theorem 1.3. *Let Γ denote a twice Q-polynomial distance-regular graph with diameter d at least 4 and valency at least 3. Then Γ is one of the following:*

(i) *the cube $H(d, 2)$ with d even*;

(ii) *the half cube $1/2H(2d + 1, 2)$*;

(iii) *the folded cube $\tilde{H}(2d + 1, 2)$*;

(iv) *the dual polar graph on $[^2A_{2d-1}(q)]$, where $q \geqslant 2$ is a prime power*;

(v) *a Hadamard graph of order 2γ with intersection array*

$$\{2\gamma, 2\gamma - 1, \gamma, 1; 1, \gamma, 2\gamma - 1, 2\gamma\}$$

with $\gamma = 1$ or γ a positive even integer.

Remark 1. A Hadamard graph of order 2γ exists if and only if a Hadamard matrix[1)] of 2γ exists [2, Subsection 1.8]. The Hadamard conjecture states that a Hadamard matrix of order n exists if and only if $n = 1, 2$, or n is a positive integer divisible by 4.

Distance-regular graphs of diameter 2 are strongly regular graphs, which possess two P-polynomial and two Q-polynomial structures. Any connected distance regular graph with valency two is an ordinary n-gon, which can have more than two Q-polynomial structures only if $n \geqslant 7$. So in the rest of this note, we restrict ourselves to distance-regular graphs with both diameter and valency at least three unless stated otherwise.

2 Definitions and preliminaries

In this paper, we use the notation adopted in the book of Brouwer et al. [2]. See also the book of Bannai and Ito [1] for more background information. For the rest of this section, we recall other definitions that will be used later.

Let $\Gamma = (X, R)$ denote a distance-regular graph with vertex set X, edge set R, valency k, and diameter d, and intersection numbers p_{ij}^k. We sometimes write $p_{i,j}^k$ in place of p_{ij}^k for the sake of clarity, and use the usual handy abbreviation: $c_i = p_{1,i-1}^i$ $(1 \leqslant i \leqslant d)$, $a_i = p_{1i}^i$ $(0 \leqslant i \leqslant d)$, $b_i = p_{1,i+1}^i$ $(0 \leqslant i \leqslant d-1)$, $k_i = p_{ii}^0$ $(0 \leqslant i \leqslant d)$. We define $c_0 = 0$, $b_d = 0$.

Let $A_0, A_1, \ldots, A_d$ be the distance matrices for Γ with rows and columns indexed by X: where (x, y)-entry of A_i is 1 if x, y have distance i and 0 otherwise. Note that $A_0 = I$ is the identity matrix and A_1 is the adjacency matrix of Γ. Let

$$E_0 = |X|^{-1}J, \quad E_1, \ldots, E_d$$

be the primitive idempotents of Γ with Krein parameters q_{ij}^k, where J is the all-one matrix. The real numbers $\theta_0, \theta_1, \ldots, \theta_d$ satisfying $A_1 = \sum_{i=0}^d \theta_i E_i$ are the eigenvalues of Γ.

1) A Hadamard matrix of order n with n a positive integer is a square $\{+1, -1\}$ matrix H of order n such that $HH^{\mathrm{T}} = nI$.

2.1 Cosines

We now recall the cosines. Let E be one of the primitive idempotents above with the associated eigenvalue θ (i.e., $A_1E = \theta E$). Let $\sigma_0, \sigma_1, \ldots, \sigma_d$ denote the real numbers satisfying

$$E = |X|^{-1} m \sum_{i=0}^{d} \sigma_i A_i, \tag{2.1}$$

where $m = \mathrm{rank}(E)$. We call $\sigma_0, \sigma_1, \ldots, \sigma_d$ the *cosine sequence* of Γ of θ (or E). By [2, Subsection 4.1.B],

