SCIENCE CHINA
Mathematics

• ARTICLES •

February 2015 Vol. 58 No. 2: 309–388
doi: 10.1007/s11425-014-4956-2

Hardy spaces H^p over non-homogeneous metric measure spaces and their applications

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Received August 7, 2014; accepted October 21, 2014; published online January 6, 2015

Abstract Let $(\mathcal{X}, d, \mu)$ be a metric measure space satisfying both the geometrically doubling and the upper doubling conditions. Let $\rho \in (1, \infty)$, $0 < p \leqslant 1 \leqslant q \leqslant \infty$, $p \neq q$, $\gamma \in [1, \infty)$ and $\epsilon \in (0, \infty)$. In this paper, the authors introduce the atomic Hardy space $\widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ and the molecular Hardy space $\widetilde{H}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)$ via the discrete coefficient $\widetilde{K}^{(\rho),\,p}_{B,\,S}$, and prove that the Calderón-Zygmund operator is bounded from $\widetilde{H}^{p,\,q,\,\gamma,\,\delta}_{\mathrm{mb},\,\rho}(\mu)$ (or $\widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$) into $L^p(\mu)$, and from $\widetilde{H}^{p,\,q,\,\gamma+1}_{\mathrm{atb},\,\rho(\rho+1)}(\mu)$ into $\widetilde{H}^{p,\,q,\,\gamma,\,\frac{1}{2}(\delta-\frac{\nu}{p}+\nu)}_{\mathrm{mb},\,\rho}(\mu)$. The boundedness of the generalized fractional integral T_β $(\beta \in (0,1))$ from $\widetilde{H}^{p_1,\,q,\,\gamma,\,\theta}_{\mathrm{mb},\,\rho}(\mu)$ (or $\widetilde{H}^{p_1,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$) into $L^{p_2}(\mu)$ with $1/p_2 = 1/p_1 - \beta$ is also established. The authors also introduce the ρ-weakly doubling condition, with $\rho \in (1, \infty)$, of the measure μ and construct a non-doubling measure μ satisfying this condition. If μ is ρ-weakly doubling, the authors further introduce the Campanato space $\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta,\,\gamma}(\mu)$ and show that $\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta,\,\gamma}(\mu)$ is independent of the choices of ρ, η, γ and q; the authors then introduce the atomic Hardy space $\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ and the molecular Hardy space $\widehat{H}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)$, which coincide with each other; the authors finally prove that $\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ is the predual of $\mathcal{E}^{1/p-1,\,1}_{\rho,\,\rho,\,1}(\mu)$. Moreover, if μ is doubling, the authors show that $\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta,\,\gamma}(\mu)$ and the Lipschitz space $\mathrm{Lip}_{\alpha,\,q}(\mu)$ $(q \in [1,\infty))$, or $\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ and the atomic Hardy space $H^{p,\,q}_{\mathrm{at}}(\mu)$ $(q \in (1,\infty])$ of Coifman and Weiss coincide. Finally, if $(\mathcal{X}, d, \mu)$ is an RD-space (reverse doubling space) with $\mu(\mathcal{X}) = \infty$, the authors prove that $\widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$, $\widetilde{H}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)$ and $H^{p,\,q}_{\mathrm{at}}(\mu)$ coincide for any $q \in (1, 2]$. In particular, when $(\mathcal{X}, d, \mu) := (\mathbb{R}^D, |\cdot|, dx)$ with dx being the D-dimensional Lebesgue measure, the authors show that spaces $\widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$, $\widetilde{H}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)$, $\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ and $\widehat{H}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)$ all coincide with $H^p(\mathbb{R}^D)$ for any $q \in (1, \infty)$.

Keywords non-homogeneous metric measure space, ρ-weakly doubling measure, Hardy space, Campanato space, Lipschitz space, Calderón-Zygmund operator, atomic block, molecular block

MSC(2010) 42B30, 42B20, 42B35, 30L99

Citation: Fu X, Lin H B, Yang D C, et al. Hardy spaces H^p over non-homogeneous metric measure spaces and their applications. Sci China Math, 2015, 58: 309–388, doi: 10.1007/s11425-014-4956-2

1 Introduction

It is well known that the real variable theory of Hardy spaces $H^p(\mathbb{R}^D)$ on the D-dimensional Euclidean space $\mathbb{R}^D$ has many important applications in various fields of analysis such as harmonic analysis and partial differential equations; see, for example, [12, 48–50]. When $p \in (1, \infty)$, $L^p(\mathbb{R}^D)$ and $H^p(\mathbb{R}^D)$ are essentially the same; however, when $p \in (0, 1]$, the space $H^p(\mathbb{R}^D)$ is much better adapted to problems

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arising in the theory of the boundedness of operators, since some of singular integrals (for example, Riesz transforms) are bounded on $H^p(\mathbb{R}^D)$, but not on $L^p(\mathbb{R}^D)$. In 1972, Fefferman and Stein [12] showed that the Hardy space $H^1(\mathbb{R}^D)$ is the predual of the bounded mean oscillation space $\mathrm{BMO}(\mathbb{R}^D)$. Later, Walsh [61] proved that the dual space of the Hardy space $H^p(\mathbb{R}^D)$ is the Campanato space introduced by Campanato [4]. From then on, various characterizations of $H^p(\mathbb{R}^D)$, including the atomic and the molecular characterizations, and their applications were studied extensively in harmonic analysis; see, for example, [5, 8, 9, 16, 32, 38, 40, 51, 64]. Moreover, the atomic and the molecular characterizations enabled the extension of the real variable theory of Hardy spaces on $\mathbb{R}^D$ to spaces of homogeneous type in the sense of Coifman and Weiss [10, 11], which is a far more general setting for function spaces and singular integrals than Euclidean spaces.

Recall that a metric space $(\mathcal{X}, d)$ equipped with a non-negative measure μ is called a *space of homogeneous type*, if $(\mathcal{X}, d, \mu)$ satisfies the *measure doubling condition*: There exists a positive constant $C_{(\mu)}$ such that, for all balls $B(x, r) := \{y \in \mathcal{X} : d(x, y) < r\}$ with $x \in \mathcal{X}$ and $r \in (0, \infty)$,

$$\mu(B(x, 2r)) \leqslant C_{(\mu)}\mu(B(x, r)). \tag{1.1}$$

This doubling condition on measures is one of the most crucial assumptions in the classical harmonic analysis. We point out that a space of homogeneous type in [10, 11] is endowed with a quasi-metric. However, for simplicity, throughout this article, we *always assume* that a space of homogeneous type is endowed with a metric.

Nevertheless, in recent years, it has been proved that many results in the classical theory of Hardy spaces and singular integrals on $\mathbb{R}^D$ remain valid with the D-dimensional Lebesgue measure replaced by a non-doubling measure (see, for example, [6, 24, 41, 53–59]). Recall that a Radon measure μ on $\mathbb{R}^D$ is called a *non-doubling measure*, if there exist positive constants C_0 and $\kappa \in (0, D]$ such that, for all $x \in \mathbb{R}^D$ and $r \in (0, \infty)$,

$$\mu(B(x, r)) \leqslant C_0 r^\kappa, \tag{1.2}$$

where $B(x, r) := \{y \in \mathbb{R}^D : |y - x| < r\}$. Tolsa [54, 55] introduced the atomic Hardy space $H^{1,q}_{\mathrm{atb}}(\mu)$, for $q \in (1, \infty]$, and its dual space, $\mathrm{RBMO}(\mu)$, the *space of functions with regularized bounded mean oscillation*, with respect to μ as in (1.2), and proved that Calderón-Zygmund operators are bounded from $H^{1,q}_{\mathrm{atb}}(\mu)$ into $L^1(\mu)$. Later, Chen et al. [6] showed that Calderón-Zygmund operators are bounded on $H^{1,q}_{\mathrm{atb}}(\mu)$. Hu et al. [24] established an equivalent characterization of $H^{1,q}_{\mathrm{atb}}(\mu)$ to obtain the $L^q(\mu)$-boundedness of commutators and their endpoint estimates. More research on function spaces, mainly on Morrey spaces, and their applications related to non-doubling measures can be found in [20, 42–47]. We point out that the analysis on such non-doubling context plays a striking role in solving several long-standing problems related to the analytic capacity, like Vitushkin's conjecture or Painlevé's problem; see [56–59].

However, as was pointed out by Hytönen [27], the measure satisfying (1.2) is different from, but not more general than, the doubling measure. Hytönen [27] introduced a new class of metric measure spaces satisfying the so-called geometrically doubling and the upper doubling conditions (see, respectively, Definitions 2.1 and 2.3 below), which are also simply called *non-homogeneous metric measure spaces*. This new class of non-homogeneous metric measure spaces includes both spaces of homogeneous type and metric spaces with non-doubling measures as special cases. It is already known that singular integrals on non-homogeneous metric measure spaces arise naturally in the study of complex and harmonic analysis questions in several complex variables (see [29, 60] for the details).

In this new setting, Hytönen [27] introduced the space $\mathrm{RBMO}(\mu)$ and established the corresponding John-Nirenberg inequality. Later, Hytönen et al. [30], and Bui and Duong [3], independently, introduced the atomic Hardy space $H^{1,q}_{\mathrm{atb}}(\mu)$ and proved that the dual space of $H^{1,q}_{\mathrm{atb}}(\mu)$ is $\mathrm{RBMO}(\mu)$. Hytönen et al. [28] and Liu et al. [37] established some equivalent characterizations for the boundedness of Calderón-Zygmund operators on $L^q(\mu)$ with $q \in (1, \infty)$ and their endpoint boundedness. Fu et al. [13] introduced a version of the atomic Hardy space $\widetilde{H}^{1,q}_{\mathrm{atb}}(\mu)$ $(\subset H^{1,q}_{\mathrm{atb}}(\mu))$ via the discrete coefficients $\widetilde{K}^{(\rho)}_{B,S}$, and showed that the Calderón-Zygmund operator is bounded on $\widetilde{H}^{1,q}_{\mathrm{atb}}(\mu)$ via establishing a molecular characterization of

$\widetilde{H}^{1,q}_{\rm atb}(\mu)$ in this context. Recently, Fu et al. [15] introduced generalized fractional integrals and established the boundedness of generalized fractional integrals and their commutators in this setting. More research on the boundedness of various operators on non-homogeneous metric measure spaces can be found in [1, 26, 34–36]. We refer the reader to the survey [62] and the monograph [63] for more progress on the theory of Hardy spaces and singular integrals over non-homogeneous metric measure spaces.

We point out that the space $\widetilde{H}^{1,q}_{\rm atb}(\mu)$ seems to be more useful in the study on the boundedness of operators, since it was shown in [13, Theorem 1.4] that Calderón-Zygmund operators are bounded on $\widetilde{H}^{1,q}_{\rm atb}(\mu)$, but the method does not work for the boundedness of Calderón-Zygmund operators on $H^{1,q}_{\rm atb}(\mu)$ over general non-homogeneous metric measure spaces defined via the continuous coefficients (see [13, Remark 2.4] or Remark 4.2(iv) below).

To the best of our knowledge, the theory of the Hardy space H^p on non-homogeneous metric measure spaces is still unknown, even on Euclidean spaces endowed with non-doubling measures. Let $(\mathcal{X}, d, \mu)$ be a non-homogeneous metric measure space in the sense of Hytönen [27]. The main purposes of this article are two-fold. First, via the discrete coefficients $\widetilde{K}^{(\rho),p}_{B,S}$, we introduce the atomic Hardy space $\widetilde{H}^{p,q,\gamma}_{{\rm atb},\rho}(\mu)$ and the molecular Hardy space $\widetilde{H}^{p,q,\gamma,\epsilon}_{{\rm mb},\rho}(\mu)$, and give their applications to the boundedness of Calderón-Zygmund operators and generalized fractional integrals. However, it is still unknown whether $\widetilde{H}^{p,q,\gamma}_{{\rm atb},\rho}(\mu)$ is independent of the choices of ρ, γ and q or not even under some additional condition, called the ρ-weakly doubling condition (see Definition 6.1 below). Moreover, the dual space of $\widetilde{H}^{p,q,\gamma}_{{\rm atb},\rho}(\mu)$ and the equivalence between $\widetilde{H}^{p,q,\gamma}_{{\rm atb},\rho}(\mu)$ and $\widetilde{H}^{p,q,\gamma,\epsilon}_{{\rm mb},\rho}(\mu)$ are also unclear. Thus, we are forced to turn to the other goal: Introduce another atomic Hardy space $\widehat{H}^{p,q,\gamma}_{{\rm atb},\rho}(\mu)$ and another molecular Hardy space $\widehat{H}^{p,q,\gamma,\epsilon}_{{\rm mb},\rho}(\mu)$, and then show that $\widehat{H}^{p,q,\gamma}_{{\rm atb},\rho}(\mu)$ is independent of the choices of ρ and γ under the ρ-weakly doubling condition. Then we study the Campanato space $\mathcal{E}^{\alpha,q}_{\rho,\eta,\gamma}(\mu)$, the dual space of $\widehat{H}^{p,q,\gamma}_{{\rm atb},\rho}(\mu)$, and the equivalence between $\widehat{H}^{p,q,\gamma}_{{\rm atb},\rho}(\mu)$ and $\widehat{H}^{p,q,\gamma,\epsilon}_{{\rm mb},\rho}(\mu)$ if μ is ρ-weakly doubling. Moreover, if μ is doubling, we show that $\mathcal{E}^{\alpha,q}_{\rho,\eta,\gamma}(\mu)$ and the Lipschitz space ${\rm Lip}_{\alpha,q}(\mu)$ $(q \in [1,\infty))$, or $\widehat{H}^{p,q,\gamma}_{{\rm atb},\rho}(\mu)$ and the atomic Hardy space $H^{p,q}_{\rm at}(\mu)$ $(q \in (1,\infty])$ introduced by Coifman and Weiss [11] coincide with equivalent quasi-norms. Finally, if $(\mathcal{X}, d, \mu)$ is an RD-space (reverse doubling space) with $\mu(\mathcal{X}) = \infty$, we prove that $\widetilde{H}^{p,q,\gamma}_{{\rm atb},\rho}(\mu)$, $\widetilde{H}^{p,q,\gamma,\epsilon}_{{\rm mb},\rho}(\mu)$ and $H^{p,q}_{\rm at}(\mu)$ coincide for any $q \in (1,2]$, which is still unknown if $q \in (2,\infty]$. In particular, when $(\mathcal{X}, d, \mu) := (\mathbb{R}^D, |\cdot|, dx)$ with dx being the D-dimensional Lebesgue measure, we show that the spaces $\widetilde{H}^{p,q,\gamma}_{{\rm atb},\rho}(\mu)$, $\widetilde{H}^{p,q,\gamma,\epsilon}_{{\rm mb},\rho}(\mu)$, $\widehat{H}^{p,q,\gamma}_{{\rm atb},\rho}(\mu)$ and $\widehat{H}^{p,q,\gamma,\epsilon}_{{\rm mb},\rho}(\mu)$ all coincide with $H^p(\mathbb{R}^D)$ for any $q \in (1,\infty)$.

The organization of this article is as follows.

In Section 2, we first recall some necessary notation and notions, including the discrete coefficient $\widetilde{K}^{(\rho),p}_{B,S}$, and give out some fundamental properties on $\widetilde{K}^{(\rho),p}_{B,S}$ which are crucial to the succeeding content.

In Section 3, we introduce the atomic Hardy space $\widetilde{H}^{p,q,\gamma}_{{\rm atb},\rho}(\mu)$ via the discrete coefficient $\widetilde{K}^{(\rho),p}_{B,S}$ $(\widetilde{K}^{(\rho),1}_{B,S} = \widetilde{K}^{(\rho)}_{B,S})$, where the dominating function of the considered measure appears in the size condition of the atomic block, which seems to be well adapted to the study of the boundedness of Caldeón-Zygmund operators and generalized fractional integrals, and establish a useful property. The key innovation in this section is the definition of $\widetilde{H}^{p,q,\gamma}_{{\rm atb},\rho}(\mu)$ as the completeness of a subspace of $L^2(\mu)$, $\widetilde{\mathbb{H}}^{p,q,\gamma}_{{\rm atb},\rho}(\mu)$, which is a suitable substitute of the classical fact that the set of all Schwartz functions having infinite order vanishing moments is dense in the Hardy space $H^p(\mathbb{R}^D)$.

In Section 4, we introduce the notion of the molecular Hardy space $\widetilde{H}^{p,q,\gamma,\epsilon}_{{\rm mb},\rho}(\mu)$, and prove that the Calderón-Zygmund operator is bounded from $\widetilde{H}^{p,q,\gamma,\delta}_{{\rm mb},\rho}(\mu)$ (or $\widetilde{H}^{p,q,\gamma}_{{\rm atb},\rho}(\mu)$) into $L^p(\mu)$ by borrowing some ideas from [30, Theorem 4.2] with much more complicated arguments, and from $\widetilde{H}^{p,q,\gamma+1}_{{\rm atb},\rho(\rho+1)}(\mu)$ into $\widetilde{H}^{p,q,\gamma,(\delta-\nu/p+\nu)/2}_{{\rm mb},\rho}(\mu)$ by using a method similar to that used in the proof of [13, Theorem 1.14] with some technical modifications.

In Section 5, we establish the boundedness of the generalized fractional integral T_β $(\beta \in (0,1))$ from $\widetilde{H}^{p_1,q,\gamma,\theta}_{{\rm mb},\rho}(\mu)$ (or $\widetilde{H}^{p_1,q,\gamma}_{{\rm atb},\rho}(\mu)$) into $L^{p_2}(\mu)$ with $1/p_2 = 1/p_1 - \beta$. The proof of the above result is parallel to that of the conclusion for Calderón-Zygmund operators in Section 4 with slight modifications. For the sake of the clearness, we present the full details there.

Section 6 is mainly devoted to the theory of Campanato spaces. We first introduce an additional assumption, called the ρ-*weakly doubling condition* (see (6.1) below), which is satisfied by spaces of homogeneous type. We also construct a non-trivial example to show that there exist some non-homogeneous metric measure spaces satisfying the ρ-weakly doubling condition (6.1); see Example 6.3 below. However, it turns out that there exist many non-homogeneous metric measure spaces which do not satisfy the ρ-weakly doubling condition; see Example 6.4 below. Then we introduce the Campanato space $\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta,\,\gamma}(\mu)$ and show that $\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta,\,\gamma}(\mu)$ is independent of the choices of ρ, η, γ and q under the assumption of ρ-weakly doubling conditions. Precisely, via a useful property of $\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta,\,\gamma}(\mu)$ (see Proposition 6.7(a) below) and the geometrically doubling condition, we prove that $\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta,\,\gamma}(\mu)$ is independent of the choices of ρ and η, where the ρ-weakly doubling condition plays a decisive role. Then, by establishing an equivalent characterization of $\mathcal{E}^{\alpha,\,q}_{\rho,\,\gamma}(\mu) := \mathcal{E}^{\alpha,\,q}_{\rho,\,\rho,\,\gamma}(\mu)$ and a useful lemma (see Lemma 6.12 below), which is analogous to [30, Lemma 2.7], and by borrowing some ideas from the proof of [30, Proposition 2.5], we show that $\mathcal{E}^{\alpha,\,q}_{\rho,\,\gamma}(\mu)$ is independent of the choice of γ. Next, by the above equivalent characterization of $\mathcal{E}^{\alpha,\,q}_{\rho,\,\gamma}(\mu)$ and the ρ-weakly doubling condition, we establish the John-Nirenberg inequality for $\mathcal{E}^{\alpha,\,q}_{\rho}(\mu) := \mathcal{E}^{\alpha,\,q}_{\rho,\,1}(\mu)$, which further implies that $\mathcal{E}^{\alpha,\,q}_{\rho}(\mu)$ is independent of the choice of q. We point out that, on spaces of homogeneous type, the independence of q of $\mathcal{E}^{\alpha,\,q}_{\rho}(\mu)$ is due to the coincidence between $\mathcal{E}^{\alpha,\,q}_{\rho}(\mu)$ and the Lipschitz space $\mathrm{Lip}_{\alpha}(\mu)$; see [39]. However, this coincidence is unknown on non-homogeneous metric measure spaces, even under the ρ-weakly doubling condition. Alternatively, we adopt the method developed by Hytönen for the proof of the John-Nirenberg inequality for the BMO type space in [27]; see also [54]. At the end of this section, we establish another useful characterization of $\mathcal{E}^{\alpha}_{\rho}(\mu) := \mathcal{E}^{\alpha,\,1}_{\rho}(\mu)$, which plays important roles in the later context.

In Section 7, we introduce the atomic Hardy space $\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ and the molecular Hardy space $\widehat{H}^{p,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)$ and investigate their relation under the ρ-weakly doubling condition. By using a method similar to that used in the proof of [13, Theorem 1.11], together with some technical modifications, we prove that $\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ and $\widehat{H}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)$ coincide with equivalent quasi-norms. It is still unclear whether the above result holds true or not on general non-homogeneous metric measure spaces, even on Euclidean spaces with non-doubling measures.

Section 8 is mainly devoted to investigating the dual space of $\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ under the ρ-weakly doubling condition. To this end, we first show that $\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ is independent of the choices of ρ and γ. Precisely, by the ρ-weakly doubling condition and borrowing some ideas from the proof of [30, Proposition 3.3(ii)], we first prove that $\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ is independent of the choice of ρ. By establishing the corresponding result (see Lemma 6.11 below) to [54, Lemma 9.2] and constructing a sequence of (ρ,β_ρ)-doubling balls which is a refinement of that appearing in the proof of [54, Lemma 9.3], we further show that $\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ is independent of γ. Finally, via the independence of ρ for $\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ and the equivalent characterization of $\mathcal{E}^{\alpha}_{\rho}(\mu) := \mathcal{E}^{\alpha,\,1}_{\rho}(\mu)$ established in Section 6 (see Proposition 6.18 below), we show that $\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ is the predual of $\mathcal{E}^{1/p-1}_{\rho}(\mu)$. It is still unknown whether the above results hold true or not on general non-homogeneous metric measure spaces, even on Euclidean spaces with non-doubling measures.

In Section 9, let $(\mathcal{X},d,\mu)$ be a space of homogeneous type in the sense of Coifman and Weiss. We investigate the relations between the Campanato space $\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta,\,\gamma}(\mu)$ and the Lipschitz space $\mathrm{Lip}_{\alpha,\,q}(\mu)$, or between $\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ and the atomic Hardy space $H^{p,\,q}_{\mathrm{at}}(\mu)$ introduced by Coifman and Weiss [11]. By carefully dividing the situation into several parts, constructing a sequence of balls via using a method similar to that used in the proof of the independence of γ for $\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ in Section 6 and adopting some ideas from [27, Proposition 4.7], we show that, if $q\in[1,\infty)$, then $\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta,\,\gamma}(\mu)$ and $\mathrm{Lip}_{\alpha,\,q}(\mu)$ coincide with equivalent norms. By a method similar to that used in the proof of this result, we also establish the coincidence of $\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ and $H^{p,\,q}_{\mathrm{at}}(\mu)$ for any $q\in(1,\infty]$ directly.

In Section 10, suppose that $(\mathcal{X},d,\mu)$ is an RD-space with $\mu(\mathcal{X})=\infty$ and $q\in(1,2]$. We show that $\widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$, $\widetilde{H}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)$ and $H^{p,\,q}_{\mathrm{at}}(\mu)$ coincide. Let $\widetilde{\mathbb{H}}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ and $\widetilde{\mathbb{H}}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)$ be dense subspaces of $\widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ and $\widetilde{H}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)$, respectively (see Definitions 3.2 and 4.1 below). We prove that

$$(H^{p,\,q}_{\mathrm{at}}(\mu)\cap L^2(\mu))\subset\widetilde{\mathbb{H}}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)\subset\widetilde{\mathbb{H}}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)\subset(H^{p,\,q}_{\mathrm{at}}(\mu)\cap L^2(\mu))$$

by two steps. In Step 1, to show that $(H^{p,\,q}_{\rm at}(\mu)\cap L^2(\mu))\subset \widetilde{\mathbb{H}}^{p,\,q,\,\gamma}_{{\rm atb},\,\rho}(\mu)$ for any $q\in(1,2]$, we first establish a technical lemma (see Lemma 10.2 below). Then we establish a useful criterion for the boundedness of some integral operators (see Lemma 10.8 below). Via this, a standard duality argument, the Calderón reproducing formula and the boundedness of the Littlewood-Paley g-function on $L^2(\mu)$ obtained in [21], we give out a key atomic decomposition for all functions from $H^{p,\,q}_{\rm at}(\mu)\cap L^2(\mu)$ in $L^2(\mu)$ (see (10.5) below), which plays an essential role in the proof of Step 1. In Step 2, via the fact that $\widetilde{\mathbb{H}}^{p,\,q,\,\gamma}_{{\rm atb},\,\rho}(\mu)\subset \widetilde{\mathbb{H}}^{p,\,q,\,\gamma,\,\epsilon}_{{\rm mb},\,\rho}(\mu)$ (see Proposition 4.3 below) and establishing the boundedness of the Littlewood-Paley S-function from $\widetilde{\mathbb{H}}^{p,\,q,\,\gamma,\,\epsilon}_{{\rm mb},\,\rho}(\mu)$ into $L^p(\mu)$, we conclude that, for any $q\in(1,\infty)$,

$$\widetilde{\mathbb{H}}^{p,\,q,\,\gamma}_{{\rm atb},\,\rho}(\mu)\subset \widetilde{\mathbb{H}}^{p,\,q,\,\gamma,\,\epsilon}_{{\rm mb},\,\rho}(\mu)\subset (H^p(\mu)\cap L^2(\mu))=(H^{p,\,q}_{\rm at}(\mu)\cap L^2(\mu)),$$

where $H^p(\mu)$ is defined by the Littlewood-Paley S-function as in [19, 21]. By these two steps and a standard density argument, we obtain the desired result. Due to the defects of the above boundedness of the Littlewood-Paley g-function and the criterion for the boundedness of some integral operators, it is still unclear whether $\widetilde{H}^{p,\,q,\,\gamma,\,\epsilon}_{{\rm mb},\,\rho}(\mu)$ (or $\widetilde{H}^{p,\,q,\,\gamma}_{{\rm atb},\,\rho}(\mu))=H^{p,\,q}_{\rm at}(\mu)$ over RD-spaces $(\mathcal{X},d,\mu)$ with $\mu(\mathcal{X})=\infty$ for $q\in(2,\infty]$. Finally, if $(\mathcal{X},d,\mu):=(\mathbb{R}^D,|\cdot|,dx)$ with dx being the D-dimensional Lebesgue measure, we prove that the spaces $\widetilde{H}^{p,\,q,\,\gamma}_{{\rm atb},\,\rho}(\mu)$, $\widetilde{H}^{p,\,q,\,\gamma,\,\epsilon}_{{\rm mb},\,\rho}(\mu)$, $\widehat{H}^{p,\,q,\,\gamma}_{{\rm atb},\,\rho}(\mu)$ and $\widehat{H}^{p,\,q,\,\gamma,\,\epsilon}_{{\rm mb},\,\rho}(\mu)$ all coincide with $H^p(\mathbb{R}^D)$ for any $q\in(1,\infty)$.

Finally, we make some conventions on notation. Throughout this article, C stands for a *positive constant* which is independent of the choices of the main parameters, but it may vary from line to line. *Constants with subscripts*, such as C_0, do not change in different occurrences. Furthermore, we use $C_{(\rho,\alpha,\ldots)}$ to denote a positive constant depending on parameters $\rho,\alpha,\ldots$ Let $\mathbb{N}:=\{1,2,\ldots\}$ and $\mathbb{Z}_+:=\{0\}\cup\mathbb{N}$. For any ball B, the center and the radius of B are denoted, respectively, by c_B and r_B. For any subset E of $\mathcal{X}$, we use χ_E to denote its *characteristic function*.

2 Preliminaries

In this section, we recall some necessary notation and notions, including the discrete coefficient $\widetilde{K}^{(\rho),\,p}_{B,\,S}$, and give out some fundamental properties on $\widetilde{K}^{(\rho),\,p}_{B,\,S}$ in the non-homogeneous context.

The following notion of the geometrically doubling is well known in analysis on metric spaces, which was originally introduced by Coifman and Weiss [10, pp. 66–67] and is also known as *metrically doubling* (see, for example, [23, p. 81]).

Definition 2.1. A metric space $(\mathcal{X},d)$ is said to be *geometrically doubling* if there exists some $N_0\in\mathbb{N}$ such that, for any ball $B(x,r)\subset\mathcal{X}$ with $x\in\mathcal{X}$ and $r\in(0,\infty)$, there exists a finite ball covering $\{B(x_i,r/2)\}_i$ of $B(x,r)$ such that the cardinality of this covering is at most N_0.

Remark 2.2. Let $(\mathcal{X},d)$ be a metric space. Hytönen [27] showed that the following statements are mutually equivalent:

(i) $(\mathcal{X},d)$ is geometrically doubling.

(ii) For any $\epsilon\in(0,1)$ and any ball $B(x,r)\subset\mathcal{X}$ with $x\in\mathcal{X}$ and $r\in(0,\infty)$, there exists a finite ball covering $\{B(x_i,\epsilon r)\}_i$ of $B(x,r)$ such that the cardinality of this covering is at most $N_0\epsilon^{-n_0}$, here and hereafter, N_0 is as in Definition 2.1 and $n_0:=\log_2 N_0$.

(iii) For every $\epsilon\in(0,1)$, any ball $B(x,r)\subset\mathcal{X}$ with $x\in\mathcal{X}$ and $r\in(0,\infty)$ contains at most $N_0\epsilon^{-n_0}$ centers of disjoint balls $\{B(x_i,\epsilon r)\}_i$.

(iv) There exists $M\in\mathbb{N}$ such that any ball $B(x,r)\subset\mathcal{X}$ with $x\in\mathcal{X}$ and $r\in(0,\infty)$ contains at most M centers $\{x_i\}_i$ of disjoint balls $\{B(x_i,r/4)\}_{i=1}^M$.

Recall that spaces of homogeneous type are geometrically doubling, which was proved by Coifman and Weiss [10, pp. 66–68].

The following notion of upper doubling metric measure spaces was originally introduced by Hytönen [27] (see also [28, 37]).

Definition 2.3. A metric measure space $(\mathcal{X}, d, \mu)$ is said to be *upper doubling* if μ is a Borel measure on $\mathcal{X}$ and there exist a *dominating function* $\lambda : \mathcal{X} \times (0, \infty) \to (0, \infty)$ and a positive constant $C_{(\lambda)}$, depending on λ, such that, for each $x \in \mathcal{X}$, $r \to \lambda(x, r)$ is non-decreasing and, for all $x \in \mathcal{X}$ and $r \in (0, \infty)$,

$$\mu(B(x, r)) \leqslant \lambda(x, r) \leqslant C_{(\lambda)}\lambda(x, r/2). \tag{2.1}$$

A metric measure space $(\mathcal{X}, d, \mu)$ is called a *non-homogeneous metric measure space* if $(\mathcal{X}, d)$ is geometrically doubling and $(\mathcal{X}, d, \mu)$ is upper doubling.

Remark 2.4. (i) Obviously, a space of homogeneous type is a special case of upper doubling spaces, where we take the dominating function $\lambda(x, r) := \mu(B(x, r))$ for all $x \in \mathcal{X}$ and $r \in (0, \infty)$. On the other hand, the D-dimensional Euclidean space $\mathbb{R}^D$ with any Radon measure μ as in (1.2) is also an upper doubling space by taking $\lambda(x, r) := C_0 r^\kappa$ for all $x \in \mathbb{R}^D$ and $r \in (0, \infty)$.

(ii) Let $(\mathcal{X}, d, \mu)$ be upper doubling with λ being the dominating function on $\mathcal{X} \times (0, \infty)$ as in Definition 2.3. It was proved in [30] that there exists another dominating function $\widetilde{\lambda}$ such that $\widetilde{\lambda} \leqslant \lambda$, $C_{(\widetilde{\lambda})} \leqslant C_{(\lambda)}$ and, for all x, $y \in \mathcal{X}$ with $d(x, y) \leqslant r$,

$$\widetilde{\lambda}(x, r) \leqslant C_{(\widetilde{\lambda})}\widetilde{\lambda}(y, r). \tag{2.2}$$

(iii) It was shown in [52] that the upper doubling condition is equivalent to the *weak growth condition*: there exist a dominating function $\lambda : \mathcal{X} \times (0, \infty) \to (0, \infty)$, with $r \to \lambda(x, r)$ non-decreasing, positive constants $C_{(\lambda)}$, depending on λ, and ϵ such that

(iii)$_1$ for all $r \in (0, \infty)$, $t \in [0, r]$, x, $y \in \mathcal{X}$ and $d(x, y) \in [0, r]$,

$$|\lambda(y, r + t) - \lambda(x, r)| \leqslant C_{(\lambda)} \left[\frac{d(x, y) + t}{r}\right]^\epsilon \lambda(x, r);$$

(iii)$_2$ for all $x \in \mathcal{X}$ and $r \in (0, \infty)$, $\mu(B(x, r)) \leqslant \lambda(x, r)$.

Based on Remark 2.4(ii), from now on, we *always assume* that $(\mathcal{X}, d, \mu)$ is a non-homogeneous metric measure space with the dominating function λ satisfying (2.2).

Though the measure doubling condition is not assumed uniformly for all balls in the non-homogeneous metric measure space $(\mathcal{X}, d, \mu)$, it was shown in [27] that there still exist many balls which have the following (α, β)-doubling property.

Definition 2.5. Let α, $\beta \in (1, \infty)$. A ball $B \subset \mathcal{X}$ is said to be (α, β)-*doubling* if $\mu(\alpha B) \leqslant \beta\mu(B)$, where, for any ball $B := B(c_B, r_B)$ and $\rho \in (0, \infty)$, $\rho B := B(c_B, \rho r_B)$.

To be precise, it was proved in [27, Lemma 3.2] that, if a metric measure space $(\mathcal{X}, d, \mu)$ is upper doubling and α, $\beta \in (1, \infty)$ with $\beta > [C_{(\lambda)}]^{\log_2 \alpha} =: \alpha^\nu$, then, for any ball $B \subset \mathcal{X}$, there exists some $j \in \mathbb{Z}_+$ such that $\alpha^j B$ is (α, β)-doubling. Moreover, let $(\mathcal{X}, d)$ be geometrically doubling, $\beta > \alpha^{n_0}$ with $n_0 := \log_2 N_0$ and μ a Borel measure on $\mathcal{X}$ which is finite on bounded sets. Hytönen [27, Lemma 3.3] also showed that, for μ-almost every $x \in \mathcal{X}$, there exist arbitrary small (α, β)-doubling balls centered at x. Furthermore, the radii of these balls may be chosen to be of the form $\alpha^{-j} r$ for $j \in \mathbb{N}$ and any preassigned number $r \in (0, \infty)$. Throughout this article, for any $\alpha \in (1, \infty)$ and ball B, the *smallest* (α, β_α)-*doubling ball of the form* $\alpha^j B$ *with* $j \in \mathbb{Z}_+$ is denoted by $\widetilde{B}^\alpha$, where

$$\beta_\alpha := \alpha^{3(\max\{n_0,\, \nu\})} + [\max\{5\alpha,\, 30\}]^{n_0} + [\max\{3\alpha, 30\}]^\nu.$$

Before we introduce the discrete coefficient $\widetilde{K}^{(\rho),\,p}_{B,\,S}$, we first give an assumption on the relation between two balls B and S, which is *supposed to hold true through the whole article*:

(A) If $B = S$, then $c_B = c_S$ and $r_B = r_S$.

Then we claim that, if $B \subset S$, then $r_B \leqslant 2r_S$. Indeed, assume that $r_B > 2r_S$. By this and $B \subset S$, together with the triangle inequality satisfied by d, we see that $S \subset B$. Thus, $B = S$, which, together with the assumption (A), implies that $r_B = r_S$. This contradicts to $r_B > 2r_S$, which completes the proof of the above claim.

On the other hand, we give a simple example to illustrate that, if $B \subsetneqq S$, then it may happen that $r_B > r_S$. Let $(\mathcal{X}, d) := (\{-1, 1, 3\}, |\cdot|)$, $B := \{x \in \{-1, 1, 3\} : |x + 1| < 3\}$ and

$$S := \left\{ x \in \{-1, 1, 3\} : |x - 1| < \frac{5}{2} \right\}.$$

Obviously, $r_B > r_S$ and $B = \{-1, 1\} \subsetneqq \{-1, 1, 3\} = S$.

Definition 2.6. For any $\rho \in (1, \infty)$, $p \in (0, 1]$ and any two balls $B \subset S \subset \mathcal{X}$, let

$$\widetilde{K}_{B,S}^{(\rho),p} := \left\{ 1 + \sum_{k=-\lfloor \log_\rho 2 \rfloor}^{N_{B,S}^{(\rho)}} \left[\frac{\mu(\rho^k B)}{\lambda(c_B, \rho^k r_B)} \right]^p \right\}^{1/p}, \tag{2.3}$$

here and hereafter, for any $a \in \mathbb{R}$, $\lfloor a \rfloor$ represents the *biggest integer which is not bigger than* a, and $N_{B,S}^{(\rho)}$ is the *smallest integer* satisfying $\rho^{N_{B,S}^{(\rho)}} r_B \geqslant r_S$.

Remark 2.7. (i) By a change of variables and (2.1), we easily conclude that

$$\widetilde{K}_{B,S}^{(\rho),p} \sim \left\{ 1 + \sum_{k=1}^{N_{B,S}^{(\rho)} + \lfloor \log_\rho 2 \rfloor + 1} \left[\frac{\mu(\rho^k B)}{\lambda(c_B, \rho^k r_B)} \right]^p \right\}^{1/p},$$

where the implicit equivalent positive constants are independent of balls $B \subset S \subset \mathcal{X}$, but depend on ρ and p.

(ii) A continuous version, $K_{B,S}$, of the coefficient in Definition 2.6 when $p = 1$ was introduced in [27,30] as follows: For any two balls $B \subset S \subset \mathcal{X}$,

$$K_{B,S} := 1 + \int_{(2S)\setminus B} \frac{1}{\lambda(c_B, d(x, c_B))} d\mu(x). \tag{2.4}$$

It was proved in [30, Lemma 2.2] that $K_{B,S}$ has all properties similar to those for $\widetilde{K}_{B,S}^{(\rho),p}$ as in Lemma 2.8 below. Unfortunately, $K_{B,S}$ and $\widetilde{K}_{B,S}^{(\rho),1}$ are usually not equivalent, but this is true for $(\mathbb{R}^D, |\cdot|, \mu)$ with μ as in (1.2); see [13] for more details on this.

Now we give some simple properties of $\widetilde{K}_{B,S}^{(\rho),p}$ defined by (2.3) adapted from [14, Lemma 3.5], in which $\rho = 6$ and $p = 1$. The arguments therein are still valid for the present case. For the sake of reader's convenience, we present some details. In what follows, for any $a \in \mathbb{R}$, $\lceil a \rceil$ represents the *smallest integer which is not smaller than* a.

Lemma 2.8. *Let $(\mathcal{X}, d, \mu)$ be a non-homogeneous metric measure space and $p \in (0, 1]$.*

(i) *For any $\rho \in (1, \infty)$, there exists a positive constant $C_{(\rho)}$, depending on ρ, such that, for all balls $B \subset R \subset S$, $[\widetilde{K}_{B,R}^{(\rho),p}]^p \leqslant C_{(\rho)} [\widetilde{K}_{B,S}^{(\rho),p}]^p$.*

(ii) *For any $\alpha \in [1, \infty)$ and $\rho \in (1, \infty)$, there exists a positive constant $C_{(\alpha,\rho)}$, depending on α and ρ, such that, for all balls $B \subset S$ with $r_S \leqslant \alpha r_B$, $[\widetilde{K}_{B,S}^{(\rho),p}]^p \leqslant C_{(\alpha,\rho)}$.*

(iii) *For any $\rho \in (1, \infty)$, there exists a positive constant $C_{(\rho,p,\nu)}$, depending on ρ, p and ν, such that, for all balls B, $[\widetilde{K}_{B,\widetilde{B}^\rho}^{(\rho),p}]^p \leqslant C_{(\rho,p,\nu)}$. Moreover, letting $\alpha, \beta \in (1, \infty)$, $B \subset S$ be any two concentric balls such that there exists no (α, β)-doubling ball in the form of $\alpha^k B$, with $k \in \mathbb{N}$, satisfying $B \subset \alpha^k B \subset S$, then there exists a positive constant $C_{(\alpha,\beta,p,\nu)}$, depending on α, β, p and ν, such that $[\widetilde{K}_{B,S}^{(\alpha),p}]^p \leqslant C_{(\alpha,\beta,p,\nu)}$.*

(iv) *For any $\rho \in (1, \infty)$, there exists a positive constant $c_{(\rho,p,\nu)}$, depending on ρ, p and ν, such that, for all balls $B \subset R \subset S$,*

$$[\widetilde{K}_{B,S}^{(\rho),p}]^p \leqslant [\widetilde{K}_{B,R}^{(\rho),p}]^p + c_{(\rho,p,\nu)} [\widetilde{K}_{R,S}^{(\rho),p}]^p.$$

(v) *For any $\rho \in (1, \infty)$, there exists a positive constant $\widetilde{c}_{(\rho,p,\nu)}$, depending on ρ, p and ν, such that, for all balls $B \subset R \subset S$, $[\widetilde{K}_{R,S}^{(\rho),p}]^p \leqslant \widetilde{c}_{(\rho,p,\nu)} [\widetilde{K}_{B,S}^{(\rho),p}]^p$.*

Proof. Fix $p \in (0,1]$, $\rho \in (1,\infty)$ and $\alpha \in [1,\infty)$. We first show (i). By $R \subset S$, we have $r_R \leqslant 2r_S$. Hence,

$$r_R \leqslant 2r_S \leqslant 2\rho^{N_{B,S}^{(\rho)}} r_B \leqslant \rho^{\lceil \log_\rho 2\rceil + N_{B,S}^{(\rho)}} r_B.$$

Thus, $N_{B,R}^{(\rho)} \leqslant \lceil \log_\rho 2\rceil + N_{B,S}^{(\rho)}$. By this and (2.1), we see that

$$\begin{aligned}[\widetilde{K}_{B,R}^{(\rho),p}]^p &\leqslant 1 + \sum_{k=-\lfloor \log_\rho 2\rfloor}^{\lceil \log_\rho 2\rceil + N_{B,S}^{(\rho)}} \left[\frac{\mu(\rho^k B)}{\lambda(c_B,\rho^k r_B)}\right]^p \\ &\leqslant 1 + \sum_{k=-\lfloor \log_\rho 2\rfloor}^{N_{B,S}^{(\rho)}} \left[\frac{\mu(\rho^k B)}{\lambda(c_B,\rho^k r_B)}\right]^p + \lceil \log_\rho 2\rceil \leqslant (1+\lceil \log_\rho 2\rceil)[\widetilde{K}_{B,S}^{(\rho),p}]^p.\end{aligned}$$

This shows (i).

Now we prove (ii). By the fact that $\rho^{N_{B,S}^{(\rho)}-1} r_B < r_S \leqslant \alpha r_B$, we have $N_{B,S}^{(\rho)} - 1 < \log_\rho \alpha$. Thus, $N_{B,S}^{(\rho)} - 1 \leqslant \lfloor \log_\rho \alpha\rfloor$. From this and (2.1), we deduce that

$$[\widetilde{K}_{B,S}^{(\rho),p}]^p \leqslant 1 + N_{B,S}^{(\rho)} + \lfloor \log_\rho 2\rfloor \leqslant 2 + \lfloor \log_\rho \alpha\rfloor + \lfloor \log_\rho 2\rfloor.$$

Thus, (ii) holds true.

Let us now prove (iii). To this end, let N be the first integer such that $\rho^N B$ is (ρ,β_ρ)-doubling. If B is (ρ,β_ρ)-doubling, the conclusion of (iii) holds true trivially. Thus, without loss of generality, we may assume that B is non-(ρ,β_ρ)-doubling, which implies that $N \geqslant 1$. For any $k \in \{-\lfloor \log_\rho 2\rfloor, \ldots, N-1\}$, we have $\mu(\rho^{k+1}B) > \beta_\rho \mu(\rho^k B)$. Thus, for any $k \in \{-\lfloor \log_\rho 2\rfloor, \ldots, N-1\}$, $\mu(\rho^k B) < \frac{\mu(\rho^N B)}{\beta_\rho^{N-k}}$. By this, together with (2.1) and the fact that $\beta_\rho > [C_{(\lambda)}]^{\log_2 2\rho} = (2\rho)^\nu$, we conclude that

$$\begin{aligned}[\widetilde{K}_{B,\widetilde{B}^\rho}^{(\rho),p}]^p = [\widetilde{K}_{B,\rho^N B}^{(\rho),p}]^p &\leqslant 2 + \sum_{k=-\lfloor \log_\rho 2\rfloor}^{N-1} \left[\frac{\mu(\rho^k B)}{\lambda(c_B,\rho^k r_B)}\right]^p \\ &\leqslant 2 + \sum_{k=-\lfloor \log_\rho 2\rfloor}^{N-1} \left[\frac{(2\rho)^\nu}{\beta_\rho}\right]^{p(N-k)} \left[\frac{\mu(\rho^N B)}{\lambda(c_B,\rho^N r_B)}\right]^p \\ &\leqslant 2 + \sum_{k=1}^{\infty} \left[\frac{(2\rho)^\nu}{\beta_\rho}\right]^{pk} \lesssim 1,\end{aligned}$$

where the implicit positive constant only depends on ρ, p and ν. Similarly, the other part of (iii) holds true, the details being omitted. This proves (iii).

Next we show (iv). By (i), $N_{B,R}^{(\rho)} \leqslant N_{B,S}^{(\rho)} + \lceil \log_\rho 2\rceil$. If $N_{B,S}^{(\rho)} \leqslant N_{B,R}^{(\rho)} \leqslant N_{B,S}^{(\rho)} + \lceil \log_\rho 2\rceil$, then there exists nothing to prove. If $N_{B,R}^{(\rho)} < N_{B,S}^{(\rho)}$, from the facts that $N_{B,S}^{(\rho)} \leqslant N_{B,R}^{(\rho)} + N_{R,S}^{(\rho)}$ (since $\rho^{N_{B,R}^{(\rho)} + N_{R,S}^{(\rho)}} r_B \geqslant \rho^{N_{R,S}^{(\rho)}} r_R \geqslant r_S$), $\rho^{N_{B,R}^{(\rho)}} r_B \geqslant r_R$, $\rho^{k+N_{B,R}^{(\rho)}+1+\lfloor \log_\rho 2\rfloor} B \subset \rho^{k+2+\lceil \log_\rho 2\rceil + \lfloor \log_\rho 2\rfloor} R$ for all $k \in \mathbb{Z} \cap [-\lfloor \log_\rho 2\rfloor, \infty)$, and (2.1), it follows that

$$\begin{aligned}[\widetilde{K}_{B,S}^{(\rho),p}]^p &\leqslant [\widetilde{K}_{B,R}^{(\rho),p}]^p + \sum_{k=N_{B,R}^{(\rho)}+1}^{N_{B,R}^{(\rho)}+N_{R,S}^{(\rho)}+1+\lfloor \log_\rho 2\rfloor} \left[\frac{\mu(\rho^k B)}{\lambda(c_B,\rho^k r_B)}\right]^p \\ &= [\widetilde{K}_{B,R}^{(\rho),p}]^p + \sum_{k=-\lfloor \log_\rho 2\rfloor}^{N_{R,S}^{(\rho)}} \left[\frac{\mu(\rho^{k+N_{B,R}^{(\rho)}+1+\lfloor \log_\rho 2\rfloor} B)}{\lambda(c_B,\rho^{k+N_{B,R}^{(\rho)}+1+\lfloor \log_\rho 2\rfloor} r_B)}\right]^p \\ &\leqslant [\widetilde{K}_{B,R}^{(\rho),p}]^p + \sum_{k=-\lfloor \log_\rho 2\rfloor}^{N_{R,S}^{(\rho)}} \left[\frac{\mu(\rho^{k+2+\lceil \log_\rho 2\rceil+\lfloor \log_\rho 2\rfloor} R)}{\lambda(c_B,\rho^{k+1+\lfloor \log_\rho 2\rfloor} r_R)}\right]^p\end{aligned}$$

$$
\begin{aligned}
&\leqslant [\widetilde{K}_{B,\,R}^{(\rho),\,p}]^p + c_{(\rho,\,p,\,\nu)} \sum_{k=-\lfloor \log_\rho 2\rfloor}^{N_{R,S}^{(\rho)}} \left[\frac{\mu(\rho^{k+2+\lceil \log_\rho 2\rceil+\lfloor \log_\rho 2\rfloor}R)}{\lambda(c_B,\rho^{k+2+\lceil \log_\rho 2\rceil+\lfloor \log_\rho 2\rfloor}r_R)}\right]^p\\
&\leqslant [\widetilde{K}_{B,\,R}^{(\rho),\,p}]^p + c_{(\rho,\,p,\,\nu)} \sum_{k=-\lfloor \log_\rho 2\rfloor}^{N_{R,S}^{(\rho)}+2+\lceil \log_\rho 2\rceil+\lfloor \log_\rho 2\rfloor} \left[\frac{\mu(\rho^k R)}{\lambda(c_R,\rho^k r_R)}\right]^p\\
&\leqslant [\widetilde{K}_{B,\,R}^{(\rho),\,p}]^p + c_{(\rho,\,p,\,\nu)}[\widetilde{K}_{R,\,S}^{(\rho),\,p}]^p,
\end{aligned}
$$

which shows (iv).

For (v), we first prove that $N_{B,R}^{(\rho)}+N_{R,S}^{(\rho)} \leqslant N_{B,\,S}^{(\rho)}+1$. Since

$$r_R = \rho^{-N_{R,S}^{(\rho)}+1}\rho^{N_{R,S}^{(\rho)}-1}r_R \leqslant \rho^{-N_{R,S}^{(\rho)}+1}r_S \leqslant \rho^{-N_{R,S}^{(\rho)}+1}\rho^{N_{B,\,S}^{(\rho)}}r_B = \rho^{N_{B,\,S}^{(\rho)}-N_{R,S}^{(\rho)}+1}r_B,$$

we obtain $N_{B,R}^{(\rho)} \leqslant N_{B,\,S}^{(\rho)} - N_{R,S}^{(\rho)}+1$. From this,

$$r_R > \rho^{N_{B,R}^{(\rho)}-1}r_B, \quad \rho^k R \subset \rho^{k+\lceil \log_\rho 2\rceil+N_{B,R}^{(\rho)}}B,$$

for all $k\in\mathbb{Z}\cap[-\lfloor \log_\rho 2\rfloor,\infty)$, (2.1) and (2.2), it follows that

$$
\begin{aligned}
[\widetilde{K}_{R,\,S}^{(\rho),\,p}]^p &\leqslant 1 + \sum_{k=-\lfloor \log_\rho 2\rfloor}^{N_{R,\,S}^{(\rho)}} \left[\frac{\mu(\rho^{k+\lceil \log_\rho 2\rceil+N_{B,R}^{(\rho)}}B)}{\lambda(c_R,\rho^{k+N_{B,R}^{(\rho)}-1}r_B)}\right]^p\\
&\lesssim 1 + \sum_{k=-\lfloor \log_\rho 2\rfloor}^{N_{R,\,S}^{(\rho)}} \left[\frac{\mu(\rho^{k+\lceil \log_\rho 2\rceil+N_{B,R}^{(\rho)}}B)}{\lambda(c_R,\rho^{k+\lceil \log_\rho 2\rceil+N_{B,R}^{(\rho)}}r_B)}\right]^p\\
&\sim 1 + \sum_{k=N_{B,R}^{(\rho)}-1-\lfloor \log_\rho 2\rfloor}^{N_{B,R}^{(\rho)}+N_{R,S}^{(\rho)}-1} \left[\frac{\mu(\rho^{k+1+\lceil \log_\rho 2\rceil}B)}{\lambda(c_B,\rho^{k+1+\lceil \log_\rho 2\rceil}r_B)}\right]^p\\
&\lesssim 1 + \sum_{k=-\lfloor \log_\rho 2\rfloor}^{N_{B,\,S}^{(\rho)}+1+\lceil \log_\rho 2\rceil} \left[\frac{\mu(\rho^k B)}{\lambda(c_B,\rho^k r_B)}\right]^p \lesssim [\widetilde{K}_{B,\,S}^{(\rho),\,p}]^p,
\end{aligned}
$$

where the implicit positive constants depend only on ρ, p and ν. This finishes the proof of (v) and hence Lemma 2.8. □

Now we show that, for any $\rho_1,\,\rho_2\in(1,\infty)$ and $p\in(0,1]$, $\widetilde{K}_{B,\,S}^{(\rho_1),\,p}\sim\widetilde{K}_{B,\,S}^{(\rho_2),\,p}$ for all balls $B\subset S$.

Lemma 2.9. *Let $(\mathcal{X},d,\mu)$ be a non-homogeneous metric measure space, ρ_1, $\rho_2\in(1,\infty)$ and $p\in(0,1]$. Then there exist positive constants $c_{(\rho_1,\,\rho_2,\,p,\,\nu)}$ and $C_{(\rho_1,\,\rho_2,\,p,\,\nu)}$, depending on ρ_1, ρ_2, ν and p, such that, for all balls $B\subset S$,*

$$c_{(\rho_1,\,\rho_2,\,p,\,\nu)}\widetilde{K}_{B,\,S}^{(\rho_2),\,p} \leqslant \widetilde{K}_{B,\,S}^{(\rho_1),\,p} \leqslant C_{(\rho_1,\,\rho_2,\,p,\,\nu)}\widetilde{K}_{B,\,S}^{(\rho_2),\,p}.$$

Proof. For the sake of simplicity, we only prove this lemma for $p=1$. With some slight modifications, the arguments here are still valid for $p\in(0,1)$. For any ρ_1, $\rho_2\in(1,\infty)$, without loss of generality, we may assume that $\rho_1>\rho_2>1$. For any two fixed balls $B\subset S$, let $N_j:=N_{B,\,S}^{(\rho_j)}$ and $\widetilde{K}_{B,\,S}^{(\rho_j)}:=\widetilde{K}_{B,\,S}^{(\rho_j),\,1}$ ($j\in\{1,2\}$). It is obvious that $N_1\leqslant N_2$. Now we consider the following two cases:

Case i. $\rho_1^{N_1}\leqslant\rho_2^{N_2}$. It is easy to see that $\rho_2^{N_2-1}\leqslant\rho_1^{N_1}$. We first prove that $\widetilde{K}_{B,\,S}^{(\rho_1)}\lesssim\widetilde{K}_{B,\,S}^{(\rho_2)}$.

Indeed, for any $n_1\in\{-\lfloor\log_{\rho_1}2\rfloor,\dots,N_1\}$, let n_2 be the smallest integer such that $\rho_2^{n_2}\geqslant\rho_1^{n_1}$. Then we have

$$n_2\in\{-\lfloor\log_{\rho_2}2\rfloor,\dots,N_2\} \quad\text{and}\quad \rho_2^{n_2-1}<\rho_1^{n_1}\leqslant\rho_2^{n_2}. \tag{2.5}$$

Consequently, $\rho_2^{n_2-1}B\subset\rho_1^{n_1}B\subset\rho_2^{n_2}B$. By some simple calculations, we see that, for any $n_2\in\{-\lfloor\log_{\rho_2}2\rfloor,\dots,N_2\}$, there exists at most one n_1 satisfying (2.5). By the above facts, $-\lfloor\log_{\rho_1}2\rfloor\geqslant$

$-\lfloor \log_{\rho_2} 2 \rfloor$ and (2.1), we obtain

$$\widetilde{K}_{B,\,S}^{(\rho_1)} \leqslant 1 + \sum_{n_1=-\lfloor \log_{\rho_1} 2 \rfloor}^{N_1} \frac{\mu(\rho_2^{n_2} B)}{\lambda(c_B, \rho_2^{n_2-1} r_B)} \lesssim 1 + \sum_{n_2=-\lfloor \log_{\rho_2} 2 \rfloor}^{N_2} \frac{\mu(\rho_2^{n_2} B)}{\lambda(c_B, \rho_2^{n_2} r_B)} \sim \widetilde{K}_{B,\,S}^{(\rho_2)},$$

where the implicit positive constants depend only on ρ_1, ρ_2 and ν.