$$\sigma_0 = 1, \quad \sigma_1 = \theta/k, \quad \theta\sigma_r = c_r\sigma_{r-1} + a_r\sigma_r + b_r\sigma_{r+1} \quad \text{for all } r = 1, \ldots, d, \tag{2.2}$$

where σ_{d+1} is indeterminate. Set $c_d = k - a_d$ and (2.2) simplifies to

$$a_d(\sigma_{d-1} - \sigma_d) = k(\sigma_d - \sigma_1\sigma_{d-1}). \tag{2.3}$$

By [8, Lemma 13.2.1], for $\theta = \theta_j$, its cosine sequence $(\sigma_i)_i$ has exactly $j-1$ sign-changes, and if $j \geqslant 2$, the sequence $\sigma_0 - \sigma_1, \sigma_1 - \sigma_2, \ldots, \sigma_d - \sigma_{d+1}$ has $j-2$ sign-changes. (The number of sign-changes in the sequence $(\gamma_i)_{i=0}^d$ is the number of indices i such that $\gamma_i\gamma_{i+1} < 0$, skipping the zero terms if any.) This implies the following lemma.

Lemma 2.1. *Let Γ denote a distance-regular graph with diameter $d \geqslant 3$, and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. Let θ denote one of θ_1, θ_d and let $\sigma_0, \sigma_1, \ldots, \sigma_d$ denote the cosine sequence for θ,*

(i) *Suppose $\theta = \theta_1$. Then $\sigma_0 > \sigma_1 > \cdots > \sigma_d$.*

(ii) *Suppose $\theta = \theta_d$. Then for each i $(0 \leqslant i \leqslant d)$, $(-1)^i\sigma_i > 0$.*

In the sequel, the second largest and smallest eigenvalues of a distance-regular graph turn out to be of particular interest.

2.2 Q-polynomial property

Let $\theta_0, \theta_1, \ldots, \theta_d$ (or $E_0, E_1, \ldots, E_d$) be a fixed ordering of the eigenvalues (or primitive idempotents) of Γ. We call this ordering is a *Q-polynomial structure* if there is a sequence $\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_d)$ and polynomial q_j of degree j, $j = 0, 1, \ldots, d$, such that

$$E_j = \sum_{i=0}^{d} q_j(\sigma_i) A_i;$$

in this case, σ is called a *Q-sequence* of Γ and E_1 is called the *primary idempotent* for this Q-sequence. In particular, the cosine sequence of a primary idempotent is a Q-sequence. The graph Γ is called *Q-polynomial* if Γ has a Q-polynomial structure. We use the standard abbreviation for the Krein parameters of a Q-polynomial structure: $a_i^* = q_{1i}^i$, $b_i^* = q_{1,i+1}^i$,, $c_i^* = q_{1,i-1}^i$.

Theorem 2.2 (See [2, Theorem 8.1.2 and Corollary 8.1.4])**.** *Let Γ be a Q-polynomial distance-regular graph. Then every Q-sequence $(\sigma_0, \sigma_1, \ldots, \sigma_d)$ of Γ satisfies the recurrence*

$$\sigma_{i+1} + \sigma_{i-1} = p\sigma_i + r, \quad i = 1, \ldots, d-1, \tag{2.4}$$

for suitable numbers p and r.

If $\theta_0, \theta_1, \ldots, \theta_d$ is the Q-polynomial structure corresponding to the above Q-sequence, then there are constants r^*, s^* such that

$$\begin{cases} \theta_{\ell+1} + \theta_{\ell-1} = p\theta_\ell + r^*, \\ \theta_{\ell+1}\theta_{\ell-1} = \theta_\ell^2 - r^*\theta_\ell - s^*, \end{cases} \qquad \ell = 1, \ldots, d-1. \tag{2.5}$$

2.3 Double Q-polynomial structures

We record a few facts about twice Q-polynomial distance-regular graphs that will be used later.

Theorem 2.3 (See [6, Theorem 5.1.2]). *Let Γ be a twice Q-polynomial distance-regular graph with intersection numbers p^i_{1i}. Suppose that $E_0, E_1, \ldots, E_d$ is a Q-polynomial structure for Γ with Krein parameters q^i_{1i}. Then we have the following implications*:

$$q^1_{11} = 0 \Rightarrow q^i_{1i} = 0, \tag{2.6}$$

$$q^1_{11} \neq 0 \Rightarrow q^i_{1i} \neq 0, \tag{2.7}$$

$$p^1_{11} = 0 \Rightarrow p^i_{1i} = 0, \tag{2.8}$$

for all $i = 1, 2, \ldots, d-1$.

The dual of (2.7), i.e., $p^1_{11} \neq 0 \Rightarrow p^i_{1i} \neq 0, 1 \leqslant i \leqslant d-1$, holds for any distance-regular graph [2, p. 178].

Lemma 2.4 (See [6, Lemma 8.2.1]). *Let Γ be distance-regular graph with diameter $d \geqslant 4$. Suppose that $E_0, E_1, \ldots, E_d$ and $\tilde{E}_0, \tilde{E}_1, \ldots, \tilde{E}_d$ are Q-polynomial structures for Γ with Krein parameters q^k_{ij} and $\tilde{q}^k_{ij}$, respectively. If $q^1_{11} \neq 0$ and $\tilde{q}^1_{11} \neq 0$, then $E_1 = \tilde{E}_d$, $E_d = \tilde{E}_1$ and $d = 4$.*

The above result played a key role in the proof of Theorem 1.1 and it will be the starting point for our investigation in this paper.

2.4 Almost dual primitivity

The graph Γ is called *imprimitive* when some i, $1 \leqslant i \leqslant d$, the distance-$i$ graph $\Gamma_i = (X, A_i)$ is disconnected. If Γ is imprimitive, then by [2, Theorem 4.2.1], Γ is *bipartite* (here Γ_2 is disconnected) or *antipodal* (here Γ_d is a union of cliques).

A Q-polynomial structure $(E_i)^d_{i=0}$ is called *dual bipartite* if $a^*_0 = a^*_1 = \cdots = a^*_d = 0$. When there is no possibility of confusion, we also say that graph Γ is dual bipartite. Similar comment applies the other concepts to follow immediately. If $c^*_i = b^*_{d-i}$ for $i = 0, 1, \ldots, d$ and $i \neq \lfloor d/2 \rfloor$, then Γ is called *dual antipodal*. An imprimitive Q-polynomial distance-regular graph is either dual bipartite or dual antipodal (or both). See Theorem 3.1 below.

The terms of almost dual bipartite/antipodal were introduced by Dickie [6]. A Q-polynomial structure $(E_i)^d_{i=0}$ is called *almost dual bipartite* if $a^*_0 = a^*_1 = \cdots = a^*_{d-1} = 0 \neq a^*_d$; it is called *almost dual antipodal* if $q^d_{1d} \neq 0 = q^d_{2d} = \cdots = q^d_{dd}$. If Γ is almost dual bipartite or antipodal, then it is called *almost dual imprimitive*.

For a classification of almost dual imprimitive distance-regular graphs, see [6, Theorem 3.1.4] for the almost dual bipartite case with $d \geqslant 4$, and [6, Theorem 2.1.2], [7] for the dual bipartite case with $d \geqslant 3$.

2.5 The tight property

Now we recall the tight property [10]. A distance-regular graph Γ is called *tight* if it is not bipartite and the following equality holds:

$$\left(\theta_1 + \frac{k}{a_1+1}\right)\left(\theta_d + \frac{k}{a_1+1}\right) = -\frac{k a_1 b_1}{(a_1^2+1)^2},$$

where θ_1 and θ_d are the second largest and smallest eigenvalues of Γ, respectively.