On the other hand, for the case $N_2 < 1$, it is obvious that $\widetilde{K}_{B,\,S}^{(\rho_2)} \lesssim 1 \lesssim \widetilde{K}_{B,\,S}^{(\rho_1)}$, which completes the proof of Case i. Thus, without loss of generality, we may assume that $N_2 \geqslant 1$. We notice that

$$\widetilde{K}_{B,\,S}^{(\rho_2)} \leqslant 2\left[1 + \sum_{n_2=-\lfloor \log_{\rho_2} 2 \rfloor}^{N_2-1} \frac{\mu(\rho_2^{n_2} B)}{\lambda(c_B, \rho_2^{n_2} r_B)}\right].$$

For any fixed $n_2 \in \{-\lfloor \log_{\rho_2} 2 \rfloor, \dots, N_2 - 1\}$, let n_1 be the smallest positive integer such that $\rho_1^{n_1} \geqslant \rho_2^{n_2}$. Then we have

$$n_1 \in \{-\lfloor \log_{\rho_1} 2 \rfloor, \dots, N_1\} \quad \text{and} \quad \rho_1^{n_1-1} < \rho_2^{n_2} \leqslant \rho_1^{n_1}. \tag{2.6}$$

Consequently, $\rho_1^{n_1-1} B \subset \rho_2^{n_2} B \subset \rho_1^{n_1} B$. By some simple calculations, we see that, for any $n_1 \in \{-\lfloor \log_{\rho_1} 2 \rfloor, \dots, N_1\}$, the number of n_2 satisfying (2.6) does not exceed $\lceil \frac{\ln \rho_1}{\ln \rho_2} \rceil$. By the above facts and (2.1), we know that

$$\begin{aligned}\widetilde{K}_{B,\,S}^{(\rho_2)} &\lesssim 1 + \sum_{n_2=-\lfloor \log_{\rho_2} 2 \rfloor}^{N_2-1} \frac{\mu(\rho_2^{n_2} B)}{\lambda(c_B, \rho_2^{n_2} r_B)} \sim 1 + \sum_{n_1=-\lfloor \log_{\rho_1} 2 \rfloor}^{N_1} \sum_{n_2: \rho_1^{n_1-1} < \rho_2^{n_2} \leqslant \rho_1^{n_1}} \frac{\mu(\rho_2^{n_2} B)}{\lambda(c_B, \rho_2^{n_2} r_B)} \\ &\lesssim 1 + \sum_{n_1=-\lfloor \log_{\rho_1} 2 \rfloor}^{N_1} \frac{\mu(\rho_1^{n_1} B)}{\lambda(c_B, \rho_1^{n_1} r_B)} \sim \widetilde{K}_{B,\,S}^{(\rho_1)},\end{aligned}$$

where the implicit positive constants depend only on ρ_1, ρ_2 and ν. This finishes the proof of Case i.

Case ii. $\rho_2^{N_2} < \rho_1^{N_1}$. The proof of this case is similar to that of Case i, the details being omitted. This finishes the proof of Lemma 2.9. □

3 Atomic Hardy spaces $\widetilde{H}_{\mathrm{atb},\,\rho}^{p,\,q,\,\gamma}(\mu)$

In this section, we introduce the atomic Hardy space $\widetilde{H}_{\mathrm{atb},\,\rho}^{p,\,q,\,\gamma}(\mu)$ and establish a useful property. Before introducing the notion of $\widetilde{H}_{\mathrm{atb},\,\rho}^{p,\,q,\,\gamma}(\mu)$, we first recall some notions related to quasi-Banach spaces; see, for example, [19].

Definition 3.1. (i) A *quasi-Banach space* $\mathcal{B}$ is a vector space endowed with a *quasi-norm* $\|\cdot\|_{\mathcal{B}}$ which is non-negative, non-degenerate (namely, $\|f\|_{\mathcal{B}} = 0$ if and only if $f = 0$), homogeneous, and obeys the quasi-triangle inequality, namely, there exists a constant $K \in [1, \infty)$ such that, for all $f, g \in \mathcal{B}$,

$$\|f + g\|_{\mathcal{B}} \leqslant K(\|f\|_{\mathcal{B}} + \|g\|_{\mathcal{B}}).$$

(ii) Let $r \in (0, 1]$. A quasi-Banach space $\mathcal{B}_r$ with the quasi-norm $\|\cdot\|_{\mathcal{B}_r}$ is called an *r-quasi-Banach space* if $\|f + g\|_{\mathcal{B}_r}^r \leqslant \|f\|_{\mathcal{B}_r}^r + \|g\|_{\mathcal{B}_r}^r$ for all $f, g \in \mathcal{B}_r$. Hereafter, $\|\cdot\|_{\mathcal{B}_r}^r$ is called the *r-quasi-norm* of the r-quasi-Banach space $\mathcal{B}_r$.

Then we introduce the notion of $\widetilde{H}_{\mathrm{atb},\,\rho}^{p,\,q,\,\gamma}(\mu)$ over general non-homogeneous metric measure spaces.

Definition 3.2. Let $\rho \in (1, \infty)$, $0 < p \leqslant 1 \leqslant q \leqslant \infty$, $p \neq q$ and $\gamma \in [1, \infty)$. A function b in $L^2(\mu)$ when $p \in (0, 1)$ and in $L^1(\mu)$ when $p = 1$ is called a *$(p, q, \gamma, \rho)_\lambda$-atomic block* if

(i) there exists a ball B such that $\operatorname{supp}(b) \subset B$;

(ii) $\int_{\mathcal{X}} b(x)\, d\mu(x) = 0$;

(iii) for any $j \in \{1, 2\}$, there exist a function a_j supported on a ball $B_j \subset B$ and a number $\lambda_j \in \mathbb{C}$ such that $b = \lambda_1 a_1 + \lambda_2 a_2$ and

$$\|a_j\|_{L^q(\mu)} \leqslant [\mu(\rho B_j)]^{1/q-1}[\lambda(c_B, r_B)]^{1-1/p}[\widetilde{K}^{(\rho),\,p}_{B_j,\,B}]^{-\gamma}.$$

Moreover, let $|b|_{\widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)} := |\lambda_1| + |\lambda_2|$.

A function f is said to belong to the *space* $\widetilde{\mathbb{H}}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ if there exists a sequence of $(p, q, \gamma, \rho)_\lambda$-atomic blocks, $\{b_i\}_{i=1}^\infty$, such that $f = \sum_{i=1}^\infty b_i$ in $L^2(\mu)$ when $p \in (0, 1)$ and in $L^1(\mu)$ when $p = 1$, and

$$\sum_{i=1}^\infty |b_i|^p_{\widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)} < \infty.$$

Moreover, define

$$\|f\|_{\widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)} := \inf\left\{\left[\sum_{i=1}^\infty |b_i|^p_{\widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)}\right]^{1/p}\right\},$$

where the infimum is taken over all possible decompositions of f as above.

The *atomic Hardy space* $\widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ is then defined as the completion of $\widetilde{\mathbb{H}}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ with respect to the p-quasi-norm $\|\cdot\|^p_{\widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)}$.

Remark 3.3. (i) By the theorem of completion of Yosida [65, p. 56], we see that $\widetilde{\mathbb{H}}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ has a completion space $\widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$, namely, for any $f \in \widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$, there exists a Cauchy sequence $\{f_k\}_{k=1}^\infty$ in $\widetilde{\mathbb{H}}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ such that

$$\lim_{k\to\infty} \|f_k - f\|^p_{\widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)} = 0. \tag{3.1}$$

Moreover, if $\{f_k\}_{k=1}^\infty$ is a Cauchy sequence in $\widetilde{\mathbb{H}}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$, then there exists a unique $f \in \widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ such that (3.1) holds true.

(ii) When $p = 1$, the space $\widetilde{\mathbb{H}}^{1,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ was introduced in [30] and proved to be a Banach space. Thus, $\widetilde{H}^{1,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu) = \widetilde{\mathbb{H}}^{1,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$; see also [3].

(iii) Fix p, ρ and γ as in Definition 3.2. For $1 \leqslant q_1 \leqslant q_2 \leqslant \infty$, we easily obtain

$$\widetilde{\mathbb{H}}^{p,\,q_2,\,\gamma}_{\mathrm{atb},\,\rho}(\mu) \subset \widetilde{\mathbb{H}}^{p,\,q_1,\,\gamma}_{\mathrm{atb},\,\rho}(\mu).$$

(iv) In Definition 3.2, it seems natural to assume $b \in L^q(\mu)$ and to require $f = \sum_{i=1}^\infty b_i$ also holds true in $L^q(\mu)$. However, if so, then it is unclear whether (iii) of this remark still holds true or not, which is crucial in applications (see, for example, Remark 10.11(i)).

Now we show that any element in $\widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ has a decomposition in terms of some $(p,\, q,\, \gamma,\, \rho)_\lambda$-atomic blocks, $\{b_i\}_{i=1}^\infty$, in $\widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$.

Proposition 3.4. *Let $(\mathcal{X}, d, \mu)$ be a non-homogeneous metric measure space, $\rho \in (1, \infty)$, $0 < p \leqslant 1 \leqslant q \leqslant \infty$, $p \neq q$ and $\gamma \in [1, \infty)$. Then $f \in \widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ if and only if there exist $(p,\, q,\, \gamma,\, \rho)_\lambda$-atomic blocks $\{b_i\}_{i=1}^\infty$ such that*

$$f = \sum_{i=1}^\infty b_i \quad \text{in} \quad \widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu) \tag{3.2}$$

and $\sum_{i=1}^\infty |b_i|^p_{\widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)} < \infty$. Moreover,

$$\|f\|^p_{\widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)} = \inf\left\{\sum_{i=1}^\infty |b_i|^p_{\widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)}\right\},$$

where the infimum is taken over all possible decompositions of f as in (3.2).

Proof. We first assume that $f \in \widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$. Observe that, if (3.2) holds true, it is easy to see that

$$\|f\|^p_{\widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)} \leqslant \inf\left\{\sum_{i=1}^{\infty} |b_i|^p_{\widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)}\right\}, \tag{3.3}$$

where the infimum is taken over all possible decompositions of f as in (3.2). It remains to prove (3.2) and the reverse inequality of (3.3). For any $f \in \widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$, we consider the following two cases.

Case i. $f \in \widetilde{\mathbb{H}}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$. By Definition 3.2, there exists a sequence of $(p,q,\gamma,\rho)_\lambda$-atomic blocks, $\{b_i\}_{i=1}^\infty$, such that $f = \sum_{i=1}^\infty b_i$ in $L^2(\mu)$ when $p \in (0,1)$ and in $L^1(\mu)$ when $p = 1$ and

$$\sum_{i=1}^{\infty} |b_i|^p_{\widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)} < \infty.$$

Now we claim that (3.2) holds true.

Indeed, for any $M \in \mathbb{N}$, $f - \sum_{i=1}^M b_i = \sum_{i=M+1}^\infty b_i$ in $L^2(\mu)$ when $p \in (0,1)$ and in $L^1(\mu)$ when $p = 1$. Then we know that

$$\left\|f - \sum_{i=1}^{M} b_i\right\|^p_{\widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)} \leqslant \sum_{i=M+1}^{\infty} |b_i|^p_{\widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)} \to 0 \quad \text{as} \quad M \to \infty.$$

Then the claim holds true. Again, by Definition 3.2 and (3.3), we obtain the desired result for Case i.

Case ii. $f \in \widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu) \backslash \widetilde{\mathbb{H}}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$. By Remark 3.3(i), there exists a Cauchy sequence $\{f_k\}_{k=1}^\infty$ in $\widetilde{\mathbb{H}}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ such that

$$\|f - f_k\|^p_{\widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)} \leqslant 2^{-k-2} \|f\|^p_{\widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)}.$$

It is easy to see that $f = \sum_{k=1}^\infty (f_k - f_{k-1})$ in $\widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$, where we let $f_0 := 0$. Since $f_k - f_{k-1} \in \widetilde{\mathbb{H}}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ for all $k \in \mathbb{N}$, by Definition 3.2 and Case i, we see that, for any $\epsilon \in (0,\infty)$ and any $k \in \mathbb{N}$, there exists a sequence of $(p,q,\gamma,\rho)_\lambda$-atomic blocks, $\{b_{k,\,i}\}_{i=1}^\infty$, such that

$$f_k - f_{k-1} = \sum_{i=1}^{\infty} b_{k,\,i} \quad \text{in both } L^2(\mu) \text{ when } p \in (0,1), \quad \text{or } L^1(\mu) \text{ when } p = 1, \quad \text{and } \widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$$

and

$$\sum_{i=1}^{\infty} |b_{k,\,i}|^p_{\widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)} < \|f_k - f_{k-1}\|^p_{\widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)} + \frac{\epsilon}{2^k}.$$

From this and $f = \sum_{k=1}^\infty (f_k - f_{k-1})$ in $\widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$, we further deduce that

$$f = \sum_{k=1}^{\infty} (f_k - f_{k-1}) = \sum_{k=1}^{\infty}\sum_{i=1}^{\infty} b_{k,\,i} \quad \text{in} \quad \widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$$

and

$$\begin{aligned}
\sum_{k=1}^{\infty}\sum_{i=1}^{\infty} |b_{k,\,i}|^p_{\widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)} &\leqslant \sum_{k=1}^{\infty} \|f_k - f_{k-1}\|^p_{\widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)} + \sum_{k=1}^{\infty} \frac{\epsilon}{2^k} \\
&\leqslant \sum_{k=1}^{\infty} [\|f_k - f\|^p_{\widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)} + \|f_{k-1} - f\|^p_{\widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)}] + \epsilon \\
&\leqslant \sum_{k=1}^{\infty} 2^{-k} \|f\|^p_{\widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)} + \epsilon = \|f\|^p_{\widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)} + \epsilon,
\end{aligned}$$

which, together with the arbitrariness of ϵ, completes the proof of Case ii and hence the necessity of Proposition 3.4.

Conversely, let $f=\sum_{i=1}^{\infty} b_i$ in $\widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ and

$$\sum_{i=1}^{\infty} |b_i|^p_{\widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)} < \infty.$$

Then, for each $k\in\mathbb{N}$, $f_k=\sum_{i=1}^{k} b_i \in \widetilde{\mathbb{H}}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ and $\lim_{k\to\infty} f_k = f$ in $\widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$. Thus, $f\in \widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$, which completes the proof of the sufficiency and hence Proposition 3.4. □

4 Boundedness of Calderón-Zygmund operators

In this section, we introduce the notion of the molecular Hardy space $\widetilde{H}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)$ and prove that the Calderón-Zygmund operator T is bounded from $\widetilde{H}^{p,\,q,\,\gamma,\,\delta}_{\mathrm{mb},\,\rho}(\mu)$ (or $\widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$) into $L^p(\mu)$, and from $\widetilde{H}^{p,\,q,\,\gamma+1}_{\mathrm{atb},\,\rho(\rho+1)}(\mu)$ into $\widetilde{H}^{p,\,q,\,\gamma,\,\frac{1}{2}(\delta-\frac{\nu}{p}+\nu)}_{\mathrm{mb},\,\rho}(\mu)$, where δ is some positive constant depending on T; see Definition 4.6 below.

We first introduce the notion of molecular Hardy spaces in a non-homogeneous metric measure space.

Definition 4.1. Let $\rho\in(1,\infty)$, $0<p\leqslant 1\leqslant q\leqslant\infty$, $p\neq q$, $\gamma\in[1,\infty)$ and $\epsilon\in(0,\infty)$. A function b in $L^2(\mu)$ when $p\in(0,1)$ and in $L^1(\mu)$ when $p=1$ is called a $(p,q,\gamma,\epsilon,\rho)_\lambda$-*molecular block* if

(i) $\int_{\mathcal{X}} b(x)d\mu(x)=0$;

(ii) there exist some ball $B:=B(c_B,r_B)$, with $c_B\in\mathcal{X}$ and $r_B\in(0,\infty)$, and some constants $\widetilde{M}$, $M\in\mathbb{N}$ such that, for all $k\in\mathbb{Z}_+$ and $j\in\{1,\ldots,M_k\}$ with $M_k:=\widetilde{M}$ if $k=0$ and $M_k:=M$ if $k\in\mathbb{N}$, there exist functions $m_{k,\,j}$ supported on some balls $B_{k,\,j}\subset U_k(B)$ for all $k\in\mathbb{Z}_+$, where $U_0(B):=\rho^2B$ and $U_k(B):=\rho^{k+2}B\backslash\rho^{k-2}B$ with $k\in\mathbb{N}$, and $\lambda_{k,\,j}\in\mathbb{C}$ such that $b=\sum_{k=0}^{\infty}\sum_{j=1}^{M_k}\lambda_{k,\,j}m_{k,\,j}$ in $L^2(\mu)$ when $p\in(0,1)$ and in $L^1(\mu)$ when $p=1$,

$$\|m_{k,\,j}\|_{L^q(\mu)}\leqslant \rho^{-k\epsilon}[\mu(\rho B_{k,\,j})]^{1/q-1}[\lambda(c_B,\rho^{k+2}r_B)]^{1-1/p}[\widetilde{K}^{(\rho),\,p}_{B_{k,\,j},\,\rho^{k+2}B}]^{-\gamma} \tag{4.1}$$

and

$$|b|^p_{\widetilde{H}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)} := \sum_{k=0}^{\infty}\sum_{j=1}^{M_k}|\lambda_{k,\,j}|^p<\infty.$$

A function f is said to belong to the *space* $\widetilde{\mathbb{H}}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)$ if there exists a sequence of $(p,q,\gamma,\epsilon,\rho)_\lambda$-molecular blocks, $\{b_i\}_{i=1}^{\infty}$, such that $f=\sum_{i=1}^{\infty}b_i$ in $L^2(\mu)$ when $p\in(0,1)$ and in $L^1(\mu)$ when $p=1$, and

$$\sum_{i=1}^{\infty}|b_i|^p_{\widetilde{H}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)}<\infty.$$

Moreover, define

$$\|f\|_{\widetilde{H}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)} := \inf\left\{\left[\sum_{i=1}^{\infty}|b_i|^p_{\widetilde{H}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)}\right]^{1/p}\right\},$$

where the infimum is taken over all possible decompositions of f as above.

The *molecular Hardy space* $\widetilde{H}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)$ is then defined as the completion of $\widetilde{\mathbb{H}}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)$ with respect to the p-quasi-norm $\|\cdot\|^p_{\widetilde{H}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)}$.

Remark 4.2. (i) From the theorem of completion of Yosida [65, p. 56], it follows that $\widetilde{\mathbb{H}}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)$ has a completion space $\widetilde{H}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)$, namely, for any $f\in\widetilde{H}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)$, there exists a Cauchy sequence $\{f_k\}_{k=1}^{\infty}$ in $\widetilde{\mathbb{H}}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)$ such that

$$\lim_{k\to\infty}\|f_k-f\|^p_{\widetilde{H}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)}=0. \tag{4.2}$$

Moreover, if $\{f_k\}_{k=1}^{\infty}$ is a Cauchy sequence in $\widetilde{\mathbb{H}}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)$, then there exists a unique $f\in\widetilde{H}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)$ such that (4.2) holds true.

(ii) It was proved, in [13, Proposition 2.2(i)], that $\widetilde{\mathbb{H}}^{1,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)$ is a Banach space and hence

$$\widetilde{H}^{1,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)=\widetilde{\mathbb{H}}^{1,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu).$$

(iii) Fix p, ρ, ϵ and γ as in Definition 4.1. For $1\leqslant q_1\leqslant q_2\leqslant\infty$, we easily have

$$\widetilde{\mathbb{H}}^{p,\,q_2,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)\subset\widetilde{\mathbb{H}}^{p,\,q_1,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu).$$

(iv) We point out that, via replacing the discrete coefficient $\widetilde{K}^{(\rho),\,1}_{B,\,S}$ in Definitions 3.2 and 4.1 by the continuous coefficient $K_{B,\,S}$ as in (2.4), the atomic Hardy space $H^{1,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ and the molecular Hardy space $H^{1,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)$ were introduced, respectively, in [13,30]. It was proved, in [30, Proposition 3.3(ii) and Theorem 3.8], that $H^{1,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ is independent of the choices of ρ, γ and q. Moreover, in [13, Remark 2.3], it was proved that $H^{1,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ and $H^{1,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)$ coincide with equivalent norms and hence $H^{1,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)$ is independent of the choices of ρ, γ, q and ϵ. However, $H^{1,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ and $\widetilde{H}^{1,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ (or $H^{1,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)$ and $\widetilde{H}^{1,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)$) may not coincide (see [13, Remark 1.9]) and the boundedness of Calderón-Zygmund operators on $H^{1,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ over general non-homogeneous metric measure spaces is also unclear (see [13, Remark 2.4]).

By [13, Theorem 1.11], we see that $\widetilde{H}^{1,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)=\widetilde{H}^{1,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)$. For $p\in(0,1)$, we have the following conclusion.

Proposition 4.3. *Suppose that $(\mathcal{X},d,\mu)$ is a non-homogeneous metric measure space. Let $p\in(0,1)$, and ρ, q, γ and ϵ be as in Definition 4.1. Then $\widetilde{\mathbb{H}}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)\subset\widetilde{\mathbb{H}}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)\subset L^p(\mu)$ and there exist positive constants C and $\widetilde{C}$ such that, for all $f\in\widetilde{\mathbb{H}}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$,*

$$C\|f\|^p_{L^p(\mu)}\leqslant\|f\|^p_{\widetilde{H}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)}\leqslant\widetilde{C}\|f\|^p_{\widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)}.$$

Proof. Let ρ, p, q, γ and ϵ be as in Proposition 4.3. By the Fatou lemma, it suffices to prove that, for any $(p,q,\gamma,\rho)_\lambda$-atomic block b, b is also a $(p,q,\gamma,\epsilon,\rho)_\lambda$-molecular block which belongs to $L^p(\mu)$ and

$$\|b\|^p_{L^p(\mu)}\lesssim|b|^p_{\widetilde{H}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)}\lesssim|b|^p_{\widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)}. \tag{4.3}$$

By Definitions 3.2 and 4.1, it is easy to see that, for any $(p,q,\gamma,\rho)_\lambda$-atomic block b, b is also a $(p,q,\gamma,\epsilon,\rho)_\lambda$-molecular block and $|b|^p_{\widetilde{H}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)}\lesssim|b|^p_{\widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)}$.

On the other hand, for any $(p,q,\gamma,\epsilon,\rho)_\lambda$-molecular block b with the same notation as in Definition 4.1, by the Fatou lemma, Hölder's inequality, (4.1), $B_{k,\,j}\subset\rho^{k+2}B$ and (2.1), we see that

$$\begin{aligned}\|b\|^p_{L^p(\mu)}&\leqslant\sum_{k=0}^{\infty}\sum_{j=1}^{M_k}|\lambda_{k,\,j}|^p\|m_{k,\,j}\|^p_{L^p(\mu)}\lesssim\sum_{k=0}^{\infty}\sum_{j=1}^{M_k}|\lambda_{k,\,j}|^p\|m_{k,\,j}\|^p_{L^q(\mu)}\,[\mu(B_{k,\,j})]^{1-p/q}\\&\lesssim\sum_{k=0}^{\infty}\sum_{j=1}^{M_k}|\lambda_{k,\,j}|^p[\mu(B_{k,\,j})]^{1-p/q}\rho^{-kp\epsilon}[\mu(\rho B_{k,\,j})]^{p/q-p}[\lambda(c_B,\rho^{k+2}r_B)]^{p-1}\\&\lesssim\sum_{k=0}^{\infty}\sum_{j=1}^{M_k}|\lambda_{k,\,j}|^p\sim|b|^p_{\widetilde{H}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)},\end{aligned}$$

which completes the proof of Proposition 4.3. □

Remark 4.4. Let $p\in(0,1)$, and ρ, q, γ and ϵ be as in Definition 4.1. By Proposition 4.3, we easily conclude that there exists a map T from $\widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ to $\widetilde{H}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)$ such that, for any $f\in\widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$, there is a unique element $\widetilde{f}\in\widetilde{H}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)$ satisfying $Tf=\widetilde{f}$ and $\|\widetilde{f}\|_{\widetilde{H}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)}\lesssim\|f\|_{\widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)}$, where the implicit positive constant is independent of f. In this sense, we say that $\widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)\subset\widetilde{H}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)$, which is different from the classical inclusion relation of spaces, since it is still unclear whether T is an injection and $\|\widetilde{f}\|_{\widetilde{H}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)}\sim\|f\|_{\widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)}$ or not.

Now we show that any element in $\widetilde{H}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)$ can be decomposed into a sum of a sequence of $(p,\,q,\,\gamma,\,\epsilon,\,\rho)_\lambda$-molecular blocks, $\{b_j\}_{j=1}^{\infty}$, in $\widetilde{H}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)$. The proof is similar to that of Proposition 3.4, the details being omitted.

Proposition 4.5. *Suppose that $(\mathcal{X},d,\mu)$ is a non-homogeneous metric measure space. Let p, ρ, q, γ and ϵ be as in Definition* 4.1*. Then $f\in\widetilde{H}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)$ if and only if there exist $(p,\,q,\,\gamma,\,\epsilon,\,\rho)_\lambda$-molecular blocks $\{b_i\}_{i=1}^{\infty}$ such that*

$$f=\sum_{i=1}^{\infty}b_i \quad \text{in} \quad \widetilde{H}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu) \tag{4.4}$$

and

$$\sum_{i=1}^{\infty}|b_i|^p_{\widetilde{H}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)}<\infty.$$

Moreover,

$$\|f\|^p_{\widetilde{H}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)}=\inf\left\{\sum_{i=1}^{\infty}|b_i|^p_{\widetilde{H}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)}\right\},$$

where the infimum is taken over all possible decompositions of f as in (4.4)*.*

Now we consider the boundedness of Calderón-Zygmund operators on these atomic and molecular Hardy spaces. To this end, we first recall the following notion of Calderón-Zygmund operators from [29].

Definition 4.6. A function $K\in L^1_{\mathrm{loc}}((\mathcal{X}\times\mathcal{X})\backslash\{(x,x):x\in\mathcal{X}\})$ is called a *Calderón-Zygmund kernel* if there exists a positive constant $C_{(K)}$, depending on K, such that

(i) for all $x,\,y\in\mathcal{X}$ with $x\neq y$,

$$|K(x,y)|\leqslant C_{(K)}\frac{1}{\lambda(x,d(x,y))}; \tag{4.5}$$

(ii) there exist positive constants $\delta\in(0,1]$ and $c_{(K)}$, depending on K, such that, for all $x,\,\widetilde{x},\,y\in\mathcal{X}$ with $d(x,y)\geqslant c_{(K)}d(x,\widetilde{x})$,

$$|K(x,y)-K(\widetilde{x},y)|+|K(y,x)-K(y,\widetilde{x})|\leqslant C_{(K)}\frac{[d(x,\widetilde{x})]^{\delta}}{[d(x,y)]^{\delta}\lambda(x,d(x,y))}. \tag{4.6}$$

A linear operator T is called a *Calderón-Zygmund operator* with kernel K satisfying (4.5) and (4.6) if, for all $f\in L^{\infty}_b(\mu):=\{f\in L^{\infty}(\mu):\ \mathrm{supp}\,(f)\text{ is bounded}\}$,

$$Tf(x):=\int_{\mathcal{X}}K(x,y)f(y)d\mu(y),\quad x\notin \mathrm{supp}\,(f). \tag{4.7}$$

A new example of operators with kernel satisfying (4.6) and (4.7) is the so-called Bergman-type operator appearing in [60]; see also [29] for an explanation.

We first recall the following useful lemma from [28].

Lemma 4.7. *Suppose that $(\mathcal{X},d,\mu)$ is a non-homogeneous metric measure space. Let T be a Calderón-Zygmund operator defined by* (4.7) *associated with kernel K satisfying* (4.5) *and* (4.6)*. Then the following statements are equivalent*:

(i) *T is bounded on $L^2(\mu)$;*

(ii) *T is bounded on $L^q(\mu)$ for all $q\in(1,\infty)$;*

(iii) *T is bounded from $L^1(\mu)$ to weak-$L^1(\mu)$.*

Now we prove the boundedness of Calderón-Zygmund operators from $\widetilde{H}^{p,\,q,\,\gamma,\,\delta}_{\mathrm{mb},\,\rho}(\mu)$ into $L^p(\mu)$. Hereafter, let $\nu:=\log_2 C_{(\lambda)}$, and δ be as in Definition 4.6.

Theorem 4.8. *Suppose that $(\mathcal{X},d,\mu)$ is a non-homogeneous metric measure space. Let $\rho\in(1,\infty)$, $\frac{\nu}{\nu+\delta}<p\leqslant 1<q<\infty$ and $\gamma\in[1,\infty)$. Assume that the Calderón-Zygmund operator T defined by* (4.7) *associated with kernel K satisfying* (4.5) *and* (4.6) *is bounded on $L^2(\mu)$. Then T is bounded from $\widetilde{H}^{p,\,q,\,\gamma,\,\delta}_{\mathrm{mb},\,\rho}(\mu)$ into $L^p(\mu)$.*

Proof. Let ρ, p, q and γ be as in the assumptions of Theorem 4.8. For the sake of simplicity, we take $\rho = 2$ and $\gamma = 1$. With some slight modifications, the arguments here are still valid for general cases. We first reduce the proof to showing that, for all $(p, q, 1, \delta, 2)_\lambda$-molecular blocks b,

$$\|Tb\|_{L^p(\mu)} \lesssim |b|_{\widetilde{H}^{p,\,q,\,1,\,\delta}_{\mathrm{mb},\,2}(\mu)}. \tag{4.8}$$

Indeed, assume that (4.8) holds true. For any $f \in \widetilde{\mathbb{H}}^{p,\,q,\,1,\,\delta}_{\mathrm{mb},\,2}(\mu)$, there exists a sequence $\{b_i\}_{i\in\mathbb{N}}$ of $(p, q, 1, \delta, 2)_\lambda$-molecular blocks such that $f = \sum_{i=1}^\infty b_i$ in $L^2(\mu)$ when $p \in (0, 1)$ and in $L^1(\mu)$ when $p = 1$ and

$$\sum_{i=1}^\infty |b_i|^p_{\widetilde{H}^{p,\,q,\,1,\,\delta}_{\mathrm{mb},\,2}(\mu)} \sim \|f\|^p_{\widetilde{H}^{p,\,q,\,1,\,\delta}_{\mathrm{mb},\,2}(\mu)}.$$

If $f = \sum_{i=1}^\infty b_i$ in $L^2(\mu)$, then, by the boundedness of T on $L^2(\mu)$, we see that, for any $N \in \mathbb{N}$,

$$\left\|\sum_{i=1}^N T(b_i) - Tf\right\|_{L^2(\mu)} = \left\|T\left(\sum_{i=1}^N b_i - f\right)\right\|_{L^2(\mu)} \lesssim \left\|\sum_{i=1}^N b_i - f\right\|_{L^2(\mu)} \to 0 \quad \text{as} \quad N \to \infty,$$

which further implies that, for all $\eta \in (0, \infty)$,

$$\mu\left(\left\{x \in \mathcal{X} : \left|\sum_{i=1}^N T(b_i)(x) - Tf(x)\right| > \eta\right\}\right) \to 0 \quad \text{as} \quad N \to \infty. \tag{4.9}$$

If $f = \sum_{i=1}^\infty b_i$ in $L^1(\mu)$, then, by the boundedness of T from $L^1(\mu)$ to weak-$L^1(\mu)$, we still know that (4.9) holds true. Thus, by the Riesz theorem, we know that there exists a subsequence of partial sums, $\{\sum_{i=1}^{N_k} T(b_i)\}_k$, such that

$$Tf = \lim_{k\to\infty} \sum_{i=1}^{N_k} T(b_i) \quad \mu\text{-almost everywhere on } \mathcal{X},$$

which, together with the Fatou lemma and (4.8), implies that

$$\begin{aligned}
\|Tf\|^p_{L^p(\mu)} &\leqslant \liminf_{k\to\infty} \int_{\mathcal{X}} \sum_{i=1}^{N_k} |T(b_i)(x)|^p d\mu(x) \leqslant \sum_{i=1}^\infty \|T(b_i)\|^p_{L^p(\mu)} \\
&\lesssim \sum_{i=1}^\infty |b_i|^p_{\widetilde{H}^{p,\,q,\,1,\,\delta}_{\mathrm{mb},\,2}(\mu)} \sim \|f\|^p_{\widetilde{H}^{p,\,q,\,1,\,\delta}_{\mathrm{mb},\,2}(\mu)}.
\end{aligned}$$

Moreover, by a standard density argument, we extend T to be a bounded linear operator from $\widetilde{H}^{p,\,q,\,1,\,\delta}_{\mathrm{mb},\,2}(\mu)$ into $L^p(\mu)$, which is the desired result.

Now we prove (4.8). Let $b = \sum_{k=0}^\infty \sum_{j=1}^{M_k} \lambda_{k,\,j} m_{k,\,j}$ be a $(p, q, 1, \delta, 2)_\lambda$-molecular block, where, for any $k \in \mathbb{Z}_+$ and $j \in \{1, \ldots, M_k\}$, $\operatorname{supp}(m_{k,\,j}) \subset B_{k,\,j} \subset U_k(B)$ for some balls B and $B_{k,\,j}$ as in Definition 4.1. Without loss of generality, we may assume that $\widetilde{M} = M$ in Definition 4.1.

By the linearity of T, we write

$$\begin{aligned}
\|Tb\|^p_{L^p(\mu)} \leqslant{}& \sum_{\ell=5}^\infty \int_{U_\ell(B)} \left|T\left(\sum_{k=0}^{\ell-5} \sum_{j=1}^M \lambda_{k,\,j} m_{k,\,j}\right)(x)\right|^p d\mu(x) \\
&+ \sum_{\ell=5}^\infty \int_{U_\ell(B)} \left|T\left(\sum_{k=\ell-4}^{\ell+4} \sum_{j=1}^M \lambda_{k,\,j} m_{k,\,j}\right)(x)\right|^p d\mu(x) \\
&+ \sum_{\ell=5}^\infty \int_{U_\ell(B)} \left|T\left(\sum_{k=\ell+5}^{\infty} \sum_{j=1}^M \lambda_{k,\,j} m_{k,\,j}\right)(x)\right|^p d\mu(x) + \sum_{\ell=0}^4 \int_{U_\ell(B)} |Tb(x)|^p d\mu(x) \\
=:{}& \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}.
\end{aligned}$$

Now we first estimate III. By (4.5), (2.1), (2.2), Hölder's inequality and (4.1), we obtain

$$\begin{aligned}
\mathrm{III} &\leqslant \sum_{\ell=5}^{\infty}\sum_{k=\ell+5}^{\infty}\sum_{j=1}^{M}|\lambda_{k,j}|^p\int_{U_\ell(B)}\left[\int_{B_{k,j}}|m_{k,j}(y)||K(x,y)|d\mu(y)\right]^p d\mu(x)\\
&\lesssim \sum_{\ell=5}^{\infty}\sum_{k=\ell+5}^{\infty}\sum_{j=1}^{M}|\lambda_{k,j}|^p\int_{U_\ell(B)}\left[\int_{B_{k,j}}\frac{|m_{k,j}(y)|}{\lambda(x,d(x,y))}d\mu(y)\right]^p d\mu(x)\\
&\lesssim \sum_{\ell=5}^{\infty}\sum_{k=\ell+5}^{\infty}\sum_{j=1}^{M}|\lambda_{k,j}|^p\int_{U_\ell(B)}\frac{1}{[\lambda(c_B,d(x,c_B))]^p}d\mu(x)\|m_{k,j}\|_{L^1(\mu)}^p\\
&\lesssim \sum_{\ell=5}^{\infty}\sum_{k=\ell+5}^{\infty}\sum_{j=1}^{M}|\lambda_{k,j}|^p\frac{\mu(2^{\ell+2}B)}{[\lambda(c_B,2^{\ell-2}r_B)]^p}\,[\mu(B_{k,j})]^{p/q'}\,\|m_{k,j}\|_{L^q(\mu)}^p\\
&\lesssim \sum_{\ell=5}^{\infty}\sum_{k=\ell+5}^{\infty}\sum_{j=1}^{M}|\lambda_{k,j}|^p\left[\mu\left(2^{\ell+2}B\right)\right]^{1-p}[\mu(B_{k,j})]^{p/q'}\\
&\quad\times 2^{-k\delta p}\,[\mu(2B_{k,j})]^{-p/q'}\left[\lambda\left(c_B,2^{k+2}r_B\right)\right]^{p-1}\\
&\lesssim \sum_{\ell=5}^{\infty}\sum_{k=\ell+5}^{\infty}\sum_{j=1}^{M}2^{-k\delta p}|\lambda_{k,j}|^p\sim\sum_{j=1}^{M}\sum_{k=10}^{\infty}\sum_{\ell=5}^{k-5}2^{-k\delta p}|\lambda_{k,j}|^p\\
&\lesssim \sum_{j=1}^{M}\sum_{k=10}^{\infty}k2^{-k\delta p}|\lambda_{k,j}|^p\lesssim\sum_{k=0}^{\infty}\sum_{j=1}^{M}|\lambda_{k,j}|^p\sim|b|^p_{\widetilde{H}^{p,q,1,\delta}_{\mathrm{mb},2}(\mu)}.
\end{aligned}$$

In order to estimate I, write

$$\begin{aligned}
\mathrm{I} &\leqslant \sum_{\ell=5}^{\infty}\int_{U_\ell(B)}\left|\int_{\mathcal{X}}\left[\sum_{k=0}^{\ell-5}\sum_{j=1}^{M}\lambda_{k,j}m_{k,j}(y)\right][K(x,y)-K(x,c_B)]d\mu(y)\right|^p d\mu(x)\\
&\quad+\sum_{\ell=5}^{\infty}\int_{U_\ell(B)}\left|\int_{\mathcal{X}}\left[\sum_{k=0}^{\ell-5}\sum_{j=1}^{M}\lambda_{k,j}m_{k,j}(y)\right]K(x,c_B)d\mu(y)\right|^p d\mu(x)=:\mathrm{I}_1+\mathrm{I}_2.
\end{aligned}$$

From (4.6), (2.2), (2.1), Hölder's inequality, (4.1) and the fact that $p\in(\frac{\nu}{\nu+\delta},1]$, it follows that

$$\begin{aligned}
\mathrm{I}_1 &\lesssim \sum_{\ell=5}^{\infty}\sum_{k=0}^{\ell-5}\sum_{j=1}^{M}|\lambda_{k,j}|^p\int_{U_\ell(B)}\left\{\int_{B_{k,j}}\frac{|m_{k,j}(y)|[d(y,c_B)]^\delta}{[d(x,c_B)]^\delta\lambda(c_B,d(x,c_B))}d\mu(y)\right\}^p d\mu(x)\\
&\lesssim \sum_{\ell=5}^{\infty}\sum_{k=0}^{\ell-5}\sum_{j=1}^{M}|\lambda_{k,j}|^p\frac{2^{(k+2)\delta p}r_B^{\delta p}\mu(2^{\ell+2}B)}{2^{(\ell-2)\delta p}r_B^{\delta p}[\lambda(c_B,2^{\ell-2}r_B)]^p}\|m_{k,j}\|_{L^1(\mu)}^p\\
&\lesssim \sum_{\ell=5}^{\infty}\sum_{k=0}^{\ell-5}\sum_{j=1}^{M}|\lambda_{k,j}|^p2^{(k-\ell)\delta p}[\mu(2^{\ell+2}B)]^{1-p}[\mu(B_{k,j})]^{p/q'}\|m_{k,j}\|_{L^q(\mu)}^p\\
&\lesssim \sum_{\ell=5}^{\infty}\sum_{k=0}^{\ell-5}\sum_{j=1}^{M}|\lambda_{k,j}|^p2^{(k-\ell)\delta p}[\mu(2^{\ell+2}B)]^{1-p}[\mu(B_{k,j})]^{p/q'}\\
&\quad\times 2^{-k\delta p}[\mu(2B_{k,j})]^{-p/q'}[\lambda(c_B,2^{k+2}r_B)]^{p-1}\\
&\lesssim \sum_{\ell=5}^{\infty}\sum_{k=0}^{\ell-5}\sum_{j=1}^{M}|\lambda_{k,j}|^p2^{-\ell\delta p}[\mu(2^{\ell+2}B)]^{1-p}[\lambda(c_B,2^{\ell+2}r_B)]^{p-1}[C_{(\lambda)}]^{(\ell-k)(1-p)}\\
&\lesssim \sum_{\ell=5}^{\infty}\sum_{k=0}^{\ell-5}\sum_{j=1}^{M}|\lambda_{k,j}|^p2^{[\nu(1-p)-\delta p]\ell}2^{-k\nu(1-p)}\lesssim\sum_{k=0}^{\infty}\sum_{j=1}^{M}|\lambda_{k,j}|^p\sim|b|^p_{\widetilde{H}^{p,q,1,\delta}_{\mathrm{mb},2}(\mu)}.
\end{aligned}$$

For I_2, the vanishing moment of b, together with (4.5), (2.1) and (2.2), implies that

$$\begin{aligned}
\mathrm{I}_2 &= \sum_{\ell=5}^{\infty}\int_{U_\ell(B)}\bigg|\int_{\mathcal{X}}\bigg[\sum_{k=\ell-4}^{\infty}\sum_{j=1}^{M}\lambda_{k,\,j}m_{k,\,j}(y)\bigg]K(x,c_B)d\mu(y)\bigg|^p d\mu(x)\\
&\lesssim \sum_{\ell=5}^{\infty}\sum_{k=\ell-4}^{\infty}\sum_{j=1}^{M}|\lambda_{k,\,j}|^p\int_{U_\ell(B)}\bigg[\int_{B_{k,\,j}}|m_{k,\,j}(y)|\frac{1}{\lambda(c_B,d(x,c_B))}d\mu(y)\bigg]^p d\mu(x)\\
&\lesssim \sum_{\ell=5}^{\infty}\sum_{k=\ell-4}^{\infty}\sum_{j=1}^{M}|\lambda_{k,\,j}|^p\frac{\mu(2^{\ell+2}B)}{[\lambda(c_B,2^{\ell-2}r_B)]^p}\|m_{k,\,j}\|_{L^1(\mu)}^p\\
&\lesssim \sum_{\ell=5}^{\infty}\sum_{k=\ell-4}^{\infty}\sum_{j=1}^{M}|\lambda_{k,\,j}|^p[\mu(2^{\ell+2}B)]^{1-p}[\mu(B_{k,\,j})]^{p/q'}\|m_{k,\,j}\|_{L^q(\mu)}^p\\
&\lesssim \sum_{\ell=5}^{\infty}\sum_{k=\ell-4}^{\infty}\sum_{j=1}^{M}|\lambda_{k,\,j}|^p[\mu(2^{\ell+2}B)]^{1-p}[\mu(B_{k,\,j})]^{p/q'}\\
&\quad\times 2^{-k\delta p}[\mu(2B_{k,\,j})]^{-p/q'}[\lambda(c_B,2^{k+2}r_B)]^{p-1}\\
&\lesssim \sum_{\ell=5}^{\infty}\sum_{k=\ell-4}^{\infty}\sum_{j=1}^{M}2^{-k\delta p}|\lambda_{k,\,j}|^p \sim \sum_{j=1}^{M}\sum_{k=1}^{\infty}\sum_{\ell=5}^{k+4}2^{-k\delta p}|\lambda_{k,\,j}|^p\\
&\lesssim \sum_{j=1}^{M}\sum_{k=0}^{\infty}k2^{-k\delta p}|\lambda_{k,\,j}|^p \lesssim \sum_{k=0}^{\infty}\sum_{j=1}^{M}|\lambda_{k,\,j}|^p \sim |b|_{\widetilde{H}_{\mathrm{mb},\,2}^{p,\,q,\,1,\,\delta}(\mu)}^p.
\end{aligned}$$

Combining I_1 and I_2, we conclude that $\mathrm{I}\lesssim |b|_{\widetilde{H}_{\mathrm{mb},\,2}^{p,\,q,\,1,\,\delta}(\mu)}^p$.

Then we turn to estimate II. We further write

$$\begin{aligned}
\mathrm{II} &\leqslant \sum_{\ell=5}^{\infty}\sum_{k=\ell-4}^{\ell+4}\sum_{j=1}^{M}|\lambda_{k,\,j}|^p\int_{U_\ell(B)}|Tm_{k,\,j}(x)|^p d\mu(x)\\
&\leqslant \sum_{\ell=5}^{\infty}\sum_{k=\ell-4}^{\ell+4}\sum_{j=1}^{M}|\lambda_{k,\,j}|^p\int_{2B_{k,\,j}}|Tm_{k,\,j}(x)|^p d\mu(x)\\
&\quad+\sum_{\ell=5}^{\infty}\sum_{k=\ell-4}^{\ell+4}\sum_{j=1}^{M}|\lambda_{k,\,j}|^p\int_{U_\ell(B)\setminus 2B_{k,\,j}}\cdots =: \mathrm{II}_1+\mathrm{II}_2.
\end{aligned}$$

By Hölder's inequality, $L^2(\mu)$-boundedness of T, Lemma 4.7, (4.1) and (2.1), we see that

$$\begin{aligned}
\mathrm{II}_1 &\leqslant \sum_{\ell=5}^{\infty}\sum_{k=\ell-4}^{\ell+4}\sum_{j=1}^{M}|\lambda_{k,\,j}|^p[\mu(2B_{k,\,j})]^{1-p/q}\|Tm_{k,\,j}\|_{L^q(\mu)}^p\\
&\lesssim \sum_{\ell=5}^{\infty}\sum_{k=\ell-4}^{\ell+4}\sum_{j=1}^{M}|\lambda_{k,\,j}|^p[\mu(2B_{k,\,j})]^{1-p/q}\|m_{k,\,j}\|_{L^q(\mu)}^p\\
&\lesssim \sum_{\ell=5}^{\infty}\sum_{k=\ell-4}^{\ell+4}\sum_{j=1}^{M}|\lambda_{k,\,j}|^p[\mu(2B_{k,\,j})]^{1-p/q}2^{-k\delta p}[\mu(2B_{k,\,j})]^{-p/q'}[\lambda(c_B,2^{k+2}r_B)]^{p-1}\\
&\lesssim \sum_{\ell=5}^{\infty}\sum_{k=\ell-4}^{\ell+4}\sum_{j=1}^{M}2^{-k\delta p}|\lambda_{k,\,j}|^p \lesssim \sum_{k=0}^{\infty}\sum_{j=1}^{M}|\lambda_{k,\,j}|^p \sim |b|_{\widetilde{H}_{\mathrm{mb},\,2}^{p,\,q,\,1,\,\delta}(\mu)}^p.
\end{aligned}$$

For II_2, from (4.5),

$$d(x,y)\geqslant d(x,c_{B_{k,\,j}})-d(y,c_{B_{k,\,j}})\geqslant \frac{1}{2}d(x,c_{B_{k,\,j}})$$

for $x \notin 2B_{k,j}$ and $y \in B_{k,j}$, (2.2), (2.1), Hölder's inequality and (4.1), we deduce that

$$
\begin{aligned}
\mathrm{II}_2 &\lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\ell+4} \sum_{j=1}^{M} |\lambda_{k,j}|^p \int_{U_\ell(B)\setminus 2B_{k,j}} \left[\int_{B_{k,j}} \frac{|m_{k,j}(y)|}{\lambda(c_{B_{k,j}}, d(x, c_{B_{k,j}}))} d\mu(y)\right]^p d\mu(x)\\
&\lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\ell+4} \sum_{j=1}^{M} |\lambda_{k,j}|^p \int_{2^{k+6}B\setminus B_{k,j}} \frac{1}{[\lambda(c_{B_{k,j}}, d(x, c_{B_{k,j}}))]^p} d\mu(x) \|m_{k,j}\|_{L^1(\mu)}^p\\
&\lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\ell+4} \sum_{j=1}^{M} |\lambda_{k,j}|^p \|m_{k,j}\|_{L^1(\mu)}^p \sum_{i=0}^{N^{(2)}_{B_{k,j},2^{k+5}B}+1} \frac{\mu(2^{i+1}B_{k,j})}{[\lambda(c_{B_{k,j}}, 2^i r_{B_{k,j}})]^p}\\
&\lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\ell+4} \sum_{j=1}^{M} |\lambda_{k,j}|^p \|m_{k,j}\|_{L^1(\mu)}^p [\mu(2^{N^{(2)}_{B_{k,j},2^{k+5}B}+2} B_{k,j})]^{1-p} [\widetilde{K}^{(2),p}_{B_{k,j},2^{k+5}B}]^p\\
&\lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\ell+4} \sum_{j=1}^{M} |\lambda_{k,j}|^p [\mu(B_{k,j})]^{p/q'} \|m_{k,j}\|_{L^q(\mu)}^p [\mu(2^{k+9}B)]^{1-p} [\widetilde{K}^{(2),p}_{B_{k,j},2^{k+5}B}]^p\\
&\lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\ell+4} \sum_{j=1}^{M} |\lambda_{k,j}|^p [\mu(B_{k,j})]^{p/q'} 2^{-k\delta p} [\mu(2B_{k,j})]^{-p/q'} [\lambda(c_B, 2^{k+2} r_B)]^{p-1}\\
&\quad \times [\widetilde{K}^{(2),p}_{B_{k,j},2^{k+2}B}]^{-p} [\mu(2^{k+9}B)]^{1-p} [\widetilde{K}^{(2),p}_{B_{k,j},2^{k+5}B}]^p\\
&\lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\ell+4} \sum_{j=1}^{M} 2^{-k\delta p} |\lambda_{k,j}|^p \lesssim \sum_{k=0}^{\infty} \sum_{j=1}^{M} |\lambda_{k,j}|^p \sim |b|^p_{\widetilde{H}^{p,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)},
\end{aligned}
$$

which, together with the estimate for II_1, implies that $\mathrm{II} \lesssim |b|^p_{\widetilde{H}^{p,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)}$.

To estimate IV, observe that

$$
\begin{aligned}
\mathrm{IV} &\leqslant \sum_{\ell=0}^{4} \int_{U_\ell(B)} \left| T\left(\sum_{k=0}^{\ell+4} \sum_{j=1}^{M} \lambda_{k,j} m_{k,j}\right)(x)\right|^p d\mu(x)\\
&\quad + \sum_{\ell=0}^{4} \int_{U_\ell(B)} \left| T\left(\sum_{k=\ell+5}^{\infty} \sum_{j=1}^{M} \lambda_{k,j} m_{k,j}\right)(x)\right|^p d\mu(x) =: \mathrm{IV}_1 + \mathrm{IV}_2.
\end{aligned}
$$

By some arguments similar to those used in the estimates for II_1 and III, we respectively obtain

$$
\mathrm{IV}_1 \lesssim |b|^p_{\widetilde{H}^{p,q,1,\delta}_{\mathrm{mb},2}(\mu)} \quad \text{and} \quad \mathrm{IV}_2 \lesssim |b|^p_{\widetilde{H}^{p,q,1,\delta}_{\mathrm{mb},2}(\mu)},
$$

which, together with the estimates for I–III, completes the proof of Theorem 4.8. □

Now we show the boundedness of Calderón-Zygmund operators from $\widetilde{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)$ into $L^p(\mu)$.

Corollary 4.9. *Suppose that* $(\mathcal{X}, d, \mu)$ *is a non-homogeneous metric measure space. Let* $\rho \in (1,\infty)$, $\frac{\nu}{\nu+\delta} < p \leqslant 1 < q < \infty$ *and* $\gamma \in [1,\infty)$. *Assume that the Calderón-Zygmund operator* T *defined by* (4.7) *associated with kernel* K *satisfying* (4.5) *and* (4.6) *is bounded on* $L^2(\mu)$. *Then* T *is bounded from* $\widetilde{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)$ *into* $L^p(\mu)$.

Proof. Let ρ, p, q, γ and δ be as in assumptions of Corollary 4.9. For the sake of simplicity, we take $\rho = 2$ and $\gamma = 1$. By an argument similar to that used in the proof of Theorem 4.8, it suffices to show that, for any $(p,q,1,2)_\lambda$-atomic block b,

$$
\|Tb\|_{L^p(\mu)} \lesssim |b|_{\widetilde{H}^{p,q,1}_{\mathrm{atb},2}(\mu)},
$$

which is an easy consequence of the facts that b is also a $(p,q,1,\delta,2)_\lambda$-molecular block and $|b|_{\widetilde{H}^{p,q,1,\delta}_{\mathrm{mb},2}(\mu)} \lesssim |b|_{\widetilde{H}^{p,q,1}_{\mathrm{atb},2}(\mu)}$ (see (4.3)), together with (4.8) from the proof of Theorem 4.8. This finishes the proof of Corollary 4.9. □

Remark 4.10. When $p=1$, Theorem 4.8 or Corollary 4.9 is a special case of [30, Theorem 4.1], since, for any $q\in(1,\infty]$, $\rho\in(1,\infty)$, $\gamma\in[1,\infty)$ and $\epsilon\in(0,\infty)$, $\widetilde{H}^{1,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)\subset H^{1,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$, where $H^{1,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ is introduced in [30] (see [13, Remark 1.9(i)]), and, by [13, Theorem 1.11 and Remark 1.9(i)], we know that $\widetilde{H}^{1,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ is independent of the choices of q, ρ and γ, and $\widetilde{H}^{1,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ and $\widetilde{H}^{1,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)$ coincide with equivalent norms.

Now we establish the $(\widetilde{H}^{p,\,q,\,\gamma+1}_{\mathrm{atb},\,\rho(\rho+1)}(\mu),\ \widetilde{H}^{p,\,q,\,\gamma,\,\frac12(\delta-\frac{\nu}{p}+\nu)}_{\mathrm{mb},\,\rho}(\mu))$-boundedness of Calderón-Zygmund operators. In what follows, for T as in Corollary 4.9, T is said to satisfy $T^*1=0$ if, for all $h\in L^\infty_b(\mu)$ with $\int_{\mathcal X}h(y)d\mu(y)=0$,

$$\int_{\mathcal X}Th(y)d\mu(y)=0.$$

Observe that, for such T and h, by Corollary 4.9, we have $Th\in L^1(\mu)$.