For any vertex x, $\Gamma(x)$ is the induced subgraph on the set of neighbours of x. If Γ is a tight distance-regular with diameter at least 3, then $\Gamma(x)$ is connected strongly regular with k vertices, valency a_1, and non-trivial eigenvalues

$$-1 - \frac{b_1}{\theta_d + 1}, \quad -1 - \frac{b_1}{\theta_1 + 1},$$

where θ_d and θ_1 are the smallest and the second largest eigenvalues of Γ, respectively. We refer $\Gamma(x)$ to as *the local graph for Γ with respect to vertex x*. See [10].

The following results are due to Pascasio [15, 16].

Theorem 2.5 (See [15, 16]). *Let Γ be a distance regular graph with diameter $d \geqslant 3$ with intersection numbers a_i. Let $\theta_0 > \theta_1 > \cdots > \theta_d$ be the eigenvalues of Γ with the respective primitive idempotents $E_0, E_1, \ldots, E_d$.*

(i) *Suppose that Γ is tight and E and F are two primitive idempotents other than E_0. Then Hadamard product $E \circ F$ is a scalar multiple of a primitive idempotent H of Γ if and only if E, F are a permutation of E_1, E_d. Moreover, $H = E_{d-1}$ and $\theta_1\theta_d = \theta_0\theta_{d-1}$. The scalar is $\frac{m_1 m_d}{|X| m_{d-1}}$, where $m_\bullet$ is the rank of $E_\bullet$.*

(ii) *If $E_0, E_1, \ldots, E_d$ is a Q-polynomial structure with Krein parameters a_i^*. Then Γ is tight if and only if Γ is not bipartite and $a_d = 0$ if and only if Γ is not bipartite and $a_d^* = 0$.*

3 Proofs of Theorems 1.2 and 1.3

We prove our main theorems in this section. In the rest of this paper, we will fix Γ to be a twice Q-polynomial distance-regular graph of diameter 4. Let $E_0, \ldots, E_4$ and $\tilde{E}_0, \ldots, \tilde{E}_4$ be Q-polynomial structures for Γ. The parameters for $(\tilde{E}_i)_i$ will be attached with a tilde.

We first prove Theorem 1.2, which is key to determine the diameter 4 case.

3.1 Proof of Theorem 1.2

Proof of Theorem 1.2. If Γ has $q_{11}^1 = 0$ (or $\tilde{q}_{11}^1 = 0$), then, by (2.6), it is dual bipartite in case $q_{1d}^d = 0$ (or $\tilde{q}_{11}^1 = 0$) and almost dual bipartite otherwise.

By Lemma 2.4, $\tilde{E}_1 = E_4$ and $\tilde{E}_4 = E_1$. Suzuki [18] classified symmetric twice Q-polynomial association schemes that are not from an ordinary polygons[2)]. By [18, Theorem 1], the Q-polynomial structures for Γ have type III. In this case, the following hold by the Q-polynomial property (see also [18, Theorem 2]):

$$q_{14}^4 = 0 = q_{34}^4, \quad q_{24}^4 \neq 0 \neq q_{23}^4.$$

Since $q_{14}^4 = 0$, $E_1 \circ E_4 = |V\Gamma|^{-1} b_3^* E_3$, where $V\Gamma$ is the vertex set of Γ. Theorem 2.5 says that θ_4 is the eigenvalue associated with E_4 or $\tilde{E}_4$. Without loss of generality, we assume that it is E_4. If Γ is tight, then E_1 is associated with θ_1. By [16, Theorem 1.5], E_0, E_1, E_2, E_3, E_4 is the natural ordering of the primitive idempotents. (The list $E_0, E_1, \ldots, E_d$ is called *natural ordering* if the adjacency matrix $A_1 = \sum_{i=0}^{d} \theta_i E_i$ with $\theta_0 > \theta_1 > \cdots > \theta_d$.) Now we denote the eigenvalues of Γ by $\theta_0 > \theta_1 > \theta_2 > \theta_3 > \theta_4$.

Now we prove Theorem 1.2 by showing that one of $a_1^*, \tilde{a}_1^*$ for Γ vanishes. We distinguishing whether Γ is bipartite or not.

3.1.1 Γ *is bipartite*

Assume that Γ is bipartite. So we have $\theta_4 = -k$ and $m_4 = 1$. Imprimitive Q-polynomial association schemes have the following characterization.

Theorem 3.1 (See [17, Theorem 3] and [3, 19]). *Let $E_0, E_1, \ldots, E_d$ be a Q-polynomial structure for association scheme $\mathfrak{X}$. Suppose that $\mathfrak{X}$ is imprimitive. More precisely, let T be a proper subset of $\{0, 1, \ldots, d\}$ with $T \neq \{0\}$ such that the linear span of $\{E_i \mid i \in T\}$ is closed under the Hadamard product. In addition, assume $m_1 > 2$. Then one of following holds*:

(i) *$T = \{0, 2, 4, \ldots\}$ and $a_i^* = 0$.*

(ii) *$T = \{0, d\}$ and $b_i^* = c_{d-i}^*$ for all $i = 0, 1, \ldots, d$ with the possible exception $i = \lfloor d/2 \rfloor$.*

An association scheme $\mathfrak{X}$ in (3.1), and (3.1) is also called *dual bipartite* and *dual antipodal*, respectively.

Now back to Γ. Let $m_1 = m_{E_1}$. Suppose $m_1 > 2$. Then Theorem 3.1 applies. If $q_{11}^1 > 0$, Γ cannot be dual bipartite and hence Γ is dual antipodal, i.e., Case (ii). So $T = \{0, 4\}$. However, E_4 is the primary idempotent for the second Q-polynomial structure, which is impossible.

Suppose $m_1 \leqslant 2$. Since $m_{E_1} < k = 3$, we have $\theta_{E_1} = \theta_1$ by [2, Theorem 4.4.4]. If $m_1 = 2$, then by [11, Theorem 13(i)] $k = 2$, this contradicts $k > 2$. By [11, Lemma 7], it is impossible for $m_1 = 1$; otherwise $m_1 + m_4 = 2 < k$.

2) The last case (V) in Suzuki's classification was recently eliminated [14].

3.1.2 Γ *is not bipartite*

Assume that Γ is not bipartite. Since $a_4^* = 0$, Γ is tight and $a_4 = 0$ by Theorem 2.5. In the literature [2, p. 247], there is an infinite series of feasible formally self-dual intersection arrays

$$\{\mu(2\mu+1), (\mu-1)(2\mu+1), \mu^2, \mu; 1, \mu, \mu(\mu-1), \mu(2\mu+1)\}. \tag{3.1}$$

This series was ruled out in [9]. Had a graph with this array existed, it would be tight with a pair of non-integral eigenvalues, and would have possessed two P-polynomial and two Q-polynomial structures. We will give an alternative proof that there are no distance-regular graphs with intersection array (3.1).

Since Γ is not bipartite, it follows from (2.7) that $a_1a_2a_3 \neq 0$. By Theorem 2.5,

$$\theta_1\theta_4 = \theta_0\theta_3. \tag{3.2}$$

Our next goal is to determine the relations among θ_i and the parameters of the local graph for Γ.

Now applying Theorem 2.2 to the two Q-polynomial structures of Γ, we obtain

$$\theta_2 - p\theta_1 + \theta_0 = \theta_3 - p\theta_2 + \theta_1 = \theta_4 - p\theta_3 + \theta_2, \tag{3.3}$$

$$\theta_2 - \tilde{p}\theta_4 + \theta_0 = \theta_3 - \tilde{p}\theta_2 + \theta_4 = \theta_1 - \tilde{p}\theta_3 + \theta_2. \tag{3.4}$$