Theorem 4.11. *Suppose that $(\mathcal X,d,\mu)$ is a non-homogeneous metric measure space. Let $\rho\in[2,\infty)$, $\frac{\nu}{\nu+\delta}<p\leqslant1<q<\infty$ and $\gamma\in[1,\infty)$. Assume that the Calderón-Zygmund operator T defined by (4.7) associated with kernel K satisfying (4.5) and (4.6) is bounded on $L^2(\mu)$ and $T^*1=0$. Then T is bounded from $\widetilde{H}^{p,\,q,\,\gamma+1}_{\mathrm{atb},\,\rho(\rho+1)}(\mu)$ into $\widetilde{H}^{p,\,q,\,\gamma,\,\frac12(\delta-\frac{\nu}{p}+\nu)}_{\mathrm{mb},\,\rho}(\mu)$.*

Proof. Observe that, when $p=1$, Theorem 4.11 is a special case of [13, Theorem 1.14], since it was shown, by [13, Theorem 1.11 and Remark 1.9(i)], that, for any $q\in(1,\infty]$, $\rho\in(1,\infty)$, $\gamma\in[1,\infty)$ and $\epsilon\in(0,\infty)$, $\widetilde{H}^{1,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ and $\widetilde{H}^{1,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)$ coincide with equivalent norms, and $\widetilde{H}^{1,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ is independent of the choices of q, ρ and γ. Thus, to show Theorem 4.11, we only need to consider the case $p\in(\frac{\nu}{\nu+\delta},1)$. Moreover, for the sake of simplicity, we assume that $\gamma=1$ and $\rho=2$. Via some slight modifications, the arguments here are still valid for general cases. We first reduce our proof to showing that, for any $(p,q,2,6)_\lambda$-atomic block, Tb is a $(p,q,1,\frac12(\delta-\frac{\nu}{p}+\nu),2)_\lambda$-molecular block and

$$|Tb|_{\widetilde{H}^{p,\,q,\,1,\,\frac12(\delta-\frac{\nu}{p}+\nu)}_{\mathrm{mb},\,2}(\mu)}\lesssim|b|_{\widetilde{H}^{p,\,q,\,2}_{\mathrm{atb},\,6}(\mu)}. \tag{4.10}$$

Indeed, assume that (4.10) holds true. For any $\widetilde{\mathbb H}^{p,\,q,\,2}_{\mathrm{atb},\,6}(\mu)$, there exists a sequence $\{b_i\}_{i\in\mathbb N}$ of $(p,q,2,6)_\lambda$-atomic blocks such that $f=\sum_{i=1}^\infty b_i$ in $L^2(\mu)$ and

$$\sum_{i=1}^\infty|b_i|^p_{\widetilde{H}^{p,\,q,\,2}_{\mathrm{atb},\,6}(\mu)}\sim\|f\|^p_{\widetilde{H}^{p,\,q,\,2}_{\mathrm{atb},\,6}(\mu)}.$$

By the boundedness of T on $L^2(\mu)$, we see that

$$\left\|\sum_{i=1}^N T(b_i)-Tf\right\|_{L^2(\mu)}=\left\|T\left(\sum_{i=1}^N b_i-f\right)\right\|_{L^2(\mu)}\lesssim\left\|\sum_{i=1}^N b_i-f\right\|_{L^2(\mu)}\to0\quad\text{as}\quad N\to\infty.$$

Thus, $Tf=\sum_{i=1}^\infty T(b_i)$ in $L^2(\mu)$. Moreover, by (4.10), $T(b_i)$ is a $(p,q,1,\frac12(\delta-\frac{\nu}{p}+\nu),2)_\lambda$-molecular block for any $i\in\mathbb N$, we know that

$$\|Tf\|^p_{\widetilde{H}^{p,\,q,\,1,\,\frac12(\delta-\frac{\nu}{p}+\nu)}_{\mathrm{mb},\,2}(\mu)}\leqslant\sum_{i=1}^\infty|T(b_i)|^p_{\widetilde{H}^{p,\,q,\,1,\,\frac12(\delta-\frac{\nu}{p}+\nu)}_{\mathrm{mb},\,2}(\mu)}\lesssim\sum_{i=1}^\infty|b_i|^p_{\widetilde{H}^{p,\,q,\,2}_{\mathrm{atb},\,6}(\mu)}\sim\|f\|^p_{\widetilde{H}^{p,\,q,\,2}_{\mathrm{atb},\,6}(\mu)}.$$

Furthermore, by a standard density argument, we extend T to be a bounded linear operator from $\widetilde{H}^{p,\,q,\,2}_{\mathrm{atb},\,6}(\mu)$ into $\widetilde{H}^{p,\,q,\,1,\,\frac12(\delta-\frac{\nu}{p}+\nu)}_{\mathrm{mb},\,2}(\mu)$.

Now we show that (4.10) holds true. Let b be a $(p,q,2,6)_\lambda$-atomic block. Then $b:=\sum_{j=1}^2\lambda_ja_j$, where, for any $j\in\{1,2\}$, $\operatorname{supp}(a_j)\subset B_j\subset B$ for some balls B_j and B as in Definition 3.2. Let $B_0:=8B$. We write

$$Tb=(Tb)\chi_{B_0}+\sum_{k=1}^\infty(Tb)\chi_{2^kB_0\setminus2^{k-1}B_0}=:\mathrm{A}_1+\mathrm{A}_2.$$

We first estimate A_1. Since $B_j \subset B$, we have $3B_j \subset 8B = B_0$. Let $N_j := N^{(2)}_{2B_j,\,B_0}$. Obviously, $N_j \geqslant -1$. Without loss of generality, we may assume that $N_j \geqslant 3$. For the case $N_j \in [-1,3)$, we easily observe that $2B_j \subset B_0 \subset 2^5 B_j$, which can be reduced to the case $N_j \geqslant 3$. We further decompose

$$\begin{aligned}\mathrm{A}_1 &= \sum_{j=1}^{2} \lambda_j (Ta_j)\chi_{2B_j} + \sum_{j=1}^{2}\sum_{i=1}^{N_j-2} \lambda_j (Ta_j)\chi_{2^{i+1}B_j\backslash 2^i B_j} + \sum_{j=1}^{2} \lambda_j (Ta_j)\chi_{B_0\backslash 2^{N_j-1}B_j}\\ &=: \mathrm{A}_{1,1} + \mathrm{A}_{1,2} + \mathrm{A}_{1,3}.\end{aligned}$$

To estimate $\mathrm{A}_{1,1}$, by Definition 3.2(iii), the boundedness of T on $L^2(\mu)$, Lemmas 4.7, 2.8(v), 2.8(iv), 2.8(ii) and 2.9, and $\widetilde{K}^{(2),\,p}_{3B_j,\,B_0} \geqslant 1$, we see that, for any $j \in \{1,2\}$,

$$\begin{aligned}\|(Ta_j)\chi_{2B_j}\|_{L^q(\mu)} &\lesssim \|a_j\|_{L^q(\mu)} \lesssim [\mu(6B_j)]^{1/q-1}[\lambda(c_B, r_B)]^{1-1/p}[\widetilde{K}^{(6),\,p}_{B_j,\,B}]^{-2}\\ &\lesssim [\mu(6B_j)]^{1/q-1}[\lambda(c_B, 8r_B)]^{1-1/p}[\widetilde{K}^{(2),\,p}_{3B_j,\,8B}]^{-2}\\ &\lesssim [\mu(6B_j)]^{1/q-1}[\lambda(c_{B_0}, r_{4B_0})]^{1-1/p}[\widetilde{K}^{(2),\,p}_{3B_j,\,4B_0}]^{-1},\end{aligned}$$

here and hereafter, c_B and c_{B_0} denote the centers of B and B_0, and r_B and r_{4B_0} denote the radii of B and $4B_0$, respectively. Let c_1, independent of a_j and j, be the implicit positive constant of the above inequality, $\sigma_{j,\,1} := c_1\lambda_j$ and $n_{j,\,1} := c_1^{-1}(Ta_j)\chi_{2B_j}$. Then $\mathrm{A}_{1,\,1} = \sum_{j=1}^{2}\sigma_{j,\,1}n_{j,\,1}$, $\operatorname{supp}(n_{j,\,1}) \subset 3B_j \subset B_0$ and

$$\|n_{j,\,1}\|_{L^q(\mu)} \leqslant [\mu(6B_j)]^{1/q-1}[\lambda(c_{B_0}, r_{4B_0})]^{1-1/p}[\widetilde{K}^{(2),\,p}_{3B_j,\,4B_0}]^{-1}.$$

For $\mathrm{A}_{1,3}$, we observe that $r_{B_0} \sim r_{2^{N_j-1}B_j}$, where r_{B_0} and $r_{2^{N_j-1}B_j}$ denote the radii of B_0 and $2^{N_j-1}B_j$, respectively. For any $j \in \{1,2\}$, let x_j and r_j be the center and the radius of B_j, respectively. By (4.5), (2.2), (2.1), Hölder's inequality, Definition 3.2(iii), $\widetilde{K}^{(6),\,p}_{B_j,\,B} \geqslant 1$, $B_0 \subset 2^{N_j+3}B_j$, Lemmas 2.8(ii) and 2.9, we obtain

$$\begin{aligned}\|(Ta_j)\chi_{B_0\backslash 2^{N_j-1}B_j}\|_{L^q(\mu)} &\lesssim \left\{\int_{8B\backslash 2^{N_j-1}B_j}\left[\int_{B_j}\frac{|a_j(y)|}{\lambda(x,d(x,y))}d\mu(y)\right]^q d\mu(x)\right\}^{1/q}\\ &\lesssim \left\{\int_{8B\backslash 2^{N_j-1}B_j}\left[\int_{B_j}\frac{|a_j(y)|}{\lambda(x_j,d(x,x_j))}d\mu(y)\right]^q d\mu(x)\right\}^{1/q}\\ &\lesssim \frac{[\mu(8B\backslash 2^{N_j-1}B_j)]^{1/q}}{\lambda(x_j, 2^{N_j-1}r_j)}[\mu(B_j)]^{1/q'}\|a_j\|_{L^q(\mu)}\\ &\lesssim [\mu(32B)]^{1/q-1}[\lambda(c_B, r_B)]^{1-1/p}[\widetilde{K}^{(6),\,p}_{B_j,\,B}]^{-2}\\ &\lesssim [\mu(4B_0)]^{1/q-1}[\lambda(c_{B_0}, r_{4B_0})]^{1-1/p}[\widetilde{K}^{(2),\,p}_{2B_0,\,4B_0}]^{-1}.\end{aligned}$$

Let c_2, independent of a_j and j, be the implicit positive constant of the above inequality, $\sigma_{j,\,3} := c_2\lambda_j$ and $n_{j,\,3} := c_2^{-1}(Ta_j)\chi_{B_0\backslash 2^{N_j-1}B_j}$. Then $\mathrm{A}_{1,\,3} = \sum_{j=1}^{2}\sigma_{j,\,3}n_{j,\,3}$, $\operatorname{supp}(n_{j,\,3}) \subset 16B = 2B_0$ and

$$\|n_{j,\,3}\|_{L^q(\mu)} \leqslant [\mu(4B_0)]^{1/q-1}[\lambda(c_{B_0}, r_{4B_0})]^{1-1/p}[\widetilde{K}^{(2),\,p}_{2B_0,\,4B_0}]^{-1}.$$

We now estimate $\mathrm{A}_{1,\,2}$. By (4.5), (2.2), (2.1), Definition 3.2(iii), Hölder's inequality, Lemmas 2.8(v), 2.8(iv), 2.8(ii) and 2.9, we conclude that

$$\begin{aligned}&\|(Ta_j)\chi_{2^{i+1}B_j\backslash 2^i B_j}\|_{L^q(\mu)}\\ &\quad\lesssim \left\{\int_{2^{i+1}B_j\backslash 2^i B_j}\left[\int_{B_j}\frac{|a_j(y)|}{\lambda(x,d(x,y))}d\mu(y)\right]^q d\mu(x)\right\}^{1/q}\\ &\quad\lesssim \left\{\int_{2^{i+1}B_j\backslash 2^i B_j}\left[\int_{B_j}\frac{|a_j(y)|}{\lambda(x_j,d(x,x_j))}d\mu(y)\right]^q d\mu(x)\right\}^{1/q}\end{aligned}$$

$$
\begin{aligned}
&\lesssim [\lambda(x_j, 2^i r_j)]^{1/q-1} \|a_j\|_{L^1(\mu)} \left[\int_{2^{i+1}B_j \backslash 2^i B_j} \frac{1}{\lambda(x_j, 2^i r_j)} d\mu(x) \right]^{1/q} \\
&\lesssim [\lambda(x_j, 2^i r_j)]^{1/q-1} \|a_j\|_{L^q(\mu)} [\mu(B_j)]^{1/q'} \left[\frac{\mu(2^{i+1}B_j)}{\lambda(x_j, 2^i r_j)} \right]^{1/q} \\
&\lesssim \frac{\mu(2^{i+3}B_j)}{\lambda(x_j, 2^i r_j)} [\widetilde{K}^{(2),\,p}_{B_j,\,B}]^{-1} [\mu(2^{i+3}B_j)]^{1/q-1} [\lambda(c_{B_0}, r_{4B_0})]^{1-1/p} [\widetilde{K}^{(2),\,p}_{2^{i+2}B_j,\,4B_0}]^{-1}.
\end{aligned}
$$

Let c_3, independent of a_j, j and i, be the implicit positive constant of the above inequality,

$$
\sigma^{(i)}_{j,\,2} := c_3 \lambda_j \frac{\mu(2^{i+3}B_j)}{\lambda(x_j, 2^i r_j)} [\widetilde{K}^{(2),\,p}_{B_j,\,B}]^{-1}
$$

and

$$
n^{(i)}_{j,\,2} := \left[c_3 \frac{\mu(2^{i+3}B_j)}{\lambda(x_j, 2^i r_j)} \right]^{-1} \widetilde{K}^{(2),\,p}_{B_j,\,B} (Ta_j) \chi_{2^{i+1}B_j \backslash 2^i B_j}.
$$

Then

$$
\mathrm{A}_{1,\,2} = \sum_{j=1}^{2} \sum_{i=1}^{N_j-2} \sigma^{(i)}_{j,\,2} n^{(i)}_{j,\,2},
$$

$\operatorname{supp}(n^{(i)}_{j,\,2}) \subset 2^{i+2}B_j \subset 2B_0$ and

$$
\|n^{(i)}_{j,\,2}\|_{L^q(\mu)} \leqslant [\mu(2(2^{i+2}B_j))]^{1/q-1} [\lambda(c_{B_0}, r_{4B_0})]^{1-1/p} [\widetilde{K}^{(2),\,p}_{2^{i+2}B_j,\,4B_0}]^{-1}.
$$

Now we turn to estimate A_2. For any $k \in \mathbb{N}$, by the geometrically doubling condition, there exists a ball covering $\{B_{k,\,j}\}_{j=1}^{M_0}$, with uniform radius $2^{k-3} r_{B_0}$, of $\widetilde{U}_k(B_0) := 2^k B_0 \backslash 2^{k-1} B_0$ such that the cardinality $M_0 \leqslant N_0 8^n$. Without loss of generality, we may assume that the centers of the balls in the covering belong to $\widetilde{U}_k(B_0)$.

Let $C_{k,\,1} := B_{k,\,1}$, $C_{k,\,l} := B_{k,\,l} \backslash \bigcup_{m=1}^{l-1} B_{k,\,m}$, $l \in \{2, \ldots, M_0\}$ and $D_{k,\,l} := C_{k,\,l} \cap \widetilde{U}_k(B_0)$ for all $l \in \{1, \ldots, M_0\}$. Then we know that $\{D_{k,\,l}\}_{l=1}^{M_0}$ is pairwise disjoint, $\widetilde{U}_k(B_0) = \bigcup_{l=1}^{M_0} D_{k,\,l}$ and, for any $l \in \{1, \ldots, M_0\}$,

$$
D_{k,\,l} \subset 2B_{k,\,l} \subset U_k(B_0) := 2^{k+2} B_0 \backslash 2^{k-2} B_0.
$$

Thus,

$$
\mathrm{A}_2 = \sum_{k=1}^{\infty} Tb \sum_{l=1}^{M_0} \chi_{D_{k,\,l}} = \sum_{k=1}^{\infty} \sum_{l=1}^{M_0} (Tb) \chi_{D_{k,\,l}}.
$$

From $\int_{\mathcal{X}} b(y) d\mu(y) = 0$, (4.6), (2.2), (2.1), Hölder's inequality, Definition 3.2(iii), $\widetilde{K}^{(2),\,p}_{B_j,\,B} \geqslant 1$, $4B_{k,\,l} \subset 2^{k+1}B_0$ and Lemma 2.8(ii), it follows that, for any $k \in \mathbb{N}$, $j \in \{1, 2\}$ and $l \in \{1, \ldots, M_0\}$,

$$
\begin{aligned}
\|(Tb)\chi_{D_{k,\,l}}\|_{L^q(\mu)} &\leqslant \left\{ \int_{D_{k,\,l}} \left[\int_B |b(y)| |K(x,y) - K(x, c_B)| d\mu(y) \right]^q d\mu(x) \right\}^{1/q} \\
&\lesssim \left[\int_{D_{k,\,l}} \left\{ \int_B |b(y)| \frac{[d(y, c_B)]^\delta}{[d(x, c_B)]^\delta \lambda(c_B, d(x, c_B))} d\mu(y) \right\}^q d\mu(x) \right]^{1/q} \\
&\lesssim \frac{r_B^\delta [\mu(D_{k,\,l})]^{1/q}}{\lambda(c_B, r_{2^{k-1}B_0}) (r_{2^{k-1}B_0})^\delta} \int_B |b(y)| d\mu(y) \\
&\lesssim 2^{-k\delta} [\mu(2^{k+1}B_0)]^{1/q-1} \sum_{j=1}^{2} |\lambda_j| [\mu(B_j)]^{1/q'} \|a_j\|_{L^q(\mu)} \\
&\lesssim 2^{-k\delta} [\mu(2^{k+1}B_0)]^{1/q-1} \sum_{j=1}^{2} |\lambda_j| [\lambda(c_B, r_B)]^{1-1/p} \\
&\lesssim 2^{-k\delta} [C_{(\lambda)}]^{(k+2)(\frac{1}{p}-1)} \sum_{j=1}^{2} |\lambda_j| [\mu(4B_{k,\,l})]^{1/q-1} [\lambda(c_B, r_{2^{k+2}B_0})]^{1-1/p}
\end{aligned}
$$

$$\lesssim 2^{-\frac{k}{2}(\delta-\frac{\nu}{p}+\nu)}2^{-\frac{k}{2}(\delta-\frac{\nu}{p}+\nu)}\sum_{j=1}^{2}|\lambda_j|[\mu(4B_{k,\,l})]^{1/q-1}$$
$$\times[\lambda(c_B, r_{2^{k+2}B_0})]^{1-1/p}[\widetilde{K}^{(2),\,p}_{2B_{k,\,l},\,2^{k+2}B_0}]^{-1}.$$

Let c_4, independent of b and k, be the implicit positive constant of the above inequality,

$$\lambda_{k,\,l} := c_4 2^{-\frac{k}{2}(\delta-\frac{\nu}{p}+\nu)}\sum_{j=1}^{2}|\lambda_j|$$

and $m_{k,\,l} := \lambda_{k,\,l}^{-1}(Tb)\chi_{D_{k,\,l}}$. Then

$$\mathrm{A}_2 = \sum_{k=1}^{\infty}\sum_{l=1}^{M_0}\lambda_{k,\,l}m_{k,\,l},$$

$\operatorname{supp}(m_{k,\,l}) \subset 2B_{k,\,l} \subset U_k(B_0)$ and

$$\|m_{k,\,l}\|_{L^q(\mu)} \leqslant 2^{-\frac{k}{2}(\delta-\frac{\nu}{p}+\nu)}[\mu(2(2B_{k,\,l}))]^{1/q-1}[\lambda(c_B, r_{2^{k+2}B_0})]^{1-1/p}[\widetilde{K}^{(2),\,p}_{2B_{k,\,l},\,2^{k+2}B_0}]^{-1}.$$

Combining the estimates for A_1 and A_2, we see that Tb is a $(p, q, 1, \frac{1}{2}(\delta-\frac{\nu}{p}+\nu), 2)_\lambda$-molecular block and

$$\begin{aligned}|Tb|^p_{\widetilde{H}^{p,\,q,\,1,\,\frac{1}{2}(\delta-\frac{\nu}{p}+\nu)}_{\mathrm{mb},\,2}(\mu)} &= \sum_{j=1}^{2}|\sigma_{j,\,1}|^p + \sum_{j=1}^{2}\sum_{i=1}^{N_j-1}|\sigma^{(i)}_{j,\,2}|^p + \sum_{j=1}^{2}|\sigma_{j,\,3}|^p + \sum_{k=1}^{\infty}\sum_{l=1}^{M_0}|\lambda_{k,\,l}|^p\\ &\lesssim \sum_{j=1}^{2}|\lambda_j|^p + \sum_{j=1}^{2}\sum_{i=1}^{N_j-2}|\lambda_j|^p\left[\frac{\mu(2^{i+3}B_j)}{\lambda(x_j, 2^i r_j)}\right]^p[\widetilde{K}^{(2),\,p}_{B_j,\,B}]^{-p}\\ &\quad+\sum_{k=1}^{\infty}\sum_{l=1}^{M_0}2^{-\frac{k}{2}(\delta-\frac{\nu}{p}+\nu)p}\sum_{j=1}^{2}|\lambda_j|^p\\ &\lesssim \sum_{j=1}^{2}|\lambda_j|^p + \sum_{k=1}^{\infty}2^{-\frac{k}{2}(\delta-\frac{\nu}{p}+\nu)p}M_0\sum_{j=1}^{2}|\lambda_j|^p \lesssim \sum_{j=1}^{2}|\lambda_j|^p \sim |b|^p_{\widetilde{H}^{p,\,q,\,2}_{\mathrm{atb},\,6}(\mu)},\end{aligned}$$

which completes the proof of Theorem 4.11. □

Remark 4.12. It is still unclear whether the range of ρ in Theorem 4.11 is sharp or not.

5 Boundedness of generalized fractional integrals

In this section, we establish the boundedness of the generalized fractional integral T_β ($\beta \in (0,1)$) from $\widetilde{H}^{p_1,\,q,\,\gamma,\,\theta}_{\mathrm{mb},\,\rho}(\mu)$ (or $\widetilde{H}^{p_1,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$) into $L^{p_2}(\mu)$ with $1/p_2 = 1/p_1 - \beta$, where θ is some positive constant depending on T_β. To this end, we first recall the notion of generalized fractional integrals from [15].

Definition 5.1. Let $\beta \in (0,1)$. A function $K_\beta \in L^1_{\mathrm{loc}}(\mathcal{X}\times\mathcal{X}\setminus\{(x,x): x \in \mathcal{X}\})$ is called a *generalized fractional integral kernel* if there exists a positive constant $C_{(K_\beta)}$, depending on K_β, such that

(i) for all x, $y \in \mathcal{X}$ with $x \neq y$,

$$|K_\beta(x,y)| \leqslant C_{(K_\beta)}\frac{1}{[\lambda(x, d(x,y))]^{1-\beta}}; \tag{5.1}$$

(ii) there exist positive constants $\theta \in (0,1]$ and $c_{(K_\beta)}$, depending on K_β, such that, for all x, $\widetilde{x}$, $y \in \mathcal{X}$ with $d(x,y) \geqslant c_{(K_\beta)}d(x,\widetilde{x})$,

$$|K_\beta(x,y) - K_\beta(\widetilde{x},y)| + |K_\beta(y,x) - K_\beta(y,\widetilde{x})| \leqslant C_{(K_\beta)}\frac{[d(x,\widetilde{x})]^\theta}{[d(x,y)]^\theta[\lambda(x, d(x,y))]^{1-\beta}}. \tag{5.2}$$

A linear operator T_β is called a *generalized fractional integral* with kernel K_β satisfying (5.1) and (5.2) if, for all $f \in L_b^\infty(\mu)$ and $x \notin \operatorname{supp}(f)$,

$$T_\beta f(x) := \int_{\mathcal{X}} K_\beta(x,y) f(y) d\mu(y). \tag{5.3}$$

Remark 5.2. It was shown in [15, Remark 1.10(iii)] that there exists a specific example of the generalized fractional integral, which is a natural variant of the so-called Bergman-type operator; see [15] for the details.

Now we show that the generalized fractional integral T_β is bounded from $\widetilde{H}^{p_1,\,q,\,\gamma,\,\theta}_{\mathrm{mb},\,\rho}(\mu)$ into $L^{p_2}(\mu)$ for $1/p_2 = 1/p_1 - \beta$. Recall that $\nu := \log_2 C_{(\lambda)}$ and θ is as in Definition 5.1.

Theorem 5.3. *Suppose that $(\mathcal{X}, d, \mu)$ is a non-homogeneous metric measure space. Let $\beta \in (0, 1/2)$, $\rho \in (1,\infty)$, $\frac{\nu}{\nu+\theta} < p_1 < p_2 \leqslant 1 < q < 1/\beta$, $1/p_2 = 1/p_1 - \beta$ and $\gamma \in [1,\infty)$. Assume that the generalized fractional integral T_β defined by (5.3) associated with kernel K_β satisfying (5.1) and (5.2) is bounded from $L^q(\mu)$ into $L^{\widetilde{q}}(\mu)$, where $1/\widetilde{q} := 1/q - \beta$. Then T_β is bounded from $\widetilde{H}^{p_1,\,q,\,\gamma,\,\theta}_{\mathrm{mb},\,\rho}(\mu)$ into $L^{p_2}(\mu)$.*

Proof. Let β, ρ, p_1, p_2, q, $\widetilde{q}$ and γ be as in assumptions of Theorem 5.3. For the sake of simplicity, we take $\rho = 2$ and $\gamma = 1$. With some minor modifications, the arguments here are still valid for general cases.

Since T_β is bounded from $L^q(\mu)$ to $L^{\widetilde{q}}(\mu)$ for $q \in (1, 1/\beta)$ and $1/\widetilde{q} = 1/q - \beta$, by [15, Theorem 1.13], we know that T_β is also bounded from $L^1(\mu)$ to weak-$L^{1/(1-\beta)}(\mu)$. By the boundedness of T_β from $L^2(\mu)$ into $L^{2/(1-2\beta)}(\mu)$ or from $L^1(\mu)$ into weak-$L^{1/(1-\beta)}(\mu)$ and an argument similar to that used in the proof of Theorem 4.8, to show Theorem 5.3, it suffices to show that, for all $(p_1, q, 1, \theta, 2)_\lambda$-molecular blocks b,

$$\|T_\beta b\|_{L^{p_2}(\mu)} \lesssim |b|_{\widetilde{H}^{p_1,\,q,\,1,\,\theta}_{\mathrm{mb},\,2}(\mu)}.$$

Let $b = \sum_{k=0}^\infty \sum_{j=1}^{M_k} \lambda_{k,\,j} m_{k,\,j}$ be a $(p_1, q, 1, \theta, 2)_\lambda$-molecular block, where, for any $k \in \mathbb{Z}_+$ and $j \in \{1, \ldots, M_k\}$, $\operatorname{supp}(m_{k,\,j}) \subset B_{k,\,j} \subset U_k(B)$ for some balls B and $B_{k,\,j}$ as in Definition 4.1. Without loss of generality, we may assume that $\widetilde{M} = M$ in Definition 4.1.

By the linearity of T_β, we write

$$\begin{aligned}
\|T_\beta b\|^{p_2}_{L^{p_2}(\mu)} &\leqslant \sum_{\ell=5}^\infty \int_{U_\ell(B)} \left| T_\beta\Bigg(\sum_{k=0}^{\ell-5} \sum_{j=1}^{M} \lambda_{k,\,j} m_{k,\,j} \Bigg)(x) \right|^{p_2} d\mu(x) \\
&\quad + \sum_{\ell=5}^\infty \int_{U_\ell(B)} \left| T_\beta\Bigg(\sum_{k=\ell-4}^{\ell+4} \sum_{j=1}^{M} \lambda_{k,\,j} m_{k,\,j} \Bigg)(x) \right|^{p_2} d\mu(x) \\
&\quad + \sum_{\ell=5}^\infty \int_{U_\ell(B)} \left| T_\beta\Bigg(\sum_{k=\ell+5}^{\infty} \sum_{j=1}^{M} \lambda_{k,\,j} m_{k,\,j} \Bigg)(x) \right|^{p_2} d\mu(x) + \sum_{\ell=0}^{4} \int_{U_\ell(B)} |T_\beta b(x)|^{p_2} d\mu(x) \\
&=: \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}.
\end{aligned}$$

Now we first estimate III. By (5.1), (2.1), (2.2), Hölder's inequality, (4.1) and $1/p_2 = 1/p_1 - \beta$, we obtain

$$\begin{aligned}
\mathrm{III} &\leqslant \sum_{\ell=5}^\infty \sum_{k=\ell+5}^\infty \sum_{j=1}^{M} |\lambda_{k,\,j}|^{p_2} \int_{U_\ell(B)} \left[\int_{B_{k,\,j}} |m_{k,\,j}(y)| |K_\beta(x,y)| d\mu(y) \right]^{p_2} d\mu(x) \\
&\lesssim \sum_{\ell=5}^\infty \sum_{k=\ell+5}^\infty \sum_{j=1}^{M} |\lambda_{k,\,j}|^{p_2} \int_{U_\ell(B)} \left\{ \int_{B_{k,\,j}} \frac{|m_{k,\,j}(y)|}{[\lambda(x, d(x,y))]^{1-\beta}} d\mu(y) \right\}^{p_2} d\mu(x) \\
&\lesssim \sum_{\ell=5}^\infty \sum_{k=\ell+5}^\infty \sum_{j=1}^{M} |\lambda_{k,\,j}|^{p_2} \int_{U_\ell(B)} \frac{1}{[\lambda(c_B, d(x, c_B))]^{p_2(1-\beta)}} d\mu(x) \|m_{k,\,j}\|^{p_2}_{L^1(\mu)}
\end{aligned}$$

$$\lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell+5}^{\infty} \sum_{j=1}^{M} |\lambda_{k,j}|^{p_2} \frac{\mu(2^{\ell+2}B)}{[\lambda(c_B, 2^{\ell-2}r_B)]^{p_2(1-\beta)}} [\mu(B_{k,j})]^{p_2/q'} \|m_{k,j}\|_{L^q(\mu)}^{p_2}$$

$$\lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell+5}^{\infty} \sum_{j=1}^{M} |\lambda_{k,j}|^{p_2} [\mu(2^{\ell+2}B)]^{1-p_2(1-\beta)} [\mu(B_{k,j})]^{p_2/q'}$$

$$\times 2^{-k\theta p_2} [\mu(2B_{k,j})]^{-p_2/q'} \left[\lambda\left(c_B, 2^{k+2}r_B\right)\right]^{p_2(1-1/p_1)}$$

$$\lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell+5}^{\infty} \sum_{j=1}^{M} 2^{-k\theta p_2} |\lambda_{k,j}|^{p_2} \sim \sum_{j=1}^{M} \sum_{k=10}^{\infty} \sum_{\ell=5}^{k-5} 2^{-k\theta p_2} |\lambda_{k,j}|^{p_2}$$

$$\lesssim \sum_{j=1}^{M} \sum_{k=10}^{\infty} k 2^{-k\theta p_2} |\lambda_{k,j}|^{p_2} \lesssim \left\{ \sum_{k=0}^{\infty} \sum_{j=1}^{M} |\lambda_{k,j}|^{p_1} \right\}^{p_2/p_1} \sim |b|^{p_2}_{\widetilde{H}^{p_1,q,1,\theta}_{\mathrm{mb},2}(\mu)}.$$

In order to estimate I, write

$$\mathrm{I} \leqslant \sum_{\ell=5}^{\infty} \int_{U_\ell(B)} \left| \int_{\mathcal{X}} \left[\sum_{k=0}^{\ell-5} \sum_{j=1}^{M} \lambda_{k,j} m_{k,j}(y) \right] [K_\beta(x,y) - K_\beta(x,c_B)] d\mu(y) \right|^{p_2} d\mu(x)$$

$$+ \sum_{\ell=5}^{\infty} \int_{U_\ell(B)} \left| \int_{\mathcal{X}} \left[\sum_{k=0}^{\ell-5} \sum_{j=1}^{M} \lambda_{k,j} m_{k,j}(y) \right] K_\beta(x,c_B) d\mu(y) \right|^{p_2} d\mu(x) =: \mathrm{I}_1 + \mathrm{I}_2.$$

From (5.2), (2.2), (2.1), Hölder's inequality, (4.1) and $p_1 \in (\frac{\nu}{\nu+\theta}, 1]$ and $1/p_2 = 1/p_1 - \beta$, it follows that

$$\mathrm{I}_1 \lesssim \sum_{\ell=5}^{\infty} \sum_{k=0}^{\ell-5} \sum_{j=1}^{M} |\lambda_{k,j}|^{p_2} \int_{U_\ell(B)} \left\{ \int_{B_{k,j}} \frac{|m_{k,j}(y)| [d(y,c_B)]^\theta}{[d(x,c_B)]^\theta [\lambda(c_B, d(x,c_B))]^{1-\beta}} d\mu(y) \right\}^{p_2} d\mu(x)$$

$$\lesssim \sum_{\ell=5}^{\infty} \sum_{k=0}^{\ell-5} \sum_{j=1}^{M} |\lambda_{k,j}|^{p_2} \frac{2^{(k+2)\theta p_2} r_B^{\theta p_2} \mu(2^{\ell+2}B)}{2^{(\ell-2)\theta p_2} r_B^{\theta p_2} [\lambda(c_B, 2^{\ell-2}r_B)]^{p_2(1-\beta)}} \|m_{k,j}\|_{L^1(\mu)}^{p_2}$$

$$\lesssim \sum_{\ell=5}^{\infty} \sum_{k=0}^{\ell-5} \sum_{j=1}^{M} |\lambda_{k,j}|^{p_2} 2^{(k-\ell)\theta p_2} [\mu(2^{\ell+2}B)]^{1-p_2(1-\beta)} [\mu(B_{k,j})]^{p_2/q'} \|m_{k,j}\|_{L^q(\mu)}^{p_2}$$

$$\lesssim \sum_{\ell=5}^{\infty} \sum_{k=0}^{\ell-5} \sum_{j=1}^{M} |\lambda_{k,j}|^{p_2} 2^{-\ell\theta p_2} [\mu(2^{\ell+2}B)]^{1-p_2(1-\beta)} [\lambda(c_B, 2^{k+2}r_B)]^{p_2(1-1/p_1)}$$

$$\lesssim \sum_{\ell=5}^{\infty} \sum_{k=0}^{\ell-5} \sum_{j=1}^{M} |\lambda_{k,j}|^{p_2} 2^{-\ell\theta p_2} [\mu(2^{\ell+2}B)]^{1-p_2(1-\beta)} [\lambda(c_B, 2^{\ell+2}r_B)]^{p_2(1-1/p_1)} [C_{(\lambda)}]^{(\ell-k)(p_2/p_1-p_2)}$$

$$\lesssim \sum_{\ell=5}^{\infty} \sum_{k=0}^{\ell-5} \sum_{j=1}^{M} |\lambda_{k,j}|^{p_2} 2^{[\nu(p_2/p_1-p_2)-\theta p_2]\ell} 2^{-k\nu(p_2/p_1-p_2)} \lesssim \sum_{k=0}^{\infty} \sum_{j=1}^{M} |\lambda_{k,j}|^{p_2} \lesssim |b|^{p_2}_{\widetilde{H}^{p_1,q,1,\theta}_{\mathrm{mb},2}(\mu)}.$$

For I_2, the vanishing moment of b, together with (5.1), (2.1), (2.2) and $1/p_2 = 1/p_1 - \beta$, implies that

$$\mathrm{I}_2 = \sum_{\ell=5}^{\infty} \int_{U_\ell(B)} \left| \int_{\mathcal{X}} \left[\sum_{k=\ell-4}^{\infty} \sum_{j=1}^{M} \lambda_{k,j} m_{k,j}(y) \right] K_\beta(x,c_B) d\mu(y) \right|^{p_2} d\mu(x)$$

$$\lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\infty} \sum_{j=1}^{M} |\lambda_{k,j}|^{p_2} \int_{U_\ell(B)} \left\{ \int_{B_{k,j}} |m_{k,j}(y)| \frac{1}{[\lambda(c_B, d(x,c_B))]^{1-\beta}} d\mu(y) \right\}^{p_2} d\mu(x)$$

$$\lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\infty} \sum_{j=1}^{M} |\lambda_{k,j}|^{p_2} \frac{\mu(2^{\ell+2}B)}{[\lambda(c_B, 2^{\ell-2}r_B)]^{p_2(1-\beta)}} \|m_{k,j}\|_{L^1(\mu)}^{p_2}$$

$$\lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\infty} \sum_{j=1}^{M} |\lambda_{k,j}|^{p_2} [\mu(2^{\ell+2}B)]^{1-p_2(1-\beta)} [\mu(B_{k,j})]^{p_2/q'} \|m_{k,j}\|_{L^q(\mu)}^{p_2}$$

$$
\begin{aligned}
&\lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\infty} \sum_{j=1}^{M} |\lambda_{k,j}|^{p_2} [\mu(2^{\ell+2}B)]^{1-p_2(1-\beta)} [\mu(B_{k,j})]^{p_2/q'} \\
&\quad \times 2^{-k\theta p_2} [\mu(2B_{k,j})]^{-p_2/q'} [\lambda(c_B, 2^{k+2} r_B)]^{p_2(1-1/p_1)} \\
&\lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\infty} \sum_{j=1}^{M} 2^{-k\theta p_2} |\lambda_{k,j}|^{p_2} \sim \sum_{j=1}^{M} \sum_{k=1}^{\infty} \sum_{\ell=5}^{k+4} 2^{-k\theta p_2} |\lambda_{k,j}|^{p_2} \\
&\lesssim \sum_{j=1}^{M} \sum_{k=0}^{\infty} k 2^{-k\theta p_2} |\lambda_{k,j}|^{p_2} \lesssim \left\{ \sum_{k=0}^{\infty} \sum_{j=1}^{M} |\lambda_{k,j}|^{p_1} \right\}^{p_2/p_1} \sim |b|^{p_2}_{\widetilde{H}^{p_1,q,1,\theta}_{\mathrm{mb},2}(\mu)}.
\end{aligned}
$$

Combining I_1 and I_2, we conclude that $\mathrm{I} \lesssim |b|^{p_2}_{\widetilde{H}^{p_1,q,1,\theta}_{\mathrm{mb},2}(\mu)}$.

Then we turn to estimate II. We further write

$$
\begin{aligned}
\mathrm{II} &\leqslant \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\ell+4} \sum_{j=1}^{M} |\lambda_{k,j}|^{p_2} \int_{U_\ell(B)} |T_\beta(m_{k,j})(x)|^{p_2} d\mu(x) \\
&\leqslant \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\ell+4} \sum_{j=1}^{M} |\lambda_{k,j}|^{p_2} \int_{2B_{k,j}} |T_\beta(m_{k,j})(x)|^{p_2} d\mu(x) \\
&\quad + \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\ell+4} \sum_{j=1}^{M} |\lambda_{k,j}|^{p_2} \int_{U_\ell(B)\setminus 2B_{k,j}} |T_\beta(m_{k,j})(x)|^{p_2} d\mu(x) =: \mathrm{II}_1 + \mathrm{II}_2.
\end{aligned}
$$

By Hölder's inequality, $(L^q(\mu), L^{\widetilde{q}}(\mu))$-boundedness of T_β, (4.1), (2.1), $1/p_2 = 1/p_1 - \beta$ and $1/\widetilde{q} = 1/q - \beta$, we see that

$$
\begin{aligned}
\mathrm{II}_1 &\leqslant \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\ell+4} \sum_{j=1}^{M} |\lambda_{k,j}|^{p_2} [\mu(2B_{k,j})]^{1-p_2/\widetilde{q}} \|T_\beta(m_{k,j})\|^{p_2}_{L^{\widetilde{q}}(\mu)} \\
&\lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\ell+4} \sum_{j=1}^{M} |\lambda_{k,j}|^{p_2} [\mu(2B_{k,j})]^{1-p_2/\widetilde{q}} \|m_{k,j}\|^{p_2}_{L^q(\mu)} \\
&\lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\ell+4} \sum_{j=1}^{M} |\lambda_{k,j}|^{p_2} [\mu(2B_{k,j})]^{1-p_2/\widetilde{q}} 2^{-k\theta p_2} [\mu(2B_{k,j})]^{-p_2/q'} [\lambda(c_B, 2^{k+2} r_B)]^{p_2(1-1/p_1)} \\
&\lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\ell+4} \sum_{j=1}^{M} 2^{-k\theta p_2} |\lambda_{k,j}|^{p_2} \lesssim \sum_{k=0}^{\infty} \sum_{j=1}^{M} |\lambda_{k,j}|^{p_2} \lesssim |b|^{p_2}_{\widetilde{H}^{p_1,q,1,\theta}_{\mathrm{mb},2}(\mu)}.
\end{aligned}
$$

For II_2, from (5.1), $d(x,y) \geqslant d(x, c_{B_{k,j}}) - d(y, c_{B_{k,j}}) \geqslant \frac{1}{2} d(x, c_{B_{k,j}})$ for $x \notin 2B_{k,j}$ and $y \in B_{k,j}$, (2.2), (2.1), $p_2(1-\beta) < p_1$, Hölder's inequality, (4.1) and $1/p_2 = 1/p_1 - \beta$, we deduce that

$$
\begin{aligned}
\mathrm{II}_2 &\lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\ell+4} \sum_{j=1}^{M} |\lambda_{k,j}|^{p_2} \int_{U_\ell(B)\setminus 2B_{k,j}} \left\{ \int_{B_{k,j}} \frac{|m_{k,j}(y)|}{[\lambda(c_{B_{k,j}}, d(x, c_{B_{k,j}}))]^{1-\beta}} d\mu(y) \right\}^{p_2} d\mu(x) \\
&\lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\ell+4} \sum_{j=1}^{M} |\lambda_{k,j}|^{p_2} \int_{2^{k+6}B\setminus B_{k,j}} \frac{1}{[\lambda(c_{B_{k,j}}, d(x, c_{B_{k,j}}))]^{p_2(1-\beta)}} d\mu(x) \|m_{k,j}\|^{p_2}_{L^1(\mu)} \\
&\lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\ell+4} \sum_{j=1}^{M} |\lambda_{k,j}|^{p_2} \left\{ \int_{2^{k+6}B\setminus B_{k,j}} \frac{1}{[\lambda(c_{B_{k,j}}, d(x, c_{B_{k,j}}))]^{p_1}} d\mu(x) \right\}^{\frac{p_2(1-\beta)}{p_1}} \\
&\quad \times [\mu(2^{k+6}B)]^{\frac{p_1 - p_2(1-\beta)}{p_1}} \|m_{k,j}\|^{p_2}_{L^1(\mu)} \\
&\lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\ell+4} \sum_{j=1}^{M} |\lambda_{k,j}|^{p_2} \left\{ \sum_{i=0}^{N^{(2)}_{B_{k,j}, 2^{k+5}B}+1} \frac{\mu(2^{i+1}B_{k,j})}{[\lambda(c_{B_{k,j}}, 2^i r_{B_{k,j}})]^{p_1}} \right\}^{\frac{p_2(1-\beta)}{p_1}}
\end{aligned}
$$

$$
\begin{aligned}
&\times [\mu(2^{k+6}B)]^{\frac{p_1-p_2(1-\beta)}{p_1}}\|m_{k,j}\|_{L^1(\mu)}^{p_2} \\
&\lesssim \sum_{\ell=5}^{\infty}\sum_{k=\ell-4}^{\ell+4}\sum_{j=1}^{M}|\lambda_{k,j}|^{p_2}[\mu(2^{N^{(2)}_{B_{k,j},2^{k+5}B}+2}B_{k,j})]^{(1-p_1)\frac{p_2(1-\beta)}{p_1}} \\
&\quad\times\left\{\sum_{i=0}^{N^{(2)}_{B_{k,j},2^{k+5}B}+1}\left[\frac{\mu(2^{i+1}B_{k,j})}{\lambda(c_{B_{k,j}},2^i r_{B_{k,j}})}\right]^{p_1}\right\}^{\frac{p_2(1-\beta)}{p_1}}[\mu(2^{k+6}B)]^{\frac{p_1-p_2(1-\beta)}{p_1}}\|m_{k,j}\|_{L^1(\mu)}^{p_2} \\
&\lesssim \sum_{\ell=5}^{\infty}\sum_{k=\ell-4}^{\ell+4}\sum_{j=1}^{M}|\lambda_{k,j}|^{p_2}\|m_{k,j}\|_{L^1(\mu)}^{p_2}[\mu(2^{N^{(2)}_{B_{k,j},2^{k+5}B}+2}B_{k,j})]^{(1-p_1)\frac{p_2(1-\beta)}{p_1}} \\
&\quad\times[\mu(2^{k+6}B)]^{\frac{p_1-p_2(1-\beta)}{p_1}}[\widetilde{K}^{(2),p_1}_{B_{k,j},2^{k+5}B}]^{p_2(1-\beta)} \\
&\lesssim \sum_{\ell=5}^{\infty}\sum_{k=\ell-4}^{\ell+4}\sum_{j=1}^{M}|\lambda_{k,j}|^{p_2}[\mu(B_{k,j})]^{p_2/q'}\|m_{k,j}\|_{L^q(\mu)}^{p_2}[\mu(2^{k+9}B)]^{1-p_2(1-\beta)}[\widetilde{K}^{(2),p_1}_{B_{k,j},2^{k+5}B}]^{p_2(1-\beta)} \\
&\lesssim \sum_{\ell=5}^{\infty}\sum_{k=\ell-4}^{\ell+4}\sum_{j=1}^{M}|\lambda_{k,j}|^{p_2}[\mu(B_{k,j})]^{p_2/q'}2^{-k\theta p_2}[\mu(2B_{k,j})]^{-p_2/q'} \\
&\quad\times[\lambda(c_B,2^{k+2}r_B)]^{p_2(1-1/p_1)}[\widetilde{K}^{(2),p_1}_{B_{k,j},2^{k+2}B}]^{-p_2}[\mu(2^{k+9}B)]^{1-p_2(1-\beta)}[\widetilde{K}^{(2),p_1}_{B_{k,j},2^{k+5}B}]^{p_2} \\
&\lesssim \sum_{\ell=5}^{\infty}\sum_{k=\ell-4}^{\ell+4}\sum_{j=1}^{M}2^{-k\theta p_2}|\lambda_{k,j}|^{p_2}\lesssim\sum_{k=0}^{\infty}\sum_{j=1}^{M}|\lambda_{k,j}|^{p_2}\lesssim|b|^{p_2}_{\widetilde{H}^{p_1,q,1,\theta}_{\mathrm{mb},2}(\mu)},
\end{aligned}
$$

which, together with the estimate for II_1, implies that

$$\mathrm{II}\lesssim|b|^{p_2}_{\widetilde{H}^{p_1,q,1,\theta}_{\mathrm{mb},2}(\mu)}.$$

To estimate IV, observe that

$$
\begin{aligned}
\mathrm{IV}&\leqslant\sum_{\ell=0}^{4}\int_{U_\ell(B)}\left|T_\beta\left(\sum_{k=0}^{\ell+4}\sum_{j=1}^{M}\lambda_{k,j}m_{k,j}\right)(x)\right|^{p_2}d\mu(x)\\
&\quad+\sum_{\ell=0}^{4}\int_{U_\ell(B)}\left|T_\beta\left(\sum_{k=\ell+5}^{\infty}\sum_{j=1}^{M}\lambda_{k,j}m_{k,j}\right)(x)\right|^{p_2}d\mu(x)=:\mathrm{IV}_1+\mathrm{IV}_2.
\end{aligned}
$$

By some arguments similar to those used in the estimates for II_1 and III, we respectively obtain

$$\mathrm{IV}_1\lesssim|b|^{p_2}_{\widetilde{H}^{p_1,q,1,\theta}_{\mathrm{mb},2}(\mu)}\quad\text{and}\quad\mathrm{IV}_2\lesssim|b|^{p_2}_{\widetilde{H}^{p_1,q,1,\theta}_{\mathrm{mb},2}(\mu)},$$

which, together with the estimates for III, I and II, completes the proof of Theorem 5.3. □

Similar to Corollary 4.9, by the proof of Theorem 5.3, we also obtain the following boundedness of the generalized fractional integral T_β from $\widetilde{H}^{p_1,q,\gamma}_{\mathrm{atb},\rho}(\mu)$ into $L^{p_2}(\mu)$ for $1/p_2=1/p_1-\beta$, the details being omitted.

Corollary 5.4. *Suppose that $(\mathcal{X},d,\mu)$ is a non-homogeneous metric measure space. Let $\beta\in(0,1/2)$, $\rho\in(1,\infty)$, $\frac{\nu}{\nu+\theta}<p_1<p_2\leqslant1<q<1/\beta$, $1/p_2=1/p_1-\beta$ and $\gamma\in[1,\infty)$. Assume that the generalized fractional integral T_β defined by* (5.3) *associated with kernel K_β satisfying* (5.1) *and* (5.2) *is bounded from $L^q(\mu)$ into $L^{\widetilde{q}}(\mu)$, where $1/\widetilde{q}:=1/q-\beta$. Then T_β is bounded from $\widetilde{H}^{p_1,q,\gamma}_{\mathrm{atb},\rho}(\mu)$ into $L^{p_2}(\mu)$.*

Remark 5.5. (a) When $p_1=1$, Theorem 5.3 or Corollary 5.4 is a special case of [15, Theorem 1.13] by the same reasons as those used in Remark 4.10.

(b) For all $\beta\in(0,1)$, $f\in L^\infty_b(\mu)$ and $x\in\mathcal{X}$, the *fractional integral* $I_\beta f(x)$ is defined by

$$I_\beta f(x):=\int_{\mathcal{X}}\frac{f(y)}{[\lambda(y,d(x,y))]^{1-\beta}}d\mu(y).$$

From [15, Section 4], we deduce that, for some $\epsilon \in (0, \infty)$, under the weak growth condition as in Remark 2.4(iii) and the following *ϵ-weak reverse doubling condition* on the dominating function λ: For all $r \in (0, 2\operatorname{diam}(\mathcal{X}))$ and $a \in (1, 2\operatorname{diam}(\mathcal{X})/r)$, there exists a number $C_{(a)} \in [1, \infty)$, depending only on a and $\mathcal{X}$, such that, for all $x \in \mathcal{X}$, $\lambda(x, ar) \geqslant C_{(a)}\lambda(x, r)$ and $\sum_{k=1}^{\infty} \frac{1}{[C_{(a^k)}]^{\epsilon}} < \infty$, the following statements hold true:

(b)$_1$ the fractional integral I_β is a special case of the generalized fractional integral, which is bounded from $L^q(\mu)$ into $L^{\widetilde{q}}(\mu)$ for all $q \in (1, 1/\beta)$ and $1/\widetilde{q} = 1/q - \beta$;

(b)$_2$ all conclusions of Theorem 5.3 and Corollary 5.4 hold true, if T_β is replaced by I_β, where I_β has the same assumptions as those of T_β in Theorem 5.3 and Corollary 5.4, respectively.

6 Campanato spaces $\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta,\,\gamma}(\mu)$

In this section, we introduce the Campanato space $\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta,\,\gamma}(\mu)$ and show that $\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta,\,\gamma}(\mu)$ is independent of the choices of ρ, η, γ and q under the following assumption of the ρ-weakly doubling condition.

Definition 6.1. Let $\rho \in (1, \infty)$. The Borel measure μ is said to satisfy the *ρ-weakly doubling condition* if, for all balls $B \subset \mathcal{X}$, there exists a positive constant $\widetilde{C}_1$, depending on ρ but independent of B, such that

$$N^{(\rho)}_{B,\,\widetilde{B}^\rho} \leqslant \widetilde{C}_1, \tag{6.1}$$

where $N^{(\rho)}_{B,\,\widetilde{B}^\rho}$ is defined as in Definition 2.6.

Remark 6.2. (i) Recall that $\widetilde{B}^\rho$ is totally determined by μ and ρ. Let $(\mathcal{X}, d, \mu)$ be a space of homogeneous type and $\lambda(x, r) := \mu(B(x, r))$ for all $x \in \mathcal{X}$ and $r \in (0, \infty)$. Then $(\mathcal{X}, d, \mu)$ satisfies (6.1), since $N^{(\rho)}_{B,\,\widetilde{B}^\rho} \sim 1$ for all balls B with equivalent positive constants depending only on $\rho \in (1, \infty)$. However, by Example 6.3 below, there exists a non-doubling measure μ on a subset of $\mathbb{R}$ satisfying (6.1); by Example 6.4 below, there exists a non-doubling measure not satisfying (6.1). In this sense, a measure satisfying (6.1) is said to be ρ-weakly doubling.

(ii) From the fact that $\rho^{N^{(\rho)}_{B,\,\widetilde{B}^\rho}} B = \widetilde{B}^\rho$ and (6.1), it follows that there exists a positive constant $C_{(\rho,\,\widetilde{C}_1)}$, depending on ρ and $\widetilde{C}_1$, such that, for any ball B,

$$r_{\widetilde{B}^\rho} \leqslant C_{(\rho,\,\widetilde{C}_1)} r_B,$$

where r_B and $r_{\widetilde{B}^\rho}$ denote the radii of balls B and $\widetilde{B}^\rho$, respectively. Obviously, we always have $r_B \leqslant r_{\widetilde{B}^\rho}$.

In the remainder of this section, we *always assume* that the Borel measure μ satisfies the ρ-weakly doubling condition.

The following example shows that there exist some non-trivial non-doubling measures satisfying (6.1).

Example 6.3. Let

$$\mathcal{X} := [0, 1] \cup \left(\bigcup_{k=1}^{\infty} \left[2\sum_{j=0}^{k-1} \mathrm{e}^{-j^2},\, 2\sum_{j=0}^{k-1} \mathrm{e}^{-j^2} + \mathrm{e}^{-k^2} \right] \right).$$

Denote $[0, 1]$ by D_0 and $[2\sum_{j=0}^{k-1} \mathrm{e}^{-j^2}, 2\sum_{j=0}^{k-1} \mathrm{e}^{-j^2} + \mathrm{e}^{-k^2}]$ by D_k for $k \in \mathbb{N}$. For any $x \in \mathcal{X}$ and $r \in (0, \infty)$, we use $B := B(x, r) := \{y \in \mathcal{X} : |y - x| < r\}$ to denote a ball of $\mathcal{X}$. Let μ be the Lebesgue measure restricted to $\mathcal{X}$. Notice that $\mu(B(x, r)) \leqslant 2r$ for all $x \in \mathcal{X}$ and $r \in (0, \infty)$. Thus, μ is an upper doubling measure with $\lambda(x, r) := 2r$ for all $x \in \mathcal{X}$ and $r \in (0, \infty)$.

Then we claim that μ is a non-doubling measure. Indeed, notice that

$$\sum_{j=k+1}^{\infty} \mathrm{e}^{-j^2} = \mathrm{e}^{-k^2} \sum_{j=1}^{\infty} \mathrm{e}^{-2kj - j^2} \leqslant \begin{cases} \dfrac{\sqrt{\pi}}{2} \mathrm{e}^{-k^2}, & \text{for all } k \in \mathbb{Z}_+, \\ \dfrac{\sqrt{\pi}}{\mathrm{e}^2 2} \mathrm{e}^{-k^2}, & \text{for all } k \in \mathbb{N}. \end{cases} \tag{6.2}$$

Let $x_k := 2\sum_{j=0}^{k-1} \mathrm{e}^{-j^2}$ and $r_k := \mathrm{e}^{-(k-1)^2}$. Then

$$\mu(B(x_k,\, r_k)) \leqslant \left(1+\frac{\sqrt{\pi}}{2}\right)\mathrm{e}^{-k^2} \quad \text{and} \quad \mu(B(x_k,\, 2r_k)) \geqslant \mathrm{e}^{-(k-1)^2}.$$

Thus,

$$\frac{\mu(B(x_k,\, 2r_k))}{\mu(B(x_k,\, r_k))} \geqslant \frac{1}{1+\frac{\sqrt{\pi}}{2}}\mathrm{e}^{2k-1} \to \infty \quad \text{as} \quad k \to \infty,$$

which implies that μ is non-doubling.

For any ball B, the smallest doubling ball of the form $2^j B$ with $j \in \mathbb{Z}_+$ is denoted by $\widetilde{B}^{(2)}$. Let $N^{(2)}_{B,\,\widetilde{B}^{(2)}}$ be the integer such that $2^{N^{(2)}_{B,\,\widetilde{B}^{(2)}}} B = \widetilde{B}^{(2)}$. We claim that

$$N^{(2)}_{B,\,\widetilde{B}^{(2)}} \leqslant 2. \tag{6.3}$$

To prove this, we consider the following two cases for k.

Case I. $k \in \{0,\, 1,\, 2\}$. In this case, for all $x \in D_k$, we have

$$\begin{cases} \mu(B(x,\, r)) \sim r \sim \mu(B(x,\, 2r)), & \text{for } r \in (0,\, 2+\sqrt{\pi}\,], \\ \mu(B(x,\, r)) \sim 1 \sim \mu(B(x,\, 2r)), & \text{for } r \in (2+\sqrt{\pi},\, \infty). \end{cases}$$

From this, it is easy to deduce that $B(x,\, r)$ with $x \in D_k$ and $r \in (0, \infty)$ is a doubling ball and hence $N^{(2)}_{B,\,\widetilde{B}^{(2)}} = 0$ in this case.