We obtain from (3.3) and (3.4)

$$p = \frac{\theta_0 - \theta_4}{\theta_1 - \theta_3}, \quad \tilde{p} = \frac{\theta_0 - \theta_1}{\theta_4 - \theta_3}, \quad p - \tilde{p} = \frac{2(\theta_1 - \theta_4)}{\theta_2 - \theta_3}.$$

From these equation and $\theta_1\theta_4 = k\theta_3$, we find

$$p = \theta_0/\theta_1, \quad \tilde{p} = \theta_0/\theta_4, \quad \theta_2 = -\theta_3. \tag{3.5}$$

Now substituting these into (3.3) leads to

$$\theta_1 + \theta_4 = 2\theta_2. \tag{3.6}$$

If we substitute (3.5) into (2.5), we find

$$r^* = \theta_2, \quad s^* = \theta_1\theta_3, \quad \tilde{r}^* = \theta_2, \quad \tilde{s}^* = \theta_4\theta_2.$$

Since Γ is tight, we have

$$\left(\theta_1 + \frac{k}{a_1+1}\right)\left(\theta_4 + \frac{k}{a_1+1}\right) = -\frac{ka_1b_1}{(a_1+1)^2}. \tag{3.7}$$

We find from this and (3.6) and (3.2) that

$$\theta_2(a_1 - 1) = b_1 + 1. \tag{3.8}$$

Now we collect some equations above that are key to the proof as follows:

$$\theta_1\theta_4 = \theta_0\theta_3, \tag{3.9}$$

$$\theta_1 + \theta_4 = 2\theta_2, \tag{3.10}$$

$$\theta_2(a_1 - 1) = b_1 + 1. \tag{3.11}$$

By Subsection 2.5, a local graph $\Gamma(x)$ is strongly regular with k vertices and valency a_1 and non-trivial eigenvalues

$$\xi = -1 - \frac{b_1}{\theta_4 + 1}, \quad \tau = -1 - \frac{b_1}{\theta_1 + 1}.$$

where $\xi \geqslant 0$ and $\tau < -1$.

The local graph $\Gamma(x)$ cannot be a conference graph. Otherwise, we have $a_1 = (k-1)/2$ and such a graph has diameter 3 by [13]. (The intersection array (3.1) has the second largest and minimal eigenvalues non-integral. Any graph with this array has a conference graph as its local graph and therefore cannot exist.) Therefore, ξ and τ are both integers and thus θ_4, θ_1 are both rational numbers. Since they are algebraic integers, θ_4, θ_1 are integers.

We find from (3.9)–(3.11) that

$$-(\theta_1+1)(\theta_4+1) = (k-2)\theta_2 - 1 = (a_1-1)\theta_2 + b_1\theta_2 - 1 = b_1 + 1 + b_1\theta_2 - 1 = b_1(\theta_2+1).$$

From this we can derive

$$\frac{-b_1^2}{(\theta_1+1)(\theta_4+1)} = \frac{b_1}{(\theta_2+1)}. \tag{3.12}$$

Since the left hand side is an integer, θ_2+1 divides b_1 and hence a_1 by (3.11). Let $a_1 = \alpha(\theta_2+1)$, $b_1 = \beta(\theta_2+1)$. By (3.9) and (3.10) we find $\theta_1, \theta_4 = \theta_2 \pm \sqrt{\theta_2(k+\theta_2)}$. Now $\theta_2(k+\theta_2) = \alpha\theta_2(\theta_2+1)^2$ and hence $k = \alpha(\theta_2+1)^2 - \theta_2$. In the rest of this proof, we will determine eigenvalues θ_i in terms of a_1 and k.

Let $(\sigma_i)_i$ be the cosine sequence of θ_1. Then by Lemma 2.1, $\sigma_i > \sigma_{i+1}$ $(0 \leqslant i \leqslant 3)$. Since $a_4 = 0$, $\sigma_3 = \sigma_1\sigma_4$ by (2.3). By the remarks preceding Lemma 2.1, the sequence $(\sigma_i)_i$ has one sign change and thus $\sigma_4 < 0$ and $\sigma_1 > 0$. Hence $\sigma_3 < 0$. It remains to determine the sign of σ_2. Let $(\tilde{\sigma}_i)_i$ be cosine sequence of θ_4. Then $(-1)^i\tilde{\sigma}_i > 0$. Let $(u_i)_i$ be the cosine sequence of θ_3. Then $u_i = \sigma_i\tilde{\sigma}_i$ by Theorem 2.5.

Fortunately, some recent results in [4] allows us to determine the sign of σ_2: It is readily deduced from [4, Propositions 1(iii) and 2] that $\sigma_2 \geqslant 0$. Hence, $u_2 \geqslant 0$. We find from this, $\theta_2 = -\theta_3$ and (2.2) (with $r=1$) that $k - \theta_2 a_1 \leqslant \theta_2^2$. As $k = (a_1-1)(\theta_2+1)+1$, we obtain $a_1 \leqslant \theta_2(\theta_2+1)$ and thus $\alpha \leqslant \theta_2$.

Recall $(\alpha(\theta_2+1)-1)\theta_2 - 1 = (a_1-1)\theta_2 - 1 = b_1 = \beta(\theta_2+1)$. We obtain $\beta = \alpha\theta_2 - 1$ and $b_1 = (\alpha\theta_2-1)(\theta_2+1)$, $\xi = -1-b_1/(\theta_4+1) = \sqrt{\alpha\theta_2}$ and $\tau = -\xi$. This means that ξ divides $a_1(= k-b_1-1)$ and hence, as $a_1 = \xi^2+\alpha$, ξ divides α. This implies that θ_2 divides α and hence $\theta_2 \leqslant \alpha$. We conclude $\alpha = \theta_2 = \xi$. Now, the local graph $\Gamma(x)$ has the following parameters:

$$k = \theta_2^2(\theta_2+2), \quad a_1 = \theta_2(\theta_2+1), \quad \theta_1 = \theta_2(\theta_2+2), \quad \theta_4 = -\theta_2^2.$$

We see that

$$\theta_1 = \frac{a_1 + \sqrt{a_1^2+4k}}{2}.$$

Now Γ is antipodal by the following result.

Lemma 3.2 (See [12, Proposition 3.5]). *Let Ω be a distance-regular graph with d at least three and distinct eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_d$. Then $\theta_1 = (a_1 + \sqrt{a_1^2+4k})/2$ if and only if one of the following holds*:

(i) $d = 3$ *and Ω is a Shilla distance-regular graph*;

(ii) $d = 4$ *and Ω is an antipodal distance-regular graph.*

Since Γ is antipodal, it is dual bipartite. So

$$a_1^* = 0, \quad \text{or} \quad \tilde{a}_1^* = 0.$$

This completes the proof of Theorem 1.2. □

3.2 Proof of Theorem 1.3

Proof of Theorem 1.3. Since the twice Q-polynomial distance-regular graphs with diameter at least 5 are classified by Theorem 1.1. It remains to treat the diameter 4 case. Let Γ be a twice Q-polynomial distance-regular graph with diameter 4 and valency at least 3. Then the Krein parameters a_i^* of Γ can be divided into the following four cases:

(i) If $a_1^* = a_4^* = 0$, then Γ is $H(4,2)$ or a Hadamard graph by [7].

(ii) If $a_1^* = 0 \neq a_4^*$, then Γ is $\frac{1}{2}H(9,2)$, or $\tilde{H}(9,2)$ by [6, Theorem 3.1.4].

(iii) Suppose $a_1^* \neq 0 \neq a_4^*$. By [18, Theorem 2], the Q-polynomial structures $(E_i)_i$ is almost dual antipodal and thus by [6, Lemma 3.1.3], the other Q-polynomial structure $(\tilde{E}_i)_i$ is almost dual bipartite. This case is implied by the previous case by treating $(\tilde{E}_i)_i$, which has $\tilde{a}_1^* = 0 \neq \tilde{a}_4^*$.