Case II. $k \in \mathbb{N} \cap (2,\, \infty)$. In this case, for all $x \in D_k$, we have

(i) for $r \in (0,\, \mathrm{e}^{-k^2}]$, $\mu(B(x,\, r)) \sim r$;

(ii) for $r \in (\mathrm{e}^{-k^2},\, \mathrm{e}^{-(k-1)^2}]$, $\mu(B(x,\, r)) \sim \mathrm{e}^{-k^2}$;

(iii) for $r \in (\mathrm{e}^{-(k-1)^2},\, 2\mathrm{e}^{-(k-1)^2}]$, $\mathrm{e}^{-k^2} \leqslant \mu(B(x,\, r)) < 2\mathrm{e}^{-(k-1)^2}$;

(iv) for $r \in (2\sum_{i=1}^{j} \mathrm{e}^{-(k-i)^2},\, 2\sum_{i=1}^{j} \mathrm{e}^{-(k-i)^2} + \mathrm{e}^{-(k-j-1)^2}]$ with $j \in \{1,\, \ldots,\, k-2\}$,

$$\mu(B(x,\, r)) \sim \mathrm{e}^{-(k-j)^2};$$

(v) for $r \in (2\sum_{i=1}^{j} \mathrm{e}^{-(k-i)^2} + \mathrm{e}^{-(k-j-1)^2},\, 2\sum_{i=1}^{j+1} \mathrm{e}^{-(k-i)^2}]$ with $j \in \{1,\, \ldots,\, k-2\}$,

$$\mathrm{e}^{-(k-j)^2} \leqslant \mu(B(x,\, r)) < 2\mathrm{e}^{-(k-j-1)^2};$$

(vi) for $r \in (2\sum_{i=1}^{k-1} \mathrm{e}^{-(k-i)^2},\, \infty)$, $\mu(B(x,\, r)) \sim 1$.

Now we show that (6.3) holds true in Case (i) through (vi).

Indeed, in the case (vi), it is easy to see that $B(x,\, r)$ is a doubling ball and hence $N^{(2)}_{B,\,\widetilde{B}^{(2)}} = 0$ in this case, i.e., (6.3) holds true in this case. Therefore, we only need to show that (6.3) holds true in Cases (i) through (v).

In the case (i), since $k \geqslant 3$, we have $2r \in (0,\, \mathrm{e}^{-(k-1)^2}]$. If $2r \in (0,\, \mathrm{e}^{-k^2}]$, then we have

$$\mu(B(x,\, r)) \sim \mu(B(x,\, 2r)) \sim r;$$

if $2r \in (\mathrm{e}^{-k^2},\, \mathrm{e}^{-(k-1)^2}]$, then $r \in (\frac{\mathrm{e}^{-k^2}}{2},\, \mathrm{e}^{-k^2}]$, which, together with (ii), leads to

$$\mu(B(x,\, r)) \sim \mu(B(x,\, 2r)) \sim \mathrm{e}^{-k^2}.$$

It then follows that $B(x,\, r)$ is a doubling ball, which shows that $N^{(2)}_{B,\,\widetilde{B}^{(2)}} = 0$ in this case, i.e., (6.3) holds true.

In the case (ii), if $2r \in (\mathrm{e}^{-k^2},\, \mathrm{e}^{-(k-1)^2}]$, then we have $\mu(B(x,\, r)) \sim \mu(B(x,\, 2r)) \sim \mathrm{e}^{-k^2}$, which implies that $\mu(B(x,\, r))$ is a doubling ball; if $2r \in (\mathrm{e}^{-(k-1)^2},\, 2\mathrm{e}^{-(k-1)^2}]$, then, for sufficiently large k, $B(x,\, r)$ may be a non-doubling ball, for example, $x = 2\sum_{j=0}^{k-1} \mathrm{e}^{-j^2}$ and $r = \mathrm{e}^{-(k-1)^2}$. In the latter case, if $B(x,\, r)$ is a non-doubling ball, we consider the ball $B(x,\, 2r)$. If $B(x,\, 2r)$ is a doubling ball, namely, there exists a positive constant C, independent of x and r, such that $\mu(B(x,4r)) \leqslant C\mu(B(x,2r))$, then there is nothing to prove. Otherwise, we consider the ball $B(x,4r)$. Notice that $2r \in (\mathrm{e}^{-(k-1)^2}, 2\mathrm{e}^{-(k-1)^2}]$. Then we have $r \in (\frac{\mathrm{e}^{-(k-1)^2}}{2}, \mathrm{e}^{-(k-1)^2}]$, which, together with $k \geqslant 3$, shows that

$$2\mathrm{e}^{-(k-1)^2} < 4r \leqslant 4\mathrm{e}^{-(k-1)^2} < 2\mathrm{e}^{-(k-1)^2} + \mathrm{e}^{-(k-2)^2}$$

and

$$2\mathrm{e}^{-(k-1)^2} < 8r \leqslant 8\mathrm{e}^{-(k-1)^2} < 2\mathrm{e}^{-(k-1)^2} + \mathrm{e}^{-(k-2)^2}.$$

It then follows, from (iv) with $j=1$, that $\mu(B(x,4r)) \sim \mu(B(x,8r)) \sim \mathrm{e}^{-(k-1)^2}$, which implies that $B(x,4r)$ is a doubling ball. From the above estimate, we conclude that $N^{(2)}_{B,\widetilde{B}^{(2)}} \leqslant 2$ in this case.

The argument of the case (iii) is similar to that used in the case of $2r \in (\mathrm{e}^{-(k-1)^2}, 2\mathrm{e}^{-(k-1)^2}]$ of (ii). Moreover, we have $N^{(2)}_{B,\widetilde{B}^{(2)}} \leqslant 1$ in this case.

Before we deal with the case (iv), we first consider the case (v). In the case (v), if $B(x,r)$ is a doubling ball, then there is nothing to prove. Otherwise, we consider the following two cases for j. If $j=k-2$, we have

$$4r > 2r > 4\sum_{i=1}^{j} \mathrm{e}^{-(k-i)^2} + 2\mathrm{e}^{-(k-j-1)^2} > 2\sum_{i=1}^{j+1} \mathrm{e}^{-(k-i)^2} = 2\sum_{i=1}^{k-1} \mathrm{e}^{-(k-i)^2},$$

which, together with (vi), shows that $\mu(B(x,4r)) \sim \mu(B(x,2r)) \sim 1$. Thus, $B(x,2r)$ is a doubling ball. If $j \leqslant k-3$, then, by (6.2), we see that

$$6\sum_{i=1}^{j+1} \mathrm{e}^{-(k-i)^2} < \mathrm{e}^{-[k-(j+1)-1]^2}.$$

It then follows that

$$2\sum_{i=1}^{j+1} \mathrm{e}^{-(k-i)^2} < 2r < 4r \leqslant 8\sum_{i=1}^{j+1} \mathrm{e}^{-(k-i)^2} < 2\sum_{i=1}^{j+1} \mathrm{e}^{-(k-i)^2} + \mathrm{e}^{-[k-(j+1)-1]^2}.$$

This via (iv) shows that $\mu(B(x,2r)) \sim \mu(B(x,4r)) \sim \mathrm{e}^{-(k-j-1)^2}$, which implies that $B(x,2r)$ is a doubling ball and hence $N^{(2)}_{B,\widetilde{B}^{(2)}} \leqslant 1$ in this case.

In the case (iv), we see that

$$2r \in \left(4\sum_{i=1}^{j} \mathrm{e}^{-(k-i)^2}, 4\sum_{i=1}^{j} \mathrm{e}^{-(k-i)^2} + 2\mathrm{e}^{-(k-j-1)^2}\right].$$

Notice that, by (6.2) and $j \leqslant k-2$, we see that

$$2\sum_{i=1}^{j} \mathrm{e}^{-(k-i)^2} = 2\sum_{i=k-j}^{k-1} \mathrm{e}^{-i^2} \leqslant \frac{2\sqrt{\pi}}{2\mathrm{e}^2}\mathrm{e}^{-(k-j-1)^2} < \mathrm{e}^{-(k-j-1)^2}.$$

Thus, we consider the following three cases for $2r$.

Case (a) $2r \in (4\sum_{i=1}^{j} \mathrm{e}^{-(k-i)^2}, 2\sum_{i=1}^{j} \mathrm{e}^{-(k-i)^2} + \mathrm{e}^{-(k-j-1)^2}]$. In this case, it is easy to see that

$$\mu(B(x,2r)) \sim \mu(B(x,r)) \sim \mathrm{e}^{-(k-j)^2},$$

which implies that $B(x,r)$ is a doubling ball.

Case (b) $2r \in (2\sum_{i=1}^{j} \mathrm{e}^{-(k-i)^2} + \mathrm{e}^{-(k-j-1)^2}, 2\sum_{i=1}^{j+1} \mathrm{e}^{-(k-i)^2}]$. In this case, by an argument similar to that used in (v), we conclude that $N^{(2)}_{B,\widetilde{B}^{(2)}} \leqslant 2$.

Case (c) $2r \in (2\sum_{i=1}^{j+1} \mathrm{e}^{-(k-i)^2}, 4\sum_{i=1}^{j} \mathrm{e}^{-(k-i)^2} + 2\mathrm{e}^{-(k-j-1)^2}]$. In this case, if $j=k-2$, we have

$$4r > 2r > 2\sum_{i=1}^{k-1} \mathrm{e}^{-(k-i)^2},$$

which, together with (vi), implies that $B(x, 2r)$ is a doubling ball; if $j \leqslant k-3$, by (6.2), we know that

$$2\sum_{i=1}^{j+1} e^{-(k-i)^2} < 2r < 4r \leqslant 8\sum_{i=1}^{j} e^{-(k-i)^2} + 4e^{-(k-j-1)^2} < 2\sum_{i=1}^{j+1} e^{-(k-i)^2} + 6\sum_{i=1}^{j+1} e^{-(k-i)^2}$$
$$< 2\sum_{i=1}^{j+1} e^{-(k-i)^2} + e^{-[k-(j+1)-1]^2}.$$

This via (iv) shows that $\mu(B(x, 2r)) \sim \mu(B(x, 4r)) \sim e^{-(k-j-1)^2}$, which implies that $B(x, 2r)$ is a doubling ball and hence $N^{(2)}_{B, \widetilde{B}^{(2)}} \leqslant 1$ in the case (iv).

Combining the above estimates, we obtain (6.3), which completes the proof of our claim and hence the example.

On the other hand, it turns out that there exist many non-homogeneous metric measure spaces which do not satisfy the ρ-weakly doubling condition (6.1). We give the following Gauss measure on $\mathbb{R}$ as an example.

Example 6.4. Let $(\mathcal{X}, |\cdot|, \mu) := (\mathbb{R}, |\cdot|, \mu)$, where $|\cdot|$ denotes the Euclidean distance and μ is the Gauss measure on $\mathbb{R}$, i.e., $d\mu(x) := \pi^{-\frac{1}{2}} e^{-x^2} dx$ for all $x \in \mathbb{R}$. As in Example 6.3, for any $x \in \mathbb{R}$ and $r \in (0, \infty)$, we use $B := B(x, r) := \{y \in \mathbb{R} : |y - x| < r\}$ to denote a ball of $\mathbb{R}$. First, we show that μ is a non-doubling measure with the dominating function $\lambda(x, r) := 2\pi^{-\frac{1}{2}} r$. Indeed, for all $x \in \mathbb{R}$ and $r \in (0, \infty)$,

$$\mu(B(x, r)) = \pi^{-\frac{1}{2}} \int_{x-r}^{x+r} e^{-y^2} dy \leqslant 2\pi^{-\frac{1}{2}} r = \lambda(x, r).$$

On the other hand, let $x_k = 2^{2k}$ and $r_{k,j} = 2^j$ with $k \in \mathbb{N}$ and $j \in \{1, \ldots, k\}$. Then, for all $k \in \mathbb{N}$ and $j \in \{1, \ldots, k\}$, we observe that

$$\mu(B(x_k, r_{k,j})) = \pi^{-\frac{1}{2}} \int_{2^{2k}-2^j}^{2^{2k}+2^j} e^{-x^2} dx \leqslant \pi^{-\frac{1}{2}} e^{-(2^{2k}-2^j)^2} 2^{j+1}$$

and

$$\mu(B(x_k, 2r_{k,j})) = \pi^{-\frac{1}{2}} \int_{2^{2k}-2^{j+1}}^{2^{2k}+2^{j+1}} e^{-x^2} dx$$
$$\geqslant \pi^{-\frac{1}{2}} \int_{2^{2k}-2^{j+1}}^{2^{2k}-3\times 2^{j-1}} e^{-x^2} dx \geqslant \pi^{-\frac{1}{2}} e^{-(2^{2k}-3\times 2^{j-1})^2} 2^{j-1}.$$

Thus,

$$\frac{\mu(B(x_k, 2r_{k,j}))}{\mu(B(x_k, r_{k,j}))} \geqslant \frac{1}{4} e^{2^{2k+j} - \frac{5}{4} 2^{2j}} \geqslant \frac{1}{4} e^{2^{2k+1} - \frac{5}{4} 2^{2k}} \geqslant \frac{1}{4} e^{2^{2k-1}} \to \infty \quad \text{as} \quad k \to \infty, \tag{6.4}$$

which implies that μ is non-doubling.

Now we claim that, for any $\rho \in (1, \infty)$, there exists a ball B such that the number $N^{(\rho)}_{B, \widetilde{B}^\rho}$ can be arbitrarily large. For the sake of simplicity, we only prove our claim for $\rho = 2$. With some simple modifications, the arguments here are still valid for all $\rho \in (1, \infty)$. Recall that a ball $B \subset \mathbb{R}$ is said to be $(2, \beta_2)$-doubling if $\mu(2B) \leqslant \beta_2 \mu(B)$ and $\widetilde{B}^{(2)}$ is the smallest $(2, \beta_2)$-doubling ball of the form $2^j B$ with $j \in \mathbb{Z}_+$. Let k_0 be the smallest positive integer such that $\frac{1}{4} e^{2^{2k_0-1}} > \beta_2$. Then, for all $k \in \mathbb{N} \cap [k_0, \infty)$, we have $\frac{1}{4} e^{2^{2k-1}} > \beta_2$. Let $B_k := B(2^{2k}, 2)$. By (6.4), it is easy to see that, for all $j \in \{0, \ldots, k-1\}$, $2^j B_k$ is not a $(2, \beta_2)$-doubling ball. It then follows, from the definition of $N^{(2)}_{B_k, \widetilde{B}^2_k}$, that

$$N^{(2)}_{B_k, \widetilde{B}^2_k} > k - 1,$$

which implies our claim and completes the proof of Example 6.4.

We now state the definition of the Campanato space $\mathcal{E}^{\alpha, q}_{\rho, \eta, \gamma}(\mu)$.

Definition 6.5. Let $\alpha \in [0,\infty)$, $\eta \in (1,\infty)$, $\rho \in [\eta,\infty)$ and $q,\gamma \in [1,\infty)$. A function $f \in L^1_{\rm loc}(\mu)$ is said to belong to the *Campanato space* $\mathcal{E}^{\alpha,q}_{\rho,\eta,\gamma}(\mu)$ if

$$\|f\|_{\mathcal{E}^{\alpha,q}_{\rho,\eta,\gamma}(\mu)} := \sup_B \left\{\frac{1}{\mu(\eta B)}\frac{1}{[\lambda(c_B,r_B)]^{\alpha q}}\int_B |f(y)-m_{\widetilde{B}^\rho}(f)|^q d\mu(y)\right\}^{1/q} + \sup_{B\subset S:B,S\ (\rho,\beta_\rho)-\text{doubling}} \frac{|m_B(f)-m_S(f)|}{[\lambda(c_S,r_S)]^\alpha[\widetilde{K}^{(\rho),1/(\alpha+1)}_{B,S}]^\gamma} < \infty,$$

here and hereafter, $m_B(f) := \frac{1}{\mu(B)}\int_B f(x)d\mu(x)$ for all balls B and $f \in L^1_{\rm loc}(\mu)$.

Remark 6.6. Arguing as in [33, Lemma 3.2], we see that $\mathcal{E}^{0,1}_{\rho,\rho,\gamma}(\mu)$ and $\widetilde{\rm RBMO}_{\rho,\gamma}(\mu)$ coincide with equivalent norms, where $\widetilde{\rm RBMO}_{\rho,\gamma}(\mu)$ was introduced in [13]; see also [27,30].

Proposition 6.7. *Suppose that $(\mathcal{X},d,\mu)$ is a non-homogeneous metric measure space satisfying* (6.1). *Let $\alpha \in [0,\infty)$, $\eta \in (1,\infty)$, $\rho \in [\eta,\infty)$ and $q,\gamma \in [1,\infty)$. The following statements hold true*:

(a) *There exists a positive constant C such that, for all balls $B \subset S$ and all functions $f \in \mathcal{E}^{\alpha,q}_{\rho,\eta,\gamma}(\mu)$,*

$$|m_{\widetilde{B}^\rho}(f)-m_{\widetilde{S}^\rho}(f)| \leqslant C[\widetilde{K}^{(\rho),1/(\alpha+1)}_{B,S}]^\gamma[\lambda(c_S,r_S)]^\alpha\|f\|_{\mathcal{E}^{\alpha,q}_{\rho,\eta,\gamma}(\mu)}.$$

(b) *If $f,g \in \mathcal{E}^{\alpha,q}_{\rho,\eta,\gamma}(\mu)$ are real-valued functions, then $\max\{f,g\} \in \mathcal{E}^{\alpha,q}_{\rho,\eta,\gamma}(\mu)$ and $\min\{f,g\} \in \mathcal{E}^{\alpha,q}_{\rho,\eta,\gamma}(\mu)$. Moreover, there exists a positive constant C, independent of f and g, such that*

$$\|\max\{f,g\}\|_{\mathcal{E}^{\alpha,q}_{\rho,\eta,\gamma}(\mu)} + \|\min\{f,g\}\|_{\mathcal{E}^{\alpha,q}_{\rho,\eta,\gamma}(\mu)} \leqslant C[\|f\|_{\mathcal{E}^{\alpha,q}_{\rho,\eta,\gamma}(\mu)} + \|g\|_{\mathcal{E}^{\alpha,q}_{\rho,\eta,\gamma}(\mu)}].$$

Proof. To show (a), we consider the following two cases:

Case (i) $r(\widetilde{B}^\rho) < r(\widetilde{S}^\rho)$. It is obvious that $\widetilde{B}^\rho \subset 2\widetilde{S}^\rho$. Let $S_0 := \widetilde{(2\widetilde{S}^\rho)}^\rho$. By Lemmas 2.8(ii)–2.8(iv) with $p = 1/(\alpha+1)$, we have

$$[\widetilde{K}^{(\rho),1/(\alpha+1)}_{\widetilde{S}^\rho,S_0}]^\gamma \lesssim [\widetilde{K}^{(\rho),1/(\alpha+1)}_{\widetilde{S}^\rho,2\widetilde{S}^\rho}]^\gamma + [\widetilde{K}^{(\rho),1/(\alpha+1)}_{2\widetilde{S}^\rho,S_0}]^\gamma \lesssim 1.$$

From this and Lemmas 2.8(ii)–2.8(v) with $p = 1/(\alpha+1)$, it follows that

$$\begin{aligned}[\widetilde{K}^{(\rho),1/(\alpha+1)}_{\widetilde{B}^\rho,S_0}]^\gamma &\lesssim [\widetilde{K}^{(\rho),1/(\alpha+1)}_{B,S_0}]^\gamma \\ &\lesssim [\widetilde{K}^{(\rho),1/(\alpha+1)}_{B,S}]^\gamma + [\widetilde{K}^{(\rho),1/(\alpha+1)}_{S,\widetilde{S}^\rho}]^\gamma + [\widetilde{K}^{(\rho),1/(\alpha+1)}_{\widetilde{S}^\rho,S_0}]^\gamma \\ &\lesssim [\widetilde{K}^{(\rho),1/(\alpha+1)}_{B,S}]^\gamma.\end{aligned}$$

Thus, by the above two inequalities, Remark 6.2(ii) and (2.1), we have

$$\begin{aligned}|m_{\widetilde{B}^\rho}(f)-m_{\widetilde{S}^\rho}(f)| &\leqslant |m_{\widetilde{B}^\rho}(f)-m_{S_0}(f)| + |m_{\widetilde{S}^\rho}(f)-m_{S_0}(f)| \\ &\lesssim \{[\widetilde{K}^{(\rho),1/(\alpha+1)}_{\widetilde{B}^\rho,S_0}]^\gamma + [\widetilde{K}^{(\rho),1/(\alpha+1)}_{\widetilde{S}^\rho,S_0}]^\gamma\}[\lambda(c_{S_0},r_{S_0})]^\alpha\|f\|_{\mathcal{E}^{\alpha,q}_{\rho,\eta,\gamma}(\mu)} \\ &\lesssim [\widetilde{K}^{(\rho),1/(\alpha+1)}_{B,S}]^\gamma[\lambda(c_{S_0},r_{S_0})]^\alpha\|f\|_{\mathcal{E}^{\alpha,q}_{\rho,\eta,\gamma}(\mu)} \\ &\lesssim [\widetilde{K}^{(\rho),1/(\alpha+1)}_{B,S}]^\gamma[\lambda(c_S,r_S)]^\alpha\|f\|_{\mathcal{E}^{\alpha,q}_{\rho,\eta,\gamma}(\mu)}.\end{aligned}$$

This finishes the proof of Case (i).

Case (ii) $r(\widetilde{S}^\rho) \leqslant r(\widetilde{B}^\rho)$. Obviously, $\widetilde{S}^\rho \subset 2\widetilde{B}^\rho$. Let $B_0 := \widetilde{(2\widetilde{B}^\rho)}^\rho$. From Lemmas 2.8(ii)–2.8(iv) with $p = 1/(\alpha+1)$, we deduce that

$$[\widetilde{K}^{(\rho),1/(\alpha+1)}_{\widetilde{B}^\rho,B_0}]^\gamma \lesssim [\widetilde{K}^{(\rho),1/(\alpha+1)}_{\widetilde{B}^\rho,2\widetilde{B}^\rho}]^\gamma + [\widetilde{K}^{(\rho),1/(\alpha+1)}_{2\widetilde{B}^\rho,B_0}]^\gamma \lesssim 1.$$

By this, $\widetilde{S}^\rho \supset S \supset B$ and Lemmas 2.8(iii)–2.8(v) with $p = 1/(\alpha+1)$, we know that

$$[\widetilde{K}^{(\rho),1/(\alpha+1)}_{\widetilde{S}^\rho,B_0}]^\gamma \lesssim [\widetilde{K}^{(\rho),1/(\alpha+1)}_{B,B_0}]^\gamma \lesssim [\widetilde{K}^{(\rho),1/(\alpha+1)}_{B,\widetilde{B}^\rho}]^\gamma + [\widetilde{K}^{(\rho),1/(\alpha+1)}_{\widetilde{B}^\rho,B_0}]^\gamma \lesssim 1.$$

Thus, combining the above two inequalities, (2.2), (2.1) and Remark 6.2(ii), we have

$$\begin{aligned}|m_{\widetilde{B}^\rho}(f)-m_{\widetilde{S}^\rho}(f)| &\leqslant |m_{\widetilde{B}^\rho}(f)-m_{B_0}(f)|+|m_{\widetilde{S}^\rho}(f)-m_{B_0}(f)|\\ &\lesssim \{[\widetilde{K}^{(\rho),\,1/(\alpha+1)}_{\widetilde{B}^\rho,\,B_0}]^\gamma+[\widetilde{K}^{(\rho),\,1/(\alpha+1)}_{\widetilde{S}^\rho,\,B_0}]^\gamma\}[\lambda(c_{B_0},r_{B_0})]^\alpha\|f\|_{\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta,\,\gamma}(\mu)}\\ &\lesssim [\lambda(c_{B_0},r_{B_0})]^\alpha\|f\|_{\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta,\,\gamma}(\mu)}\sim[\lambda(c_S,r_{B_0})]^\alpha\|f\|_{\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta,\,\gamma}(\mu)}\\ &\lesssim [\widetilde{K}^{(\rho),\,1/(\alpha+1)}_{B,\,S}]^\gamma[\lambda(c_S,r_B)]^\alpha\|f\|_{\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta,\,\gamma}(\mu)}\\ &\lesssim [\widetilde{K}^{(\rho),\,1/(\alpha+1)}_{B,\,S}]^\gamma[\lambda(c_S,r_S)]^\alpha\|f\|_{\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta,\,\gamma}(\mu)}.\end{aligned}$$

This finishes the proof of Case (ii) and hence (a).

To prove (b), since $\max\{f,\,g\}=\frac{f+g+|f-g|}{2}$ and $\min\{f,\,g\}=\frac{f+g-|f-g|}{2}$, it suffices to show that, for any real-valued function $h\in\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta,\,\gamma}(\mu)$, $|h|\in\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta,\,\gamma}(\mu)$ and

$$\||h|\|_{\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta,\,\gamma}(\mu)}\lesssim\|h\|_{\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta,\,\gamma}(\mu)}.$$

To this end, by Definition 6.5, Hölder's inequality, Remark 6.2(ii) and (2.1), we see that, for any ball B,

$$\begin{aligned}&\left\{\frac{1}{\mu(\eta B)}\int_B ||h(y)|-m_{\widetilde{B}^\rho}(|h|)|^q d\mu(y)\right\}^{1/q}\\ &\quad\leqslant\left\{\frac{1}{\mu(\eta B)}\int_B ||h(y)|-|m_{\widetilde{B}^\rho}(h)||^q d\mu(y)\right\}^{1/q}+||m_{\widetilde{B}^\rho}(h)|-m_{\widetilde{B}^\rho}(|h|)|\\ &\quad\leqslant\left\{\frac{1}{\mu(\eta B)}\int_B |h(y)-m_{\widetilde{B}^\rho}(h)|^q d\mu(y)\right\}^{1/q}+m_{\widetilde{B}^\rho}(|h-m_{\widetilde{B}^\rho}(h)|)\\ &\quad\lesssim\{[\lambda(c_B,r_B)]^\alpha+[\lambda(c_B,r_{\widetilde{B}^\rho})]^\alpha\}\|h\|_{\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta,\,\gamma}(\mu)}\lesssim[\lambda(c_B,r_B)]^\alpha\|h\|_{\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta,\,\gamma}(\mu)}.\end{aligned}$$

On the other hand, by Definition 6.5, (2.1) and (2.2), we find that, for all (ρ,β_ρ)-doubling balls $B\subset S$,

$$\begin{aligned}&|m_B(|h|)-m_S(|h|)|\\ &\quad\leqslant|m_B(|h|)-|m_B(h)||+||m_B(h)|-|m_S(h)||+||m_S(h)|-m_S(|h|)|\\ &\quad\leqslant m_B(|h-m_B(h)|)+|m_B(h)-m_S(h)|+m_S(|h-m_S(h)|)\\ &\quad\lesssim\{[\lambda(c_B,r_B)]^\alpha+[\lambda(c_S,r_S)]^\alpha[\widetilde{K}^{(\rho),\,1/(\alpha+1)}_{B,\,S}]^\gamma+[\lambda(c_S,r_S)]^\alpha\}\|h\|_{\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta,\,\gamma}(\mu)}\\ &\quad\lesssim[\lambda(c_S,r_S)]^\alpha[\widetilde{K}^{(\rho),\,1/(\alpha+1)}_{B,\,S}]^\gamma\|h\|_{\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta,\,\gamma}(\mu)}.\end{aligned}$$

The above two inequalities imply that $|h|\in\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta,\,\gamma}(\mu)$ and $\||h|\|_{\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta,\,\gamma}(\mu)}\lesssim\|h\|_{\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta,\,\gamma}(\mu)}$, which completes the proof of (b) and hence Proposition 6.7. □

We now show that the space $\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta,\,\gamma}(\mu)$ is independent of the choices of ρ and η.

Proposition 6.8. *Suppose that $(\mathcal{X},d,\mu)$ is a non-homogeneous metric measure space satisfying* (6.1). *Let $\alpha\in[0,\infty)$ and $q,\,\gamma\in[1,\infty)$. The following statements hold true*:

(i) *for any η_1, η_2 and ρ satisfying $1<\eta_1<\eta_2\leqslant\rho<\infty$, $\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta_1,\,\gamma}(\mu)$ and $\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta_2,\,\gamma}(\mu)$ coincide with equivalent norms*;

(ii) *for any ρ_1, ρ_2 and η satisfying $1<\eta\leqslant\rho_1,\,\rho_2<\infty$, $\mathcal{E}^{\alpha,\,q}_{\rho_1,\,\eta,\,\gamma}(\mu)$ and $\mathcal{E}^{\alpha,\,q}_{\rho_2,\,\eta,\,\gamma}(\mu)$ coincide with equivalent norms.*

Proof. We first prove (i). Fix $\alpha\in[0,\infty)$ and $q,\gamma\in[1,\infty)$. Let η_1, η_2 and ρ satisfy $1<\eta_1<\eta_2\leqslant\rho<\infty$. It is obvious that $\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta_1,\,\gamma}(\mu)\subset\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta_2,\,\gamma}(\mu)$ and, for all $f\in\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta_1,\,\gamma}(\mu)$, $\|f\|_{\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta_2,\,\gamma}(\mu)}\leqslant\|f\|_{\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta_1,\,\gamma}(\mu)}$.

Conversely, let $f\in\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta_2,\,\gamma}(\mu)$. We show that $f\in\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta_1,\,\gamma}(\mu)$ and $\|f\|_{\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta_1,\,\gamma}(\mu)}\lesssim\|f\|_{\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta_2,\,\gamma}(\mu)}$. To this end, it suffices to show that, for any ball B,

$$\left\{\frac{1}{\mu(\eta_1 B)}\int_B |f(y)-m_{\widetilde{B}^\rho}(f)|^q d\mu(y)\right\}^{1/q}\lesssim[\lambda(c_B,r_B)]^\alpha\|f\|_{\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta_2,\,\gamma}(\mu)}. \tag{6.5}$$

To do so, for any $x \in B$, let B_x be the ball centered at x with radius $\frac{\eta_1-1}{10\eta_2}r_B$. Then $r_{\eta_2 B_x} = \frac{\eta_1-1}{10}r_B$ and $\eta_2 B_x \subset \eta_1 B$.

By the geometrically doubling condition and Remark 2.2(ii), we see that there exist $N_1 \in \mathbb{N}$, depending on η_1, η_2 and $(\mathcal{X}, d, \mu)$, and a finite sequence $\{B_{x_i}\}_{i=1}^{N_1} =: \{B_i\}_{i=1}^{N_1}$ of balls such that $x_i \in B$ for all $i \in \{1, \ldots, N_1\}$ and $B \subset \bigcup_{i=1}^{N_1} B_i$. By this, $\eta_2 B_i \subset \eta_1 B$ for all $i \in \{1, \ldots, N_1\}$, (2.1), (2.2), Proposition 6.7(a) and Lemma 2.8(ii), we conclude that

$$\begin{aligned}
&\frac{1}{\mu(\eta_1 B)}\int_B |f(y) - m_{\widetilde{B}^\rho}(f)|^q d\mu(y)\\
&\quad\leqslant \sum_{i=1}^{N_1} \frac{1}{\mu(\eta_1 B)}\int_{B_i} |f(y) - m_{\widetilde{B}^\rho}(f)|^q d\mu(y)\\
&\quad\lesssim \sum_{i=1}^{N_1} \frac{\mu(\eta_2 B_i)}{\mu(\eta_1 B)}\left\{\frac{1}{\mu(\eta_2 B_i)}\int_{B_i} |f(y) - m_{\widetilde{B}_i^\rho}(f)|^q d\mu(y) + |m_{\widetilde{B}_i^\rho}(f) - m_{\widetilde{B}^\rho}(f)|^q\right\}\\
&\quad\lesssim \sum_{i=1}^{N_1} \{[\lambda(c_{B_i}, r_{B_i})]^{\alpha q}\|f\|^q_{\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta_2,\,\gamma}(\mu)} + |m_{\widetilde{B}_i^\rho}(f) - m_{\widetilde{(\eta_1 B)}^\rho}(f)|^q + |m_{\widetilde{(\eta_1 B)}^\rho}(f) - m_{\widetilde{B}^\rho}(f)|^q\}\\
&\quad\lesssim [\lambda(c_B, \eta_1 r_B)]^{\alpha q}\|f\|^q_{\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta_2,\,\gamma}(\mu)}\{[\widetilde{K}^{(\rho),\,1/(\alpha+1)}_{B_i,\,\eta_1 B}]^{\gamma q} + [\widetilde{K}^{(\rho),\,1/(\alpha+1)}_{B,\,\eta_1 B}]^{\gamma q}\}\\
&\quad\lesssim [\lambda(c_B, r_B)]^{\alpha q}\|f\|^q_{\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta_2,\,\gamma}(\mu)},
\end{aligned}$$

which completes the proof of (6.5) and hence (i).

To show (ii), fix $\alpha \in [0, \infty)$ and $q, \gamma \in [1, \infty)$. Let ρ_1, ρ_2 and η satisfy $1 < \eta \leqslant \rho_1, \rho_2 < \infty$. By the symmetry of ρ_1 and ρ_2, it suffices to show that $\mathcal{E}^{\alpha,\,q}_{\rho_2,\,\eta,\,\gamma}(\mu) \subset \mathcal{E}^{\alpha,\,q}_{\rho_1,\,\eta,\,\gamma}(\mu)$ and $\|f\|_{\mathcal{E}^{\alpha,\,q}_{\rho_1,\,\eta,\,\gamma}(\mu)} \lesssim \|f\|_{\mathcal{E}^{\alpha,\,q}_{\rho_2,\,\eta,\,\gamma}(\mu)}$ for all $f \in \mathcal{E}^{\alpha,\,q}_{\rho_2,\,\eta,\,\gamma}(\mu)$. Assume that $f \in \mathcal{E}^{\alpha,\,q}_{\rho_2,\,\eta,\,\gamma}(\mu)$. From the Minkowski inequality, Hölder's inequality, Proposition 6.7, $\rho_1 \geqslant \eta$, Remark 6.2(ii), Lemmas 2.9 and 2.8(iii), we deduce that

$$\begin{aligned}
&\left\{\frac{1}{\mu(\eta B)}\int_B |f(y) - m_{\widetilde{B}^{\rho_1}}(f)|^q d\mu(y)\right\}^{1/q}\\
&\quad\leqslant \left\{\frac{1}{\mu(\eta B)}\int_B |f(y) - m_{\widetilde{B}^{\rho_2}}(f)|^q d\mu(y)\right\}^{1/q} + |m_{\widetilde{B}^{\rho_1}}(f) - m_{\widetilde{B}^{\rho_2}}(f)|\\
&\quad\leqslant [\lambda(c_B, r_B)]^\alpha\|f\|_{\mathcal{E}^{\alpha,\,q}_{\rho_2,\,\eta,\,\gamma}(\mu)} + |m_{\widetilde{B}^{\rho_1}}(f) - m_{\widetilde{\widetilde{B}^{\rho_1}}^{\rho_2}}(f)| + |m_{\widetilde{\widetilde{B}^{\rho_1}}^{\rho_2}}(f) - m_{\widetilde{B}^{\rho_2}}(f)|\\
&\quad\lesssim [\lambda(c_B, r_B)]^\alpha\|f\|_{\mathcal{E}^{\alpha,\,q}_{\rho_2,\,\eta,\,\gamma}(\mu)} + \left\{\frac{1}{\mu(\eta\widetilde{B}^{\rho_1})}\int_{\widetilde{B}^{\rho_1}} |f(y) - m_{\widetilde{\widetilde{B}^{\rho_1}}^{\rho_2}}(f)|^q d\mu(y)\right\}^{1/q}\\
&\qquad + [\widetilde{K}^{(\rho_2),\,1/(\alpha+1)}_{B,\,\widetilde{B}^{\rho_1}}]^\gamma[\lambda(c_B, r_{\widetilde{B}^{\rho_1}})]^\alpha\|f\|_{\mathcal{E}^{\alpha,\,q}_{\rho_2,\,\eta,\,\gamma}(\mu)}\\
&\quad\lesssim \{[\lambda(c_B, r_B)]^\alpha + [\lambda(c_B, r_{\widetilde{B}^{\rho_1}})]^\alpha\}\|f\|_{\mathcal{E}^{\alpha,\,q}_{\rho_2,\,\eta,\,\gamma}(\mu)} \lesssim [\lambda(c_B, r_B)]^\alpha\|f\|_{\mathcal{E}^{\alpha,\,q}_{\rho_2,\,\eta,\,\gamma}(\mu)}.
\end{aligned}$$

On the other hand, for all (ρ_1, β_{ρ_1})-doubling balls $B \subset S$, by Hölder's inequality, Proposition 6.7, $\rho_1 \geqslant \eta$, Lemma 2.9, (2.1) and (2.2), we have

$$\begin{aligned}
&|m_B(f) - m_S(f)|\\
&\quad\leqslant |m_B(f) - m_{\widetilde{B}^{\rho_2}}(f)| + |m_{\widetilde{B}^{\rho_2}}(f) - m_{\widetilde{S}^{\rho_2}}(f)| + |m_{\widetilde{S}^{\rho_2}}(f) - m_S(f)|\\
&\quad\leqslant \left\{\frac{1}{\mu(B)}\int_B |f(y) - m_{\widetilde{B}^{\rho_2}}(f)|^q d\mu(y)\right\}^{1/q} + [\widetilde{K}^{(\rho_2),\,1/(\alpha+1)}_{B,\,S}]^\gamma[\lambda(c_S, r_S)]^\alpha\|f\|_{\mathcal{E}^{\alpha,\,q}_{\rho_2,\,\eta,\,\gamma}(\mu)}\\
&\qquad + \left\{\frac{1}{\mu(S)}\int_S |f(y) - m_{\widetilde{S}^{\rho_2}}(f)|^q d\mu(y)\right\}^{1/q}\\
&\quad\lesssim [\lambda(c_B, r_B)]^\alpha\|f\|_{\mathcal{E}^{\alpha,\,q}_{\rho_2,\,\eta,\,\gamma}(\mu)} + [\widetilde{K}^{(\rho_1),\,1/(\alpha+1)}_{B,\,S}]^\gamma[\lambda(c_S, r_S)]^\alpha\|f\|_{\mathcal{E}^{\alpha,\,q}_{\rho_2,\,\eta,\,\gamma}(\mu)}\\
&\quad\lesssim [\widetilde{K}^{(\rho_1),\,1/(\alpha+1)}_{B,\,S}]^\gamma[\lambda(c_S, r_S)]^\alpha\|f\|_{\mathcal{E}^{\alpha,\,q}_{\rho_2,\,\eta,\,\gamma}(\mu)},
\end{aligned}$$

which completes the proof of (ii) and hence Proposition 6.8. □

Remark 6.9. (i) By Proposition 6.8, we know that the space $\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta,\,\gamma}(\mu)$ is independent of the choices of ρ and η. From now on, unless explicitly pointed out, we *always assume* that $\rho=\eta$ in $\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta,\,\gamma}(\mu)$ and *write* $\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta,\,\gamma}(\mu)$ *simply by* $\mathcal{E}^{\alpha,\,q}_{\rho,\,\gamma}(\mu)$ and its *norm* $\|\cdot\|_{\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta,\,\gamma}(\mu)}$ *simply by* $\|\cdot\|_{\mathcal{E}^{\alpha,\,q}_{\rho,\,\gamma}(\mu)}$.

(ii) It is still unknown whether $\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta,\,\gamma}(\mu)$ is independent of the choices of ρ and η or not on general non-homogeneous metric measure spaces without the assumption (6.1).

Before we show that $\mathcal{E}^{\alpha,\,q}_{\rho,\,\gamma}(\mu)$ is independent of the choices of γ and q, we first give a useful characterization of $\mathcal{E}^{\alpha,\,q}_{\rho,\,\gamma}(\mu)$ which is a variant of [30, Proposition 2.10].

Proposition 6.10. *Suppose that* $(\mathcal{X},d,\mu)$ *is a non-homogeneous metric measure space satisfying* (6.1). *Let* $\rho\in(1,\infty)$, $\alpha\in[0,\infty)$ *and* $q,\,\gamma\in[1,\infty)$. *The following statements are equivalent*:

(a) $f\in\mathcal{E}^{\alpha,\,q}_{\rho,\,\gamma}(\mu)$;

(b) *there exists a sequence* $\{f_B\}_B$ *of complex numbers associated with balls* $B:=B(c_B,r_B)$, *with* $c_B\in\mathcal{X}$ *and* $r_B\in(0,\infty)$, *such that*

$$\|f\|^{(q)}_{*,\,\rho}:=\sup_B\left\{\frac{1}{\mu(\rho B)}\frac{1}{[\lambda(c_B,\,r_B)]^{\alpha q}}\int_B|f(y)-f_B|^q d\mu(y)\right\}^{1/q}+\sup_{B\subset S}\frac{|f_B-f_S|}{[\lambda(c_S,\,r_S)]^{\alpha}[\widetilde{K}^{(\rho),\,1/(\alpha+1)}_{B,\,S}]^{\gamma}}<\infty,$$

where $c_S\in\mathcal{X}$ *and* $r_S\in(0,\infty)$ *denote, respectively, the center and the radius of the ball* S, *and the first supremum is taking over all balls* $B\subset\mathcal{X}$ *and the second one all balls* $B\subset S\subset\mathcal{X}$.

Moreover, the norms $\|\cdot\|_{\mathcal{E}^{\alpha,\,q}_{\rho,\,\gamma}(\mu)}$ *and* $\|\cdot\|^{(q)}_{*,\,\rho}$ *are equivalent.*

Proof. Fix $\rho\in(1,\infty)$, $\alpha\in[0,\infty)$ and $q,\,\gamma\in[1,\infty)$. Let $f\in\mathcal{E}^{\alpha,\,q}_{\rho,\,\gamma}(\mu)$. We first show that $\|f\|^{(q)}_{*,\,\rho}\lesssim\|f\|_{\mathcal{E}^{\alpha,\,q}_{\rho,\,\gamma}(\mu)}$. Indeed, for any ball B, let $f_B:=m_{\widetilde{B}^\rho}(f)$. Then Proposition 6.7 implies that, for any two balls $B\subset S$,

$$|f_B-f_S|\lesssim[\lambda(c_S,r_S)]^{\alpha}[\widetilde{K}^{(\rho),\,1/(\alpha+1)}_{B,\,S}]^{\gamma}\|f\|_{\mathcal{E}^{\alpha,\,q}_{\rho,\,\gamma}(\mu)}.$$

This, together with the fact that, for any ball B,

$$\left\{\frac{1}{\mu(\rho B)}\frac{1}{[\lambda(c_B,\,r_B)]^{\alpha q}}\int_B|f(y)-f_B|^q d\mu(y)\right\}^{1/q}\leqslant\|f\|_{\mathcal{E}^{\alpha,\,q}_{\rho,\,\gamma}(\mu)}$$

implies that $\|f\|^{(q)}_{*,\,\rho}\lesssim\|f\|_{\mathcal{E}^{\alpha,\,q}_{\rho,\,\gamma}(\mu)}$.

Conversely, assume that $\|f\|^{(q)}_{*,\,\rho}<\infty$. If B is a (ρ,β_ρ)-doubling ball, then, by Hölder's inequality, we have

$$|f_B-m_B(f)|\lesssim\left\{\frac{1}{\mu(\rho B)}\int_B|f(y)-f_B|^q d\mu(y)\right\}^{1/q}\lesssim[\lambda(c_B,r_B)]^{\alpha}\|f\|^{(q)}_{*,\,\rho},\tag{6.6}$$

which, together with the Minkowski inequality, implies that

$$\begin{aligned}&\left\{\frac{1}{\mu(B)}\int_B|f(y)-m_B(f)|^q d\mu(y)\right\}^{1/q}\\&\quad\lesssim\left\{\frac{1}{\mu(\rho B)}\int_B|f(y)-f_B|^q d\mu(y)\right\}^{1/q}+|f_B-m_B(f)|\lesssim[\lambda(c_B,r_B)]^{\alpha}\|f\|^{(q)}_{*,\,\rho}.\end{aligned}$$

Thus, by this, the Minkowski inequality, (6.6), Remark 6.2(ii) and Lemma 2.8(iii), we obtain

$$\begin{aligned}&\left\{\frac{1}{\mu(\rho B)}\int_B|f(y)-m_{\widetilde{B}^\rho}(f)|^q d\mu(y)\right\}^{1/q}\\&\quad\leqslant\left\{\frac{1}{\mu(\rho B)}\int_B|f(y)-f_B|^q d\mu(y)\right\}^{1/q}+|f_B-f_{\widetilde{B}^\rho}|+|f_{\widetilde{B}^\rho}-m_{\widetilde{B}^\rho}(f)|\\&\quad\lesssim\{[\lambda(c_B,r_B)]^{\alpha}+[\lambda(c_B,r_{\widetilde{B}^\rho})]^{\alpha}[\widetilde{K}^{(\rho),\,1/(\alpha+1)}_{B,\,\widetilde{B}^\rho}]^{\gamma}\}\|f\|^{(q)}_{*,\,\rho}\end{aligned}$$

$$\lesssim [\lambda(c_B, r_B)]^\alpha \|f\|_{*,\rho}^{(q)}.$$

Moreover, by (6.6), (2.1) and (2.2), we conclude that, for all (ρ, β_ρ)-doubling balls $B \subset S$,

$$\begin{aligned}|m_B(f) - m_S(f)| &\leqslant |m_B(f) - f_B| + |f_B - f_S| + |f_S - m_S(f)| \\ &\lesssim \{[\lambda(c_B, r_B)]^\alpha + [\lambda(c_S, r_S)]^\alpha [\widetilde{K}_{B,S}^{(\rho),1/(\alpha+1)}]^\gamma\} \|f\|_{*,\rho}^{(q)} \\ &\lesssim [\lambda(c_S, r_S)]^\alpha \|f\|_{*,\rho}^{(q)} [\widetilde{K}_{B,S}^{(\rho),1/(\alpha+1)}]^\gamma,\end{aligned}$$

which completes the proof of Proposition 6.10. □

To show that $\mathcal{E}_{\rho,\gamma}^{\alpha,q}(\mu)$ is independent of the choice of $\gamma \in [1, \infty)$, we need the following technical lemma, which is similar to [30, Lemma 2.6] (see also [54, Lemma 9.2]).

Lemma 6.11. *Suppose that $(\mathcal{X}, d, \mu)$ is a non-homogeneous metric measure space. Let $m \in \mathbb{N} \cap (1, \infty)$, $\rho \in (1, \infty)$, $p \in (0, 1]$ and $B := B_1 \subset \cdots \subset B_m$ be concentric balls with center c_B and radii of the form $\rho^N r_B$, where $N \in \mathbb{Z}_+$. If $\widetilde{K}_{B_i,B_{i+1}}^{(\rho),p} > (3 + \lfloor \log_\rho 2 \rfloor)^{1/p}$ for any $i \in \{1, \ldots, m-1\}$, then*

$$\sum_{i=1}^{m-1} [\widetilde{K}_{B_i,B_{i+1}}^{(\rho),p}]^p < (3 + \lfloor \log_\rho 2 \rfloor) [\widetilde{K}_{B_1,B_m}^{(\rho),p}]^p. \tag{6.7}$$

Proof. Fix $m \in \mathbb{N}$, $\rho \in (1, \infty)$ and $p \in (0, 1]$. Assume that, for any $i \in \{1, \ldots, m\}$, $r_{B_i} := \rho^{N_i} r_B$ for some $N_i \in \mathbb{Z}_+$. For any $i \in \{1, \ldots, m-1\}$, by $\widetilde{K}_{B_i,B_{i+1}}^{(\rho),p} > (3 + \lfloor \log_\rho 2 \rfloor)^{1/p}$, it is easy to see that $N_{i+1} - N_i = N_{B_i,B_{i+1}}^{(\rho)} \geqslant 1$, $1 < \sum_{k=1}^{N_{B_i,B_{i+1}}^{(\rho)}} [\frac{\mu(\rho^k B_i)}{\lambda(c_B, \rho^k r_{B_i})}]^p$ and $N_m = N_{B_1,B_m}^{(\rho)}$. From these facts and (2.1), we deduce that, for any $i \in \{1, \ldots, m-1\}$,

$$\begin{aligned}[\widetilde{K}_{B_i,B_{i+1}}^{(\rho),p}]^p &\leqslant 2 + \lfloor \log_\rho 2 \rfloor + \sum_{k=1}^{N_{B_i,B_{i+1}}^{(\rho)}} \left[\frac{\mu(\rho^k B_i)}{\lambda(c_B, \rho^k r_{B_i})}\right]^p \\ &< (3 + \lfloor \log_\rho 2 \rfloor) \sum_{k=1}^{N_{B_i,B_{i+1}}^{(\rho)}} \left[\frac{\mu(\rho^k B_i)}{\lambda(c_B, \rho^k r_{B_i})}\right]^p \\ &= (3 + \lfloor \log_\rho 2 \rfloor) \sum_{k=N_i+1}^{N_{i+1}} \left[\frac{\mu(\rho^k B)}{\lambda(c_B, \rho^k r_B)}\right]^p.\end{aligned}$$

Notice that $p \in (0, 1]$ and $N_m = N_{B_1,B_m}^{(\rho)}$. It then follows that

$$\sum_{i=1}^{m-1} [\widetilde{K}_{B_i,B_{i+1}}^{(\rho),p}]^p < (3 + \lfloor \log_\rho 2 \rfloor) [\widetilde{K}_{B_1,B_m}^{(\rho),p}]^p,$$

which implies (6.7) and hence completes the proof of Lemma 6.11. □

The following lemma is an analogue of [30, Lemma 2.7], whose proof needs to use Lemma 6.11, the details being omitted.

Lemma 6.12. *Suppose that $(\mathcal{X}, d, \mu)$ is a non-homogeneous metric measure space. Let $\alpha \in [0, \infty)$, $\rho \in (1, \infty)$ and $p \in (0, 1]$. For a large positive constant C, the following statement holds true: Let $x \in \mathcal{X}$ be a fixed point, and $\{f_B\}_{B \ni x}$ some collection of complex numbers associated with balls $B \ni x$. If there exists a positive constant C_x, depending on x, such that, for all balls B and S with $x \in B \subset S$ and $\widetilde{K}_{B,S}^{(\rho),p} \leqslant C$, $|f_B - f_S| \leqslant C_x \widetilde{K}_{B,S}^{(\rho),p} [\lambda(c_S, r_S)]^\alpha$, then, for all balls B and S with $x \in B \subset S$,*

$$|f_B - f_S| \leqslant C C_x \widetilde{K}_{B,S}^{(\rho),p} [\lambda(c_S, r_S)]^\alpha.$$

By Lemma 6.12, now we are ready to state the result that $\mathcal{E}_{\rho,\gamma}^{\alpha,q}(\mu)$ is independent of the choice of γ, whose proof is similar to that of [30, Proposition 2.5], the details being omitted.

Proposition 6.13. *Suppose that $(\mathcal{X}, d, \mu)$ is a non-homogeneous metric measure space satisfying* (6.1). *Let $\alpha \in [0,\infty)$, $\rho, \gamma \in (1,\infty)$ and $q \in [1,\infty)$. Then $\mathcal{E}^{\alpha,q}_{\rho,\gamma}(\mu)$ and $\mathcal{E}^{\alpha,q}_{\rho,1}(\mu)$ coincide with equivalent norms.*

Remark 6.14. (i) By Proposition 6.13, we know that the space $\mathcal{E}^{\alpha,q}_{\rho,\gamma}(\mu)$ is independent of the choice of γ. From now on, unless explicitly pointed out, we *always assume* that $\gamma = 1$ in $\mathcal{E}^{\alpha,q}_{\rho,\gamma}(\mu)$ and write $\mathcal{E}^{\alpha,q}_{\rho,\gamma}(\mu)$ simply by $\mathcal{E}^{\alpha,q}_{\rho}(\mu)$.

(ii) It is still unknown whether $\mathcal{E}^{\alpha,q}_{\rho,\eta,\gamma}(\mu)$ is independent of the choice of γ or not on general non-homogeneous metric measure spaces without the assumption (6.1), even on Euclidean spaces endowed with non-doubling measures.

In order to show that $\mathcal{E}^{\alpha,q}_{\rho}(\mu)$ is independent of q, we establish the following John-Nirenberg type inequality which is a generalization of [27, Proposition 6.1]. Hereafter, *$\mathcal{E}^{\alpha,1}_{\rho}(\mu)$ is simply denoted by* $\mathcal{E}^{\alpha}_{\rho}(\mu)$ and its *equivalent norm $\|\cdot\|^{(1)}_{*,\rho}$ simply by $\|\cdot\|_{*,\rho}$.*

Proposition 6.15. *Suppose that $(\mathcal{X}, d, \mu)$ is a non-homogeneous metric measure space satisfying* (6.1). *Let $\alpha \in [0,\infty)$ and $\rho \in (1,\infty)$. Then there exists a positive constant c such that, for any $f \in \mathcal{E}^{\alpha}_{\rho}(\mu)$, $t \in (0,\infty)$ and every ball $B_0 := B(x_0, r)$ with $x_0 \in \mathcal{X}$ and $r \in (0,\infty)$,*

$$\mu\left(\left\{x \in B_0 : \frac{|f(x) - f_{B_0}|}{[\lambda(x_0,r)]^\alpha} > t\right\}\right) \leqslant 2\mu(\rho B_0)\mathrm{e}^{-\frac{ct}{\|f\|_{*,\rho}}},$$

where f_{B_0} is as in Proposition 6.10(ii) *with B replaced by B_0.*

Proof. Fix $\alpha \in [0,\infty)$ and $\rho \in (1,\infty)$. Let $\sigma := 5\rho$, $f \in \mathcal{E}^{\alpha}_{\rho}(\mu)$ and L be a large positive constant whose value will be determined later. We first claim that, for μ-almost every $x \in B_0$ with $\frac{|f(x)-f_{B_0}|}{[\lambda(x_0,r)]^\alpha} > 2L$, there exists a (σ, β_σ)-doubling ball $\widehat{B}^\sigma_x$ of the form $B(x, \sigma^{-i}r)$, $i \in \mathbb{N}$, satisfying

$$\widehat{B}^\sigma_x \subset \sqrt{\rho}B_0 \quad \text{and} \quad \frac{|f_{\widehat{B}^\sigma_x} - f_{B_0}|}{[\lambda(x_0,r)]^\alpha} > L. \tag{6.8}$$

Indeed, from $\frac{|f(x)-f_{B_0}|}{[\lambda(x_0,r)]^\alpha} > 2L$ and [27, Corollary 3.6], it follows that there exists a (σ, β_σ)-doubling ball $\widehat{B}^\sigma_x$ of the form $B(x, \sigma^{-i}r)$, $i \in \mathbb{N}$, such that $\widehat{B}^\sigma_x \subset \sqrt{\rho}B_0$ and $\frac{|m_{\widehat{B}^\sigma_x}(f)-f_{B_0}|}{[\lambda(x_0,r)]^\alpha} > 2L$. Thus, by this, Propositions 6.8 and 6.10, (2.1) and (2.2), we conclude that

$$\begin{aligned}
\frac{|f_{\widehat{B}^\sigma_x} - f_{B_0}|}{[\lambda(x_0,r)]^\alpha} &\geqslant \frac{|m_{\widehat{B}^\sigma_x}(f) - f_{B_0}|}{[\lambda(x_0,r)]^\alpha} - \frac{|f_{\widehat{B}^\sigma_x} - m_{\widehat{B}^\sigma_x}(f)|}{[\lambda(x_0,r)]^\alpha} \\
&> 2L - \frac{1}{[\lambda(x_0,r)]^\alpha}\frac{1}{\mu(\widehat{B}^\sigma_x)}\int_{\widehat{B}^\sigma_x}|f(y) - f_{\widehat{B}^\sigma_x}|d\mu(y) \\
&\geqslant 2L - \frac{[\lambda(x,\sqrt{\rho}r)]^\alpha}{[\lambda(x_0,r)]^\alpha}\beta_\sigma\|f\|_{*,\sqrt{\rho}} \\
&\geqslant 2L - C_1\|f\|_{*,\rho} \geqslant L,
\end{aligned}$$

provided that $L \geqslant C_1\|f\|_{*,\rho}$ and C_1 is a positive constant, which implies the claim.