(iv) If $a_1^* \neq 0 = a_4^*$, then $\tilde{a}_1^* = 0$ by Theorem 1.2 and Cases (i) and (ii) apply. So the proof of Theorem 1.3 is completed. □

Remark 2. The following graphs are twice Q-polynomial distance-regular graphs of diameter 3: Graphs in Theorem 1.1(ii)–(iv) with $d = 3$, antipodal distance-regular graphs with intersection array $\{k, \mu, 1; 1, \mu, k\}$ ($\mu < k - 1$), and bipartite (but not antipodal) distance-regular graphs with intersection array $\{k, k-1, k-\mu; 1, \mu, k\}$ ($\mu < k - 1$). These antipodal ones are called Taylor graphs, and these bipartite ones are the incidence graphs of nontrivial symmetric 2-designs. See [2, p. 431].

Acknowledgements This work was supported by Natural Science Foundation of Hebei Province (Grant No. A2012205079) and Science Foundation of Hebei Normal University (Grant No. L2011B02). The second author thanks the 100 Talents Program of the Chinese Academy of Sciences for support.

References

1 Bannai E, Ito T. Algebraic Combinatorics I: Association Schemes. Menlo Park, CA: Benjamin/Cummings, 1984
2 Brouwer A E, Cohen A M, Neumaier A. Distance-Regular Graphs. New York: Springer, 1989
3 Cerzo D R, Suzuki H. Non-existence of imprimitive Q-polynomial schemes of exceptional type with $d = 4$. European J Combin, 2009, 30: 674–681
4 Cioabă S M, Koolen J H. On the connectedness of the complement of a ball in distance-regular graphs. J Algebraic Combin, 2013, 38: 191–195
5 Van Dam E R, Koolen J H, Tanaka H. Distance-regular graphs. ArXiv:1410.6294, 2014
6 Dickie G A. Q-polynomial structures for association schemes and distance regular graphs. PhD thesis. Madison: University of Wisconsin-Madison, 1995
7 Dickie G A, Terwilliger P. Dual bipartite Q-polynomial distance-regular graphs. European J Combin, 1996, 17: 613–623
8 Godsil C D. Algebraic Combinatorics. New York: Chapman and Hall, 1993
9 Godsil C D, Koolen J H. On the multiplicity of eigenvalues of distance-regular graphs. Linear Algebra Appl, 1995, 226–228: 273–275
10 Jurišić A, Koolen J H, Terwilliger P. Tight distance-regular graphs. J Algebraic Combin, 2000, 12: 163–197
11 Koolen J H, Kim J, Park J. Distance-regular graphs with a relatively small eigenvalue multiplicity. Elec J Combin, 2013, 20: 1
12 Koolen J H, Park J, et al. An inequality involving the second largest and smallest eigenvalue of a distance-regular graph. Linear Algebra Appl, 2011, 434: 2404–2412
13 Koolen J H, Park J. Distance-regular graphs with a_1 or c_2 at least half the valency. J Combin Theory Ser A, 2012, 119: 546–55
14 Ma J, Wang K. The nonexistence of exceptional 5-class association schemes with two Q-polynomial structures. Linear Algebra Appl, 2014, 440: 278–285
15 Pascasio A. Tight graphs and their primitive idempotents. J Algebraic Combin, 1999, 10: 47–59
16 Pascasio A. Tight distance-regular graphs and the Q-polynomial property. Graph Combin, 2001, 17: 149–169
17 Suzuki H. Imprimitive Q-polynomial association schemes. J Algebraic Combin, 1998, 7: 165–180
18 Suzuki H. Association schemes with multiple Q-polynomial structures. J Algebraic Combin, 1998, 7: 181–196
19 Tanaka H, Tanaka R. Nonexistence of exceptional imprimitive Q-polynomial association schemes with six classes. European J Combin, 2011, 32: 155–161