Now we let $\widehat{B}^\sigma_x$ be the biggest (σ, β_σ)-doubling ball of the form $B(x, \sigma^{-i}r)$, $i \in \mathbb{N}$, satisfying (6.8). By (6.8), (2.1) and (2.2), we know that

$$\begin{aligned}
&\frac{1}{\mu(\widehat{B}^\sigma_x)}\int_{\widehat{B}^\sigma_x}\frac{|f(y) - f_{B_0}|}{[\lambda(x_0,r)]^\alpha}d\mu(y) \\
&\quad\geqslant \frac{|f_{\widehat{B}^\sigma_x} - f_{B_0}|}{[\lambda(x_0,r)]^\alpha} - \frac{1}{\mu(\widehat{B}^\sigma_x)}\int_{\widehat{B}^\sigma_x}\frac{|f(y) - f_{\widehat{B}^\sigma_x}|}{[\lambda(x_0,r)]^\alpha}d\mu(y) \\
&\quad> L - \frac{[\lambda(x,\sqrt{\rho}r)]^\alpha}{[\lambda(x_0,r)]^\alpha}\beta_\sigma\|f\|_{*,\sqrt{\rho}} \geqslant L - C_1\|f\|_{*,\rho} \geqslant L/2,
\end{aligned} \tag{6.9}$$

provided that $L \geqslant 2C_1\|f\|_{*,\rho}$.

Then we show that the ball $\widetilde{(\sigma\widehat{B}_x^\sigma)}^\sigma =: \widehat{\widehat{B}}_x^\sigma$ satisfies

$$\widehat{\widehat{B}}_x^\sigma \not\subset \sqrt{\rho}B_0 \quad \text{or} \quad \frac{|f_{\widehat{\widehat{B}}_x^\sigma} - f_{B_0}|}{[\lambda(x_0,r)]^\alpha} \leqslant L. \tag{6.10}$$

Indeed, it suffices to prove that, if $\widehat{\widehat{B}}_x^\sigma \subset \sqrt{\rho}B_0$, then $\frac{|f_{\widehat{\widehat{B}}_x^\sigma} - f_{B_0}|}{[\lambda(x_0,r)]^\alpha} \leqslant L$. From Lemma 2.9, $B(x,r) \subset 2\sqrt{\rho}B_0$, Lemmas 2.8(ii)–2.8(v), it follows that, if $\widehat{\widehat{B}}_x^\sigma \subset \sqrt{\rho}B_0 \subset 2\sqrt{\rho}B_0$, then

$$\begin{aligned}
\frac{|f_{\widehat{\widehat{B}}_x^\sigma} - f_{B_0}|}{[\lambda(x_0,r)]^\alpha} &\leqslant \frac{|f_{\widehat{\widehat{B}}_x^\sigma} - f_{2\sqrt{\rho}B_0}|}{[\lambda(x_0,r)]^\alpha} + \frac{|f_{2\sqrt{\rho}B_0} - f_{B_0}|}{[\lambda(x_0,r)]^\alpha}\\
&\lesssim \|f\|_{*,\rho}\frac{[\lambda(x_0,2\sqrt{\rho}r)]^\alpha}{[\lambda(x_0,r)]^\alpha}[\widetilde{K}^{(\rho),1/(\alpha+1)}_{\widehat{\widehat{B}}_x^\sigma,2\sqrt{\rho}B_0} + \widetilde{K}^{(\rho),1/(\alpha+1)}_{B_0,2\sqrt{\rho}B_0}]\\
&\lesssim \|f\|_{*,\rho}\widetilde{K}^{(\sigma),1/(\alpha+1)}_{\widehat{B}_x^\sigma,2\sqrt{\rho}B_0} \lesssim \|f\|_{*,\rho}[\widetilde{K}^{(\sigma),1/(\alpha+1)}_{\widehat{B}_x^\sigma,B(x,r)} + \widetilde{K}^{(\sigma),1/(\alpha+1)}_{B(x,r),2\sqrt{\rho}B_0}]\\
&\leqslant C_2\|f\|_{*,\rho} \leqslant L,
\end{aligned}$$

provided that $L \geqslant C_2\|f\|_{*,\rho}$ and C_2 is a positive constant, which shows (6.10).

Moreover, if $\widehat{\widehat{B}}_x^\sigma \not\subset \sqrt{\rho}B_0$, let $\sigma^j\widehat{B}_x^\sigma$ be the smallest ball of the form $\sigma^k\widehat{B}_x^\sigma$ ($k \in \mathbb{N}$) satisfying $\sigma^j\widehat{B}_x^\sigma \not\subset \sqrt{\rho}B_0$. We easily obtain

$$r_{\sigma^j\widehat{B}_x^\sigma} \sim r_{B_0} \quad \text{and} \quad \widehat{\widehat{B}}_x^\sigma = \widetilde{(\sigma^j\widehat{B}_x^\sigma)}^\sigma,$$

where $r_{\sigma^j\widehat{B}_x^\sigma}$ and r_{B_0} denote the radii of balls $\sigma^j\widehat{B}_x^\sigma$ and B_0, respectively. By this, $\sigma^j\widehat{B}_x^\sigma \subset 3\sigma\sqrt{\rho}B_0$, $\widehat{B}_x^\sigma \subset \sqrt{\rho}B_0$, Remark 6.2(ii), (2.1), (2.2), Lemmas 2.9, 2.8(ii) and 2.8(iii), we have

$$\begin{aligned}
\frac{|f_{\widehat{\widehat{B}}_x^\sigma} - f_{B_0}|}{[\lambda(x_0,r)]^\alpha} &\leqslant \frac{|f_{\widehat{\widehat{B}}_x^\sigma} - f_{\sigma^j\widehat{B}_x^\sigma}|}{[\lambda(x_0,r)]^\alpha} + \frac{|f_{\sigma^j\widehat{B}_x^\sigma} - f_{3\sigma\sqrt{\rho}B_0}|}{[\lambda(x_0,r)]^\alpha} + \frac{|f_{3\sigma\sqrt{\rho}B_0} - f_{B_0}|}{[\lambda(x_0,r)]^\alpha}\\
&\lesssim \frac{[\lambda(x,r_{\widehat{\widehat{B}}_x^\sigma})]^\alpha}{[\lambda(x_0,r)]^\alpha}\|f\|_{*,\rho} + \frac{[\lambda(x_0,3\sigma\sqrt{\rho}r)]^\alpha}{[\lambda(x_0,r)]^\alpha}\|f\|_{*,\rho}[\widetilde{K}^{(\rho),1/(\alpha+1)}_{\sigma^j\widehat{B}_x^\sigma,3\sigma\sqrt{\rho}B_0} + \widetilde{K}^{(\rho),1/(\alpha+1)}_{B_0,3\sigma\sqrt{\rho}B_0}]\\
&\lesssim \frac{[\lambda(x,r_{\sigma\widehat{B}_x^\sigma})]^\alpha}{[\lambda(x_0,r)]^\alpha}\|f\|_{*,\rho} + \|f\|_{*,\rho} \leqslant C_3\|f\|_{*,\rho} \leqslant L,
\end{aligned}$$

provided that $L \geqslant C_3\|f\|_{*,\rho}$ and C_3 is a positive constant.

Thus, in any case, we have

$$\frac{|f_{\widehat{\widehat{B}}_x^\sigma} - f_{B_0}|}{[\lambda(x_0,r)]^\alpha} \leqslant L, \tag{6.11}$$

provided that $L \geqslant \max\{C_2, C_3\}\|f\|_{*,\rho}$.

Furthermore, by [23, Theorem 1.2] and [27, Lemma 2.5], we see that there exists a sequence $\{\widehat{B}_{x_i}^\sigma\}_{i\in I}$ of disjoint balls such that $x_i \in B_0$ for any $i \in I$ and $B_0 \subset \bigcup_{x\in B_0}\widehat{B}_x^\sigma \subset \bigcup_{i\in I}5\widehat{B}_{x_i}^\sigma$. Let $B^{(i)} := 5\widehat{B}_{x_i}^\sigma$ for any $i \in I$. Observe that, for any $n \in \mathbb{N}\cap[2,\infty)$, if $x \in B_0$ and $\frac{|f(x)-f_{B_0}|}{[\lambda(x_0,r)]^\alpha} > nL$, then there exists $i \in I$ such that $x \in B^{(i)}$ and, from (6.11), Remark 6.2(ii), (2.1), Lemmas 2.9 and 2.8(ii)–2.8(v), it follows that

$$\begin{aligned}
\frac{|f(x) - f_{B^{(i)}}|}{[\lambda(x_0,r)]^\alpha} &\geqslant \frac{|f(x) - f_{B_0}|}{[\lambda(x_0,r)]^\alpha} - \frac{|f_{B_0} - f_{\widehat{\widehat{B}}_{x_i}^\sigma}|}{[\lambda(x_0,r)]^\alpha} - \frac{|f_{\widehat{\widehat{B}}_{x_i}^\sigma} - f_{5\widehat{B}_{x_i}^\sigma}|}{[\lambda(x_0,r)]^\alpha}\\
&> nL - L - \frac{[\lambda(x_i, r_{\widehat{\widehat{B}}_{x_i}^\sigma})]^\alpha}{[\lambda(x_0,r)]^\alpha}\|f\|_{*,\rho}\widetilde{K}^{(\rho),1/(\alpha+1)}_{5\widehat{B}_{x_i}^\sigma,\widehat{\widehat{B}}_{x_i}^\sigma}\\
&\geqslant (n-1)L - C_4\|f\|_{*,\rho} \geqslant (n-2)L,
\end{aligned} \tag{6.12}$$

provided that $L \geqslant C_4\|f\|_{*,\rho}$ and C_4 is a positive constant.

By (6.9), the disjointness of $\{\widehat{B}^{\sigma}_{x_i}\}_{i\in I}$, $\widehat{B}^{\sigma}_{x_i}\subset\sqrt{\rho}B_0$ for all $i\in I$, Propositions 6.8(ii), 6.10(b), Lemma 2.8(ii) and (2.1), we further see that

$$\begin{aligned}\sum_{i\in I}\mu(\rho B^{(i)})&\leqslant\beta_\sigma\sum_{i\in I}\mu(\widehat{B}^{\sigma}_{x_i})\leqslant\frac{2\beta_\sigma}{L}\sum_{i\in I}\int_{\widehat{B}^{\sigma}_{x_i}}\frac{|f(y)-f_{B_0}|}{[\lambda(x_0,r)]^\alpha}d\mu(y)\\&\leqslant\frac{2\beta_\sigma}{L}\left\{\int_{\sqrt{\rho}B_0}\frac{|f(y)-f_{\sqrt{\rho}B_0}|}{[\lambda(x_0,r)]^\alpha}d\mu(y)+\frac{|f_{\sqrt{\rho}B_0}-f_{B_0}|\mu(\sqrt{\rho}B_0)}{[\lambda(x_0,r)]^\alpha}\right\}\\&\lesssim\frac{1}{L}\frac{[\lambda(x_0,\sqrt{\rho}r)]^\alpha}{[\lambda(x_0,r)]^\alpha}\|f\|_{*,\rho}\{\mu(\rho B_0)+\mu(\sqrt{\rho}B_0)\widetilde{K}^{(\rho),\,1/(\alpha+1)}_{B_0,\,\sqrt{\rho}B_0}\}\\&\leqslant\frac{C_5}{L}\|f\|_{*,\rho}\mu(\rho B_0)\leqslant\frac{1}{2}\mu(\rho B_0),\end{aligned}\tag{6.13}$$

provided that $L\geqslant 2C_5\|f\|_{*,\rho}$ and C_5 is a positive constant.

Moreover, for any $t\in(0,\infty)$, there exists $n\in\mathbb{Z}_+$ such that $2nL\leqslant t<2(n+1)L$. By this and (6.12), we know that

$$\begin{aligned}\left\{x\in B_0:\frac{|f(x)-f_{B_0}|}{[\lambda(x_0,r)]^\alpha}>t\right\}&\subset\left\{x\in B_0:\frac{|f(x)-f_{B_0}|}{[\lambda(x_0,r)]^\alpha}>2nL\right\}\\&\subset\bigcup_{i\in I}\left\{x\in B^{(i)}:\frac{|f(x)-f_{B^{(i)}}|}{[\lambda(x_0,r)]^\alpha}>2(n-1)L\right\}.\end{aligned}\tag{6.14}$$

Finally, by (6.13), (6.14), iterating with the balls $B^{(i)}$ in place of B_0 and an argument similar to that used in the proof of [27, Proposition 6.1], we conclude that

$$\mu\left(\left\{x\in B_0:\frac{|f(x)-f_{B_0}|}{[\lambda(x_0,r)]^\alpha}>t\right\}\right)\leqslant 2\mu(\rho B_0)\mathrm{e}^{-\frac{ct}{\|f\|_{*,\rho}}}$$

with $c:=\frac{\ln 2}{2L}\|f\|_{*,\rho}$ and $L:=2\max\{C_i:i\in\{1,\dots,5\}\}$. This finishes the proof of Proposition 6.15. □

By Proposition 6.15, we easily obtain the following conclusion.

Corollary 6.16. *Suppose that $(\mathcal{X},d,\mu)$ is a non-homogeneous metric measure space satisfying* (6.1). *Let $\alpha\in[0,\infty)$, $\rho\in(1,\infty)$ and $q\in[1,\infty)$. Then there exists a positive constant C such that, for any $f\in\mathcal{E}^\alpha_\rho(\mu)$ and every ball $B:=B(c_B,r_B)$ with $c_B\in\mathcal{X}$ and $r_B\in(0,\infty)$,*

$$\left\{\frac{1}{\mu(\rho B)}\int_B|f(x)-f_B|^q d\mu(x)\right\}^{1/q}\leqslant C\|f\|_{*,\rho}[\lambda(c_B,r_B)]^\alpha,$$

where f_B is as in Proposition 6.10(ii).

Remark 6.17. (i) By Corollary 6.16 and Proposition 6.10, together with Hölder's inequality, we know that $\mathcal{E}^{\alpha,\,q}_\rho(\mu)$ is independent of the choice of q, the details being omitted. From now on, unless explicitly pointed out, we *always assume* that $q=1$ in $\mathcal{E}^{\alpha,\,q}_\rho(\mu)$ and *write* $\mathcal{E}^{\alpha,\,1}_\rho(\mu)$ *simply by* $\mathcal{E}^\alpha_\rho(\mu)$ and its *norm* $\|\cdot\|_{\mathcal{E}^{\alpha,\,1}_\rho(\mu)}$ *simply by* $\|\cdot\|_{\mathcal{E}^\alpha_\rho(\mu)}$.

(ii) It is still unknown whether $\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta,\,\gamma}(\mu)$ is independent of the choice of q or not on general non-homogeneous metric measure spaces without the assumption (6.1), even on Euclidean spaces endowed with non-doubling measures.

We establish another characterization of $\mathcal{E}^\alpha_\rho(\mu)$ which is needed in the later context. To this end, we first recall the so-called median value of a function on balls in [25,30]. Precisely, let f be a measurable function. The *median value* of f on any ball B, denoted by $m_f(B)$, is defined as follows. If f is real-valued, then, for any ball B with $\mu(B)\neq 0$, let $m_f(B)$ be some *real number* such that $\inf_{c\in\mathbb{R}}\frac{1}{\mu(B)}\int_B|f(x)-c|d\mu(x)$ is attained. It is known that $m_f(B)$ satisfies

$$\mu(\{x\in B:f(x)>m_f(B)\})\leqslant\mu(B)/2$$

and

$$\mu(\{x\in B: f(x)<m_f(B)\})\leqslant \mu(B)/2.$$

For all balls B with $\mu(B)=0$, let $m_f(B)=0$. If f is complex-valued, we take

$$m_f(B):=[m_{\Re f}(B)]+\mathrm{i}[m_{\Im f}(B)],$$

where $\mathrm{i}^2=-1$ and, for any complex number z, denote by $\Re z$ and $\Im z$ its *real part* and *imaginary part*, respectively.

Let $\alpha\in[0,\infty)$, $\rho\in[2,\infty)$ and $q,\gamma\in[1,\infty)$. The norm $\|f\|_{\circ,\rho}$ of a suitable function f is defined by

$$\begin{aligned}\|f\|_{\circ,\rho}:=&\sup_{B:B\ (\rho,\beta_\rho)\text{-doubling ball}}\frac{1}{\mu(B)}\frac{1}{[\lambda(c_B,r_B)]^\alpha}\int_B|f(y)-m_f(B)|d\mu(y)\\&+\sup_{B\subset S:B,\,S\ (\rho,\beta_\rho)\text{-doubling balls}}\frac{|m_f(B)-m_f(S)|}{[\lambda(c_S,r_S)]^\alpha[\widetilde{K}_{B,S}^{(\rho),\,1/(\alpha+1)}]^\gamma}.\end{aligned}$$

Then we have the following equivalent characterization of $\mathcal{E}_\rho^\alpha(\mu)$.

Proposition 6.18. *Suppose that $(\mathcal{X},d,\mu)$ is a non-homogeneous metric measure space satisfying* (6.1). *Let $\alpha\in[0,\infty)$, $\rho\in[6/5,\infty)$ and $q,\gamma\in[1,\infty)$. Then the norms $\|\cdot\|_{\circ,\rho}$ and $\|\cdot\|_{\mathcal{E}_\rho^\alpha(\mu)}$ are equivalent.*

Proof. Fix $\alpha\in[0,\infty)$ and $\rho\in[6/5,\infty)$. For the sake of simplicity, we assume that $\gamma=1$. The arguments here are still valid for the general case with some minor modifications. Let $f\in\mathcal{E}_\rho^\alpha(\mu)$. Now we show that $\|f\|_{\circ,\rho}\lesssim\|f\|_{\mathcal{E}_\rho^\alpha(\mu)}$. For any (ρ,β_ρ)-doubling ball B, by the definition of $m_f(B)$, we conclude that

$$\begin{aligned}|m_f(B)-m_B(f)|&\leqslant\frac{1}{\mu(B)}\int_B|f(y)-m_f(B)|d\mu(y)\\&\leqslant\frac{1}{\mu(B)}\int_B|f(y)-m_B(f)|d\mu(y)\lesssim\|f\|_{\mathcal{E}_\rho^\alpha(\mu)}[\lambda(c_B,r_B)]^\alpha,\end{aligned}\tag{6.15}$$

which implies that, for any (ρ,β_ρ)-doubling ball B,

$$\frac{1}{\mu(B)}\int_B|f(y)-m_f(B)|d\mu(y)\lesssim\|f\|_{\mathcal{E}_\rho^\alpha(\mu)}[\lambda(c_B,r_B)]^\alpha.$$

On the other hand, by (6.15), (2.1) and (2.2), we know that, for all (ρ,β_ρ)-doubling balls $B\subset S$,

$$\begin{aligned}|m_f(B)-m_f(S)|&\leqslant|m_f(B)-m_B(f)|+|m_B(f)-m_S(f)|+|m_S(f)-m_f(S)|\\&\lesssim\|f\|_{\mathcal{E}_\rho^\alpha(\mu)}[\lambda(c_S,r_S)]^\alpha\widetilde{K}_{B,S}^{(\rho),\,1/(\alpha+1)}+\|f\|_{\mathcal{E}_\rho^\alpha(\mu)}\{[\lambda(c_B,r_B)]^\alpha+[\lambda(c_S,r_S)]^\alpha\}\\&\lesssim\|f\|_{\mathcal{E}_\rho^\alpha(\mu)}[\lambda(c_S,r_S)]^\alpha\widetilde{K}_{B,S}^{(\rho),\,1/(\alpha+1)}.\end{aligned}$$

Combining these two inequalities, we conclude that $\|f\|_{\circ,\rho}\lesssim\|f\|_{\mathcal{E}_\rho^\alpha(\mu)}$.

Conversely, let $\|f\|_{\circ,\rho}<\infty$. We now prove that $\|f\|_{\mathcal{E}_{\rho,\gamma}^{\alpha,q}(\mu)}\lesssim\|f\|_{\circ,\rho}$. For any ball B, if B is (ρ,β_ρ)-doubling, we see that

$$\begin{aligned}\frac{1}{\mu(\rho B)}\int_B|f(y)-m_B(f)|d\mu(y)&\leqslant\frac{1}{\mu(B)}\int_B|f(y)-m_f(B)|d\mu(y)+|m_f(B)-m_B(f)|\\&\lesssim\frac{1}{\mu(B)}\int_B|f(y)-m_f(B)|d\mu(y)\lesssim\|f\|_{\circ,\rho}[\lambda(c_B,r_B)]^\alpha.\end{aligned}$$

Thus, we only need to consider the case that B is non-(ρ,β_ρ)-doubling.

Assume that B is a non-(ρ,β_ρ)-doubling ball. For any $x\in B$, let B_x be the biggest $(5\rho,\beta_\rho)$-doubling ball centered at x with radius $(5\rho)^{-k}r_B$ for some $k\in\mathbb{N}$ (since $\beta_\rho>(5\rho)^{n_0}$ and [27, Lemma 3.3]). From $\rho\geqslant 6/5$, it follows easily that $5B_x\subset 2B\subset(6/5)B\subset\widetilde{B}^\rho$. Moreover, by [23, Theorem 1.2] and [27, Lemma 2.5], we see that there exists a countable disjoint subfamily $\{B_{x_i}\}_i=:\{B_i\}_i$ of $\{B_x\}_x$ such that $x_i \in B$ for all i and $B \subset \bigcup_{x\in B} B_x \subset \bigcup_i 5B_i$. For any i, by $5B_i \subset B(x_i, r_B) \subset (6/5)B \subset \rho B \subset \widetilde{B}^\rho$, and Lemmas 2.8(ii)–2.8(v), we see that

$$\widetilde{K}^{(\rho),\,1/(\alpha+1)}_{5B_i,\,\widetilde{B}^\rho} \lesssim \widetilde{K}^{(\rho),\,1/(\alpha+1)}_{5B_i,\,B(x_i,r_B)} + \widetilde{K}^{(\rho),\,1/(\alpha+1)}_{B(x_i,r_B),\,\rho B} + \widetilde{K}^{(\rho),\,1/(\alpha+1)}_{\rho B,\,\widetilde{B}^\rho} \lesssim 1.$$

From this, together with the fact that $5B_i$ is a (ρ, β_ρ)-doubling ball for any i, (2.1) and (2.2), Remark 6.2(ii), $5B_i \subset (6/5)B \subset \rho B \subset \widetilde{B}^\rho$ for any i and the disjointness of $\{B_i\}_i$, it follows that

$$\begin{aligned}
&\int_B |f(y) - m_{\widetilde{B}^\rho}(f)|d\mu(y)\\
&\quad\leqslant \sum_i \int_{5B_i} |f(y) - m_f(5B_i)|d\mu(y) + \sum_i \mu(5B_i)[|m_f(5B_i) - m_f(\widetilde{B}^\rho)| + |m_f(\widetilde{B}^\rho) - m_{\widetilde{B}^\rho}(f)|]\\
&\quad\lesssim \|f\|_{\circ,\,\rho} \sum_i \mu(5\rho B_i)[\lambda(c_{B_i}, r_{B_i})]^\alpha + \sum_i \mu(5B_i)\bigg\{\|f\|_{\circ,\,\rho}[\lambda(c_B, r_{\widetilde{B}^\rho})]^\alpha[\widetilde{K}^{(\rho),\,1/(\alpha+1)}_{5B_i,\,\widetilde{B}^\rho}]\\
&\qquad + \frac{1}{\mu(\widetilde{B}^\rho)} \int_{\widetilde{B}^\rho} |f(y) - m_f(\widetilde{B}^\rho)|d\mu(y)\bigg\}\\
&\quad\lesssim \|f\|_{\circ,\,\rho} \sum_i \mu(B_i)\{[\lambda(c_B, r_B)]^\alpha + [\lambda(c_B, r_{\widetilde{B}^\rho})]^\alpha\}\\
&\quad\lesssim \|f\|_{\circ,\,\rho} \sum_i \mu(B_i)[\lambda(c_B, r_B)]^\alpha \lesssim \|f\|_{\circ,\,\rho}\mu(\rho B)[\lambda(c_B, r_B)]^\alpha.
\end{aligned}$$

On the other hand, for all (ρ, β_ρ)-doubling balls $B \subset S$, by (2.1) and (2.2), we have

$$\begin{aligned}
&|m_B(f) - m_S(f)|\\
&\quad\leqslant |m_B(f) - m_f(B)| + |m_f(B) - m_f(S)| + |m_f(S) - m_S(f)|\\
&\quad\lesssim \{[\lambda(c_B, r_B)]^\alpha + [\lambda(c_S, r_S)]^\alpha \widetilde{K}^{(\rho),\,1/(\alpha+1)}_{B,\,S} + [\lambda(c_S, r_S)]^\alpha\}\|f\|_{\circ,\,\rho}\\
&\quad\lesssim \|f\|_{\circ,\,\rho}[\lambda(c_S, r_S)]^\alpha \widetilde{K}^{(\rho),\,1/(\alpha+1)}_{B,\,S}.
\end{aligned}$$

These two inequalities show that $\|f\|_{\mathcal{E}^\alpha_\rho(\mu)} \lesssim \|f\|_{\circ,\,\rho}$, which completes the proof of Proposition 6.18. □

We point out that it is still unclear whether the range of ρ in Proposition 6.18 is sharp or not.

7 Atomic Hardy spaces $\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ and molecular Hardy spaces $\widehat{H}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)$

In this section, under the assumption of ρ-weakly doubling conditions, we introduce the atomic Hardy space $\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ and the molecular Hardy space $\widehat{H}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)$, and show that the spaces $\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ and $\widehat{H}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)$ coincide with equivalent quasi-norms.

Definition 7.1. Let $\rho \in (1, \infty)$, $0 < p \leqslant 1 \leqslant q \leqslant \infty$, $p \neq q$ and $\gamma \in [1, \infty)$. A function $b \in L^1(\mu)$ is called a $(p, q, \gamma, \rho)_{\lambda,\,1}$-*atomic block* if b satisfies Definitions 3.2(i)–3.2(iii). Moreover, let

$$|b|_{\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)} := |\lambda_1| + |\lambda_2|.$$

Remark 7.2. It is easy to see that any $(1, q, \gamma, \rho)_{\lambda,\,1}$-atomic block is also a $(1, q, \gamma, \rho)_\lambda$-atomic block and vice versa. We point out that the difference between the $(p, q, \gamma, \rho)_\lambda$-atomic block and the $(p, q, \gamma, \rho)_{\lambda,\,1}$-atomic blocks exists in that the former is an $L^2(\mu)$ function when $p \in (0, 1)$, while the latter is only an $L^1(\mu)$ function.

Observe that, for any $(p, q, \gamma, \rho)_{\lambda,\,1}$-atomic block b, there exist some balls B_j ($j \in \{1, 2\}$) and B, and some numbers $\lambda_j \in \mathbb{C}$ ($j \in \{1, 2\}$) such that $\operatorname{supp}(b) \subset B$, $b = \lambda_1 a_1 + \lambda_2 a_2$ and $\operatorname{supp}(a_j) \subset B_j \subset B$, $j \in \{1, 2\}$. By $\int_{\mathcal{X}} b(x)d\mu(x) = 0$, Proposition 6.7(a), Definition 3.2(iii), (2.1) and (2.2), we know that

$$\bigg|\int_{\mathcal{X}} f(x)b(x)d\mu(x)\bigg| = \bigg|\int_{\mathcal{X}} [f(x) - m_{\widetilde{B}^\rho}(f)]b(x)d\mu(x)\bigg|$$

$$
\begin{aligned}
&\leqslant \sum_{j=1}^{2}|\lambda_j|\int_{B_j}|f(x)-m_{\widetilde{B}^\rho}(f)||a_j(x)|d\mu(x)\\
&\leqslant \sum_{j=1}^{2}|\lambda_j|\left[\int_{B_j}|a_j(x)|^q d\mu(x)\right]^{1/q}\left[\int_{B_j}|f(x)-m_{\widetilde{B}^\rho}(f)|^{q'}d\mu(x)\right]^{1/q'}\\
&\leqslant \sum_{j=1}^{2}|\lambda_j|[\mu(\rho B_j)]^{1/q-1}[\lambda(c_B,r_B)]^{1-1/p}[\widetilde{K}^{(\rho),\,p}_{B_j,\,B}]^{-\gamma}\\
&\quad\times\left\{\left[\int_{B_j}|f(x)-m_{\widetilde{B}_j^\rho}(f)|^{q'}d\mu(x)\right]^{1/q'}+[\mu(B_j)]^{1/q'}|m_{\widetilde{B}_j^\rho}(f)-m_{\widetilde{B}^\rho}(f)|\right\}\\
&\lesssim \sum_{j=1}^{2}|\lambda_j|[\mu(\rho B_j)]^{1/q-1}[\lambda(c_B,r_B)]^{1-1/p}[\widetilde{K}^{(\rho),\,p}_{B_j,\,B}]^{-\gamma}\\
&\quad\times\{[\mu(\rho B_j)]^{1/q'}+[\mu(B_j)]^{1/q'}[\widetilde{K}^{(\rho),\,p}_{B_j,\,B}]^{\gamma}\}[\lambda(c_B,r_B)]^{1/p-1}\|f\|_{\mathcal{E}_\rho^{1/p-1}(\mu)}\\
&\lesssim \sum_{j=1}^{2}|\lambda_j|\|f\|_{\mathcal{E}_\rho^{1/p-1}(\mu)}\sim|b|_{\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)}\|f\|_{\mathcal{E}_\rho^{1/p-1}(\mu)}. \qquad (7.1)
\end{aligned}
$$

Thus, a $(p,q,\gamma,\rho)_{\lambda,\,1}$-atomic block b can be seen as an element in the *dual space* $(\mathcal{E}_\rho^{1/p-1}(\mu))^*$ of $\mathcal{E}_\rho^{1/p-1}(\mu)$.

Definition 7.3. Let $0<p\leqslant 1\leqslant q\leqslant\infty$ and $p\neq q$. The *atomic Hardy space* $\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ is defined as the subspace of $(\mathcal{E}_\rho^{1/p-1}(\mu))^*$ when $p<1$ and of $L^1(\mu)$ when $p=1$, consisting of those linear functional admitting an atomic decomposition

$$f=\sum_{i=1}^{\infty}b_i \qquad (7.2)$$

in $(\mathcal{E}_\rho^{1/p-1}(\mu))^*$ when $p<1$ and in $L^1(\mu)$ when $p=1$, where $\{b_i\}_{i=1}^\infty$ are $(p,q,\gamma,\rho)_{\lambda,\,1}$-atomic blocks such that $\sum_{i=1}^\infty|b_i|^p_{\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)}<\infty$. Moreover, define

$$\|f\|_{\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)}:=\inf\left\{\left[\sum_{i=1}^{\infty}|b_i|^p_{\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)}\right]^{1/p}\right\},$$

where the infimum is taken over all possible decompositions of f as above.

Remark 7.4. (i) It follows from Remark 7.2 that $\widehat{H}^{1,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ is the atomic Hardy space defined via the discrete coefficients $\widetilde{K}^{(\rho)}_{B,\,S}$ introduced in [13], where it was shown that $\widehat{H}^{1,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ is independent of the choices of q, ρ and γ. Hereafter, $\widehat{H}^{1,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ is simply denoted by $\widehat{H}^1_{\mathrm{atb}}(\mu)$.

(ii) Let $\rho\in(1,\infty)$, $\gamma\in[1,\infty)$ and $q\in(1,\infty)$. By Remarks 3.3(ii) and 7.4(i), we know that $\widehat{H}^{1,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)=\widetilde{H}^{1,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ over general non-homogeneous metric measure spaces.

(iii) Fix p, ρ and γ as in Definition 7.1. For $1\leqslant q_1\leqslant q_2\leqslant\infty$ and $q_1>p$, we notice that $\widehat{H}^{p,\,q_2,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)\subset\widehat{H}^{p,\,q_1,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$.

(iv) By the results in [13], we know that the Calderón-Zygmund operator is bounded on $\widehat{H}^1_{\mathrm{atb}}(\mu)$. However, when $p\in(0,1)$, it is still unclear whether the Calderón-Zygmund operator is bounded on $\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ or not.

We now introduce the notion of the molecular Hardy space $\widehat{H}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)$ in the non-homogeneous setting by first presenting the following notion of $(p,q,\gamma,\epsilon,\rho)_{\lambda,\,1}$-molecular blocks.

Definition 7.5. Let $\rho\in(1,\infty)$, $0<p\leqslant 1\leqslant q\leqslant\infty$, $p\neq q$, $\gamma\in[1,\infty)$ and $\epsilon\in(0,\infty)$. A function $b\in L^1(\mu)$ is called a $(p,q,\gamma,\epsilon,\rho)_{\lambda,\,1}$*-molecular block* if

(i) $\int_{\mathcal{X}}b(x)d\mu(x)=0$;

(ii) there exist some ball B and some constants $\widetilde{M}$, $M\in\mathbb{N}$ such that, for all $k\in\mathbb{Z}_+$ and $j\in\{1,\dots,M_k\}$ with $M_k:=\widetilde{M}$ if $k=0$ and $M_k:=M$ if $k\in\mathbb{N}$, there exist functions $m_{k,\,j}$ supported on some balls

$B_{k,j} \subset U_k(B)$ for all $k \in \mathbb{Z}_+$, where $U_0(B) := \rho^2 B$ and $U_k(B) := \rho^{k+2}B \backslash \rho^{k-2}B$ with $k \in \mathbb{N}$, and $\lambda_{k,j} \in \mathbb{C}$ such that $b = \sum_{k=0}^{\infty}\sum_{j=1}^{M_k} \lambda_{k,j} m_{k,j}$ in $L^1(\mu)$ when $p = 1$ and in both $L^1(\mu)$ and $(\mathcal{E}_\rho^{1/p-1}(\mu))^*$ when $p \in (0,1)$,

$$\|m_{k,j}\|_{L^q(\mu)} \leqslant \rho^{-k\epsilon}[\mu(\rho B_{k,j})]^{1/q-1}[\lambda(c_B, \rho^{k+2}r_B)]^{1-1/p}[\widetilde{K}^{(\rho),p}_{B_{k,j},\rho^{k+2}B}]^{-\gamma} \tag{7.3}$$

and

$$|b|^p_{\widehat{H}^{p,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)} := \sum_{k=0}^{\infty}\sum_{j=1}^{M_k} |\lambda_{k,j}|^p < \infty.$$

Remark 7.6. Observe that any $(1,q,\gamma,\epsilon,\rho)_{\lambda,1}$-molecular block is also a $(1,q,\gamma,\epsilon,\rho)_\lambda$-molecular block and vice versa.

Definition 7.7. Let $0 < p \leqslant 1 \leqslant q \leqslant \infty$, $p \neq q$ and $\epsilon \in (0,\infty)$. The *molecular Hardy space* $\widehat{H}^{p,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)$ is defined as the subspace of $(\mathcal{E}_\rho^{1/p-1}(\mu))^*$ when $p < 1$ and of $L^1(\mu)$ when $p = 1$, consisting of those linear functional admitting a molecular decomposition

$$f = \sum_{i=1}^{\infty} b_i \tag{7.4}$$

in $(\mathcal{E}_\rho^{1/p-1}(\mu))^*$ when $p < 1$ and in $L^1(\mu)$ when $p = 1$, where $\{b_i\}_{i=1}^{\infty}$ are $(p,q,\gamma,\epsilon,\rho)_{\lambda,1}$-molecular blocks such that $\sum_{i=1}^{\infty} |b_i|^p_{\widehat{H}^{p,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)} < \infty$. Moreover, define

$$\|f\|_{\widehat{H}^{p,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)} := \inf\left\{\left[\sum_{i=1}^{\infty} |b_i|^p_{\widehat{H}^{p,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)}\right]^{1/p}\right\},$$

where the infimum is taken over all possible decompositions of f as above.

Remark 7.8. (i) It follows from Remark 7.6 that $H^{1,q,\gamma}_{\mathrm{atb},\rho}(\mu)$ is the molecular Hardy space defined via the discrete coefficients $\widetilde{K}^{(\rho)}_{B,S}$ introduced in [13].

(ii) Let ρ, p, q, γ and ϵ be as in Definition 7.5. Then each $(p,q,\gamma,\rho)_{\lambda,1}$-atomic block is a $(p,q,\gamma,\epsilon,\rho)_{\lambda,1}$-molecular block and hence $\widehat{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu) \subset \widehat{H}^{p,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)$ and, for all $f \in \widehat{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)$,

$$\|f\|_{\widehat{H}^{p,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)} \leqslant \|f\|_{\widehat{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)}.$$

Moreover, we have the following relation between $\widehat{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)$ and $\widehat{H}^{p,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)$.

Theorem 7.9. *Suppose that* $(\mathcal{X},d,\mu)$ *is a non-homogeneous metric measure space satisfying* (6.1). *Let* $\rho \in (1,\infty)$, $0 < p \leqslant 1 \leqslant q \leqslant \infty$, $p \neq q$, $\gamma \in [1,\infty)$ *and* $\epsilon \in (0,\infty)$. *Then* $\widehat{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)$ *and* $\widehat{H}^{p,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)$ *coincide with equivalent quasi-norms.*

Proof. When $p = 1$, the conclusion of Theorem 7.9 was obtained in [13, Theorem 1.11] without the assumption (6.1). Thus, we only need to consider the case $p \in (0,1)$. Fix $\rho \in (1,\infty)$, $\gamma \in [1,\infty)$, $\epsilon \in (0,\infty)$, $0 < p < 1 \leqslant q \leqslant \infty$ and $p \neq q$. Let $I_B^{(\rho)} := N^{(\rho)}_{B,\widetilde{B}^\rho}$ for any ball B. By Remark 7.8(ii), to show Theorem 7.9 in this case, it suffices to prove that $\widehat{H}^{p,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu) \subset \widehat{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)$ and that, for any $f \in \widehat{H}^{p,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)$, $f \in \widehat{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)$ and $\|f\|_{\widehat{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)} \lesssim \|f\|_{\widehat{H}^{p,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)}$. To this end, we first show that any $(p,q,\gamma,\epsilon,\rho)_{\lambda,1}$-molecular block b can be decomposed into a sum of some $(p,q,\gamma,\rho)_{\lambda,1}$-atomic blocks and $(p,\infty,\gamma,\rho)_{\lambda,1}$-atomic blocks and $\|b\|_{\widehat{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)} \lesssim |b|_{\widehat{H}^{p,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)}$.

For any $(p,q,\gamma,\epsilon,\rho)_{\lambda,1}$-molecular block b, by Definition 7.5, we know that

$$b = \sum_{k=0}^{\infty}\sum_{j=1}^{M_k} \lambda_{k,j} m_{k,j} \quad \text{in } L^1(\mu) \text{ and } (\mathcal{E}_\rho^{1/p-1}(\mu))^*, \tag{7.5}$$

where, for any $k \in \mathbb{Z}_+$ and $j \in \{1, \ldots, M_k\}$, $\lambda_{k,j} \in \mathbb{C}$ and $\operatorname{supp}(m_{k,j}) \subset B_{k,j} \subset U_k(B)$ with the same notation as in Definition 7.5. Moreover, observe that, by (7.5), Hölder's inequality and (7.3), we have

$$\sum_{k=0}^{\infty}\sum_{j=1}^{M_k}\|\lambda_{k,j}m_{k,j}\|_{L^1(\mu)} \leqslant \sum_{k=0}^{\infty}\sum_{j=1}^{M_k}|\lambda_{k,j}|\rho^{-k\epsilon}[\lambda(c_B,\rho^{k+2}r_B)]^{1-1/p}[\widetilde{K}^{(\rho),p}_{B_{k,j},\rho^{k+2}B}]^{-\gamma}$$
$$\lesssim \left(\sum_{k=0}^{\infty}\sum_{j=1}^{M_k}|\lambda_{k,j}|^p\right)^{1/p}[\lambda(c_B,r_B)]^{1-1/p} < \infty. \tag{7.6}$$

For each $k \in \mathbb{Z}_+$, let $b_k := \sum_{j=1}^{M_k}\lambda_{k,j}m_{k,j}$, $B^{\rho}_{k+2} := \rho^{k+2}B$ and $\widetilde{B}^{\rho}_{k+2} := (\widetilde{\rho^{k+2}B})^{2\rho}$. By (7.5), we write

$$\begin{aligned}
b &= \sum_{k=0}^{\infty}\left[b_k - \frac{\chi_{\widetilde{B}^{\rho}_{k+2}}}{\mu(\widetilde{B}^{\rho}_{k+2})}\int_{\mathcal{X}} b_k(y)d\mu(y)\right] + \sum_{k=0}^{\infty}\frac{\chi_{\widetilde{B}^{\rho}_{k+2}}}{\mu(\widetilde{B}^{\rho}_{k+2})}\int_{\mathcal{X}} b_k(y)d\mu(y)\\
&= \sum_{k=0}^{\infty}\sum_{j=1}^{M_k}\lambda_{k,j}\left[m_{k,j} - \frac{\chi_{\widetilde{B}^{\rho}_{k+2}}}{\mu(\widetilde{B}^{\rho}_{k+2})}\int_{B_{k,j}} m_{k,j}(y)d\mu(y)\right] + \sum_{k=0}^{\infty}\frac{\chi_{\widetilde{B}^{\rho}_{k+2}}}{\mu(\widetilde{B}^{\rho}_{k+2})}\int_{\mathcal{X}} b_k(y)d\mu(y)\\
&=: \sum_{k=0}^{\infty}\sum_{j=1}^{M_k} b_{k,j} + \sum_{k=0}^{\infty}\chi_k\widetilde{M}_k =: \mathrm{I} + \mathrm{II},
\end{aligned}$$

where, for all $k \in \mathbb{Z}_+$ and $j \in \{1, \ldots, M_k\}$,

$$b_{k,j} := \lambda_{k,j}\left[m_{k,j} - \frac{\chi_{\widetilde{B}^{\rho}_{k+2}}}{\mu(\widetilde{B}^{\rho}_{k+2})}\int_{B_{k,j}} m_{k,j}(y)d\mu(y)\right],$$

$\chi_k := \frac{\chi_{\widetilde{B}^{\rho}_{k+2}}}{\mu(\widetilde{B}^{\rho}_{k+2})}$ and $\widetilde{M}_k := \int_{\mathcal{X}} b_k(y)d\mu(y)$. From (7.6), it follows that $\sum_{k=0}^{\infty}\sum_{j=1}^{M_k} b_{k,j}$ and $\sum_{k=0}^{\infty}\chi_k\widetilde{M}_k$ both converge in $L^1(\mu)$.

To estimate I, we first show that, for any $k \in \mathbb{Z}_+$ and $j \in \{1, \ldots, M_k\}$, $b_{k,j}$ is a $(p, q, \gamma, \rho)_{\lambda,1}$-atomic block. Noticing that $\operatorname{supp}(b_{k,j}) \subset 2\widetilde{B}^{\rho}_{k+2}$ and $\int_{\mathcal{X}} b_{k,j}(y)d\mu(y) = 0$, to show this, it only needs to show that $b_{k,j}$ satisfies Definition 3.2(iii). To this end, we further decompose $b_{k,j}$ into

$$\begin{aligned}
b_{k,j} &= \lambda_{k,j}\left[m_{k,j} - \frac{\chi_{\operatorname{supp}(m_{k,j})}}{\mu(\widetilde{B}^{\rho}_{k+2})}\int_{B_{k,j}} m_{k,j}(y)d\mu(y)\right]\\
&\quad - \lambda_{k,j}\frac{\chi_{\widetilde{B}^{\rho}_{k+2}\setminus\operatorname{supp}(m_{k,j})}}{\mu(\widetilde{B}^{\rho}_{k+2})}\int_{B_{k,j}} m_{k,j}(y)d\mu(y) =: \mathrm{A}^{(1)}_{k,j} - \mathrm{A}^{(2)}_{k,j}.
\end{aligned}$$

By the Minkowski inequality, Hölder's inequality, (7.3), (2.1), Remark 6.2(ii), Lemmas 2.8(iv) and 2.8(iii), we know that

$$\begin{aligned}
\|\mathrm{A}^{(1)}_{k,j}\|_{L^q(\mu)} &\leqslant |\lambda_{k,j}|\left\{\|m_{k,j}\|_{L^q(\mu)} + \frac{[\mu(\operatorname{supp}(m_{k,j}))]^{1/q}}{\mu(\widetilde{B}^{\rho}_{k+2})}\left|\int_{B_{k,j}} m_{k,j}(y)d\mu(y)\right|\right\}\\
&\lesssim |\lambda_{k,j}|\left\{\|m_{k,j}\|_{L^q(\mu)} + \frac{[\mu(\operatorname{supp}(m_{k,j}))]^{1/q}[\mu(B_{k,j})]^{1/q'}}{\mu(\widetilde{B}^{\rho}_{k+2})}\|m_{k,j}\|_{L^q(\mu)}\right\}\\
&\lesssim |\lambda_{k,j}|\|m_{k,j}\|_{L^q(\mu)} \lesssim |\lambda_{k,j}|\rho^{-k\epsilon}[\mu(\rho B_{k,j})]^{1/q-1}[\lambda(c_B, r_{B^{\rho}_{k+2}})]^{1-1/p}[\widetilde{K}^{(\rho),p}_{B_{k,j},B^{\rho}_{k+2}}]^{-\gamma}\\
&\lesssim |\lambda_{k,j}|\rho^{-k\epsilon}[\mu(\rho B_{k,j})]^{1/q-1}[\lambda(c_B, r_{2\widetilde{B}^{\rho}_{k+2}})]^{1-1/p}[\widetilde{K}^{(\rho),p}_{B_{k,j},2\widetilde{B}^{\rho}_{k+2}}]^{-\gamma}.
\end{aligned}$$

Let c_5, independent of k and j, be the implicit positive constant of the above inequality, $\mu^{(1)}_{k,j} := c_5|\lambda_{k,j}|\rho^{-k\epsilon}$ and $a^{(1)}_{k,j} := \frac{1}{\mu^{(1)}_{k,j}}\mathrm{A}^{(1)}_{k,j}$. Then $\mathrm{A}^{(1)}_{k,j} = \mu^{(1)}_{k,j}a^{(1)}_{k,j}$, $\operatorname{supp}(a^{(1)}_{k,j}) \subset B_{k,j} \subset 2\widetilde{B}^{\rho}_{k+2}$ and

$$\|a^{(1)}_{k,j}\|_{L^q(\mu)} \leqslant [\mu(\rho B_{k,j})]^{1/q-1}[\lambda(c_B, r_{2\widetilde{B}^{\rho}_{k+2}})]^{1-1/p}[\widetilde{K}^{(\rho),p}_{B_{k,j},2\widetilde{B}^{\rho}_{k+2}}]^{-\gamma}.$$

From Hölder's inequality, (7.3), the fact that $\widetilde{K}^{(\rho),p}_{B_{k,j},B^{\rho}_{k+2}} \geqslant 1$, the $(2\rho,\beta_{2\rho})$-doubling property of $\widetilde{B}^{\rho}_{k+2}$, Remark 6.2(ii) and Lemma 2.8(ii), it follows that

$$\begin{aligned}\|\mathrm{A}^{(2)}_{k,j}\|_{L^q(\mu)} &= |\lambda_{k,j}|\frac{[\mu(\widetilde{B}^{\rho}_{k+2}\setminus \operatorname{supp}(m_{k,j}))]^{1/q}}{\mu(\widetilde{B}^{\rho}_{k+2})}\left|\int_{B_{k,j}} m_{k,j}(y)d\mu(y)\right| \\ &\leqslant |\lambda_{k,j}|[\mu(\widetilde{B}^{\rho}_{k+2})]^{1/q-1}[\mu(B_{k,j})]^{1/q'}\|m_{k,j}\|_{L^q(\mu)} \\ &\lesssim |\lambda_{k,j}|[\mu(\widetilde{B}^{\rho}_{k+2})]^{1/q-1}[\mu(B_{k,j})]^{1/q'}\rho^{-k\epsilon}[\mu(\rho B_{k,j})]^{1/q-1}[\lambda(c_B, r_{B^{\rho}_{k+2}})]^{1-1/p} \\ &\lesssim |\lambda_{k,j}|[\mu(2\rho\widetilde{B}^{\rho}_{k+2})]^{1/q-1}\rho^{-k\epsilon}[\lambda(c_B, r_{B^{\rho}_{k+2}})]^{1-1/p} \\ &\lesssim |\lambda_{k,j}|\rho^{-k\epsilon}[\mu(2\rho\widetilde{B}^{\rho}_{k+2})]^{1/q-1}[\lambda(c_B, r_{2\widetilde{B}^{\rho}_{k+2}})]^{1-1/p}[\widetilde{K}^{(\rho),p}_{2\widetilde{B}^{\rho}_{k+2},2\widetilde{B}^{\rho}_{k+2}}]^{-\gamma}.\end{aligned}$$

Let c_6, independent of k and j, be the implicit positive constant of the above inequality, $\mu^{(2)}_{k,j} := c_6|\lambda_{k,j}|\rho^{-k\epsilon}$ and $a^{(2)}_{k,j} := \frac{1}{\mu^{(2)}_{k,j}}\mathrm{A}^{(2)}_{k,j}$. Then $\mathrm{A}^{(2)}_{k,j} = \mu^{(2)}_{k,j}a^{(2)}_{k,j}$, $\operatorname{supp}(a^{(2)}_{k,j}) \subset 2\widetilde{B}^{\rho}_{k+2}$ and

$$\|a^{(2)}_{k,j}\|_{L^q(\mu)} \leqslant [\mu(2\rho\widetilde{B}^{\rho}_{k+2})]^{1/q-1}[\lambda(c_B, r_{2\widetilde{B}^{\rho}_{k+2}})]^{1-1/p}[\widetilde{K}^{(\rho),p}_{2\widetilde{B}^{\rho}_{k+2},2\widetilde{B}^{\rho}_{k+2}}]^{-\gamma}.$$

Thus, $b_{k,j} = \mu^{(1)}_{k,j}a^{(1)}_{k,j} + \mu^{(2)}_{k,j}a^{(2)}_{k,j}$ is a $(p,q,\gamma,\rho)_{\lambda,1}$-atomic block and

$$|b_{k,j}|_{\widehat{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)} \lesssim |\lambda_{k,j}|\rho^{-k\epsilon}.$$

Moreover, we have

$$\|\mathrm{I}\|^p_{\widehat{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)} \lesssim \sum_{k=0}^{\infty}\sum_{j=1}^{M_k}|\lambda_{k,j}|^p\rho^{-kp\epsilon} \lesssim \sum_{k=0}^{\infty}\sum_{j=1}^{M_k}|\lambda_{k,j}|^p \sim |b|^p_{\widehat{H}^{p,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)}. \tag{7.7}$$

Now we turn to estimate II. Observe that, by (7.6) and Hölder's inequality, we have

$$\sum_{k=0}^{\infty}|\widetilde{M}_k| \leqslant \sum_{k=0}^{\infty}\|b_k\|_{L^1(\mu)} < \infty.$$

For each $k \in \mathbb{Z}_+$, let $N_k := \sum_{i=k}^{\infty}\widetilde{M}_i$. From Hölder's inequality and (7.3), it follows that

$$\begin{aligned}\sum_{k=0}^{\infty}\|\chi_k N_k\|_{L^1(\mu)} &\leqslant \sum_{k=0}^{\infty}\sum_{i=k}^{\infty}\|\chi_k\widetilde{M}_i\|_{L^1(\mu)} \leqslant \sum_{k=0}^{\infty}\sum_{i=k}^{\infty}\|b_i\|_{L^1(\mu)} \\ &\leqslant \sum_{k=0}^{\infty}\sum_{i=k}^{\infty}\sum_{j=1}^{M_i}|\lambda_{i,j}|\|m_{i,j}\|_{L^q(\mu)}[\mu(B_{i,j})]^{1/q'} \\ &\leqslant \sum_{k=0}^{\infty}\sum_{i=k}^{\infty}\sum_{j=1}^{M_i}|\lambda_{i,j}|\rho^{-i\epsilon}[\lambda(c_B,\rho^{i+2}r_B)]^{1-1/p} \\ &\leqslant \sum_{k=0}^{\infty}\rho^{-k\epsilon}\sum_{i=0}^{\infty}\sum_{j=1}^{M_i}|\lambda_{i,j}|[\lambda(c_B,r_B)]^{1-1/p} \\ &\leqslant \left(\sum_{i=0}^{\infty}\sum_{j=1}^{M_i}|\lambda_{i,j}|^p\right)^{1/p}[\lambda(c_B,r_B)]^{1-1/p} < \infty.\end{aligned}$$

Similarly, $\sum_{k=0}^{\infty}\|\chi_k N_{k+1}\|_{L^1(\mu)} < \infty$. By the above facts, we have

$$\sum_{k=0}^{\infty}\chi_k\widetilde{M}_k = \sum_{k=0}^{\infty}\chi_k(N_k - N_{k+1}) = \sum_{k=0}^{\infty}(\chi_{k+1}-\chi_k)N_{k+1} + \chi_0 N_0$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} (\chi_{k+1} - \chi_k) N_{k+1} \\
&= \sum_{k=0}^{\infty} \sum_{i=k+1}^{\infty} \sum_{j=1}^{M_i} \lambda_{i,\,j} (\chi_{k+1} - \chi_k) \int_{B_{i,\,j}} m_{i,\,j}(y) d\mu(y) \\
&=: \sum_{k=0}^{\infty} \sum_{i=k+1}^{\infty} \sum_{j=1}^{M_i} b_{k,\,j,\,i},
\end{aligned}
$$

where the summation in the last equality holds in $L^1(\mu)$.

Now we prove that, for any $k \in \mathbb{Z}_+$, $i \in \{k+1, k+2, \ldots\}$ and $j \in \{1, \ldots, M_k\}$, $b_{k,\,j,\,i}$ is a $(p, \infty, \gamma, \rho)_{\lambda,1}$-atomic block. Observing that $\operatorname{supp}(b_{k,\,j,\,i}) \subset 2\widetilde{B}^{\rho}_{k+3}$ and $\int_{\mathcal{X}} b_{k,\,j,\,i}(y) d\mu(y) = 0$, we only need to show that $b_{k,\,j,\,i}$ satisfies Definition 3.2(iii).

To this end, we further write

$$
b_{k,\,j,\,i} = \lambda_{i,\,j} \chi_{k+1} \int_{B_{i,\,j}} m_{i,\,j}(y) d\mu(y) - \lambda_{i,\,j} \chi_k \int_{B_{i,\,j}} m_{i,\,j}(y) d\mu(y) =: \mathrm{A}^{(1)}_{k,\,j,\,i} - \mathrm{A}^{(2)}_{k,\,j,\,i}.
$$

From Hölder's inequality, (7.3), the fact that $\widetilde{K}^{(\rho),\,p}_{B_{i,\,j},\,B^{\rho}_{i+2}} \geqslant 1$, Remark 6.2(ii) and Lemma 2.8(ii), we deduce that

$$
\begin{aligned}
\|\mathrm{A}^{(1)}_{k,\,j,\,i}\|_{L^\infty(\mu)} &\leqslant |\lambda_{i,\,j}| \frac{[\mu(B_{i,\,j})]^{1/q'}}{\mu(\widetilde{B}^{\rho}_{k+3})} \|m_{i,\,j}\|_{L^q(\mu)} \\
&\leqslant |\lambda_{i,\,j}| \frac{[\mu(B_{i,\,j})]^{1/q'}}{\mu(\widetilde{B}^{\rho}_{k+3})} \rho^{-i\epsilon} [\mu(\rho B_{i,\,j})]^{1/q-1} [\lambda(c_{B_i}, r_{B^{\rho}_{i+2}})]^{1-1/p} [\widetilde{K}^{(\rho),\,p}_{B_{i,\,j},\,B^{\rho}_{i+2}}]^{-\gamma} \\
&\lesssim |\lambda_{i,\,j}| \rho^{-i\epsilon} [\mu(2\rho \widetilde{B}^{\rho}_{k+3})]^{-1} [\lambda(c_B, r_{B^{\rho}_{k+3}})]^{1-1/p} \\
&\lesssim |\lambda_{i,\,j}| \rho^{-i\epsilon} [\mu(2\rho \widetilde{B}^{\rho}_{k+3})]^{-1} [\lambda(c_B, r_{2\widetilde{B}^{\rho}_{k+3}})]^{1-1/p} [\widetilde{K}^{(\rho),\,p}_{2\widetilde{B}^{\rho}_{k+3},\,2\widetilde{B}^{\rho}_{k+3}}]^{-\gamma}.
\end{aligned}
$$

Let c_7, independent of k and j, be the implicit positive constant of the above inequality, $\mu^{(1)}_{k,\,j,\,i} := c_7 |\lambda_{i,\,j}| \rho^{-i\epsilon}$ and $a^{(1)}_{k,\,j,\,i} := \frac{1}{\mu^{(1)}_{k,\,j,\,i}} \mathrm{A}^{(1)}_{k,\,j,\,i}$. Then we see that $\mathrm{A}^{(1)}_{k,\,j,\,i} = \mu^{(1)}_{k,\,j,\,i} a^{(1)}_{k,\,j,\,i}$, $\operatorname{supp}(a^{(1)}_{k,\,j,\,i}) \subset 2\widetilde{B}^{\rho}_{k+3}$ and $\|a^{(1)}_{k,\,j,\,i}\|_{L^\infty(\mu)} \leqslant [\mu(2\rho \widetilde{B}^{\rho}_{k+3})]^{-1} [\lambda(c_B, r_{2\widetilde{B}^{\rho}_{k+3}})]^{1-1/p} [\widetilde{K}^{(\rho),\,p}_{2\widetilde{B}^{\rho}_{k+3},\,2\widetilde{B}^{\rho}_{k+3}}]^{-\gamma}$.

By an argument similar to that used in the estimate for $\mathrm{A}^{(1)}_{k,\,j,\,i}$, we conclude that

$$
\|\mathrm{A}^{(2)}_{k,\,j,\,i}\|_{L^\infty(\mu)} \leqslant c_8 |\lambda_{i,\,j}| \rho^{-i\epsilon} [\mu(2\rho \widetilde{B}^{\rho}_{k+2})]^{-1} [\lambda(c_B, r_{2\widetilde{B}^{\rho}_{k+3}})]^{1-1/p} [\widetilde{K}^{(\rho),\,p}_{2\widetilde{B}^{\rho}_{k+2},\,2\widetilde{B}^{\rho}_{k+3}}]^{-\gamma},
$$

where c_8 is a positive constant independent of k, j and i. Let $\mu^{(2)}_{k,\,j,\,i} := c_8 |\lambda_{i,\,j}| \rho^{-i\epsilon}$ and

$$
a^{(2)}_{k,\,j,\,i} := \frac{1}{\mu^{(2)}_{k,\,j,\,i}} \mathrm{A}^{(2)}_{k,\,j,\,i}.
$$

Then $\mathrm{A}^{(2)}_{k,\,j,\,i} = \mu^{(2)}_{k,\,j,\,i} a^{(2)}_{k,\,j,\,i}$, $\operatorname{supp}(a^{(2)}_{k,\,j,\,i}) \subset 2\widetilde{B}^{\rho}_{k+2} \subset 2\widetilde{B}^{\rho}_{k+3}$ and

$$
\|a^{(2)}_{k,\,j,\,i}\|_{L^\infty(\mu)} \leqslant [\mu(2\rho \widetilde{B}^{\rho}_{k+2})]^{-1} [\lambda(c_B, r_{2\widetilde{B}^{\rho}_{k+3}})]^{1-1/p} [\widetilde{K}^{(\rho),\,p}_{2\widetilde{B}^{\rho}_{k+2},\,2\widetilde{B}^{\rho}_{k+3}}]^{-\gamma}.
$$

Thus, $b_{k,\,j,\,i} = \mu^{(1)}_{k,\,j,\,i} a^{(1)}_{k,\,j,\,i} + \mu^{(2)}_{k,\,j,\,i} a^{(2)}_{k,\,j,\,i}$ is a $(p, \infty, \gamma, \rho)_{\lambda,\,1}$-atomic block and

$$
|b_{k,\,j,\,i}|_{\widehat{H}^{p,\,\infty,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)} \lesssim |\lambda_{i,\,j}| \rho^{-i\epsilon}.
$$

Moreover, we have

$$
\|\mathrm{II}\|^p_{H^{p,\,\infty,\,\gamma}_{\mathrm{atb}}(\mu)} \lesssim \sum_{k=0}^{\infty} \sum_{i=k+1}^{\infty} \sum_{j=1}^{M_i} |b_{k,\,j,\,i}|^p_{\widehat{H}^{p,\,\infty,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)}
$$

$$\lesssim \sum_{k=0}^{\infty}\sum_{i=k+1}^{\infty}\rho^{-ip\epsilon}\sum_{j=1}^{M_i}|\lambda_{i,j}|^p \sim \sum_{i=1}^{\infty}\rho^{-ip\epsilon}\sum_{k=0}^{i-1}\sum_{j=1}^{M_i}|\lambda_{i,j}|^p$$
$$\lesssim \sum_{i=1}^{\infty}\sum_{j=1}^{M_i}|\lambda_{i,j}|^p \sim |b|^p_{\widehat{H}^{p,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)}.$$

From this fact and (7.7), we deduce that both $\sum_{k=0}^{\infty}\sum_{j=1}^{M_k} b_{k,j}$ and $\sum_{k=0}^{\infty}\sum_{i=k+1}^{\infty}\sum_{j=1}^{M_i} b_{k,j,i}$ converge in $(\mathcal{E}^{1/p-1}_{\rho}(\mu))^*$.

Now we claim that

$$b=\sum_{k=0}^{\infty}\sum_{j=1}^{M_k} b_{k,j}+\sum_{k=0}^{\infty}\sum_{i=k+1}^{\infty}\sum_{j=1}^{M_i} b_{k,j,i} \quad \text{in} \quad (\mathcal{E}^{1/p-1}_{\rho}(\mu))^*.$$

Indeed, by $b=\sum_{k=0}^{\infty}\sum_{j=1}^{M_k}\lambda_{k,j}m_{k,j}$ in $(\mathcal{E}^{1/p-1}_{\rho}(\mu))^*$, we see that, for any $g\in \mathcal{E}^{1/p-1}_{\rho}(\mu)$,

$$\begin{aligned}\int_{\mathcal{X}} b(x)g(x)d\mu(x) &= \lim_{K\to\infty}\sum_{k=0}^{K}\sum_{j=1}^{M_k}\int_{\mathcal{X}}\lambda_{k,j}m_{k,j}(x)g(x)d\mu(x)\\ &= \lim_{K\to\infty}\sum_{k=0}^{K}\sum_{j=1}^{M_k}\int_{\mathcal{X}} b_{k,j}(x)g(x)d\mu(x)+\lim_{K\to\infty}\sum_{k=0}^{K}\widetilde{M}_k\int_{\mathcal{X}}\chi_k(x)g(x)d\mu(x).\end{aligned}$$

Moreover, by the fact that $|N_k|<\infty$ for any $k\in\mathbb{Z}_+$ and $N_0=0$, we further write

$$\begin{aligned}&\lim_{K\to\infty}\sum_{k=0}^{K}\widetilde{M}_k\int_{\mathcal{X}}\chi_k(x)g(x)d\mu(x)\\ &=\lim_{K\to\infty}\int_{\mathcal{X}}\sum_{k=0}^{K}(N_k-N_{k+1})\chi_k(x)g(x)d\mu(x)\\ &=\lim_{K\to\infty}\int_{\mathcal{X}}\left[\chi_0(x)N_0-\chi_K(x)N_{K+1}+\sum_{k=1}^{K}N_k\chi_k(x)-\sum_{k=0}^{K-1}N_{k+1}\chi_k(x)\right]g(x)d\mu(x)\\ &=-\lim_{K\to\infty}\int_{\mathcal{X}}\chi_K(x)N_{K+1}g(x)d\mu(x)+\lim_{K\to\infty}\int_{\mathcal{X}}\sum_{k=0}^{K-1}N_{k+1}[\chi_{k+1}(x)-\chi_k(x)]g(x)d\mu(x)\\ &=: \mathrm{A}+\lim_{K\to\infty}\sum_{k=0}^{K-1}\sum_{i=K+1}^{\infty}\widetilde{M}_i\int_{\mathcal{X}}[\chi_{k+1}(x)-\chi_k(x)]g(x)d\mu(x)\\ &=\mathrm{A}+\lim_{K\to\infty}\sum_{k=0}^{K-1}\sum_{i=K+1}^{\infty}\sum_{j=1}^{M_i}\int_{\mathcal{X}} b_{k,j,i}(x)g(x)d\mu(x).\end{aligned}$$

Thus, to prove the above claim, it suffices to show that $\mathrm{A}=0$. To this end, by Hölder's inequality and (7.3), we conclude that, for any $K\in\mathbb{N}$,

$$\begin{aligned}|N_{K+1}| &\leqslant \sum_{i=K+1}^{\infty}|\widetilde{M}_i| \leqslant \sum_{i=K+1}^{\infty}\int_{\mathcal{X}}|b_i(y)|d\mu(y)\\ &\leqslant \sum_{i=K+1}^{\infty}\sum_{j=1}^{M_i}|\lambda_{i,j}|\int_{B_{i,j}}|m_{i,j}(y)|d\mu(y)\\ &\leqslant \sum_{i=K+1}^{\infty}\sum_{j=1}^{M_i}|\lambda_{i,j}|[\mu(B_{i,j})]^{1/q'}\|m_{i,j}\|_{L^q(\mu)}\\ &\leqslant \sum_{i=K+1}^{\infty}\sum_{j=1}^{M_i}|\lambda_{i,j}|[\mu(B_{i,j})]^{1/q'}\rho^{-i\epsilon}[\mu(\rho B_{i,j})]^{1/q-1}[\lambda(c_B,\rho^{i+2}r_B)]^{1-1/p}\end{aligned}$$

$$\leqslant \sum_{i=K+1}^{\infty}\sum_{j=1}^{M_i}|\lambda_{i,j}|\rho^{-K\epsilon}[\lambda(c_B,\rho^{K+2}r_B)]^{1-1/p}$$
$$\leqslant \rho^{-K\epsilon}[\lambda(c_B,\rho^{K+2}r_B)]^{1-1/p}\left(\sum_{i=K+1}^{\infty}\sum_{j=1}^{M_i}|\lambda_{i,j}|^p\right)^{1/p}. \tag{7.8}$$

We write

$$\begin{aligned}\left|\int_{\mathcal{X}}\chi_K(x)N_{K+1}g(x)d\mu(x)\right| &= |N_{K+1}||m_{\widetilde{B}_K^\rho}(g)|\\ &\leqslant |N_{K+1}||m_{\widetilde{B}_K^\rho}(g)-m_{\widetilde{B}_0^\rho}(g)|+|N_{K+1}||m_{\widetilde{B}_0^\rho}(g)|\\ &=: \mathrm{I}_K+\mathrm{II}_K.\end{aligned}$$

By (7.8), $g\in L^1_{\rm loc}(\mu)$ and $\sum_{k=0}^{\infty}\sum_{j=1}^{M_k}|\lambda_{k,j}|^p<\infty$, we know that

$$\mathrm{II}_K\leqslant |m_{\widetilde{B}_0^\rho}(g)|\rho^{-K\epsilon}[\lambda(c_B,r_B)]^{1-1/p}\left(\sum_{i=K+1}^{\infty}\sum_{j=1}^{M_i}|\lambda_{i,j}|^p\right)^{1/p}\to 0\quad\text{as}\quad K\to\infty.$$

From Proposition 6.7(a), $g\in\mathcal{E}_\rho^{1/p-1}(\mu)$, (7.8) and $\sum_{k=0}^{\infty}\sum_{j=1}^{M_k}|\lambda_{k,j}|^p<\infty$, we further deduce that

$$\begin{aligned}\mathrm{I}_K&\leqslant |N_{K+1}|[\widetilde{K}^{(\rho),p}_{\rho^2B,\rho^{K+2}B}]^\gamma[\lambda(c_B,\rho^{K+2}r_B)]^{1/p-1}\|g\|_{\mathcal{E}_\rho^{1/p-1}(\mu)}\\ &\lesssim |N_{K+1}|K^{\gamma/p}[\lambda(c_B,\rho^{K+2}r_B)]^{1/p-1}\|g\|_{\mathcal{E}_\rho^{1/p-1}(\mu)}\\ &\lesssim \rho^{-K\epsilon}K^{\gamma/p}\left(\sum_{i=K+1}^{\infty}\sum_{j=1}^{M_i}|\lambda_{i,j}|^p\right)^{1/p}\|g\|_{\mathcal{E}_\rho^{1/p-1}(\mu)}\to 0\quad\text{as}\quad K\to\infty,\end{aligned}$$

which, together with the estimate of I_K, completes the proof of the above claim.

By Remark 7.4(ii) and the estimates for I and II, we see that $b\in\widehat{H}^{p,q,\gamma}_{\rm atb,\rho}(\mu)$ and

$$\|b\|^p_{\widehat{H}^{p,q,\gamma}_{\rm atb,\rho}(\mu)}\leqslant\|\mathrm{I}\|^p_{\widehat{H}^{p,q,\gamma}_{\rm atb,\rho}(\mu)}+\|\mathrm{II}\|^p_{\widehat{H}^{p,q,\gamma}_{\rm atb,\rho}(\mu)}\lesssim\|\mathrm{I}\|^p_{\widehat{H}^{p,q,\gamma}_{\rm atb,\rho}(\mu)}+\|\mathrm{II}\|^p_{H^{p,\infty,\gamma}_{\rm atb}(\mu)}\lesssim|b|^p_{\widehat{H}^{p,q,\gamma,\epsilon}_{\rm mb,\rho}(\mu)},$$

which, together with some standard arguments, then completes the proof of Theorem 7.9. □

Remark 7.10. (i) As was pointed out in the proof of Theorem 7.9, if $\rho\in(1,\infty)$, $q\in(1,\infty]$, $\gamma\in[1,\infty)$ and $\epsilon\in(0,\infty)$, then $\widehat{H}^{1,q,\gamma}_{\rm atb,\rho}(\mu)$ and $\widehat{H}^{1,q,\gamma,\epsilon}_{\rm mb,\rho}(\mu)$ coincide with equivalent norms, which is just [13, Theorem 1.11]; namely, in this case, the assumption (6.1) is superfluous. However, when $p\in(0,1)$, without (6.1), it is still unclear whether Theorem 7.9 holds true or not.

(ii) By Theorem 7.9, we see that $\widehat{H}^{p,q,\gamma,\epsilon}_{\rm mb,\rho}(\mu)$ is independent of the choice of ϵ under the assumption (6.1).

The following result is an easy consequence of Theorem 7.9 and Remark 6.2(i), the details being omitted.

Corollary 7.11. *Let $(\mathcal{X},d,\mu)$ be a space of homogeneous type with the dominating function*

$$\lambda(x,r):=\mu(B(x,r))\quad\text{for all } x\in\mathcal{X}\text{ and } r\in(0,\infty),$$

and ρ, p, q, γ and ϵ be as in Theorem 7.9. Then the conclusions in Theorem 7.9 and Remark 7.10 also hold true in this setting.

8 Duality between $\widehat{H}^{p,q,\gamma}_{\rm atb,\rho}(\mu)$ and $\mathcal{E}_\rho^{1/p-1}(\mu)$

In this section, we show that $\mathcal{E}_\rho^{1/p-1}(\mu)$ is the dual space of $\widehat{H}^{p,q,\gamma}_{\rm atb,\rho}(\mu)$. To this end, assuming that $(\mathcal{X},d,\mu)$ satisfies the assumption (6.1), we show that $\widehat{H}^{p,q,\gamma}_{\rm atb,\rho}(\mu)$ is independent of the choices of ρ and γ. We point out that *all conclusions in this section hold true for the case $p=1$ without the assumption* (6.1); see [13,30] for the details. Thus, we mainly focus on $p\in(0,1)$ in this section.

Proposition 8.1. *Suppose that $(\mathcal{X}, d, \mu)$ is a non-homogeneous metric measure space satisfying (6.1). Let $\rho \in (1, \infty)$, $0 < p < 1 \leqslant q \leqslant \infty$ and $\gamma \in [1, \infty)$. Then the space $\widehat{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)$ is independent of the choice of $\rho \in (1, \infty)$.*

Proof. Let $0 < p < 1 \leqslant q \leqslant \infty$ and $\gamma \in [1, \infty)$. Assume that $\rho \geqslant \rho_1 > \rho_2 > 1$. It is easy to see that $\widehat{H}^{p,q,\gamma}_{\mathrm{atb},\rho_1}(\mu) \subset \widehat{H}^{p,q,\gamma}_{\mathrm{atb},\rho_2}(\mu)$ and, for all $f \in \widehat{H}^{p,q,\gamma}_{\mathrm{atb},\rho_1}(\mu)$,

$$\|f\|^p_{\widehat{H}^{p,q,\gamma}_{\mathrm{atb},\rho_2}(\mu)} \leqslant \|f\|^p_{\widehat{H}^{p,q,\gamma}_{\mathrm{atb},\rho_1}(\mu)}.$$

On the other hand, to show that $\widehat{H}^{p,q,\gamma}_{\mathrm{atb},\rho_2}(\mu) \subset \widehat{H}^{p,q,\gamma}_{\mathrm{atb},\rho_1}(\mu)$, let

$$b = \sum_{j=1}^{2} \lambda_j a_j \in \widehat{H}^{p,q,\gamma}_{\mathrm{atb},\rho_2}(\mu)$$

be a $(p, q, \gamma, \rho_2)_{\lambda,1}$-atomic block, where, for any $j \in \{1, 2\}$, a_j is a function supported on $B_j \subset B$ for some balls B_j and B as in Definition 7.1.

Now we claim that, without loss of generality, we may assume that B is (ρ^2, β_{ρ^2})-doubling. The reasons are as follows: If B is non-(ρ^2, β_{ρ^2})-doubling, by Lemmas 2.9, 2.8(ii) and 2.8(iv), (2.1) and Remark 6.2(ii), we see that

$$\begin{aligned}\|a_j\|_{L^q(\mu)} &\leqslant [\mu(\rho_2 B_j)]^{1/q-1}[\lambda(c_B, r_B)]^{1-1/p}[\widetilde{K}^{(\rho_2),p}_{B_j,B}]^{-\gamma}\\ &\lesssim [\mu(\rho_2 B_j)]^{1/q-1}[\lambda(c_B, \rho^{2N^{(\rho^2)}_{B,\widetilde{B}^{\rho^2}}} r_B)]^{1-1/p}[\widetilde{K}^{(\rho_2),p}_{B_j,\widetilde{B}^{\rho^2}}]^{-\gamma}\\ &\lesssim [\mu(\rho_2 B_j)]^{1/q-1}[\lambda(c_B, r_{\widetilde{B}^{\rho^2}})]^{1-1/p}[\widetilde{K}^{(\rho_2),p}_{B_j,\widetilde{B}^{\rho^2}}]^{-\gamma}.\end{aligned}$$

Thus, we can replace B by $\widetilde{B}^{\rho^2}$, which shows the claim.

Then, for each $j \in \{1, 2\}$, we have

$$\|a_j\|_{L^q(\mu)} \leqslant [\mu(\rho_2 B_j)]^{1/q-1}[\lambda(c_B, r_B)]^{1-1/p}[\widetilde{K}^{(\rho_2),p}_{B_j,B}]^{-\gamma}. \tag{8.1}$$

From Remark 2.2(ii), it follows that there exists a sequence $\{B_{k,j}\}_{k=1}^N$ of balls such that

$$B_j \subset \bigcup_{k=1}^{N} B_{k,j} := \bigcup_{k=1}^{N} B\left(c_{B_{k,j}}, \frac{\rho_2 - 1}{10\rho_0(\rho_1 + 1)} r_{B_j}\right)$$

and $c_{B_{k,j}} \in B_j$ for all $k \in \{1, \dots, N\}$, where $\rho_0 \in (1, \rho_1)$. Observe that $\rho_1\rho_0 B_{k,j} \subset \rho_2 B_j$. For any $k \in \{1, \dots, N\}$, define $a_{k,j} := a_j \frac{\chi_{B_{k,j}}}{\sum_{k=1}^N \chi_{B_{k,j}}}$ and $\lambda_{k,j} := \lambda_j$. Then we have

$$\mathrm{supp}\,(a_{k,j}) \subset \rho_0 B_{k,j} \quad \text{and} \quad b = \sum_{j=1}^{2} \lambda_j a_j = \sum_{j=1}^{2}\sum_{k=1}^{N} \lambda_{k,j} a_{k,j}.$$

Moreover, by (8.1), the fact that $\rho_2 B_j \subset 3\rho B$, (2.1), Lemmas 2.9, 2.8(i), 2.8(ii), 2.8(iv) and 2.8(v), and the fact that $\rho_0 B_{k,j} \subset \rho B$, we know that

$$\begin{aligned}\|a_{k,j}\|_{L^q(\mu)} \leqslant \|a_j\|_{L^q(\mu)} &\leqslant [\mu(\rho_2 B_j)]^{1/q-1}[\lambda(c_B, r_B)]^{1-1/p}[\widetilde{K}^{(\rho_2),p}_{B_j,B}]^{-\gamma}\\ &\lesssim [\mu(\rho_1\rho_0 B_{k,j})]^{1/q-1}[\lambda(c_B, \rho r_B)]^{1-1/p}[\widetilde{K}^{(\rho_1),p}_{B_j,3\rho B}]^{-\gamma}\\ &\lesssim [\mu(\rho_1\rho_0 B_{k,j})]^{1/q-1}[\lambda(c_B, \rho r_B)]^{1-1/p}[\widetilde{K}^{(\rho_1),p}_{\rho_0 B_{k,j},\rho B}]^{-\gamma}.\end{aligned} \tag{8.2}$$

Let $C_{k,j} := \lambda_{k,j}(a_{k,j} + \gamma_{k,j}\chi_B)$, where $\gamma_{k,j} := -\frac{1}{\mu(B)}\int_{\mathcal{X}} a_{k,j}(x)d\mu(x)$. Now we claim that $C_{k,j}$ is a $(p, q, \gamma, \rho_1)_{\lambda,1}$-atomic block. Indeed, $\mathrm{supp}\,(C_{k,j}) \subset \rho B$ and $\int_{\mathcal{X}} C_{k,j}(x)d\mu(x) = 0$. Moreover, since

$B_{k,j} \subset \rho B$, Hölder's inequality, (8.2), B is (ρ^2, β_{ρ^2})-doubling, $\rho > \rho_1$, (2.1) and Lemma 2.8(ii), we conclude that

$$\begin{aligned}\|\gamma_{k,j}\chi_B\|_{L^q(\mu)} &\leqslant [\mu(B)]^{1/q-1}\|a_{k,j}\|_{L^q(\mu)}[\mu(B_{k,j})]^{1-1/q} \\ &\lesssim [\mu(\rho_1\rho B)]^{1/q-1}[\lambda(c_B,\rho r_B)]^{1-1/p}[\widetilde{K}^{(\rho_1),p}_{B_{k,j},\rho B}]^{-\gamma} \\ &\lesssim [\mu(\rho_1\rho B)]^{1/q-1}[\lambda(c_B,\rho r_B)]^{1-1/p}[\widetilde{K}^{(\rho_1),p}_{\rho B,\rho B}]^{-\gamma}.\end{aligned}$$

This, together with (8.2), (2.1), Lemma 2.8(ii), implies that $|C_{k,j}|_{\widehat{H}^{p,q,\gamma}_{\mathrm{atb},\rho_1}(\mu)} \lesssim |\lambda_{k,j}|$. Thus, the claim holds true.

By the above claim and $\int_{\mathcal{X}} b(x)d\mu(x) = 0$, we see that

$$b = \sum_{j=1}^{2}\sum_{k=1}^{N} C_{k,j} \in \widehat{H}^{p,q,\gamma}_{\mathrm{atb},\rho_1}(\mu) \tag{8.3}$$

and

$$\|b\|^p_{\widehat{H}^{p,q,\gamma}_{\mathrm{atb},\rho_1}(\mu)} \lesssim \sum_{j=1}^{2}\sum_{k=1}^{N} |C_{k,j}|^p_{\widehat{H}^{p,q,\gamma}_{\mathrm{atb},\rho_1}(\mu)} \lesssim \sum_{j=1}^{2}|\lambda_j|^p \sim |b|^p_{\widehat{H}^{p,q,\gamma}_{\mathrm{atb},\rho_2}(\mu)}. \tag{8.4}$$

For all $f \in \widehat{H}^{p,q,\gamma}_{\mathrm{atb},\rho_2}(\mu)$, by Proposition 6.8, we know that there exists a sequence $\{b_i\}_i$ of $(p,q,\gamma,\rho_2)_{\lambda,1}$-atomic blocks such that $f = \sum_{i=1}^{\infty} b_i$ in $(\mathcal{E}^{1/p-1}_{\rho_2}(\mu))^* = (\mathcal{E}^{1/p-1}_{\rho_1}(\mu))^*$ and

$$\sum_{i=1}^{\infty}|b_i|^p_{\widehat{H}^{p,q,\gamma}_{\mathrm{atb},\rho_2}(\mu)} \lesssim \|f\|^p_{\widehat{H}^{p,q,\gamma}_{\mathrm{atb},\rho_2}(\mu)}.$$

From this fact, (8.3) and (8.4), we further deduce that $f = \sum_{i=1}^{\infty}\sum_{j=1}^{2}\sum_{k=1}^{N} C^i_{k,j}$ in $(\mathcal{E}^{1/p-1}_{\rho_1}(\mu))^*$, where $\{C^i_{k,j}\}_{i,j,k}$ are all $(p,q,\gamma,\rho_1)_{\lambda,1}$-atomic blocks as in (8.3) satisfying

$$\sum_{i=1}^{\infty}\sum_{j=1}^{2}\sum_{k=1}^{N}|C^i_{k,j}|^p_{\widehat{H}^{p,q,\gamma}_{\mathrm{atb},\rho_1}(\mu)} \lesssim \sum_{i=1}^{\infty}|b_i|^p_{\widehat{H}^{p,q,\gamma}_{\mathrm{atb},\rho_2}(\mu)} \lesssim \|f\|^p_{\widehat{H}^{p,q,\gamma}_{\mathrm{atb},\rho_2}(\mu)},$$

which implies that $f \in \widehat{H}^{p,q,\gamma}_{\mathrm{atb},\rho_1}(\mu)$ and

$$\|f\|_{\widehat{H}^{p,q,\gamma}_{\mathrm{atb},\rho_1}(\mu)} \lesssim \|f\|_{\widehat{H}^{p,q,\gamma}_{\mathrm{atb},\rho_2}(\mu)}.$$

This finishes the proof of Proposition 8.1. □

Proposition 8.2. *Let $\rho \in (1,\infty)$, $0 < p < 1 \leqslant q \leqslant \infty$ and $\gamma \in [1,\infty)$. Then the space $\widehat{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)$ is independent of the choice of $\gamma \in [1,\infty)$.*

Proof. Assume that $1 \leqslant \gamma_1 < \gamma_2$. Notice that $[\widetilde{K}^{(\rho),p}_{B,S}]^{-\gamma_2} \leqslant [\widetilde{K}^{(\rho),p}_{B,S}]^{-\gamma_1}$ for all balls $B \subset S$. From this, we deduce that $\widehat{H}^{p,q,\gamma_2}_{\mathrm{atb},\rho}(\mu) \subset \widehat{H}^{p,q,\gamma_1}_{\mathrm{atb},\rho}(\mu)$ and, for all $f \in \widehat{H}^{p,q,\gamma_2}_{\mathrm{atb},\rho}(\mu)$, $f \in \widehat{H}^{p,q,\gamma_1}_{\mathrm{atb},\rho}(\mu)$ and

$$\|f\|_{\widehat{H}^{p,q,\gamma_1}_{\mathrm{atb},\rho}(\mu)} \leqslant \|f\|_{\widehat{H}^{p,q,\gamma_2}_{\mathrm{atb},\rho}(\mu)}.$$

Now we consider the following converse inclusion that $\widehat{H}^{p,q,\gamma_1}_{\mathrm{atb},\rho}(\mu) \subset \widehat{H}^{p,q,\gamma_2}_{\mathrm{atb},\rho}(\mu)$. Let

$$b = \sum_{j=1}^{2}\lambda_j a_j \in \widehat{H}^{p,q,\gamma_1}_{\mathrm{atb},\rho}(\mu)$$

be a $(p,\, q,\, \gamma_1,\, \rho)_{\lambda,1}$-atomic block, where, for any $j \in \{1,2\}$, a_j is a function supported on $B_j \subset B$ for some balls B_j and B as in Definition 7.1. We first show that any $(p,q,\gamma_1,\rho)_{\lambda,1}$-atomic block can be decomposed into a sum of some $(p,q,\gamma_2,\rho)_{\lambda,1}$-atomic blocks and

$$\|b\|_{\widehat{H}^{p,q,\gamma_2}_{\mathrm{atb},\rho}(\mu)} \lesssim |b|_{\widehat{H}^{p,q,\gamma_1}_{\mathrm{atb},\rho}(\mu)}. \tag{8.5}$$

To prove (8.5), we consider the following four cases:

Case (I) For any $j \in \{1, 2\}$, $\widetilde{K}^{(\rho), p}_{B_j, B} \leqslant [(3 + \lfloor \log_\rho 2 \rfloor) C_{(\rho)}]^{1/p}$, where $C_{(\rho)}$ is as in Lemma 2.8(i).

Case (II) $\widetilde{K}^{(\rho), p}_{B_1, B} > [(3 + \lfloor \log_\rho 2 \rfloor) C_{(\rho)}]^{1/p}$ and $\widetilde{K}^{(\rho), p}_{B_2, B} \leqslant [(3 + \lfloor \log_\rho 2 \rfloor) C_{(\rho)}]^{1/p}$.

Case (III) $\widetilde{K}^{(\rho), p}_{B_1, B} \leqslant [(3 + \lfloor \log_\rho 2 \rfloor) C_{(\rho)}]^{1/p}$ and $\widetilde{K}^{(\rho), p}_{B_2, B} > [(3 + \lfloor \log_\rho 2 \rfloor) C_{(\rho)}]^{1/p}$.

Case (IV) For any $j \in \{1, 2\}$, $\widetilde{K}^{(\rho), p}_{B_j, B} > [(3 + \lfloor \log_\rho 2 \rfloor) C_{(\rho)}]^{1/p}$.

In Case (I), for any $j \in \{1, 2\}$, we have

$$\begin{aligned} [\widetilde{K}^{(\rho), p}_{B_j, B}]^{-\gamma_1} < 1 &= [(3 + \lfloor \log_\rho 2 \rfloor) C_{(\rho)}]^{\gamma_2/p} [(3 + \lfloor \log_\rho 2 \rfloor) C_{(\rho)}]^{-\gamma_2/p} \\ &\leqslant [(3 + \lfloor \log_\rho 2 \rfloor) C_{(\rho)}]^{\gamma_2/p} [\widetilde{K}^{(\rho), p}_{B_j, B}]^{-\gamma_2}. \end{aligned}$$

For any $j \in \{1, 2\}$, let $\widetilde{\lambda}_j := [(3 + \lfloor \log_\rho 2 \rfloor) C_{(\rho)}]^{\gamma_2/p} \lambda_j$ and $\widetilde{a}_j := [(3 + \lfloor \log_\rho 2 \rfloor) C_{(\rho)}]^{-\gamma_2/p} a_j$. Then $b = \widetilde{\lambda}_1 \widetilde{a}_1 + \widetilde{\lambda}_2 \widetilde{a}_2$. From this, it is easy to see that b is a $(p, q, \gamma_2, \rho)_{\lambda, 1}$-atomic block, which implies that $b \in \widehat{H}^{p, q, \gamma_2}_{\mathrm{atb}, \rho}(\mu)$ and

$$\|b\|_{\widehat{H}^{p, q, \gamma_2}_{\mathrm{atb}, \rho}(\mu)} \leqslant [(3 + \lfloor \log_\rho 2 \rfloor) C_{(\rho)}]^{\gamma_2/p} (|\lambda_1| + |\lambda_2|) \sim |b|_{\widehat{H}^{p, q, \gamma_1}_{\mathrm{atb}, \rho}(\mu)}$$

in this case.

The proofs of Cases (II)–(IV) are similar. For brevity, we only prove Case (II).

In Case (II), we have $\widetilde{K}^{(\rho), p}_{B_1, B} > [(3 + \lfloor \log_\rho 2 \rfloor) C_{(\rho)}]^{1/p}$. We now choose a sequence $\{B_1^{(i)}\}_{i=0}^m$ of balls with certain $m \in \mathbb{N}$ as follows. Let $B_1^{(0)} := B_1$ and $B_0 := (\widetilde{\rho^{N^{(\rho)}_{B_1, B}} B_1})^{\rho^2}$. To choose $B_1^{(1)}$, let N_1 be the smallest positive integer satisfying $\widetilde{K}^{(\rho), p}_{B_1^{(0)}, \rho^{N_1} B_1^{(0)}} > [(3 + \lfloor \log_\rho 2 \rfloor) C_{(\rho)}]^{1/p}$. If $r_{(\widetilde{\rho^{N_1} B_1^{(0)}})^{\rho^2}} \geqslant r_{B_0}$, then we let $B_1^{(1)} := B_0$ and the selection process terminates. Otherwise, we let $B_1^{(1)} := (\widetilde{\rho^{N_1} B_1^{(0)}})^{\rho^2}$. To choose $B_1^{(2)}$, if, for any $N \in \mathbb{N}$, $\widetilde{K}^{(\rho), p}_{B_1^{(1)}, \rho^N B_1^{(1)}} \leqslant [(3 + \lfloor \log_\rho 2 \rfloor) C_{(\rho)}]^{1/p}$, let $B_1^{(2)} := B_0$ and the selection process terminates. Otherwise, let N_2 be the smallest positive integer satisfying $\widetilde{K}^{(\rho), p}_{B_1^{(1)}, \rho^{N_2} B_1^{(1)}} > [(3 + \lfloor \log_\rho 2 \rfloor) C_{(\rho)}]^{1/p}$. If $r_{(\widetilde{\rho^{N_2} B_1^{(1)}})^{\rho^2}} \geqslant r_{B_0}$, then we let $B_1^{(2)} := B_0$ and the selection process terminates. Otherwise, we let $B_1^{(2)} := (\widetilde{\rho^{N_2} B_1^{(1)}})^{\rho^2}$. We continue as long as this selection process is possible; clearly, finally the condition $r_{(\widetilde{\rho^{N_{i+1}} B_1^{(i)}})^{\rho^2}} < r_{B_0}$ is violated after finitely many steps. Without loss of generality, we may assume that the process will stop after m ($m \in \mathbb{N} \cap (1, \infty)$) steps. Now we conclude that $\{B_1^{(i)}\}_{i=0}^m$ have the following properties:

(i) $B_1^{(0)} := B_1$, $B_1^{(i)} := (\widetilde{\rho^{N_i} B_1^{(i-1)}})^{\rho^2}$ for any $i \in \{1, \ldots, m-1\}$, and $B_1^{(m)} := B_0$;

(ii) for any $i \in \{1, \ldots, m-1\}$, by Lemma 2.8(i) and the definition of N_i, we have

$$\widetilde{K}^{(\rho), p}_{B_1^{(i-1)}, B_1^{(i)}} \geqslant \left[C_{(\rho)}\right]^{-1/p} \widetilde{K}^{(\rho), p}_{B_1^{(i-1)}, \rho^{N_i} B_1^{(i-1)}} > \left(3 + \lfloor \log_\rho 2 \rfloor\right)^{1/p};$$

(iii) there exists a positive constant C such that, for any $i \in \{1, \ldots, m\}$, $\widetilde{K}^{(\rho), p}_{B_1^{(i-1)}, B_1^{(i)}} \leqslant C$. Indeed, if, for any $N \in \mathbb{N}$, $\widetilde{K}^{(\rho), p}_{B_1^{(m-1)}, \rho^N B_1^{m-1}} \leqslant [(3 + \lfloor \log_\rho 2 \rfloor) C_{(\rho)}]^{1/p}$, then, from the choice of $B_1^{(m)}$, we have $\widetilde{K}^{(\rho), p}_{B_1^{(m-1)}, B_1^{(m)}} \lesssim 1$. Otherwise, by Lemmas 2.9, 2.8(ii)–2.8(iv), and the definition of N_i, we see that, for any $i \in \{1, \ldots, m\}$,

$$\begin{aligned} \widetilde{K}^{(\rho), p}_{B_1^{(i-1)}, B_1^{(i)}} &\leqslant 2^{1-p} [\widetilde{K}^{(\rho), p}_{B_1^{(i-1)}, \rho^{N_i} B_1^{(i-1)}} + c_{(\rho, p, \nu)} \widetilde{K}^{(\rho), p}_{\rho^{N_i} B_1^{(i-1)}, (\widetilde{\rho^{N_i} B_1^{(i-1)}})^{\rho^2}}] \\ &\leqslant 2^{1-p} [K^{(\rho), p}_{B_1^{(i-1)}, \rho^{N_i - 1} B_1^{(i-1)}} + c_{(\rho, p, \nu)} \widetilde{K}^{(\rho), p}_{\rho^{N_i - 1} B_1^{(i-1)}, \rho^{N_i} B_1^{(i-1)}} + c_{(\rho, p, \nu)}] \leqslant C; \end{aligned}$$

(iv) by (ii), Lemma 6.11, the fact that $B_1^{m-1}\subset B_1^{(m)}\subset 2\rho^{2\widetilde{C}_1+1}B$, Lemmas 2.8(i), 2.8(ii) and 2.8(iv), where $\widetilde{C}_1$ is as in (6.1), we know that

$$\begin{aligned} m=(m-2)+2 &\leqslant \sum_{i=1}^{m-2}[\widetilde{K}^{(\rho),\,p}_{B_1^{(i)},\,B_1^{i+1}}]^p+2<(3+\lfloor\log_\rho 2\rfloor)[\widetilde{K}^{(\rho),\,p}_{B_1,\,B_1^{m-1}}]^p+2\\ &\lesssim [\widetilde{K}^{(\rho),\,p}_{B_1,\,2\rho^{2\widetilde{C}_1+1}B}]^p\lesssim[\widetilde{K}^{(\rho),\,p}_{B_1,\,B}]^p+[\widetilde{K}^{(\rho),\,p}_{B,\,2\rho^{2\widetilde{C}_1+1}B}]^p\lesssim[\widetilde{K}^{(\rho),\,p}_{B_1,\,B}]^p. \end{aligned}$$

Let C be the implicit positive constant of the above inequality, $(\widetilde{C}_b)^p:=C[\widetilde{K}^{(\rho),\,p}_{B_1,\,B}]^p$ and $\widetilde{c}_0:=\widetilde{C}_b a_1$. For any $i\in\{1,\ldots,m\}$, let

$$\widetilde{c}_i:=\frac{\chi_{B_1^{(i)}}}{\mu(B_1^{(i)})}\int_{\mathcal{X}}\widetilde{c}_{i-1}(y)d\mu(y).$$

If $i=0$, by Definition 3.2(iii), (2.2), $r_{B_1^{(1)}}\leqslant\rho^{2\widetilde{C}_1+1}r_B$, (2.1) and (iii), we have

$$\begin{aligned} \|\widetilde{c}_0\|_{L^q(\mu)} &\lesssim [\mu(\rho B_1)]^{1/q-1}[\lambda(c_B,r_B)]^{1-1/p}[\widetilde{K}^{(\rho),\,p}_{B_1,\,B}]^{-\gamma_1+1}\\ &\lesssim [\mu(\rho B_1^{(0)})]^{1/q-1}[\lambda(c_{B_1},r_B)]^{1-1/p}\\ &\lesssim [\mu(\rho B_1^{(0)})]^{1/q-1}[\lambda(c_{B_1},r_{\rho B_1^{(1)}})]^{1-1/p}[\widetilde{K}^{(\rho),\,p}_{B_1^0,\,\rho B_1^{(1)}}]^{-\gamma_2}, \end{aligned} \tag{8.6}$$

where the implicit positive constant is independent of $\widetilde{K}^{(\rho),\,p}_{B_1,\,B}$. For $i=1$, by Hölder's inequality, Definition 3.2(iii), (2.2), the facts that $r_{B_1^{(1)}}\leqslant\rho^{2\widetilde{C}_1+1}r_B$ and $B_1^{(1)}$ is doubling, (2.1) and Lemma 2.8(ii), we conclude that

$$\begin{aligned} \|\widetilde{c}_1\|_{L^q(\mu)} &\leqslant [\mu(B_1^{(1)})]^{1/q-1}[\mu(B_1)]^{1-1/q}\|\widetilde{C}_b a_1\|_{L^q(\mu)}\\ &\lesssim [\mu(B_1^{(1)})]^{1/q-1}[\mu(B_1)]^{1-1/q}[\mu(\rho B_1)]^{1/q-1}[\lambda(c_B,r_B)]^{1-1/p}[\widetilde{K}^{(\rho),\,p}_{B_1,\,B}]^{-\gamma_1+1}\\ &\lesssim [\mu(B_1^{(1)})]^{1/q-1}[\lambda(c_{B_1},r_B)]^{1-1/p}\\ &\lesssim [\mu(\rho^2 B_1^{(1)})]^{1/q-1}[\lambda(c_{B_1},r_{\rho B_1^{(1)}})]^{1-1/p}[\widetilde{K}^{(\rho),\,p}_{\rho B_1^{(1)},\,\rho B_1^{(1)}}]^{-\gamma_2}. \end{aligned} \tag{8.7}$$

Similar to (8.6) and (8.7), respectively, for any $i\in\{2,\ldots,m\}$, we have

$$\|\widetilde{c}_{i-1}\|_{L^q(\mu)}\lesssim[\mu(\rho^2 B_1^{(i-1)})]^{1/q-1}[\lambda(c_{B_1},r_{\rho B_1^i})]^{1-1/p}[\widetilde{K}^{(\rho),\,p}_{\rho B_1^{(i-1)},\,\rho B_1^i}]^{-\gamma_2} \tag{8.8}$$

and

$$\|\widetilde{c}_i\|_{L^q(\mu)}\lesssim[\mu(\rho^2 B_1^{(i)})]^{1/q-1}[\lambda(c_{B_1},r_{\rho B_1^i})]^{1-1/p}[\widetilde{K}^{(\rho),\,p}_{\rho B_1^{(i)},\,\rho B_1^i}]^{-\gamma_2}. \tag{8.9}$$

For any $i\in\{1,\ldots,m\}$, let $c_i:=\frac{\lambda_1}{\widetilde{C}_b}(\widetilde{c}_{i-1}-\widetilde{c}_i)$. Then $\operatorname{supp}(c_i)\subset\rho B_1^{(i)}$ and

$$\int_{\mathcal{X}}c_i(x)d\mu(x)=0,$$

which, together with (8.8) and (8.9), implies that c_i is a $(p,\,q,\,\gamma_2,\,\rho)_{\lambda,\,1}$-atomic block associated with the ball $\rho B_1^{(i)}$ and

$$|c_i|_{\widehat{H}^{p,\,q,\,\gamma_2}_{\mathrm{atb},\,\rho}(\mu)}\lesssim\frac{|\lambda_1|}{\widetilde{C}_b}. \tag{8.10}$$

Now we see that

$$b=\sum_{i=1}^m c_i+\frac{\lambda_1}{\widetilde{C}_b}\widetilde{c}_m+\lambda_2 a_2.$$

Notice that $\int_{\mathcal{X}}b(x)d\mu(x)=0$ and $\int_{\mathcal{X}}c_i(x)d\mu(x)=0$. It then follows that

$$\int_{\mathcal{X}}\left[\frac{\lambda_1}{\widetilde{C}_b}\widetilde{c}_m(x)+\lambda_2 a_2(x)\right]d\mu(x)=0.$$

On the other hand, we have $\mathrm{supp}\,(\widetilde{c}_m) \subset \rho B_1^{(m)} \subset 2\rho^{\widetilde{C}_1+2}B =: B'$, $r_{B'} = 2\rho^{\widetilde{C}_1+2}r_B \leqslant 2\rho^{\widetilde{C}_1+2}r_{B_1^{(m)}}$ and $\mathrm{supp}\,(a_2) \subset B_2 \subset B'$. An argument similar to that used in the estimate of (8.7) shows that

$$\|\widetilde{c}_m\|_{L^q(\mu)} \lesssim [\mu(\rho B_1^{(m)})]^{1/q-1}[\lambda(c_{B_1}, r_{B'})]^{1-1/p}[\widetilde{K}^{(\rho),\,p}_{B_1^m,\,B'}]^{-\gamma_2}.$$

From Definition 3.2(iii), (2.1), $\widetilde{K}^{(\rho),\,p}_{B_2,\,B} \leqslant [(3+\lfloor \log_\rho 2\rfloor)C_{(\rho)}]^{1/p}$, $B' = 2\rho^{\widetilde{C}_1+2}B$, Lemmas 2.8(ii) and 2.8(iv), it follows that

$$\begin{aligned}\|a_2\|_{L^q(\mu)} &\lesssim [\mu(\rho B_2)]^{1/q-1}[\lambda(c_B, r_B)]^{1-1/p}[\widetilde{K}^{(\rho),\,p}_{B_2,\,B}]^{-\gamma_2}\\ &\lesssim [\mu(\rho B_2)]^{1/q-1}[\lambda(c_B, r_{B'})]^{1-1/p}[\widetilde{K}^{(\rho),\,p}_{B_2,\,B'}]^{-\gamma_2}.\end{aligned}$$

Thus, $c_{m+1} := \frac{\lambda_1}{\widetilde{C}_b}\widetilde{c}_m + \lambda_2 a_2$ is a $(p,\, q,\, \gamma_2,\, \rho)_{\lambda,\,1}$-atomic block associated with the ball B' and

$$|c_{m+1}|_{\widehat{H}^{p,\,q,\,\gamma_2}_{\mathrm{atb},\,\rho}(\mu)} \lesssim |\lambda_1| + |\lambda_2|.$$

By this fact, the definition of $\widetilde{C}_b$, (8.10) and (iv), we obtain $b = \sum_{i=1}^{m+1} c_i \in \widehat{H}^{p,\,q,\,\gamma_2}_{\mathrm{atb},\,\rho}(\mu)$ and

$$\|b\|^p_{\widehat{H}^{p,\,q,\,\gamma_2}_{\mathrm{atb},\,\rho}(\mu)} \lesssim \sum_{i=1}^{m+1} |c_i|^p_{\widehat{H}^{p,\,q,\,\gamma_2}_{\mathrm{atb},\,\rho}(\mu)} \lesssim (|\lambda_1| + |\lambda_2|)^p \sim |b|^p_{\widehat{H}^{p,\,q,\,\gamma_1}_{\mathrm{atb},\,\rho}(\mu)},$$

where the implicit positive constant is independent of m. This finishes the proof of (8.5).

Let $f \in \widehat{H}^{p,\,q,\,\gamma_1}_{\mathrm{atb},\,\rho}(\mu)$. Then, by Proposition 6.13 and Definition 7.1, there exists a sequence $\{b_j\}_j$ of $(p, q, \gamma_1, \rho)_{\lambda,\,1}$-atomic blocks such that $f = \sum_{j=1}^{\infty} b_j$ in $(\mathcal{E}^{\alpha,\,q}_{\rho,\,\gamma_1}(\mu))^* = (\mathcal{E}^{\alpha,\,q}_{\rho,\,\gamma_2}(\mu))^*$ and

$$\sum_{j=1}^{\infty} |b_j|^p_{\widehat{H}^{p,\,q,\,\gamma_1}_{\mathrm{atb},\,\rho}(\mu)} \lesssim \|f\|^p_{\widehat{H}^{p,\,q,\,\gamma_1}_{\mathrm{atb},\,\rho}(\mu)}.$$

From this fact and (8.5), we further deduce that $f = \sum_{j=1}^{\infty}\sum_{i=1}^{m_j+1} c_{j,\,i}$ in $(\mathcal{E}^{\alpha,\,q}_{\rho,\,\gamma_2}(\mu))^*$, where $\{c_{j,\,i}\}_{j,\,i}$ are all $(p, q, \gamma_2, \rho)_{\lambda,\,1}$-atomic blocks as in (8.5) satisfying

$$\sum_{j=1}^{\infty}\sum_{i=1}^{m_j+1} |c_{j,\,i}|^p_{\widehat{H}^{p,\,q,\,\gamma_2}_{\mathrm{atb},\,\rho}(\mu)} \lesssim \sum_{j=1}^{\infty} |b_j|^p_{\widehat{H}^{p,\,q,\,\gamma_1}_{\mathrm{atb},\,\rho}(\mu)} \lesssim \|f\|^p_{\widehat{H}^{p,\,q,\,\gamma_1}_{\mathrm{atb},\,\rho}(\mu)},$$

which implies that $f \in \widehat{H}^{p,\,q,\,\gamma_2}_{\mathrm{atb},\,\rho}(\mu)$ and

$$\|f\|_{\widehat{H}^{p,\,q,\,\gamma_2}_{\mathrm{atb},\,\rho}(\mu)} \lesssim \|f\|_{\widehat{H}^{p,\,q,\,\gamma_1}_{\mathrm{atb},\,\rho}(\mu)}.$$

This finishes the proof of Proposition 8.2. □

Now we are ready to show that $\mathcal{E}^{1/p-1}_{\rho}(\mu)$ is the dual space of $\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$.

Theorem 8.3. *Let $p \in (0, 1]$, $\rho \in (1, \infty)$, $\gamma \in [1, \infty)$ and $q \in (1, \infty)$. Then*

$$\mathcal{E}^{1/p-1}_{\rho}(\mu) = (\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu))^*.$$

Proof. When $p = 1$, by [13, Remark 2.6(iii)], the conclusion of Theorem 8.3 holds true without the assumption (6.1). Thus, it remains to consider the case when $p \in (0, 1)$. Let $p \in (0, 1)$, $\rho \in (1, \infty)$, $\gamma \in [1, \infty)$ and $q \in (1, \infty)$. We first show that $\mathcal{E}^{1/p-1}_{\rho}(\mu) \subset (\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu))^*$. To this end, let $f \in \mathcal{E}^{1/p-1}_{\rho}(\mu)$. Recall that any $h \in \widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ is, by Definition 7.3, a continuous linear functional in $(\mathcal{E}^{1/p-1}_{\rho}(\mu))^*$. Let us write $\langle h, f\rangle$ to denote the value of the linear functional h at $f \in \mathcal{E}^{1/p-1}_{\rho}(\mu)$. Then the mapping $\ell_f : h \to \langle h, f\rangle$ is a well-defined linear functional on $\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$. If $h = \sum_{i=1}^{\infty} b_i$ is an atomic decomposition of h in terms of $(p, q, \gamma, \rho)_{\lambda,\,1}$-atomic blocks $\{b_i\}_i$ such that $\sum_{i=1}^{\infty} |b_i|^p_{\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)} \lesssim \|h\|^p_{\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)}$, by (7.1), we then have

$$|\langle h, f\rangle| = \left|\sum_{i=1}^{\infty}\int_{\mathcal{X}} b_i(x)f(x)d\mu(x)\right|$$

$$\lesssim \left[\sum_{i=1}^{\infty}|b_i|^p_{\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)}\right]^{1/p}\|f\|_{\mathcal{E}^{1/p-1}_{\rho}(\mu)} \lesssim \|h\|_{\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)}\|f\|_{\mathcal{E}^{1/p-1}_{\rho}(\mu)}.$$

Therefore, we conclude that

$$|\ell_f(h)| \lesssim \|f\|_{\mathcal{E}^{1/p-1}_{\rho}(\mu)}\|h\|_{\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)}.$$

This shows that $\mathcal{E}^{1/p-1}_{\rho}(\mu) \subset (\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu))^*$.

To see the converse, for any $q \in (1,\infty)$, we first claim that, if $\ell \in (\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu))^*$, then there exists a function $f \in L^{q'}_{\mathrm{loc}}(\mu)$ such that, for all $g \in \bigcup_B L^q_0(B)$,

$$\ell(g) = \int_{\mathcal{X}} f(x)g(x)d\mu(x),$$

where, for all balls $B \subset \mathcal{X}$, $L^q_0(B)$ denotes the *subspace* of $L^q(B)$ consisting of functions having integral zero. Indeed, let $\{B_k\}_k$ be an increasing sequence of balls which exhausts $\mathcal{X}$. For each k, let $\mathbb{C}(B_k)$ denote the space of functions those are constants on B_k. Suppose that $q \in (1,\infty)$ and $\ell \in (\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu))^*$. Then $\ell \in (L^q_0(B_k))^* = L^{q'}(B_k)/\mathbb{C}(B_k)$. Indeed, if $g \in L^q_0(B_k)$, then $g \in \widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ and

$$\|g\|_{\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)} \leqslant [\mu(\rho B_k)]^{1-1/q}\left[\lambda\left(c_{B_k}, r_{B_k}\right)\right]^{1/p-1}\|g\|_{L^q(\mu)}.$$

We further see that

$$\begin{aligned}|\ell(g)| &\leqslant \|\ell\|_{(\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu))^*}\|g\|_{\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)}\\ &\leqslant \|\ell\|_{(\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu))^*}[\mu(\rho B_k)]^{1-1/q}[\lambda(c_{B_k}, r_{B_k})]^{1/p-1}\|g\|_{L^q(\mu)}.\end{aligned}$$

Hence, by the Riesz representation theorem, there exists a unique $f_k \in L^{q'}(B_k)/\mathbb{C}(B_k)$ such that, for every $g \in L^q_0(B_k)$,

$$\ell(g) = \int_{B_k} f_k(x)g(x)d\mu(x).$$

Since $\{B_k\}_k$ is increasing, by a standard argument, we see that there exists a unique function $f \in L^{q'}_{\mathrm{loc}}(\mu)$ such that, for all $g \in \bigcup_B L^q_0(B)$,

$$\ell(g) = \int_{\mathcal{X}} f(x)g(x)d\mu(x).$$

This proves the claim.

We now show that, if $f \in L^{q'}_{\mathrm{loc}}(\mu)$ such that $\ell \in (\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu))^*$, then $f \in \mathcal{E}^{1/p-1}_{\rho}(\mu)$ and

$$\|f\|_{\mathcal{E}^{1/p-1}_{\rho}(\mu)} \lesssim \|\ell\|_{(\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu))^*}.$$

To this end, by Proposition 6.18, it suffices to show that, for any (ρ,β_ρ)-doubling ball B,

$$\frac{1}{\mu(\rho B)}\frac{1}{[\lambda(c_B,\, r_B)]^{1/p-1}}\int_B |f(x) - m_f(B)|d\mu(x) \lesssim \|\ell\|_{(\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu))^*} \tag{8.11}$$

and, for all (ρ,β_ρ)-doubling balls $B \subset S$,

$$|m_f(B) - m_f(S)| \lesssim \|\ell\|_{(\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu))^*}[\widetilde{K}^{(\rho),\,p}_{B,\,S}]^{\gamma}[\lambda(c_S,\, r_S)]^{1/p-1}. \tag{8.12}$$

We first prove (8.11). Let B be a (ρ,β_ρ)-doubling ball. Assume that

$$\begin{aligned}&\int_{\{x\in B:\, f(x)>m_f(B)\}}|f(x) - m_f(B)|^{q'}d\mu(x)\\ &\quad\geqslant \int_{\{x\in B:\, f(x)<m_f(B)\}}|f(x) - m_f(B)|^{q'}d\mu(x).\end{aligned} \tag{8.13}$$

Consider

$$a(x) := \begin{cases} |f(x) - m_f(B)|^{q'-1}, & \text{if } x \in \{x \in B : f(x) > m_f(B)\}, \\ \widetilde{C}_B, & \text{if } x \in \{x \in B : f(x) < m_f(B)\}, \\ 0, & \text{otherwise}, \end{cases}$$

where $\widetilde{C}_B$ denotes the constant such that $\int_{\mathcal{X}} a(x)d\mu(x) = 0$. By the definition of $m_f(B)$, we have

$$\mu(\{x \in B : f(x) > m_f(B)\}) \leqslant \mu(B)/2 \leqslant \mu(\{x \in B : f(x) \leqslant m_f(B)\}). \tag{8.14}$$

From this fact, we deduce that $\operatorname{supp}(a) \subset \sqrt{\rho}B$, a is a $(p, q, \gamma, \sqrt{\rho})_\lambda$-atomic block and

$$\begin{aligned} \|a\|_{\widetilde{H}^{p,q,\gamma}_{\mathrm{atb},\sqrt{\rho}}(\mu)} &\leqslant \|a\|_{L^q(\mu)}[\mu(\sqrt{\rho} \times \sqrt{\rho}B)]^{1/q'}[\lambda(c_B, r_B)]^{1/p-1} \\ &\leqslant [\mu(\rho B)]^{1/q'}[\lambda(c_B, r_B)]^{1/p-1}\bigg[\int_{\{x\in B: f(x)>m_f(B)\}} |f(x) - m_f(B)|^{q'} d\mu(x) \\ &\quad + \int_{\{x\in B: f(x)\leqslant m_f(B)\}} |\widetilde{C}_B|^q d\mu(x)\bigg]^{1/q}. \end{aligned}$$

By (8.14), the definition of $\widetilde{C}_B$ and Hölder's inequality, we have

$$\begin{aligned} &\int_{\{x\in B: f(x)\leqslant m_f(B)\}} |\widetilde{C}_B|^q d\mu(x) \\ &\quad = \bigg|\int_{\{x\in B: f(x)\leqslant m_f(B)\}} \widetilde{C}_B d\mu(x)\bigg|^q [\mu(\{x \in B : f(x) \leqslant m_f(B)\})]^{1-q} \\ &\quad \lesssim \bigg|\int_{\{x\in B: f(x)>m_f(B)\}} |f(x) - m_f(B)|^{q'-1} d\mu(x)\bigg|^q [\mu(B)]^{1-q} \\ &\quad \lesssim \int_{\{x\in B: f(x)>m_f(B)\}} |f(x) - m_f(B)|^{q'} d\mu(x). \end{aligned}$$

From this, $\operatorname{supp}(a) \subset \sqrt{\rho}B$ and Proposition 8.1, it follows that

$$\begin{aligned} \|a\|_{\widehat{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)} &\sim \|a\|_{\widehat{H}^{p,q,\gamma}_{\mathrm{atb},\sqrt{\rho}}(\mu)} \\ &\lesssim [\mu(\rho B)]^{1/q'}[\lambda(c_B, r_B)]^{1/p-1}\bigg[\int_{\{x\in B: f(x)>m_f(B)\}} |f(x) - m_f(B)|^{q'} d\mu(x)\bigg]^{1/q}. \end{aligned} \tag{8.15}$$

On the other hand, by the definition of a and (8.13), we see that

$$\begin{aligned} \int_B f(x)a(x)d\mu(x) &= \int_B [f(x) - m_f(B)]a(x)d\mu(x) \\ &\geqslant \int_{\{x\in B: f(x)>m_f(B)\}} |f(x) - m_f(B)|^{q'} d\mu(x) \\ &\geqslant \frac{1}{2}\int_B |f(x) - m_f(B)|^{q'} d\mu(x), \end{aligned}$$

which, together with (8.15), implies that

$$\begin{aligned} &\bigg[\int_B |f(x) - m_f(B)|^{q'} d\mu(x)\bigg]^{1/q'} \|a\|_{\widehat{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)} \\ &\quad \lesssim [\mu(\rho B)]^{1/q'}[\lambda(c_B, r_B)]^{1/p-1}\int_B |f(x) - m_f(B)|^{q'} d\mu(x) \\ &\quad \lesssim [\mu(\rho B)]^{1/q'}[\lambda(c_B, r_B)]^{1/p-1}\int_B f(x)a(x)d\mu(x) \\ &\quad \lesssim [\mu(\rho B)]^{1/q'}[\lambda(c_B, r_B)]^{1/p-1}\|\ell\|_{(\widehat{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu))^*}\|a\|_{\widehat{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)}. \end{aligned}$$

From this and Hölder's inequality, it then follows that

$$\begin{aligned}&\frac{1}{\mu(\rho B)}\frac{1}{[\lambda(c_B,\,r_B)]^{1/p-1}}\int_B|f(x)-m_f(B)|d\mu(x)\\&\quad\leqslant\frac{[\mu(\rho B)]^{-1/q'}}{[\lambda(c_B,\,r_B)]^{1/p-1}}\left[\int_B|f(x)-m_f(B)|^{q'}d\mu(x)\right]^{1/q'}\lesssim\|\ell\|_{(\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu))^*}.\end{aligned}$$

Thus, (8.11) holds true.

To show (8.12), for all (ρ,β_ρ)-doubling balls $B\subset S$, let

$$a_1:=\frac{|f-m_f(S)|^{q'}}{f-m_f(S)}\chi_{\{x\in S:f(x)\neq m_f(S)\}}$$

and $a_2:=\widetilde{C}_S\chi_S$, where $\widetilde{C}_S$ denotes the constant such that $\int_{\mathcal{X}}[a_1(x)+a_2(x)]d\mu(x)=0$. Observe that

$$|\widetilde{C}_S|\leqslant[\mu(S)]^{-1}[\mu(B)]^{1-1/q}\left[\int_B|f(x)-m_f(S)|^{q'}d\mu(x)\right]^{1/q}.\tag{8.16}$$

From this, together with the fact that B and S are (ρ,β_ρ)-doubling and Proposition 8.1, it follows that $\mathrm{supp}\,(b)\subset(2\sqrt{\rho}+1)S$,

$$b:=\lambda_1\widetilde{a}_1+\lambda_2\widetilde{a}_2\in\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\sqrt{\rho}}(\mu)\subset\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$$

and

$$\begin{aligned}\|b\|_{\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)}\sim\|b\|_{\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\sqrt{\rho}}(\mu)}&\lesssim[\mu(\sqrt{\rho}\times\sqrt{\rho}B)]^{1-1/q}[\lambda(c_S,\,r_{(2\sqrt{\rho}+1)S})]^{1/p-1}\\&\quad\times[\widetilde{K}^{(\rho),\,p}_{\sqrt{\rho}B,\,(2\sqrt{\rho}+1)S}]^{\gamma}\left[\int_B|f(x)-m_f(S)|^{q'}d\mu(x)\right]^{1/q},\end{aligned}\tag{8.17}$$

where

$$\begin{aligned}\widetilde{a}_1&:=a_1[\mu(\rho B)]^{1/q-1}\left[\int_B|f(x)-m_f(S)|^{q'}d\mu(x)\right]^{-1/q}\\&\quad\times[\widetilde{K}^{(\rho),\,p}_{\sqrt{\rho}B,\,(2\sqrt{\rho}+1)S}]^{-\gamma}[\lambda(c_S,\,r_{(2\sqrt{\rho}+1)S})]^{1-1/p},\\\widetilde{a}_2&:=[\mu(\rho S)]^{-1}[\lambda(c_S,r_{(2\sqrt{\rho}+1)S})]^{1-1/p}[\widetilde{K}^{(\rho),\,p}_{\sqrt{\rho}S,\,(2\sqrt{\rho}+1)S}]^{-\gamma}\chi_S,\\\lambda_1&:=\left[\int_B|f(x)-m_f(S)|^{q'}d\mu(x)\right]^{1/q}[\widetilde{K}^{(\rho),\,p}_{\sqrt{\rho}B,\,(2\sqrt{\rho}+1)S}]^{\gamma}[\mu(\rho B)]^{1-1/q}\\&\quad\times[\lambda(c_S,\,r_{(2\sqrt{\rho}+1)S})]^{1/p-1}\end{aligned}$$

and

$$\lambda_2:=\widetilde{C}_S\mu(\rho S)[\lambda(x_S,r_{(2\sqrt{\rho}+1)S})]^{1/p-1}[\widetilde{K}^{(\rho),\,p}_{\sqrt{\rho}S,\,(2\sqrt{\rho}+1)S}]^{\gamma}.$$

Then $\mathrm{supp}\,(\widetilde{a}_1)\subset\sqrt{\rho}B\subset(2\sqrt{\rho}+1)S$ and $\mathrm{supp}\,(\widetilde{a}_2)\subset\sqrt{\rho}S\subset(2\sqrt{\rho}+1)S$. By the definition of a_1, the (ρ,β_ρ)-doubling property of B and S, the vanishing moment of b, (8.16), (8.11) and (8.17), we see that

$$\begin{aligned}&\int_B|f(x)-m_f(S)|^{q'}d\mu(x)\\&\quad=\int_B a_1(x)[f(x)-m_f(S)]d\mu(x)\\&\quad\leqslant\left|\int_{\mathcal{X}}f(x)b(x)d\mu(x)\right|+|\widetilde{C}_S|\int_S|f(x)-m_f(S)|d\mu(x)\\&\quad\lesssim\|\ell\|_{(\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu))^*}\|b\|_{\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)}+|\widetilde{C}_S|\|\ell\|_{(\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu))^*}\mu(\rho S)[\lambda(c_S,r_S)]^{1/p-1}\\&\quad\lesssim\|\ell\|_{(\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu))^*}\left[\int_B|f(x)-m_f(S)|^{q'}d\mu(x)\right]^{1/q}[\widetilde{K}^{(\rho),\,p}_{B,\,S}]^{\gamma}[\mu(B)]^{1-1/q}[\lambda(c_S,\,r_S)]^{1/p-1}.\end{aligned}$$

This implies that

$$\left[\int_B |f(x)-m_f(S)|^{q'}d\mu(x)\right]^{1/q'} \lesssim \|\ell\|_{(\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu))^*}[\widetilde{K}^{(\rho),\,p}_{B,\,S}]^{\gamma}[\mu(B)]^{1-1/q}[\lambda(c_S,\,r_S)]^{1/p-1}.$$

Thus, from this, (8.11), the (ρ,β_ρ)-doubling property of B and S, and Hölder's inequality, it follows that

$$\begin{aligned}|m_f(B)-m_f(S)| &= \frac{1}{\mu(B)}\int_B |m_f(B)-m_f(S)|d\mu(x)\\ &\leqslant \frac{1}{\mu(B)}\int_B |f(x)-m_f(B)|d\mu(x)+\frac{1}{\mu(B)}\int_B |f(x)-m_f(S)|d\mu(x)\\ &\leqslant \left[\frac{1}{\mu(B)}\int_B |f(x)-m_f(B)|^{q'}d\mu(x)\right]^{1/q'}+\left[\frac{1}{\mu(B)}\int_B |f(x)-m_f(S)|^{q'}d\mu(x)\right]^{1/q'}\\ &\lesssim \|\ell\|_{(\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu))^*}[\lambda(c_B,\,r_B)]^{1/p-1}+[\lambda(c_S,\,r_S)]^{1/p-1}[\widetilde{K}^{(\rho),\,p}_{B,\,S}]^{\gamma}\|\ell\|_{(\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu))^*}\\ &\lesssim [\lambda(c_S,\,r_S)]^{1/p-1}[\widetilde{K}^{(\rho),\,p}_{B,\,S}]^{\gamma}\|\ell\|_{(\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu))^*},\end{aligned}$$

which implies (8.12), and hence completes the proof of Theorem 8.3. □

Remark 8.4. It is still unclear whether Theorem 8.3 holds true or not for $q=1$ and $p\in(0,1)$, or $q=\infty$ and $p\in(0,1]$ on non-homogeneous metric measure spaces satisfying the ρ-weakly doubling condition (6.1).

9 Relations between $\mathcal{E}^{\alpha}_{\rho}(\mu)$ and $\mathrm{Lip}_{\alpha,\,q}(\mu)$ or between $\widehat{H}^{p,\,q}_{\mathrm{atb}}(\mu)$ and $H^{p,\,q}_{\mathrm{at}}(\mu)$

In this section, we investigate the relations between $\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta,\,\gamma}(\mu)$ and $\mathrm{Lip}_{\alpha,\,q}(\mu)$, and between $\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ and the atomic Hardy space $H^{p,\,q}_{\mathrm{at}}(\mu)$ introduced by Coifman and Weiss [11] over spaces of homogeneous type.

Let $(\mathcal{X},d,\mu)$ be a space of homogeneous type with $\lambda(x,r):=\mu(B(x,r))$ for all $x\in\mathcal{X}$ and $r\in(0,\infty)$. Recall that (6.1) holds true in spaces of homogeneous type. Thus, all the results obtained in Sections 6–8 are still valid in this setting and we denote $\mathcal{E}^{\alpha,\,q}_{\rho,\,\eta,\,\gamma}(\mu)$ simply by $\mathcal{E}^{\alpha}_{\rho}(\mu)$. We first establish an equivalent characterization of $\mathcal{E}^{\alpha}_{\rho}(\mu)$. To this end, we recall the notions of spaces $\mathrm{Lip}_{\alpha,\,q}(\mu)$ and $\mathrm{Lip}_{\alpha}(\mu)$ in [39]. To be precise, let $\alpha\in[0,\infty)$, $q\in[1,\infty)$, $\rho\in(1,\infty)$ and δ be a quasi-distance on $\mathcal{X}$. A function ϕ is said to be in the *Lipschitz space* $\mathrm{Lip}_{\alpha,\,q}(\mu;\delta)$ if

$$\|\phi\|^{(\delta)}_{\alpha,\,q}:=\sup_{B_\delta}\left\{\frac{1}{[\mu(B_\delta)]^{1+q\alpha}}\int_{B_\delta}|f(y)-m_{B_\delta}(f)|^q d\mu(y)\right\}^{1/q}<\infty,$$

where the supremum is taken over all balls B_δ from $(\mathcal{X},\delta,\mu)$, and a function ψ is said to be in the *space* $\mathrm{Lip}_{\alpha}(\mu;\delta)$ if

$$\|\psi\|^{(\delta)}_{\alpha}:=\sup_{x\neq y}\frac{|\psi(x)-\psi(y)|}{[\delta(x,y)]^{\alpha}}<\infty. \tag{9.1}$$

Then we let $\mathrm{Lip}_{\alpha,\,q}(\mu):=\mathrm{Lip}_{\alpha,\,q}(\mu;d)$ and $\mathrm{Lip}_{\alpha}(\mu):=\mathrm{Lip}_{\alpha}(\mu;d)$, respectively.

Remark 9.1. By [39, Theorem 5], we see that, for any $\alpha\in(0,\infty)$, there exists a quasi-distance δ on $\mathcal{X}$, defined by setting, for all $x,\,y\in\mathcal{X}$,

$$\delta(x,y):=\inf\{\mu(B):B \text{ is a ball containing } x \text{ and } y\}$$

such that $(\mathcal{X},\delta,\mu)$ is a normal space. Namely, there exist two positive constants, c_9 and c_{10}, such that $c_9r\leqslant\mu(B_\delta(x,r))\leqslant c_{10}r$ for every $x\in\mathcal{X}$, $r\in(\mu(x),\mu(\mathcal{X}))$ and

$$B_\delta(x,r):=\{x\in\mathcal{X}:\delta(x,y)<r\}$$

and, for any $q \in [1,\infty)$, a function ϕ is in the Lipschitz space $\mathrm{Lip}_{\alpha,q}(\mu)$ of $(\mathcal{X},d,\mu)$ if and only if there exists a function ψ in the space $\mathrm{Lip}_{\alpha}(\mu;\delta)$ of $(\mathcal{X},\delta,\mu)$ such that $\phi=\psi$ for μ-almost every $x\in\mathcal{X}$. Moreover, $\|\phi\|^{(d)}_{\alpha,q}\sim\|\psi\|^{(\delta)}_{\alpha}$.

Hereafter, we *always let* δ *be as in Remark* 9.1.

Now we discuss the relation between $\mathcal{E}^{\alpha}_{\rho}(\mu)$ and $\mathrm{Lip}_{\alpha,q}(\mu)$.

Proposition 9.2. *Suppose that* $(\mathcal{X},d,\mu)$ *is a space of homogeneous type,* $\alpha\in[0,\infty)$, $\rho\in(1,\infty)$ *and* $q\in[1,\infty)$. *Then* $\mathcal{E}^{\alpha}_{\rho}(\mu)$ *and* $\mathrm{Lip}_{\alpha,q}(\mu)$ *coincide with equivalent norms.*

Proof. Fix $\alpha\in[0,\infty)$ and $q\in[1,\infty)$. By Proposition 6.8(ii), without loss of generality, we may assume that $\rho=2$. By Definition 6.5, we know that $\mathcal{E}^{\alpha}_{\rho}(\mu)\subset\mathrm{Lip}_{\alpha,q}(\mu)$ and, for all $f\in\mathcal{E}^{\alpha}_{\rho}(\mu)$, $\|f\|^{(d)}_{\alpha,q}\leqslant\|f\|_{\mathcal{E}^{\alpha}_{\rho}(\mu)}$.

Conversely, by Definition 6.5, it suffices to prove that, for all $f\in\mathrm{Lip}_{\alpha,q}(\mu)$ and balls $B\subset S$,

$$|m_B(f)-m_S(f)|\lesssim\|f\|^{(d)}_{\alpha,q}\widetilde{K}^{(\rho),p}_{B,S}[\mu(S)]^{\alpha}. \tag{9.2}$$

To this end, we consider the following two cases.

Case (I) $\mu(S)\leqslant 4C_{(\mu)}\mu(B)$, where $C_{(\mu)}$ is as in (1.1). Thus, by this and Hölder's inequality, we have

$$\begin{aligned}|m_B(f)-m_S(f)|&\leqslant\frac{1}{\mu(B)}\int_B|f(x)-m_S(f)|d\mu(x)\lesssim\frac{1}{\mu(S)}\int_S|f(x)-m_S(f)|d\mu(x)\\&\lesssim\frac{1}{[\mu(S)]^{1/q}}\left[\int_S|f(x)-m_S(f)|^q d\mu(x)\right]^{1/q}\lesssim[\mu(S)]^{\alpha}\|f\|^{(d)}_{\alpha,q},\end{aligned}$$

which implies (9.2) in Case (I).

Case (II)

$$\mu(S)>4C_{(\mu)}\mu(B). \tag{9.3}$$

Now we show (9.2). Let N be the smallest integer such that $2^N r_B\geqslant r_S$. Let $B_*:=2^{N+1}B$. Then $S\subset B_*\subset 6S$, which implies that

$$\mu(S)\leqslant\mu(B_*)\leqslant\mu(6S)\leqslant[C_{(\mu)}]^3\mu(S). \tag{9.4}$$

Furthermore, let $B^{(0)}:=B$. By (9.3) and (9.4), we see that

$$\mu(2^{N+1}B)=\mu(B_*)\geqslant\mu(S)>4C_{(\mu)}\mu(B)>2\mu(B^{(0)}).$$

Thus, let $B^{(1)}:=2^{N_1}B^{(0)}$, $N_1\in\mathbb{N}$, be the smallest ball in the form of $2^kB^{(0)}$ $(k\in\mathbb{N})$ such that $\mu(2^kB^{(0)})>2\mu(B^{(0)})$. Moreover, by (9.3) and (9.4), we know that $r_{B^{(1)}}\leqslant r_{B_*}$ and

$$\mu(B^{(1)})\leqslant C_{(\mu)}\mu(2^{-1}B^{(1)})\leqslant 2C_{(\mu)}\mu(B^{(0)}).$$

We further consider the following two subcases.

Subcase (i) There exists $k\in\mathbb{N}$ such that $\mu(2^kB^{(1)})>2\mu(B^{(1)})$. In this case, we let $2^{N_2}B^{(1)}$, $N_2\in\mathbb{N}$, be the smallest ball in the form of $2^kB^{(1)}$, $k\in\mathbb{N}$, such that $\mu(2^kB^{(1)})>2\mu(B^{(1)})$. Now we divide this subcase into two parts:

(a) $r_{2^{N_2}B^{(1)}}\leqslant r_{B_*}$. Let $B^{(2)}:=2^{N_2}B^{(1)}$. Then $\mu(B^{(2)})\leqslant C_{(\mu)}\mu(2^{-1}B^{(2)})\leqslant 2C_{(\mu)}\mu(B^{(1)})$.

(b) $r_{2^{N_2}B^{(1)}}>r_{B_*}$. Let $B^{(2)}:=B_*$. Then $\mu(B^{(2)})\leqslant 2\mu(B^{(1)})$, where we terminate the construction in this subcase.

Subcase (ii) For any $k\in\mathbb{N}$, $\mu(2^kB^{(1)})\leqslant 2\mu(B^{(1)})$. Let $B^{(2)}:=B_*$. Then $\mu(B^{(2)})\leqslant 2\mu(B^{(1)})$ and we terminate the construction in this subcase.

We continue to choose the balls $\{B^{(i)}\}_i$ in this way. Clearly, finally the condition $r_{2^{N_{i+1}}B^{(i)}}\leqslant r_{B_*}$ $(i\in\mathbb{N})$ is violated after finitely many steps. Without loss of generality, we may assume that the process stops after m $(m\in\mathbb{N}\cap(1,\infty))$ steps. Then we obtain a sequence of balls, $\{B^{(i)}\}^m_{i=0}$, such that

(i) $B=:B^{(0)}\subset\cdots\subset B^{(m)}:=B_*$;

(ii) for any $i \in \{1, \dots, m-1\}$, $2\mu(B^{(i-1)}) < \mu(B^{(i)}) \leqslant 2C_{(\mu)}\mu(B^{(i-1)})$;

(iii) $\mu(B^{(m)}) \leqslant 2C_{(\mu)}\mu(B^{(m-1)})$ and $\mu(S) \leqslant \mu(B^{(m)}) \leqslant [C_{(\mu)}]^3\mu(S)$.

Observe that $m \leqslant N+1$. Thus, by the fact that $S \subset B_* \subset 6S$, Hölder's inequality and (i)–(iii), we have

$$
\begin{aligned}
&|m_B(f) - m_S(f)| \\
&\quad \leqslant \sum_{i=1}^{m} |m_{B^{(i-1)}}(f) - m_{B^{(i)}}(f)| + |m_{B^{(m)}}(f) - m_S(f)| \\
&\quad \leqslant \sum_{i=1}^{m} \frac{1}{\mu(B^{(i-1)})} \int_{B^{(i-1)}} |f(x) - m_{B^{(i)}}(f)|d\mu(x) + \frac{1}{\mu(S)} \int_S |f(x) - m_{B^{(m)}}(f)|d\mu(x) \\
&\quad \lesssim \sum_{i=1}^{m} \frac{1}{\mu(B^{(i)})} \int_{B^{(i)}} |f(x) - m_{B^{(i)}}(f)|d\mu(x) + \frac{1}{\mu(B^{(m)})} \int_{B^{(m)}} |f(x) - m_{B^{(m)}}(f)|d\mu(x) \\
&\quad \lesssim \sum_{i=1}^{m} \left[\frac{1}{\mu(B^{(i)})} \int_{B^{(i)}} |f(x) - m_{B^{(i)}}(f)|^q d\mu(x)\right]^{1/q} + \left[\frac{1}{\mu(B^{(m)})} \int_{B^{(m)}} |f(x) - m_{B^{(m)}}(f)|^q d\mu(x)\right]^{1/q} \\
&\quad \lesssim \sum_{i=1}^{m} [\mu(B^{(i)})]^\alpha \|f\|_{\alpha,q}^{(d)} + [\mu(B^{(m)})]^\alpha \|f\|_{\alpha,q}^{(d)} \\
&\quad \lesssim (1+m)[\mu(B^{(m)})]^\alpha \|f\|_{\alpha,q}^{(d)} \lesssim (1+m)[\mu(6S)]^\alpha \|f\|_{\alpha,q}^{(d)} \\
&\quad \lesssim (1+N)[\mu(6S)]^\alpha \|f\|_{\alpha,q}^{(d)} \sim [\widetilde{K}_{B,S}^{(\rho),p}]^p [\mu(S)]^\alpha \|f\|_{\alpha,q}^{(d)}.
\end{aligned}
$$

This finishes the proof of (9.2) in Case (II) and hence Proposition 9.2. □

Remark 9.3. When $\alpha = 0$, Proposition 9.2 is just [27, Proposition 4.7] with $\lambda(x,r) = \mu(B(x,r))$ for all $x \in \mathcal{X}$ and $r \in (0,\infty)$.

Now we recall the notion of the atomic Hardy space $H_{\rm at}^{p,q}(\mu)$ from [11]. Suppose that $p \in (0,1]$ and $q \in [1,\infty] \cap (p,\infty]$. A function a on $\mathcal{X}$ is called a (p,q)-*atom* if

(i) $\operatorname{supp}(a) \subset B$ for some ball $B \subset \mathcal{X}$;

(ii) $\|a\|_{L^q(\mu)} \leqslant [\mu(B)]^{1/q-1/p}$;

(iii) $\int_{\mathcal{X}} a(x)\, d\mu(x) = 0$.

A function $f \in L^1(\mu)$ or a linear functional $f \in (\mathrm{Lip}_{1/p-1}(\mu))^*$ when $p \in (0,1)$ is said to be in the *Hardy space* $H_{\rm at}^{1,q}(\mu)$ when $p = 1$ or $H_{\rm at}^{p,q}(\mu)$ when $p \in (0,1)$ if there exist (p,q)-atoms $\{a_j\}_{j=1}^\infty$ and $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$ such that

$$f = \sum_{j\in\mathbb{N}} \lambda_j a_j,$$

which converges in $L^1(\mu)$ when $p = 1$ or in $(\mathrm{Lip}_{1/p-1}(\mu))^*$ when $p \in (0,1)$, and

$$\sum_{j\in\mathbb{N}} |\lambda_j|^p < \infty.$$

Moreover, the *norm* of f in $H_{\rm at}^{p,q}(\mu)$ with $p \in (0,1]$ and $q \in [1,\infty] \cap (p,\infty]$ is defined by

$$\|f\|_{H_{\rm at}^{p,q}(\mu)} := \inf\left\{\left(\sum_{j\in\mathbb{N}} |\lambda_j|^p\right)^{1/p}\right\},$$

where the infimum is taken over all possible decompositions of f as above.

Coifman and Weiss [11] proved that $H_{\rm at}^{p,q}(\mu)$ and $H_{\rm at}^{p,\infty}(\mu)$ coincide with equivalent norms for all $p \in (0,1]$ and $q \in [1,\infty) \cap (p,\infty)$. Thus, we denote $H_{\rm at}^{p,q}(\mu)$ simply by $H_{\rm at}^p(\mu)$.

Let $p \in (0,1]$, $q \in (1,\infty]$, $\gamma \in [1,\infty)$ and $\rho \in (1,\infty)$. Recall that the space $\widehat{H}_{{\rm atb},\rho}^{p,q,\gamma}(\mu)$ is independent of the choices of $\gamma \in [1,\infty)$ and $\rho \in (1,\infty)$; see Propositions 8.1 and 8.2. Denote $\widehat{H}_{{\rm atb},\rho}^{p,q,\gamma}(\mu)$ *simply by* $\widehat{H}_{\rm atb}^{p,q}(\mu)$. Moreover, without loss of generality, we may let $\gamma = 1/p$ and $\rho = 2$.

Now we show that $\widehat{H}_{\rm atb}^{p,q}(\mu)$ and $H_{\rm at}^p(\mu)$ coincide with equivalent quasi-norms.

Theorem 9.4. *Let $(\mathcal{X}, d, \mu)$ be a space of homogeneous type, $p \in (0, 1]$ and $q \in (1, \infty]$. Then the spaces $\widehat{H}^{p,q}_{\rm atb}(\mu)$ and $H^p_{\rm at}(\mu)$ coincide with equivalent quasi-norms.*

Proof. Let $p \in (0, 1]$ and $q \in (1, \infty]$. We first show that $H^p_{\rm at}(\mu) \subset \widehat{H}^{p,q}_{\rm atb}(\mu)$. To this end, let $f \in H^p_{\rm at}(\mu)$. Then there exist sequences of (p, q)-atoms, $\{b_k\}_{k=1}^\infty$, and numbers, $\{\lambda_k\}_{k=1}^\infty \subset \mathbb{C}$, such that $f = \sum_{k=1}^\infty \lambda_k b_k$ in $(\mathrm{Lip}_{1/p-1}(\mu; \delta))^*$ and

$$\sum_{k=1}^{\infty} |\lambda_k|^p \lesssim \|f\|^p_{H^p_{\rm at}(\mu)}. \tag{9.5}$$

We then claim that, for each k, $\lambda_k b_k$ is a $(p, q, \frac{1}{p}, 2)_{\lambda,1}$-atomic block and

$$|\lambda_k b_k|_{\widehat{H}^{p,q}_{\rm atb}(\mu)} \lesssim |\lambda_k|. \tag{9.6}$$

Indeed, let $\rho \in (1, \infty)$ and $\gamma \in [1, \infty)$. It suffices to prove that, if b is a (p, q)-atom, then b is a $(p, q, 1/p, 2)_{\lambda,1}$-atomic block. Suppose that $\mathrm{supp}\,(b) \subset B(c_B, r_B) =: B$, then

$$\|b\|_{L^q(\mu)} \leqslant [\mu(B)]^{1/q-1/p}.$$

Let

$$a_1 = a_2 := \left[\frac{\mu(\rho B)}{\mu(B)}\right]^{1/q-1} [\widetilde{K}^{(\rho),p}_{B,B}]^{-1/p} b$$

and

$$\lambda_1 = \lambda_2 := \frac{1}{2}\left[\frac{\mu(B)}{\mu(\rho B)}\right]^{1/q-1} [\widetilde{K}^{(\rho),p}_{B,B}]^{1/p}.$$

It then follows that $\mathrm{supp}\,(a_1) \subset B_1 := B$, $\mathrm{supp}\,(a_2) \subset B_2 := B$, $b = \lambda_1 a_1 + \lambda_2 a_2$ and, for $j \in \{1, 2\}$,

$$\|a_j\|_{L^q(\mu)} \leqslant [\mu(\rho B_j)]^{1/q-1} [\mu(B)]^{1-1/p} [\widetilde{K}^{(\rho),p}_{B_j,B}]^{-1/p},$$

which further implies that b is a $(p, q, 1/p, 2)_{\lambda,1}$-atomic block and, moreover,

$$|b|_{\widehat{H}^{p,q,1/p}_{\rm atb,2}(\mu)} = |\lambda_1| + |\lambda_2| \lesssim 1.$$

This finishes the proof of the above claim. Moreover, from Remark 9.1 and Proposition 9.2, we deduce that $f = \sum_{k=1}^\infty \lambda_k b_k$ in $(\mathcal{E}^{1/p-1}_\rho(\mu))^*$, which, together with (9.5) and (9.6), further implies that $f \in \widehat{H}^{p,q}_{\rm atb}(\mu)$ and $\|f\|_{\widehat{H}^{p,q}_{\rm atb}(\mu)} \lesssim \|f\|_{H^p_{\rm at}(\mu)}$.

Now we consider the converse inclusion that $\widehat{H}^{p,q}_{\rm atb}(\mu) \subset H^p_{\rm at}(\mu)$. Let $b = \sum_{j=1}^2 \lambda_j a_j$ be a $(p, q, 1/p, 2)_{\lambda,1}$-atomic block, where, for any $j \in \{1, 2\}$, a_j is a function supported on $B_j \subset B$ for some balls B_j and B as in Definition 7.1, and

$$\|a_j\|_{L^q(\mu)} \leqslant [\mu(2B_j)]^{1/q-1} [\mu(B)]^{1-1/p} [\widetilde{K}^{(2),p}_{B_j,B}]^{-1/p}. \tag{9.7}$$

We consider the following four cases:

Case (I) For any $j \in \{1, 2\}$, $\frac{\mu(B)}{\mu(B_j)} \leqslant 4C_{(\mu)}$, where $C_{(\mu)}$ is as in (1.1);

Case (II) $\frac{\mu(B)}{\mu(B_1)} > 4C_{(\mu)}$ and $\frac{\mu(B)}{\mu(B_2)} \leqslant 4C_{(\mu)}$;

Case (III) $\frac{\mu(B)}{\mu(B_1)} \leqslant 4C_{(\mu)}$ and $\frac{\mu(B)}{\mu(B_2)} > 4C_{(\mu)}$;

Case (IV) For any $j \in \{1, 2\}$, $\frac{\mu(B)}{\mu(B_j)} > 4C_{(\mu)}$.

In Case (I), we see that

$$\|b\|_{L^q(\mu)} \leqslant |\lambda_1| \|a_1\|_{L^q(\mu)} + |\lambda_2| \|a_2\|_{L^q(\mu)} \leqslant [4C_{(\mu)}]^{1-1/q} (|\lambda_1| + |\lambda_2|) [\mu(B)]^{1/q-1/p},$$

which implies that $[4C_{(\mu)}]^{1/q-1} (|\lambda_1| + |\lambda_2|)^{-1} b$ is a (p, q)-atom and

$$\|[4C_{(\mu)}]^{1/q-1} (|\lambda_1| + |\lambda_2|)^{-1} b\|_{H^p_{\rm at}(\mu)} \leqslant 1.$$

The proofs of Cases (II)–(IV) are similar. For brevity, we only prove Case (II).

In Case (II), $\frac{\mu(B)}{\mu(B_1)} > 4C_{(\mu)}$ and $\frac{\mu(B)}{\mu(B_2)} \leqslant 4C_{(\mu)}$. We now choose a sequence of balls, $\{B_1^{(i)}\}_{i=0}^m$ with certain $m \in \mathbb{N}$, as follows. Let $B_1^{(0)} := B_1$ and $B_0 := 2^{N_{B_1,B}^{(2)}+1}B_1$. Then $B \subset B_0 \subset 6B$, which, together with (1.1), shows that

$$\mu(B) \leqslant \mu(B_0) \leqslant \mu(6B) \leqslant [C_{(\mu)}]^3\mu(B). \tag{9.8}$$

To choose $B_1^{(1)}$, let N_1 be the smallest positive integer satisfying $\mu(2^{N_1}B_1^{(0)}) > 2\mu(B_1^{(0)})$. We let $B_1^{(1)} := 2^{N_1}B_1^{(0)}$. By (9.8), (1.1) and the choice of $B_1^{(1)}$, we have $r_{B_1^{(1)}} < r_{B_0}$ and

$$2\mu(B_1^{(0)}) < \mu(B_1^{(1)}) \leqslant C_{(\mu)}\mu(2^{-1}B_1^{(1)}) \leqslant 2C_{(\mu)}\mu(B_1^{(0)}).$$

To choose $B_1^{(2)}$, if, for any $N \in \mathbb{N}$, $\mu(2^NB_1^{(1)}) \leqslant 2\mu(B_1^{(1)})$, let $B_1^{(2)} := B_0$ and the selection process terminates. Otherwise, let N_2 be the smallest positive integer satisfying $\mu(2^{N_2}B_1^{(1)}) > 2\mu(B_1^{(1)})$. If $r_{2^{N_2}B_1^{(1)}} \geqslant r_{B_0}$, then we let $B_1^{(2)} := B_0$ and the selection process terminates. Otherwise, we let $B_1^{(2)} := 2^{N_2}B_1^{(1)}$. We continue as long as this selection process is possible; clearly, finally the condition $r_{2^{N_{i+1}}B_1^{(i)}} < r_{B_0}$ is violated after finitely many steps. Without loss of generality, we may assume that the process stops after m ($m \in \mathbb{N} \cap (1, N_{B_1,B}^{(2)}+1]$) steps. Then we obtain a family of balls, $\{B_1^{(i)}\}_{i=1}^m$, such that

(i) $B_1^{(0)} := B_1 \subset B$, $B_1^{(i)} := 2^{N_i}B_1^{(i-1)} \subset B_1^{(m)} := B_0 \subset 6B$ for any $i \in \{1,\dots,m-1\}$;

(ii) for any $i \in \{1,\dots,m-1\}$, by (1.1) and the definition of N_i, we have

$$2\mu(B_1^{(i-1)}) < \mu(B_1^{(i)}) \leqslant 2C_{(\mu)}\mu(B_1^{(i-1)});$$

(iii) $\mu(B_1^{(m)}) \leqslant 2C_{(\mu)}\mu(B_1^{m-1})$ and $\mu(B) \leqslant \mu(B_1^{(m)}) \leqslant [C_{(\mu)}]^3\mu(B)$;

(iv) from the above selection process and the definition of $\widetilde{K}_{B_1,B}^{(2),p}$, we conclude that

$$m \leqslant N_{B_1,B}^{(2)} + 1 \leqslant [1 + N_{B_1,B}^{(2)}]^{1/p} \leqslant \widetilde{K}_{B_1,B}^{(2),p} =: [\widehat{C}_b]^p. \tag{9.9}$$

Let $\widetilde{c}_0 := \widehat{C}_b a_1$. For any $i \in \{1,\dots,m\}$, let

$$\widetilde{c}_i := \frac{\chi_{B_1^{(i)}}}{\mu(B_1^{(i)})}\int_{\mathcal{X}} \widetilde{c}_{i-1}(y)d\mu(y).$$

For $i \in \{1,\dots,m\}$, we claim that

$$\|\widetilde{c}_{i-1}\|_{L^q(\mu)} \lesssim [\mu(B_1^i)]^{1/q-1/p} \tag{9.10}$$

and

$$\|\widetilde{c}_i\|_{L^q(\mu)} \lesssim [\mu(B_1^i)]^{1/q-1/p}, \tag{9.11}$$

where the implicit positive constant is independent of $\widehat{C}_b$ and hence B_1, B_2 and B. Indeed, we prove (9.10) and (9.11) by induction. By (9.7), $B_1^{(1)} \subsetneqq B_0 \subset 6B$, (1.1) and (ii), we have

$$\|\widetilde{c}_0\|_{L^q(\mu)} \lesssim [\mu(2B_1^{(0)})]^{1/q-1}[\mu(B)]^{1-1/p}[\widetilde{K}_{B_1,B}^{(2),p}]^{-1/p+1/p} \lesssim [\mu(B_1^{(1)})]^{1/q-1/p}.$$

For $i = 1$, by Hölder's inequality, (9.7), $B_1^{(1)} \subsetneqq B_0 \subset 6B$, (1.1) and (ii), we conclude that

$$\begin{aligned}\|\widetilde{c}_1\|_{L^q(\mu)} &\leqslant [\mu(B_1^{(1)})]^{1/q-1}[\mu(B_1^{(0)})]^{1-1/q}\|\widehat{C}_b a_1\|_{L^q(\mu)}\\ &\lesssim [\mu(B_1^{(1)})]^{1/q-1}[\mu(B_1^{(0)})]^{1-1/q}[\mu(2B_1^{(0)})]^{1/q-1}[\mu(B)]^{1-1/p}[\widetilde{K}_{B_1,B}^{(2),p}]^{-1/p+1/p}\\ &\lesssim [\mu(B_1^{(1)})]^{1/q-1/p}.\end{aligned}$$

Moreover, by (ii) (if $m = 2$, we use (iii)), we have

$$\|\widetilde{c}_1\|_{L^q(\mu)} \lesssim [\mu(B_1^{(2)})]^{1/q-1/p}.$$

Now, we assume that (9.10) and (9.11) hold true for $i \in \mathbb{N} \cap [1, m)$. It then follows, from Hölder's inequality, (9.11) and (ii) (if $i+1=m$, we use (iii)), that

$$\begin{aligned}\|\widetilde{c}_{i+1}\|_{L^q(\mu)} &\lesssim [\mu(B_1^{i+1})]^{1/q-1}[\mu(B_1^{(i)})]^{1-1/q}\|\widetilde{c}_i\|_{L^q(\mu)}\\ &\lesssim [\mu(B_1^{i+1})]^{1/q-1}[\mu(B_1^{(i)})]^{1-1/q}[\mu(B_1^{(i)})]^{1/q-1/p} \lesssim [\mu(B_1^{i+1})]^{1/q-1/p}\end{aligned}$$

and, moreover, if $i+1<m$, we further have

$$\|\widetilde{c}_{i+1}\|_{L^q(\mu)} \lesssim [\mu(B_1^{i+2})]^{1/q-1/p}.$$

By induction, we conclude that (9.10) and (9.11) hold true.

For $i \in \{1,\dots,m\}$, let $c_i := [\widetilde{C}]^{-1}(\widetilde{c}_{i-1}-\widetilde{c}_i)$, where $\widetilde{C}$ is a positive constant to be fixed later. Then $\operatorname{supp}(c_i) \subset 2B_1^{(i)}$ and $\int_{\mathcal{X}} c_i(x)d\mu(x)=0$, which, together with (9.10), (9.11) and (1.1), implies that c_i is a multiple of a (p,q)-atom associated with the ball $2B_1^{(i)}$ provided that $\widetilde{C}$ is large enough. We now write

$$b = \frac{\widetilde{C}\lambda_1}{\widehat{C}_b}\sum_{i=1}^m c_i + \frac{\lambda_1}{\widehat{C}_b}\widetilde{c}_m + \lambda_2 a_2.$$

Let $c_{m+1} := [\widetilde{C}(|\lambda_1|+|\lambda_2|)]^{-1}(\frac{\lambda_1}{\widehat{C}_b}\widetilde{c}_m+\lambda_2 a_2)$. Notice that $\int_{\mathcal{X}} b(x)d\mu(x)=0$ and, for each $i\in\{1,\dots,m\}$, $\int_{\mathcal{X}} c_i(x)d\mu(x)=0$. It then follows that $\int_{\mathcal{X}} c_{m+1}(x)d\mu(x)=0$. On the other hand, we have $\operatorname{supp}(\widetilde{c}_m) \subset 2B_1^{(m)} \subset 12B =: B'$, $\mu(B')\sim\mu(B)$ and $\operatorname{supp}(a_2)\subset B_2\subset B'$. By (9.11), (iii) and $\mu(B')\sim\mu(B)$, we have

$$\|\widetilde{c}_m\|_{L^q(\mu)} \lesssim [\mu(B_1^{(m)})]^{1/q-1/p} \lesssim [\mu(B')]^{1/q-1/p}.$$

From (9.7), $\frac{\mu(B)}{\mu(B_2)} \leqslant 4C_{(\mu)}$ and $\mu(B')\sim\mu(B)$, it follows that

$$\|a_2\|_{L^q(\mu)} \lesssim [\mu(2B_2)]^{1/q-1}[\mu(B)]^{1-1/p}[\widetilde{K}^{(2),\,p}_{B_2,B}]^{-1/p} \lesssim [\mu(B')]^{1/q-1/p}.$$

Let $\widetilde{C}$ be a positive constant, which is independent of $\widehat{C}_b$ and m, such that

$$\|c_i\|_{L^q(\mu)} \leqslant \widetilde{C}[\mu(B_1^{i+2})]^{1/q-1/p}$$

for each $i\in\{1,\dots,m\}$, and

$$\left\|\frac{\lambda_1}{\widehat{C}_b}\widetilde{c}_m+\lambda_2 a_2\right\|_{L^q(\mu)} \leqslant \widetilde{C}(|\lambda_1|+|\lambda_2|)[\mu(B')]^{1/q-1/p}.$$

Then we see that c_{m+1} is a $(p,\,q)$-atom associated with the ball B'. From this and (iv), we conclude that

$$b = \frac{\widetilde{C}\lambda_1}{\widehat{C}_b}\sum_{i=1}^m c_i + \widetilde{C}(|\lambda_1|+|\lambda_2|)c_{m+1} \in H^p_{\rm at}(\mu)$$

and

$$\|b\|^p_{H^p_{\rm at}(\mu)} \lesssim m\frac{|\lambda_1|^p}{[\widehat{C}_b]^p} + (|\lambda_1|+|\lambda_2|)^p \lesssim (|\lambda_1|+|\lambda_2|)^p \sim |b|^p_{\widehat{H}^{p,\,q}_{\rm atb}(\mu)}, \tag{9.12}$$

where the implicit positive constant is independent of $\widehat{C}_b$ and m.

Moreover, for any $f\in\widehat{H}^{p,\,q}_{\rm atb}(\mu)$, by Definition 7.3 with $\gamma=1/p$ and $\rho=2$, we know that there exists a sequence $\{b_k\}_{k\in\mathbb{N}}$ of $(p,q,1/p,2)_{\lambda,\,1}$-atomic blocks such that $f=\sum_{k\in\mathbb{N}} b_k$ in $(\mathcal{E}_\rho^{1/p-1}(\mu))^*$ and

$$\sum_{k\in\mathbb{N}} |b_k|^p_{\widehat{H}^{p,\,q}_{\rm atb}(\mu)} \lesssim \|f\|^p_{\widehat{H}^{p,\,q}_{\rm atb}(\mu)}. \tag{9.13}$$

For each k, assume that $b_k=\lambda_{k,\,1}a_{k,\,1}+\lambda_{k,\,2}a_{k,\,2}$, where $\operatorname{supp}(b_k)\subset B_k$, $\operatorname{supp}(a_{k,\,j})\subset B_{k,\,j}$ for $j\in\{1,\,2\}$. Let $[\widehat{C}_k]^p := \widetilde{K}^{(2),\,p}_{B_{k,\,1},\,B_k}$. From Remark 9.1 and Proposition 9.2, (9.9), (9.12) and (9.13), we deduce that

$$f=\sum_{k\in\mathbb{N}} b_k = \sum_{k\in\mathbb{N}}\left[\sum_{i=1}^{m_k}\frac{\widetilde{C}\lambda_{k,\,1}}{\widehat{C}_k}c_{k,\,i}+\widetilde{C}(|\lambda_{k,\,1}|+|\lambda_{k,\,2}|)c_{m_k+1}\right]$$

in $(\mathcal{E}_\rho^{1/p-1}(\mu))^* = (\mathrm{Lip}_{1/p-1}(\mu;\delta))^*$ and

$$\sum_{k\in\mathbb{N}}\left[\sum_{i=1}^{m_k}\left|\frac{\lambda_{k,1}}{\widehat{C}_k}\right|^p + (|\lambda_{k,1}|+|\lambda_{k,2}|)^p\right] \lesssim \sum_{k\in\mathbb{N}}|b_k|^p_{\widehat{H}^{p,q}_{\mathrm{atb}}(\mu)} \lesssim \|f\|^p_{\widehat{H}^{p,q}_{\mathrm{atb}}(\mu)}.$$

This implies that $f\in H^p_{\mathrm{at}}(\mu)$ and $\|f\|_{H^p_{\mathrm{at}}(\mu)} \lesssim \|f\|_{\widehat{H}^{p,q}_{\mathrm{atb}}(\mu)}$, which completes the proof of Theorem 9.4. □

Remark 9.5. (i) Theorem 9.4 implies that, if $(\mathcal{X}, d, \mu)$ is a space of homogeneous type with

$$\lambda(x,r) := \mu(B(x,r))$$

for all $x\in\mathcal{X}$ and $r\in(0,\infty)$, then $\widehat{H}^1_{\mathrm{atb}}(\mu)$ and $H^1_{\mathrm{at}}(\mu)$ coincide with equivalent norms. We notice that it is not a paradox of the example given by Tolsa [54, Example 5.6] since $\lambda(x,r)\sim r$ for all $x\in\mathcal{X}$ and $r\in(0,\sqrt{2}]$, which is not equivalent to $\mu(B(x,r))\sim r^2$, in that example.

(ii) Let $\rho\in(1,\infty)$, $\gamma\in[1,\infty)$ and $q\in(1,\infty]$. Combining Theorem 9.4 and Remark 7.4(ii), we obtain

$$\widehat{H}^{1,q,\gamma}_{\mathrm{atb},\rho}(\mu) = H^1_{\mathrm{at}}(\mu) = \widetilde{H}^{1,q,\gamma}_{\mathrm{atb},\rho}(\mu)$$

over spaces of homogeneous type.

(ii) From Theorem 9.4 and [11, Theorem A], it follows that $\widehat{H}^{p,q}_{\mathrm{atb}}(\mu)$ is independent of the choice of q in spaces of homogeneous type with $\lambda(x,r) := \mu(B(x,r))$ for all $x\in\mathcal{X}$ and $r\in(0,\infty)$.

10 Relation between $\widetilde{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)$ and $H^p_{\mathrm{at}}(\mu)$ over RD-spaces

In this section, we investigate the relations among $\widetilde{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)$, $\widetilde{H}^{p,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)$ and $H^{p,q}_{\mathrm{at}}(\mu)$ over RD-spaces. As a corollary, the relations among $\widetilde{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)$, $\widetilde{H}^{p,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)$, $H^p(\mathbb{R}^D)$, $\widehat{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)$ and $\widehat{H}^{p,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)$ over Euclidean spaces $(\mathbb{R}^D, |\cdot|)$ endowed with the D-dimensional Lebesgue measure dx are also presented.

Let $(\mathcal{X}, d, \mu)$ be a space of homogeneous type with $\lambda(x,r) := \mu(B(x,r))$ for all $x\in\mathcal{X}$ and $r\in(0,\infty)$. The following notion of RD-spaces was introduced by Han et al. [22]. A space of homogeneous type is called an RD-*space* if there exist constants $\kappa\in(0,\nu]$ and $C\in[1,\infty)$ such that, for all $x\in\mathcal{X}$, $r\in(0,\ \mathrm{diam}\,(\mathcal{X})/2)$ and $\lambda\in[1,\ \mathrm{diam}\,(\mathcal{X})/(2r))$,

$$C^{-1}\lambda^\kappa\mu(B(x,r)) \leqslant \mu(B(x,\lambda r)) \leqslant C\lambda^\nu\mu(B(x,r)), \tag{10.1}$$

where $\mathrm{diam}\,(\mathcal{X}) := \sup_{x,y\in\mathcal{X}} d(x,y)$ and $\nu := \log_2 C_{(\lambda)}$ is as in Section 1. We point out that the RD-space is also a space of homogeneous type. In the remainder of this section, we *always assume* that $(\mathcal{X}, d, \mu)$ is an RD-space with $\mu(\mathcal{X})=\infty$ and let $V_r(x) := \mu(B(x,r))$ and $V(x,y) := \mu(B(x,d(x,y)))$ for all $x, y\in\mathcal{X}$ and $r\in(0,\infty)$.

The following *space of test functions on* $\mathcal{X}$ was introduced by Han et al. [21, 22]. Throughout this section, we fix $x_1\in\mathcal{X}$.

Let $\beta\in(0,1]$ and $\gamma\in(0,\infty)$. A function f on $\mathcal{X}$ is said to belong to the *space of test functions,* $\mathcal{G}(\beta,\gamma)$, if there exists a non-negative constant $\widetilde{C}$ such that

(A1) for all $x\in\mathcal{X}$,

$$|f(x)| \leqslant \widetilde{C}\frac{1}{V_1(x_1)+V(x_1,x)}\left[\frac{1}{1+d(x_1,x)}\right]^\gamma;$$

(A2) for all $x, y\in\mathcal{X}$ satisfying $d(x,y)\leqslant(1+d(x_1,x))/2$,

$$|f(x)-f(y)| \leqslant \widetilde{C}\left[\frac{d(x,y)}{1+d(x_1,x)}\right]^\beta\frac{1}{V_1(x_1)+V(x_1,x)}\left[\frac{1}{1+d(x_1,x)}\right]^\gamma.$$

Moreover, for $f\in\mathcal{G}(\beta,\gamma)$, its *norm* is defined by

$$\|f\|_{\mathcal{G}(\beta,\gamma)} := \inf\{\widetilde{C} : \widetilde{C} \text{ satisfies (A1) and (A2)}\}.$$

The *space* $\mathring{\mathcal{G}}(\beta,\gamma)$ is defined as the set of all functions $f \in \mathcal{G}(\beta,\gamma)$ satisfying $\int_{\mathcal{X}} f(x)d\mu(x) = 0$. Moreover, we endow the space $\mathring{\mathcal{G}}(\beta,\gamma)$ with the same norm as the space $\mathcal{G}(\beta,\gamma)$. Furthermore, $\mathring{\mathcal{G}}(\beta,\gamma)$ is a Banach space.

For any given $\epsilon \in (0, 1]$, let $\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma)$ be the completion of the set $\mathring{\mathcal{G}}(\epsilon, \epsilon)$ in $\mathring{\mathcal{G}}(\beta,\gamma)$ when $\beta,\gamma \in (0, \epsilon]$. Moreover, if $f \in \mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma)$, we then define $\|f\|_{\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma)} := \|f\|_{\mathring{\mathcal{G}}(\beta,\gamma)}$. We define the *dual space* $(\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma))^*$ to be the set of all continuous linear functionals $\mathcal{L}$ from $\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma)$ to $\mathbb{C}$, and endow it with the weak* topology.

Suppose that $\epsilon_1 \in (0,1]$ and $\epsilon_2, \epsilon_3 \in (0,\infty)$. Let $\{D_t\}_{t\in(0,\infty)}$ be a family of bounded linear operators on $L^2(\mu)$ such that, for all $t \in (0,\infty)$, $D_t(x,y)$, the kernel of D_t, is a measurable function from $\mathcal{X}\times\mathcal{X}$ to $\mathbb{C}$ satisfying the following estimates: There exists a positive constant L_0 such that, for all $t \in (0,\infty)$ and all $x, \widetilde{x}, y \in \mathcal{X}$ with $d(x,\widetilde{x}) \leqslant [t + d(x,y)]/2$,

(A3) $|D_t(x,y)| \leqslant L_0 \frac{1}{V_t(x)+V_t(y)+V(x,y)}[\frac{t}{t+d(x,y)}]^{\epsilon_2}$;

(A4) $|D_t(x,y) - D_t(\widetilde{x},y)| \leqslant L_0[\frac{d(x,\widetilde{x})}{t+d(x,y)}]^{\epsilon_1} \frac{1}{V_t(x)+V_t(y)+V(x,y)}[\frac{t}{t+d(x,y)}]^{\epsilon_3}$;

(A5) Property (A4) also holds true with the roles of x and y interchanged;

(A6) $\int_{\mathcal{X}} D_t(x,y)d\mu(x) = 0$;

(A7) $\int_{\mathcal{X}} D_t(x,y)d\mu(y) = 0$.

Now we recall the following *Calderón reproducing formula* which is a continuous variant of [22, Theorem 3.10]. Hereafter, we let $a\wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$ for all $a, b \in \mathbb{R}$.

Lemma 10.1. *Let $\epsilon_1 := 1$, $\epsilon_2, \epsilon_3 \in (0,\infty)$, $\epsilon \in (0,\epsilon_1 \wedge \epsilon_2)$ and $\{D_t\}_{t\in(0,\infty)}$ be as above. Then there exists a family $\{\widetilde{D}_t\}_{t\in(0,\infty)}$ of linear operators such that, for all $f \in \mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma)$ with $\beta, \gamma \in (0,\epsilon)$,*

$$f = \int_0^\infty \widetilde{D}_t D_t(f)\,\frac{dt}{t}$$

in $\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma)$ and in $L^q(\mu)$ for all $q \in (1,\infty)$. Moreover, the kernels of the operators $\widetilde{D}_t$ satisfy the conditions (A3), (A4), (A6) *and* (A7) *with ϵ_1 and ϵ_2 replaced by $\widetilde{\epsilon} \in (\epsilon,\epsilon_1\wedge\epsilon_2)$.*

To the best of our knowledge, the following useful property is well known but there exists no complete proof. We present full details here.

Lemma 10.2. *Let ϵ_1 be as in* (A4)*, $\epsilon \in (0,\epsilon_1]$, $\beta, \gamma \in (0,\epsilon]$ and $q \in (1,\infty)$. Then $\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma)$ is dense in $L^q(\mu)$.*

To prove this lemma, we need the following two technical conclusions. Hereafter, for any $\epsilon \in (0,\infty)$, we denote $\|\cdot\|_\epsilon^{(d)}$, which is as in (9.1) with δ and α replaced respectively by d and ϵ, simply by $\|\cdot\|_\epsilon$.

Lemma 10.3. *Let $\epsilon \in (0,1]$, F be a nonempty closed set and G an open set containing F. Then there exists $f \in \mathrm{Lip}_\epsilon(\mu)$ such that*

$$f = 1 \ \text{ on } \ F, \quad \operatorname{supp}(f) \subset \overline{G}, \quad 0 \leqslant f \leqslant 1 \ \text{ on } \ \mathcal{X},$$

where, for any set $A \subset \mathcal{X}$, $\overline{A}$ represents the smallest closed set containing A.

Proof. For $x \in \mathcal{X}$, let

$$f(x) := \left[\frac{d(x, G^\complement)}{d(x, F) + d(x, G^\complement)}\right]^\epsilon,$$

here and hereafter, for any two sets $A, B \subset \mathcal{X}$, $A^\complement := \mathcal{X}\backslash A$ and

$$d(A,B) := \inf\{d(a,b) : a \in A \text{ and } b \in B\}.$$

It is easy to show that f has all the required properties in Lemma 10.3 with $\|f\|_\epsilon \leqslant \frac{1}{[d(F,G^\complement)]^\epsilon}$, which completes the proof of Lemma 10.3. □

Moreover, we have the following conclusion.

Lemma 10.4. *Let $\epsilon \in (0,1]$. For any ball $B(x_0,r_0)$ and $\eta \in (0,\infty)$, there exists*

$$h \in \mathrm{Lip}_{\epsilon,b}(\mu) := \{f \in \mathrm{Lip}_\epsilon(\mu) : \operatorname{supp}(f) \text{ is bounded}\}$$

such that

$$\operatorname{supp}(h) \subset (B(x_0, r_0))^{\complement}, \quad \int_{\mathcal{X}} h(x)d\mu(x) = 1 \quad \text{and} \quad \|h\|_{L^q(\mu)} < \eta.$$

Proof. For any $k \in \mathbb{N}$, let $F_k := \overline{B(x_0, k + r_0 + 1)} \backslash B(x_0, r_0 + 1)$ and

$$G_k := B(x_0, k + r_0 + 2) \backslash \overline{B(x_0, r_0)}.$$

From [22, Remark 1.2(i)], it follows that $G_k \supset F_k \neq \emptyset$ for some sufficiently large k. By Lemma 10.3 with F and G replaced by F_k and G_k, respectively, we conclude that there exists $\widetilde{h}_k \in \mathrm{Lip}_\epsilon(\mu)$ ($\|\widetilde{h}_k\|_\epsilon \leqslant 1$) such that $\widetilde{h}_k = 1$ on F_k, $\operatorname{supp}(\widetilde{h}_k) \subset \overline{G_k}$ and $0 \leqslant \widetilde{h}_k \leqslant 1$ on $\mathcal{X}$. Let $h_k := \frac{\widetilde{h}_k}{\boldsymbol{J}_k}$, where $\boldsymbol{J}_k := \int_{\mathcal{X}} \widetilde{h}_k(x)d\mu(x)$. Then $\int_{\mathcal{X}} h_k(x)d\mu(x) = 1$ and

$$\boldsymbol{J}_k \geqslant \int_{F_k} d\mu(x) \geqslant \mu(B(x_0, k + r_0 + 1) \backslash B(x_0, r_0 + 1)). \tag{10.2}$$

We only need to show that $\lim_{k\to\infty} \|h_k\|^q_{L^q(\mu)} = 0$. By (10.2) and $0 \leqslant \widetilde{h}_k \leqslant 1$ on $\mathcal{X}$, we have

$$\|h_k\|^q_{L^q(\mu)} \leqslant \frac{\mu(B(x_0, k + r_0 + 3) \backslash B(x_0, r_0))}{\mu(B(x_0, k + r_0 + 1) \backslash B(x_0, r_0 + 1))} \frac{1}{[\mu(B(x_0, k + r_0 + 1) \backslash B(x_0, r_0 + 1))]^{q-1}}.$$

Noticing that $\mu(\mathcal{X}) = \infty$, we obtain

$$\lim_{k\to\infty} \frac{1}{[\mu(B(x_0, k + r_0 + 1) \backslash B(x_0, r_0 + 1))]^{q-1}} = 0,$$

which reduces the proof to the fact that

$$\limsup_{k\to\infty} \frac{\mu(B(x_0, k + r_0 + 3) \backslash B(x_0, r_0))}{\mu(B(x_0, k + r_0 + 1) \backslash B(x_0, r_0 + 1))} \leqslant C,$$

where C is as in (10.1).

Indeed, from $\mu(\mathcal{X}) = \infty$ and (10.1), we deduce that

$$\begin{aligned}
&\limsup_{k\to\infty} \frac{\mu(B(x_0, k + r_0 + 3) \backslash B(x_0, r_0))}{\mu(B(x_0, k + r_0 + 1) \backslash B(x_0, r_0 + 1))} \\
&\quad \leqslant 1 + \limsup_{k\to\infty} \frac{\mu(B(x_0, k + r_0 + 3) \backslash B(x_0, k + r_0 + 1))}{\mu(B(x_0, k + r_0 + 1) \backslash B(x_0, r_0 + 1))} \\
&\quad \leqslant 1 + \limsup_{k\to\infty} \frac{[C(\frac{k+r_0+3}{k+r_0+1})^\nu - 1]\mu(B(x_0, k + r_0 + 1))}{\mu(B(x_0, k + r_0 + 1) \backslash B(x_0, r_0 + 1))} \leqslant C,
\end{aligned}$$

where ν and C are as in (10.1). This finishes the proof of Lemma 10.4. □

Now we are ready to prove Lemma 10.2.

Proof of Lemma 10.2. By some arguments as in [22, p. 19], we know that

$$\mathring{\mathrm{Lip}}_{\epsilon,\, b}(\mu) := \left\{ f \in \mathrm{Lip}_{\epsilon,\, b}(\mu) : \int_{\mathcal{X}} f(x)d\mu(x) = 0 \right\} \subset \mathring{\mathcal{G}}(\epsilon, \epsilon) \subset \mathring{\mathcal{G}}^\epsilon_0(\beta, \gamma) \subset L^2(\mu).$$

Thus, to show Lemma 10.2, it suffices to prove that $\mathring{\mathrm{Lip}}_{\epsilon,\, b}(\mu)$ is dense in $L^q(\mu)$. By the fact that $\mathrm{Lip}_{\epsilon,\, b}(\mu)$ is dense in $L^q(\mu)$ (see, for example, [22, Corollary 2.11(ii)]), we know that, for any $\eta \in (0, \infty)$ and $f \in L^q(\mu)$, there exists $g \in \mathrm{Lip}_{\epsilon,\, b}(\mu)$ such that $\|g - f\|_{L^q(\mu)} < \eta/2$. Now we show that there exists $\widetilde{g} \in \mathring{\mathrm{Lip}}_{\epsilon,\, b}(\mu)$ such that $\|\widetilde{g} - f\|_{L^q(\mu)} < \eta$. We consider the following two cases:

Case (i) $\int_{\mathcal{X}} g(x)d\mu(x) = 0$. The result holds true immediately.

Case (ii) $\int_{\mathcal{X}} g(x)d\mu(x) = A \neq 0$. Let a ball $B(x_0, r_0) \supset \operatorname{supp}(g)$. By Lemma 10.4, there exists $h \in \mathrm{Lip}_{\epsilon,\, b}(\mu)$ such that $\operatorname{supp}(h) \subset (B(x_0, r_0))^{\complement}$, $\int_{\mathcal{X}} h(x)d\mu(x) = 1$ and $\|h\|_{L^q(\mu)} < \frac{\eta}{2|A|}$, which implies that $\widetilde{g} := g - Ah \in \mathring{\mathrm{Lip}}_{\epsilon,\, b}(\mu)$ and

$$\|f - \widetilde{g}\|_{L^q(\mu)} \leqslant \|f - g\|_{L^q(\mu)} + |A|\|h\|_{L^q(\mu)} < \frac{\eta}{2} + |A|\frac{\eta}{2|A|} = \eta.$$

This, together with Case (i), finishes the proof of Lemma 10.2. □

Recall that the *Littlewood-Paley S-function* $S(f)(x)$ for any $f \in L^q(\mu)$ with $q \in (1,\infty)$ and $x \in \mathcal{X}$ is defined by

$$S(f)(x) := \left\{ \int_{\Gamma(x)} |D_t(f)(y)|^2 \frac{d\mu(y)dt}{V_t(x)\,t} \right\}^{1/2},$$

where

$$\Gamma(x) := \{(y,t) \in \mathcal{X} \times (0,\infty) : d(y,x) < t\}$$

generalizes the notion of a cone with vertex at x and aperture 1.

In RD-spaces, Han et al. [21] introduced the *Hardy space* $H^p(\mu)$ defined by

$$H^p(\mu) := \{f \in (\mathring{\mathcal{G}}_0^{\epsilon}(\beta,\gamma))^* : S(f) \in L^p(\mu)\}$$

endowed with the quasi-norm

$$\|f\|_{H^p(\mu)} := \|S(f)\|_{L^p(\mu)}.$$

Moreover, Grafakos et al. [19] proved that $H^p_{\rm at}(\mathcal{X})$ and $H^p(\mu)$ coincide with equivalent quasi-norms.

Before dealing with the relation between $\widetilde{H}^{p,\,q,\,\gamma}_{{\rm atb},\,\rho}(\mu)$ and $H^p_{\rm at}(\mu)$, we need the following construction of dyadic cubes on spaces of homogeneous type from [7]; see also [21].

Lemma 10.5. *Let $\mathcal{X}$ be a space of homogeneous type. Then there exist a collection*

$$\{Q^k_\beta \subset \mathcal{X} : k \in \mathbb{Z},\ \beta \in I_k\}$$

of open subsets, where I_k is some index set, and positive constants $\delta \in (0,1)$ and L_1, L_2 such that

(i) $\mu(\mathcal{X}\setminus \bigcup_\beta Q^k_\beta) = 0$ for each fixed k, and $Q^k_\beta \cap Q^k_\gamma = \emptyset$ if $\beta \neq \gamma$;

(ii) for any β, γ, k and l with $l \geqslant k$, either $Q^l_\gamma \subset Q^k_\beta$ or $Q^l_\gamma \cap Q^k_\beta = \emptyset$;

(iii) for each (k,β) and each $l < k$, there exists a unique γ such that $Q^k_\beta \subset Q^l_\gamma$;

(iv) $\operatorname{diam}(Q^k_\beta) \leqslant L_1\delta^k$;

(v) each Q^k_β contains some ball $B(z^k_\beta, L_2\delta^k)$, where $z^k_\beta \in \mathcal{X}$.

We further introduce some notation from [21]. Let

$$\mathcal{R} := \left\{Q^k_\beta \subset \mathcal{X} : k \in \mathbb{Z},\ \beta \in I_k\right\}.$$

For any $k \in \mathbb{Z}$, let $\Omega_k := \{x \in \mathcal{X} : S(f)(x) > 2^k\}$ and

$$\mathcal{R}_k := \left\{Q \in \mathcal{R} : \mu(Q \cap \Omega_k) > \frac{1}{2}\mu(Q) \text{ and } \mu(Q \cap \Omega_{k+1}) \leqslant \frac{1}{2}\mu(Q)\right\}.$$

Moreover, for any $Q^k_\beta \in \mathcal{R}$, let

$$\widehat{Q}^k_\beta := \{(x,t) \in \mathcal{X} \times (0,\infty) : x \in Q^k_\beta \text{ and } L_1\delta^k < t \leqslant L_1\delta^{k-1}\},$$
$$\mathcal{R}^{\rm mc}_k := \{Q \in \mathcal{R}_k : \text{if } \widetilde{Q} \supset Q \text{ and } \widetilde{Q} \in \mathcal{R}, \text{ then } \widetilde{Q} \notin \mathcal{R}_k\},$$

and, for any $Q^{\rm mc}_k \in \mathcal{R}^{\rm mc}_k$,

$$\widetilde{Q}^{\rm mc}_k := \bigcup_{Q \in \mathcal{R}_k,\, Q \subset Q^{\rm mc}_k} \widehat{Q},$$

here and hereafter, "mc" means *maximal cubes*.

We need the following useful lemma.

Lemma 10.6. *Let k, $j \in \mathbb{Z}$ and $k < j$. Then*

$$\left(\bigcup_{Q^{\rm mc}_k \in \mathcal{R}^{\rm mc}_k} \widetilde{Q}^{\rm mc}_k\right) \cap \left(\bigcup_{Q^{\rm mc}_j \in \mathcal{R}^{\rm mc}_j} \widetilde{Q}^{\rm mc}_j\right) = \emptyset.$$

Proof. Let $k,\, j \in \mathbb{Z}$ and $k < j$. To prove this lemma, it suffices to prove that, for any two dyadic cubes Q and P satisfying $Q \in \mathcal{R}_k$, $Q \subset Q_k^{\rm mc} \in \mathcal{R}_k^{\rm mc}$, $P \in \mathcal{R}_j$ and $P \subset Q_j^{\rm mc} \in \mathcal{R}_j^{\rm mc}$, it holds true that $\widehat{Q} \cap \widehat{P} = \emptyset$. We prove it by contradiction. Suppose that $\widehat{Q} \cap \widehat{P} \neq \emptyset$. Then $Q \cap P \neq \emptyset$. By Lemma 10.5(ii), $Q \subset P$ or $P \subset Q$. Without loss of generality, we may assume that $Q \subset P$. Let $Q = Q_\beta^m$ and $P = Q_\gamma^n$ for some $m,\, n \in \mathbb{Z}$, $\beta \in I_m$ and $\gamma \in I_n$. If $m \neq n$, then $\widehat{Q} \cap \widehat{P} = \emptyset$, which contradicts to the assumption that $\widehat{Q} \cap \widehat{P} \neq \emptyset$. Thus, $m = n$, which, together with Lemma 10.5(i), implies that $Q_\beta^m = Q_\gamma^n$. Moreover, from $Q \in \mathcal{R}_k$, it follows that $\mu(Q \cap \Omega_k) > \frac{1}{2}\mu(Q)$ and $\mu(Q \cap \Omega_{k+1}) \leqslant \frac{1}{2}\mu(Q)$. On the other hand, by $P \in \mathcal{R}_j$, we see that $\mu(P \cap \Omega_j) > \frac{1}{2}\mu(P)$ and $\mu(P \cap \Omega_{j+1}) \leqslant \frac{1}{2}\mu(P)$. Meanwhile, $k < j$ implies that $\Omega_j \subset \Omega_{k+1}$ and hence $\mu(P \cap \Omega_{k+1}) > \frac{1}{2}\mu(P)$. Thus, $P \neq Q$, which contradicts to the fact that $Q = Q_\beta^m = Q_\gamma^n = P$. This finishes the proof of Lemma 10.6. □

Now we introduce some useful decompositions of $\widetilde{D}_t(x, y)$ in Lemma 10.1 which are easy consequences of [22, Proposition 2.9], the details being omitted.

Lemma 10.7. *Let $\epsilon_1 \in (0, 1]$, $\epsilon_2 \in (0, \infty)$, $\epsilon \in (0, \epsilon_1 \wedge \epsilon_2)$, $\widetilde{\epsilon} \in (\epsilon, \epsilon_1 \wedge \epsilon_2)$ and $\{\widetilde{D}_t\}_{t\in(0,\infty)}$ be as in Lemma 10.1. Then, for any $N \in (0, \widetilde{\epsilon}]$, $t \in (0, \infty)$ and $x,\, y \in \mathcal{X}$,*

$$\widetilde{D}_t(x, y) = \sum_{\ell=0}^{\infty} 2^{-N\ell} \varphi_{2^\ell t}(x, y),$$

where $\varphi_{2^\ell t}(x, y)$ is an adjust bump function in x associated with the ball $B(y, 2^\ell t)$, which means that there exists a positive constant C such that, for all $t \in (0, \infty)$ and $y \in \mathcal{X}$,

(i) $\operatorname{supp}(\varphi_{2^\ell t}(\cdot, y)) \subset B(y, 2^\ell t)$;

(ii) $|\varphi_{2^\ell t}(x, y)| \leqslant C \frac{1}{V_{2^\ell t}(y)}$ *for all* $x \in \mathcal{X}$;

(iii) $\|\varphi_{2^\ell t}(\cdot, y)\|_\epsilon \leqslant C(2^\ell t)^{-\eta} \frac{1}{V_{2^\ell t}(y)}$ *for all* $0 < \eta \leqslant \widetilde{\epsilon}$;

(iv) $\int_{\mathcal{X}} \varphi_{2^\ell t}(x, y) d\mu(x) = 0$.

Then we introduce a useful criterion for the boundedness of some integral operators. Hereafter, we denote the inner product of $L^2(\mu)$ by $(\cdot, \cdot)$.

Lemma 10.8. *Let $K_t(\cdot, \cdot)$ for $t \in (0, \infty)$ be a measurable function from $\mathcal{X} \times \mathcal{X}$ to $\mathbb{C}$ and $\{K_t\}_{t\in(0,\infty)}$ a set of $L^2(\mu)$-bounded linear operators defined by*

$$K_t(f)(x) := \int_{\mathcal{X}} K_t(x, y) f(y) d\mu(y) \quad \textit{for all } t \in (0, \infty),\ x \in \mathcal{X} \textit{ and } f \in L^2(\mu).$$

If there exist positive constants ϵ_1, ϵ_2 and C such that, for all $x,\, y \in \mathcal{X}$ and $s,\, t \in (0, \infty)$,

$$|K_t(x, y)| \leqslant C \frac{1}{V_t(x) + V_t(y) + V(x, y)} \left[\frac{t}{t + d(x, y)}\right]^{\epsilon_2}, \tag{10.3}$$

and

$$|K_t K_s^*(x, y)| \leqslant C \left(\frac{t}{s} \wedge \frac{s}{t}\right)^{\epsilon_1} \frac{1}{V_{t\vee s}(x) + V_{t\vee s}(y) + V(x, y)} \left[\frac{t \vee s}{t \vee s + d(x, y)}\right]^{\epsilon_2}, \tag{10.4}$$

where K_t^* denotes the adjoint operator of K_t. Then there exists a positive constant $\widetilde{C}$ such that, for all $f \in L^2(\mu)$,

$$\|G(f)\|_{L^2(\mu)} \leqslant \widetilde{C} \|f\|_{L^2(\mu)},$$

where, for all $x \in \mathcal{X}$,

$$G(f)(x) := \left\{ \int_0^\infty |K_t(f)(x)|^2 \frac{dt}{t} \right\}^{1/2}.$$

Proof. For any $f \in L^2(\mu)$, by Fubini's theorem, we write

$$\|G(f)\|_{L^2(\mu)}^2 = (G(f), G(f)) = \int_{\mathcal{X}} \int_0^\infty |K_t(f)(x)|^2 \frac{dt}{t} d\mu(x)$$

$$
\begin{aligned}
&= \int_0^\infty \int_{\mathcal{X}} |K_t(f)(x)|^2 d\mu(x) \frac{dt}{t} = \lim_{N\to\infty} \int_{1/N}^N (K_t(f), K_t(f)) \frac{dt}{t} \\
&= \lim_{N\to\infty} \int_{1/N}^N \int_{\mathcal{X}} K_t^* K_t(f)(x) f(x) d\mu(x) \frac{dt}{t} \\
&= \lim_{N\to\infty} \int_{\mathcal{X}} \int_{1/N}^N K_t^* K_t(f)(x) \frac{dt}{t} f(x) d\mu(x) = \lim_{N\to\infty} \left(\int_{1/N}^N K_t^* K_t(f) \frac{dt}{t}, f \right).
\end{aligned}
$$

Moreover, from (10.3), (10.4) and the Schur lemma (see [17, p. 457]), we deduce that, for all $s,\, t \in (0,\infty)$, K_t^* and $K_s^{**} = K_s$ are bounded on $L^2(\mu)$ and

$$
\|K_t^* K_t K_s^* K_s\|_{L^2(\mu)\to L^2(\mu)} \lesssim \|K_t K_s^*\|_{L^2(\mu)\to L^2(\mu)} \lesssim \left(\frac{t}{s} \wedge \frac{s}{t}\right)^{\epsilon_1}.
$$

By the above two inequalities and [18, p. 237, Exercise 8.5.8], we conclude that

$$
\|G(f)\|_{L^2(\mu)}^2 \leqslant \liminf_{N\to\infty} \left\| \int_{1/N}^N K_t^* K_t(f) \frac{dt}{t} \right\|_{L^2(\mu)} \|f\|_{L^2(\mu)} \lesssim \|f\|_{L^2(\mu)}^2,
$$

which completes the proof of Lemma 10.8. □

Before showing the main result of this section, we introduce another technical lemma which gives a sufficient condition to the fact that $f = g$ in $L^2(\mu)$ for all $f,\, g \in (\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma))^*$.

Lemma 10.9. *Let ϵ_1 be as in* (A4) *and $\beta,\, \gamma \in (0,\epsilon_1)$. If $f,\, g \in L^2(\mu)$ and $f = g$ in $(\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma))^*$, then $f = g$ in $L^2(\mu)$.*

Proof. For any $f,\, g \in L^2(\mu)$, let

$$
(f,g) := \int_{\mathcal{X}} f(x) g(x) d\mu(x).
$$

Now we claim that $(f,\cdot)$ is a bounded functional on $\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma) \subset L^2(\mu)$. Indeed, by (A1) and Hölder's inequality, we conclude that, for any $f \in L^2(\mu)$ and $h \in \mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma)$,

$$
\begin{aligned}
|(f,h)| &\leqslant \int_{\mathcal{X}} |f(x)||h(x)| d\mu(x) \\
&\lesssim \|h\|_{\mathcal{G}(\beta,\gamma)} \int_{\mathcal{X}} \frac{1}{V_1(x_1) + V(x_1,x)} \left[\frac{1}{1+d(x,x_1)}\right]^\gamma |f(x)| d\mu(x) \\
&\lesssim \|h\|_{\mathcal{G}(\beta,\gamma)} \|f\|_{L^2(\mu)} \left[\int_{\mathcal{X}} \left\{ \frac{1}{V_1(x_1) + V(x_1,x)} \left[\frac{1}{1+d(x,x_1)}\right]^\gamma \right\}^2 d\mu(x) \right]^{\frac{1}{2}} \\
&\lesssim \|h\|_{\mathcal{G}(\beta,\gamma)} \|f\|_{L^2(\mu)} \\
&\quad \times \left\{ \int_{B(x_1,1)} \left[\frac{1}{V_1(x_1)}\right]^2 d\mu(x) + \frac{1}{V_1(x_1)} \int_{\mathcal{X}\setminus B(x_1,1)} \frac{1}{V(x_1,x)} \left[\frac{1}{d(x,x_1)}\right]^{2\gamma} d\mu(x) \right\}^{\frac{1}{2}} \\
&\lesssim \|h\|_{\mathcal{G}(\beta,\gamma)} \|f\|_{L^2(\mu)} \frac{1}{V_1(x_1)},
\end{aligned}
$$

which implies the claim.

From this, a density argument, Lemma 10.2 and $f = g$ in $(\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma))^*$, it follows that

$$
\begin{aligned}
\|f-g\|_{L^2(\mu)} &= \sup\{|(f-g,h)| : \|h\|_{L^2(\mu)} \leqslant 1\} \\
&= \sup\{|(f-g,h)| : h \in \mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma) \text{ and } \|h\|_{L^2(\mu)} \leqslant 1\} \\
&= \sup\{|(f,h)-(g,h)| : h \in \mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma) \text{ and } \|h\|_{L^2(\mu)} \leqslant 1\} = 0,
\end{aligned}
$$

which completes the proof of Lemma 10.9. □

Now we are ready to prove the following main result of this section.

Theorem 10.10. *Let $(\mathcal{X}, d, \mu)$ be an* RD*-space with $\mu(\mathcal{X}) = \infty$, $\frac{\nu}{\nu+1} < p \leqslant 1 < q \leqslant 2$, $\rho \in (1, \infty)$, $\gamma \in [1, \infty)$ and $\epsilon \in (0, \infty)$, where ν is as in* (10.1)*. Then $\widetilde{H}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)$, $\widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$ and $H^p_{\mathrm{at}}(\mu)$ coincide with equivalent quasi-norms.*

Proof. Let p, q, ρ, γ and ϵ be as in assumptions of Theorem 10.10. We first claim that $H^p_{\mathrm{at}}(\mu) \cap L^2(\mu)$ is dense in $H^p_{\mathrm{at}}(\mu)$. Indeed, for any $f \in H^p_{\mathrm{at}}(\mu)$, by Theorem 9.4, we have $f = \sum_{i=1}^{\infty} \lambda_i b_i$ in $(\mathrm{Lip}_{1/p-1}(\mu))^*$, where $\{b_i\}_i$ is a sequence of $(p, 2)$-atoms, $\mathrm{supp}\,(b_i) \subset B_i$ for some ball B_i and $\|b_i\|_{L^2(\mu)} \leqslant [\mu(B_i)]^{1/2-1/p}$. Let $f_N := \sum_{i=1}^{N} \lambda_i b_i$, $N \in \mathbb{N}$. Then $f_N \in L^2(\mu)$ for all $N \in \mathbb{N}$. Meanwhile, $f - f_N = \sum_{i=N+1}^{\infty} \lambda_i b_i$ in $(\mathrm{Lip}_{1/p-1}(\mu))^*$, and $\|f - f_N\|^p_{H^p_{\mathrm{at}}(\mu)} \leqslant \sum_{i=N+1}^{\infty} |\lambda_i|^p \to 0$ as $N \to \infty$. Thus, $H^p_{\mathrm{at}}(\mu) \cap L^2(\mu)$ is dense in $H^p_{\mathrm{at}}(\mu)$, which completes the proof of this claim.

We easily observe that $L^2(\mu) \subset (\mathring{\mathcal{G}}^{\epsilon}_0(\beta, \gamma))^*$. By [19, Remark 5.5(ii)], we know that $H^p_{\mathrm{at}}(\mu) = H^p(\mu)$ with equivalent quasi-norms. By this, the above claim and a standard density argument, to show Theorem 10.10, it suffices to prove that

$$(H^p_{\mathrm{at}}(\mu) \cap L^2(\mu)) \subset \widetilde{\mathbb{H}}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu) \subset \widetilde{\mathbb{H}}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu) \subset (H^p(\mu) \cap L^2(\mu)).$$

We show this by two steps.

Step 1. Now we show that $(H^p_{\mathrm{at}}(\mu) \cap L^2(\mu)) \subset \widetilde{\mathbb{H}}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$. By Remark 3.3(iii) and $q \in (1, 2]$, it suffices to show that $(H^p_{\mathrm{at}}(\mu) \cap L^2(\mu)) \subset \widetilde{\mathbb{H}}^{p,\,2,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$. To this end, by Lemma 10.1 and an argument similar to that used in the proof of [22, p. 1524, (2.30)], we know that, for any $f \in (\mathring{\mathcal{G}}^{\epsilon}_0(\beta, \gamma))^*$ with $0 < \beta, \gamma < \epsilon < 1 \wedge \epsilon_2$, $\epsilon_3 \in (0, \infty)$ and $\widetilde{\epsilon} \in (\epsilon, 1 \wedge \epsilon_2)$ (ϵ_2 and ϵ_3 are as in (A4)),

$$f(x) = \sum_{\ell=0}^{\infty} 2^{-N\ell} \sum_{k \in \mathbb{Z}} \sum_{Q^{\mathrm{mc}}_k \in \mathcal{R}^{\mathrm{mc}}_k} \int_{\widetilde{Q}^{\mathrm{mc}}_k} \varphi_{2^\ell t}(x, y) D_t(f)(y) \,\frac{d\mu(y)dt}{t} \quad \text{in} \quad (\mathring{\mathcal{G}}^{\epsilon}_0(\beta, \gamma))^*,$$

where $\varphi_{2^\ell t}(x, y)$ is as in Lemma 10.7. Then we show that, for any $f \in H^p_{\mathrm{at}}(\mu) \cap L^2(\mu)$,

$$f(x) = \sum_{\ell=0}^{\infty} 2^{-N\ell} \sum_{k \in \mathbb{Z}} \sum_{Q^{\mathrm{mc}}_k \in \mathcal{R}^{\mathrm{mc}}_k} \int_{\widetilde{Q}^{\mathrm{mc}}_k} \varphi_{2^\ell t}(x, y) D_t(f)(y) \,\frac{d\mu(y)dt}{t} \quad \text{in} \quad L^2(\mu). \tag{10.5}$$

By Lemma 10.9, it suffices to prove that

$$\left\| \sum_{\ell=0}^{\infty} 2^{-N\ell} \sum_{k \in \mathbb{Z}} \sum_{Q^{\mathrm{mc}}_k \in \mathcal{R}^{\mathrm{mc}}_k} \int_{\widetilde{Q}^{\mathrm{mc}}_k} \varphi_{2^\ell t}(\cdot, y) D_t(f)(y) \,\frac{d\mu(y)dt}{t} \right\|_{L^2(\mu)} < \infty.$$

To this end, for any $x, y \in \mathcal{X}$, $t, s \in (0, \infty)$ and $f \in L^2(\mu)$, let

$$\varphi_{2^\ell t}(f)(x) := \int_{\mathcal{X}} \varphi_{2^\ell t}(x, z) f(z) d\mu(z).$$

Then

$$\varphi^*_{2^\ell t}(f)(x) := \int_{\mathcal{X}} \varphi_{2^\ell t}(z, x) f(z) d\mu(z)$$

and

$$\varphi^*_{2^\ell t} \varphi_{2^\ell s}(x, y) := \int_{\mathcal{X}} \varphi_{2^\ell t}(z, x) \varphi_{2^\ell t}(z, y) d\mu(z).$$

By a duality method, Lemma 10.2, Fubini's theorem, Hölder's inequality, Lemma 10.6 and [21, Proposition 2.14], we obtain

$$\left\| \sum_{\ell=0}^{\infty} 2^{-N\ell} \sum_{k \in \mathbb{Z}} \sum_{Q^{\mathrm{mc}}_k \in \mathcal{R}^{\mathrm{mc}}_k} \int_{\widetilde{Q}^{\mathrm{mc}}_k} \varphi_{2^\ell t}(\cdot, y) D_t(f)(y) \,\frac{d\mu(y)dt}{t} \right\|_{L^2(\mu)}$$

$$
\begin{aligned}
&= \sup_{\substack{\|h\|_{L^2(\mu)}\leqslant 1\\ h\in\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma)}} \left|\left\langle \sum_{\ell=0}^{\infty} 2^{-N\ell} \sum_{k\in\mathbb{Z}} \sum_{Q_k^{\rm mc}\in\mathcal{R}_k^{\rm mc}} \int_{\widetilde{Q}_k^{\rm mc}} \varphi_{2^\ell t}(\cdot,y) D_t(f)(y)\,\frac{d\mu(y)dt}{t}, h\right\rangle\right| \\
&= \sup_{\substack{\|h\|_{L^2(\mu)}\leqslant 1\\ h\in\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma)}} \left|\sum_{\ell=0}^{\infty} 2^{-N\ell} \sum_{k\in\mathbb{Z}} \sum_{Q_k^{\rm mc}\in\mathcal{R}_k^{\rm mc}} \left\langle \int_{\widetilde{Q}_k^{\rm mc}} \varphi_{2^\ell t}(\cdot,y) D_t(f)(y)\,\frac{d\mu(y)dt}{t}, h\right\rangle\right| \\
&= \sup_{\substack{\|h\|_{L^2(\mu)}\leqslant 1\\ h\in\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma)}} \left|\sum_{\ell=0}^{\infty} 2^{-N\ell} \sum_{k\in\mathbb{Z}} \sum_{Q_k^{\rm mc}\in\mathcal{R}_k^{\rm mc}} \int_{\widetilde{Q}_k^{\rm mc}} \left[\int_{\mathcal{X}} \varphi_{2^\ell t}(x,y)h(x)d\mu(x)\right] D_t(f)(y)\,\frac{d\mu(y)dt}{t}\right| \\
&\leqslant \sup_{\|h\|_{L^2(\mu)}\leqslant 1} \sum_{\ell=0}^{\infty} 2^{-N\ell} \sum_{k\in\mathbb{Z}} \sum_{Q_k^{\rm mc}\in\mathcal{R}_k^{\rm mc}} \left|\int_{\widetilde{Q}_k^{\rm mc}} \left[\int_{\mathcal{X}} \varphi_{2^\ell t}(x,y)h(x)d\mu(x)\right] D_t(f)(y)\,\frac{d\mu(y)dt}{t}\right| \\
&\leqslant \sup_{\|h\|_{L^2(\mu)}\leqslant 1} \sum_{\ell=0}^{\infty} 2^{-N\ell} \left\{\int_{\mathcal{X}}\int_0^\infty |\varphi^*_{2^\ell t}(h)(y)|^2\,\frac{dt}{t}d\mu(y)\right\}^{1/2} \left\{\int_{\mathcal{X}}\int_0^\infty |D_t(f)(y)|^2\,\frac{dt}{t}d\mu(y)\right\}^{1/2} \\
&\lesssim \sup_{\|h\|_{L^2(\mu)}\leqslant 1} \sum_{\ell=0}^{\infty} 2^{-N\ell/2} \left\{\int_{\mathcal{X}}\int_0^\infty |2^{-N\ell/2}\varphi^*_{2^\ell t}(h)(y)|^2\,\frac{dt}{t}d\mu(y)\right\}^{1/2} \|f\|_{L^2(\mu)},
\end{aligned}
$$

where, in the third equality of the above equation, we used the fact that, for any $Q_k^{\rm mc}\in\mathcal{R}_k^{\rm mc}$, $f\in L^2(\mu)$ and $h\in\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma)$,

$$
\mathrm{O} := \int_{\mathcal{X}}\int_{\widetilde{Q}_k^{\rm mc}} |\varphi_{2^\ell t}(x,y)|\,|h(x)||D_t(f)(y)|\,\frac{dt}{t}d\mu(y)d\mu(x) < \infty.
$$

Indeed, let $Q_k^{\rm mc} := Q_{\beta_0}^{k_0}$ for some $k_0\in\mathbb{Z}$ and some $\beta_0\in I_{k_0}$. By Fubini-Tonelli theorem, Lemmas 10.7(i) and 10.7(ii), and (A1), $\widetilde{Q}_k^{\rm mc}\subset Q_{\beta_0}^{k_0}\times(0, L_1\delta^{k_0-1}]$ (see [21, p. 1524]) and [21, Proposition 2.14], we conclude that

$$
\begin{aligned}
\mathrm{O} &= \int_{\widetilde{Q}_k^{\rm mc}} \left[\int_{\mathcal{X}} |\varphi_{2^\ell t}(x,y)||h(x)|d\mu(x)\right] |D_t(f)(y)|\,\frac{dt}{t}d\mu(y) \\
&\lesssim \int_{\widetilde{Q}_k^{\rm mc}} \left[\int_{B(y,2^\ell t)} \frac{|h(x)|}{V_{2^\ell t}(y)}d\mu(x)\right] |D_t(f)(y)|\,\frac{dt}{t}d\mu(y) \\
&\lesssim \|h\|_{\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma)} \frac{1}{V_1(x_1)} \int_{\widetilde{Q}_k^{\rm mc}} |D_t(f)(y)|\,\frac{dt}{t}d\mu(y) \\
&\lesssim \|h\|_{\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma)} \frac{1}{V_1(x_1)} \left[\int_{\widetilde{Q}_k^{\rm mc}} \frac{dt}{t}d\mu(y)\right]^{1/2} \left[\int_{\widetilde{Q}_k^{\rm mc}} |D_t(f)(y)|^2\,\frac{dt}{t}d\mu(y)\right]^{1/2} \\
&\lesssim \|h\|_{\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma)} \frac{1}{V_1(x_1)} [\mu(Q_{\beta_0}^{k_0})L_1\delta^{k_0-1}]^{1/2} \left[\int_{\mathcal{X}}\int_0^\infty |D_t(f)(y)|^2\,\frac{dt}{t}d\mu(y)\right]^{1/2} \\
&\lesssim \|h\|_{\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma)} \frac{1}{V_1(x_1)} [\mu(Q_{\beta_0}^{k_0})L_1\delta^{k_0-1}]^{1/2} \|f\|_{L^2(\mu)} < \infty,
\end{aligned}
$$

which implies the desired result.

Let

$$
\Phi_\ell(h)(y) := \left\{\int_0^\infty |2^{-N\ell/2}\varphi^*_{2^\ell t}(h)(y)|^2\,\frac{dt}{t}\right\}^{1/2}
$$

for all $y\in\mathcal{X}$. To prove (10.5), we only need to show that

$$
\|\Phi_\ell(h)\|_{L^2(\mu)} \lesssim \|h\|_{L^2(\mu)}, \tag{10.6}
$$

where the implicit positive constant is independent of ℓ.

By Lemma 10.8, we need to show that $\{\varphi_{2^\ell t}\}_{t\in(0,\infty)}$ satisfy (10.3) and (10.4). From Lemma 10.7, we easily deduce that (10.3) holds true for $\{\varphi_{2^\ell t}\}_{t\in(0,\infty)}$. Thus, it suffices to show that, for all x, $y\in\mathcal{X}$, s, $t\in(0,\infty)$,

$$\begin{aligned}&|(2^{-N\ell/2}\varphi_{2^\ell t})^*(2^{-N\ell/2}\varphi_{2^\ell s})(x,y)|\\&\quad\lesssim\left(\frac{t}{s}\wedge\frac{s}{t}\right)^\eta\frac{1}{V_{t\vee s}(x)+V_{t\vee s}(y)+V(x,y)}\left[\frac{t\vee s}{t\vee s+d(x,y)}\right]^{N/2}.\end{aligned}$$

Due to the symmetry of t and s, without loss of generality, we may assume that $s<t$. Thus, we only need to show that, for all x, $y\in\mathcal{X}$, $0<s<t<\infty$,

$$|\varphi^*_{2^\ell t}\varphi_{2^\ell s}(x,y)|\lesssim 2^{N\ell}\left(\frac{s}{t}\right)^\eta\frac{1}{V_t(x)+V_t(y)+V(x,y)}\left[\frac{t}{t+d(x,y)}\right]^{N/2}.$$

By Lemma 10.7(iv), we write

$$\begin{aligned}|\varphi^*_{2^\ell t}\varphi_{2^\ell s}(x,y)|&\leqslant\int_{\mathcal{X}}|\varphi_{2^\ell t}(z,x)-\varphi_{2^\ell t}(y,x)||\varphi_{2^\ell s}(z,y)|d\mu(z)\\&\leqslant\int_{\{z\in\mathcal{X}:d(z,y)\leqslant\frac{2^\ell t+d(x,y)}{2}\}}|\varphi_{2^\ell t}(z,x)-\varphi_{2^\ell t}(y,x)||\varphi_{2^\ell s}(z,y)|d\mu(z)\\&\quad+\int_{\{z\in\mathcal{X}:d(z,y)>\frac{2^\ell t+d(x,y)}{2}\}}|\varphi_{2^\ell t}(z,x)||\varphi_{2^\ell s}(z,y)|d\mu(z)\\&\quad+|\varphi_{2^\ell t}(y,x)|\int_{\{z\in\mathcal{X}:d(z,y)>\frac{2^\ell t+d(x,y)}{2}\}}|\varphi_{2^\ell s}(z,y)|d\mu(z)\\&=:\mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3.\end{aligned}$$

We first estimate I_1. Observe that, if $z\in B(x,2^\ell t)$ and $\frac{2^\ell t+d(x,y)}{2}\geqslant d(y,z)\geqslant d(x,y)-d(x,z)$, then $y\in B(x,2^{\ell+2}t)$; if $d(x,y)<2^{\ell+2}t$, then

$$\left(\frac{t}{t+d(x,y)}\right)^{N/2}\geqslant\left(\frac{t}{t+d(x,y)}\right)^{N}\gtrsim 2^{-N\ell}.$$

From the above two facts, Lemmas 10.7(i)–10.7(iii) and Remark 2.4(ii), it follows that

$$\begin{aligned}\mathrm{I}_1&\lesssim\int_{\{z\in\mathcal{X}:d(z,y)\leqslant\frac{2^\ell t+d(x,y)}{2}\}}\|\varphi_{2^\ell t}(\cdot,x)\|_\eta[d(y,z)]^\eta\chi_{B(x,2^{\ell+2}t)}(y)\frac{\chi_{B(y,2^\ell s)}(z)}{V_{2^\ell s}(y)}d\mu(z)\\&\lesssim\int_{\{z\in\mathcal{X}:d(z,y)\leqslant\frac{2^\ell t+d(x,y)}{2}\}}\frac{1}{(2^\ell t)^\eta}\frac{\chi_{B(x,2^{\ell+2}t)}(y)}{V_{2^\ell t}(x)}(2^\ell s)^\eta\frac{\chi_{B(y,2^\ell s)}(z)}{V_{2^\ell s}(y)}d\mu(z)\\&\lesssim\left(\frac{s}{t}\right)^\eta\int_{\{z\in\mathcal{X}:d(z,y)\leqslant\frac{2^\ell t+d(x,y)}{2}\}}\frac{\chi_{B(x,2^{\ell+2}t)}(y)}{V_{2^{\ell+2}t}(x)}\frac{\chi_{B(y,2^\ell s)}(z)}{V_{2^\ell s}(y)}d\mu(z)\\&\lesssim\left(\frac{s}{t}\right)^\eta\frac{\chi_{B(x,2^{\ell+2}t)}(y)}{V_{2^{\ell+2}t}(x)+V_{2^{\ell+2}t}(y)+V(x,y)}\\&\lesssim 2^{N\ell}\left(\frac{s}{t}\right)^\eta\frac{1}{V_t(x)+V_t(y)+V(x,y)}\left[\frac{t}{t+d(x,y)}\right]^{N/2}.\end{aligned}$$

Now we turn to estimate I_2. Observe that, if $\{z\in\mathcal{X}:d(z,y)>\frac{2^\ell t+d(x,y)}{2}\}\neq\emptyset$ and $z\in B(x,2^\ell t)\cap B(y,2^\ell s)$, then $y\in B(x,2^{\ell+1}t)$ and $1\lesssim(\frac{s}{t})^\eta$. By the above facts, Lemmas 10.7(i) and 10.7(ii), and some arguments similar to those used in the estimate for I_1, we further have

$$\mathrm{I}_2\lesssim\int_{\left\{z\in\mathcal{X}:d(z,y)>\frac{2^\ell t+d(x,y)}{2}\right\}}\frac{\chi_{B(x,2^\ell t)}(z)}{V_{2^\ell t}(x)}\frac{\chi_{B(y,2^\ell s)}(z)}{V_{2^\ell s}(y)}d\mu(z)$$

$$\begin{aligned}&\lesssim \frac{\chi_{B(x,2^{\ell+1}t)}(y)}{V_{2^{\ell+1}t}(x)}\int_{\{z\in\mathcal{X}:d(z,y)>\frac{2^\ell t+d(x,y)}{2}\}}\frac{\chi_{B(y,2^\ell s)}(z)}{V_{2^\ell s}(y)}d\mu(z)\\&\lesssim\left(\frac{s}{t}\right)^\eta\frac{1}{V_{2^{\ell+1}t}(x)+V_{2^{\ell+1}t}(y)+V(x,y)}\\&\lesssim 2^{N\ell}\left(\frac{s}{t}\right)^\eta\frac{1}{V_t(x)+V_t(y)+V(x,y)}\left[\frac{t}{t+d(x,y)}\right]^{N/2}.\end{aligned}$$

For I_3, by some arguments similar to those used in the estimate for I_2, we see that

$$\begin{aligned}\mathrm{I}_3&\lesssim \frac{\chi_{B(x,2^{\ell}t)}(y)}{V_{2^{\ell}t}(x)}\int_{\{z\in\mathcal{X}:d(z,y)>\frac{2^\ell t+d(x,y)}{2}\}}\frac{\chi_{B(y,2^\ell t)}(z)}{V_{2^\ell s}(y)}d\mu(z)\\&\lesssim 2^{N\ell}\left(\frac{s}{t}\right)^\eta\frac{1}{V_t(x)+V_t(y)+V(x,y)}\left[\frac{t}{t+d(x,y)}\right]^{N/2}.\end{aligned}$$

Combining the estimates for I_1, I_2 and I_3, we finish the proof of (10.6) and hence (10.5).

Moreover, from the proof of [21, Theorem 2.21], it follows that, for any $f\in H^p_{\mathrm{at}}(\mu)\cap L^2(\mu)$, there exists a positive constant L_3 such that

$$f=\sum_{\ell=0}^\infty 2^{-N\ell}\sum_{k\in\mathbb{Z}}\sum_{Q_k^{\mathrm{mc}}\in\mathcal{R}_k^{\mathrm{mc}}}\lambda^\ell_{Q_k^{\mathrm{mc}}}a^\ell_{Q_k^{\mathrm{mc}}}\quad\text{in}\quad L^2(\mu),$$

where

$$\lambda^\ell_{Q_k^{\mathrm{mc}}}=L_3[\mu(B_k^{\mathrm{mc}})]^{1/p-1/2}\left[\int_{\widetilde{Q}_k^{\mathrm{mc}}}|D_t(f)(y)|^2\,\frac{d\mu(y)dt}{t}\right]^{1/2},$$

$Q_k^{\mathrm{mc}}:=Q^{k_0}_{\beta_0}$, $B_k^{\mathrm{mc}}:=B(z^{k_0}_{\beta_0},(\frac{1}{\delta}+1)L_1 2^\ell\delta^{k_0})$ and, for all $x\in\mathcal{X}$,

$$a^\ell_{Q_k^{\mathrm{mc}}}(x):=\frac{1}{\lambda^\ell_{Q_k^{\mathrm{mc}}}}\int_{\widetilde{Q}_k^{\mathrm{mc}}}\varphi_{2^\ell t}(x,y)D_t(f)(y)\,\frac{d\mu(y)dt}{t}$$

is a $(p,2)$-atom supported on B_k^{mc}. By an argument similar to that used in the proof of (9.6), we further conclude that $a^\ell_{Q_k^{\mathrm{mc}}}$ is also a $(p,2,\gamma,\rho)_\lambda$-atomic block and $|a^\ell_{Q_k^{\mathrm{mc}}}|_{\widetilde{H}^{p,2,\gamma}_{\mathrm{atb},\rho}(\mu)}\lesssim|\lambda^\ell_{Q_k^{\mathrm{mc}}}|$. Thus, $f\in\widetilde{\mathbb{H}}^{p,2,\gamma}_{\mathrm{atb},\rho}(\mu)$ and

$$\|f\|_{\widetilde{H}^{p,2,\gamma}_{\mathrm{atb},\rho}(\mu)}\lesssim\|f\|_{H^p_{\mathrm{at}}(\mu)},$$

which completes the proof of Step 1.

Step 2. In this step, we show that $\widetilde{\mathbb{H}}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)\subset\widetilde{\mathbb{H}}^{p,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)\subset(H^p(\mu)\cap L^2(\mu))$ for any $q\in(1,\infty)$. By Proposition 4.3, we see that $\widetilde{\mathbb{H}}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)\subset\widetilde{\mathbb{H}}^{p,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)$ for any $q\in(1,\infty)$. Thus, to prove the desired conclusion, it suffices to show that $\widetilde{\mathbb{H}}^{p,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)\subset(H^p(\mu)\cap L^2(\mu))$ for any $q\in(1,\infty)$.

We first reduce the proof to showing that, if b is a $(p,q,\gamma,\epsilon,\rho)_\lambda$-molecular block, then

$$S(b)\in L^p(\mu)\quad\text{and}\quad\|S(b)\|_{L^p(\mu)}\lesssim|b|_{\widetilde{H}^{p,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)}.\tag{10.7}$$

Indeed, assume that (10.7) holds true. For any $f\in\widetilde{\mathbb{H}}^{p,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)$, by Definition 4.1, we know that there exists a sequence $\{b_i\}_{i\in\mathbb{N}}$ of $(p,q,\gamma,\epsilon,\rho)_\lambda$-molecular blocks such that $f=\sum_{i=1}^\infty b_i$ in $L^2(\mu)$ and

$$\sum_{i=1}^\infty|b_i|^p_{\widetilde{H}^{p,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)}\sim\|f\|^p_{\widetilde{H}^{p,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)}.\tag{10.8}$$

Notice that $D_t(y,\cdot)\in L^2(\mu)$ for any $y\in\mathcal{X}$ and $t\in(0,\infty)$. Thus, for any $y\in\mathcal{X}$, we have

$$|D_t(f)(y)|=|(D_t(y,\cdot),f)|\leqslant\sum_{i=0}^\infty|(D_t(y,\cdot),b_i)|=\sum_{i=0}^\infty|D_t(b_i)(y)|.$$

From this, the Fatou lemma, (10.7) and (10.8), we deduce that

$$\|S(f)\|_{L^p(\mu)}^p \leqslant \sum_{i=1}^{\infty} \|S(b_i)\|_{L^p(\mu)}^p \lesssim \sum_{i=1}^{\infty} |b_i|_{\widetilde{H}_{\mathrm{mb},\,\rho}^{p,\,q,\,\gamma,\,\epsilon}(\mu)}^p \sim \|f\|_{\widetilde{H}_{\mathrm{mb},\,\rho}^{p,\,q,\,\gamma,\,\epsilon}(\mu)}^p,$$

which completes the proof of Step 2.

Now we prove (10.7) by following the ideas of the proof of Theorem 4.8. For the sake of simplicity, we assume that $\gamma = 1$ and $\rho = 2$. Let $b = \sum_{k=0}^{\infty} \sum_{j=1}^{M_k} \lambda_{k,\,j} a_{k,\,j}$ be a $(p, q, 1, \epsilon, 2)_\lambda$-molecular block as in Definition 4.1 with $\gamma = 1$ and $\rho = 2$, where, for any $k \in \mathbb{Z}_+$ and $j \in \{1, \ldots, M_k\}$, $\operatorname{supp}(a_{k,\,j}) \subset B_{k,\,j} \subset U_k(B)$ for some $B_{k,\,j}$ and $U_k(B)$ as in Definition 4.1. Without loss of generality, we may assume that $\widetilde{M} = M$ in Definition 4.1. Since S is sublinear, we write

$$\begin{aligned}
\|S(b)\|_{L^p(\mu)}^p &\leqslant \sum_{\ell=5}^{\infty} \int_{U_\ell(B)} \left| S\bigg(\sum_{k=0}^{\ell-5} \sum_{j=1}^{M} \lambda_{k,\,j} m_{k,\,j} \bigg)(x) \right|^p d\mu(x) \\
&\quad + \sum_{\ell=5}^{\infty} \int_{U_\ell(B)} \left| S\bigg(\sum_{k=\ell-4}^{\ell+4} \sum_{j=1}^{M} \lambda_{k,\,j} m_{k,\,j} \bigg)(x) \right|^p d\mu(x) \\
&\quad + \sum_{\ell=5}^{\infty} \int_{U_\ell(B)} \left| S\bigg(\sum_{k=\ell+5}^{\infty} \sum_{j=1}^{M} \lambda_{k,\,j} m_{k,\,j} \bigg)(x) \right|^p d\mu(x) + \sum_{\ell=0}^{4} \int_{U_\ell(B)} |S(b)(x)|^p d\mu(x) \\
&=: \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}.
\end{aligned}$$

Now we first estimate III. For any $x \in U_\ell(B)$ and $\ell \in \mathbb{N} \cap [5, \infty)$, by the Minkowski inequality, we see that

$$\begin{aligned}
&S\bigg(\sum_{k=\ell+5}^{\infty} \sum_{j=1}^{M} \lambda_{k,\,j} m_{k,\,j} \bigg)(x) \\
&\quad \leqslant \sum_{k=\ell+5}^{\infty} \sum_{j=1}^{M} |\lambda_{k,\,j}| \bigg\{ \int_{\Gamma(x)} \bigg[\int_{\mathcal{X}} |m_{k,\,j}(z)| |D_t(y,z)| d\mu(z) \bigg]^2 \frac{d\mu(y)dt}{V_t(x)t} \bigg\}^{1/2} \\
&\quad \leqslant \sum_{k=\ell+5}^{\infty} \sum_{j=1}^{M} |\lambda_{k,\,j}| \int_{B_{k,\,j}} \bigg[\int_{\Gamma(x)} |m_{k,\,j}(z)|^2 |D_t(y,z)|^2 \frac{d\mu(y)dt}{V_t(x)t} \bigg]^{1/2} d\mu(z) \\
&\quad \leqslant \sum_{k=\ell+5}^{\infty} \sum_{j=1}^{M} |\lambda_{k,\,j}| \int_{B_{k,\,j}} |m_{k,\,j}(z)| [\mathrm{M}_1(x,z) + \mathrm{M}_2(x,z)] d\mu(z),
\end{aligned}$$

where, for all $x \in \mathcal{X}$ and $z \in B_{k,\,j}$ with $k \in \mathbb{N} \cap [\ell+5, \infty)$ and $j \in \{1, \ldots, M\}$,

$$\mathrm{M}_1(x,z) := \bigg[\int_{\substack{\Gamma(x) \cap \\ \{(y,\,t) \in \mathcal{X} \times (0,\infty):\, t \leqslant \frac{d(x,z)}{2}\}}} |D_t(y,z)|^2 \frac{d\mu(y)dt}{V_t(x)t} \bigg]^{1/2}$$

and

$$\mathrm{M}_2(x,z) := \bigg[\int_{\substack{\Gamma(x) \cap \\ \{(y,\,t) \in \mathcal{X} \times (0,\infty):\, t > \frac{d(x,z)}{2}\}}} |D_t(y,z)|^2 \frac{d\mu(y)dt}{V_t(x)t} \bigg]^{1/2}.$$

For any $x,\, y,\, z \in \mathcal{X}$ satisfying $d(y,x) < t$ and $t \leqslant d(x,z)/2$, it is easy to see that

$$d(y,z) \geqslant d(x,z) - d(y,x) \geqslant \frac{1}{2} d(x,z).$$

It then follows, from this, (A3) and (10.1), that

$$\mathrm{M}_1(x,z) \lesssim \bigg[\frac{1}{[V(x,z)]^2} \int_0^{d(x,z)/2} \bigg(\int_{B(x,t)} \frac{d\mu(y)}{V_t(x)} \bigg) \bigg(\frac{t}{d(x,z)} \bigg)^{\epsilon_2} \frac{dt}{t} \bigg]^{1/2} \lesssim \frac{1}{V(x,z)}$$

and

$$\mathrm{M}_2(x,z) \lesssim \left[\int_{d(x,z)/2}^{\infty}\left(\int_{B(x,t)}\frac{d\mu(y)}{V_t(x)}\right)\frac{dt}{[V_{2t}(z)]^2 t}\right]^{1/2}$$
$$\lesssim \frac{1}{V(x,z)}\left[\int_{d(x,z)/2}^{\infty}\frac{[d(x,z)]^{2\kappa}}{(2t)^{2\kappa}t}dt\right]^{1/2} \lesssim \frac{1}{V(x,z)}.$$

Moreover, by $z \in B_{k,j} \subset 2^{k+2}B\backslash 2^{k-2}B$, $k \geqslant \ell+5$, $x \in 2^{\ell+2}B\backslash 2^{\ell-2}B$, we have $d(x,c_B) < 2^{\ell+2}r_B$, $d(z,c_B) \geqslant 2^{k-2}r_B \geqslant 2^{\ell+3}r_B$ and

$$d(x,z) \geqslant d(z,c_B) - d(x,c_B) \geqslant 2^{\ell+3}r_B - 2^{\ell+2}r_B = 2^{\ell+2}r_B > d(x,c_B),$$

where c_B and r_B denote the center and the radius of B, respectively. Thus, for all $x \in \mathcal{X}$,

$$S\left(\sum_{k=\ell+5}^{\infty}\sum_{j=1}^{M}\lambda_{k,j}m_{k,j}\right)(x) \lesssim \sum_{k=\ell+5}^{\infty}\sum_{j=1}^{M}|\lambda_{k,j}|\frac{1}{V(x,c_B)}\|m_{k,j}\|_{L^1(\mu)}. \tag{10.9}$$

From (10.9), Hölder's inequality, (4.1) and (10.1), we deduce that

$$\begin{aligned}
\mathrm{III} &\lesssim \sum_{\ell=5}^{\infty}\sum_{k=\ell+5}^{\infty}\sum_{j=1}^{M}|\lambda_{k,j}|^p\int_{U_\ell(B)}\frac{1}{[V(x,c_B)]^p}d\mu(x)\|m_{k,j}\|_{L^1(\mu)}^p\\
&\lesssim \sum_{\ell=5}^{\infty}\sum_{k=\ell+5}^{\infty}\sum_{j=1}^{M}|\lambda_{k,j}|^p\frac{V_{2^{\ell+2}r_B}(c_B)}{[V_{2^{\ell-2}r_B}(c_B)]^p}\|m_{k,j}\|_{L^q(\mu)}^p[\mu(B_{k,j})]^{p/q'}\\
&\lesssim \sum_{\ell=5}^{\infty}\sum_{k=\ell+5}^{\infty}\sum_{j=1}^{M}|\lambda_{k,j}|^p[V_{2^{\ell+2}r_B}(c_B)]^{1-p}2^{-k\epsilon p}[\lambda(c_B,2^{k+2}r_B)]^{p-1}\\
&\lesssim \sum_{\ell=5}^{\infty}\sum_{k=\ell+5}^{\infty}\sum_{j=1}^{M}2^{-k\epsilon p}|\lambda_{k,j}|^p \sim \sum_{j=1}^{M}\sum_{k=10}^{\infty}\sum_{\ell=5}^{k-5}2^{-k\epsilon p}|\lambda_{k,j}|^p\\
&\lesssim \sum_{j=1}^{M}\sum_{k=10}^{\infty}k2^{-k\epsilon p}|\lambda_{k,j}|^p \lesssim \sum_{k=0}^{\infty}\sum_{j=1}^{M}|\lambda_{k,j}|^p \sim |b|_{\widetilde{H}_{\mathrm{mb},2}^{p,q,1,\epsilon}(\mu)}^p.
\end{aligned}$$

In order to estimate I, for all $x \in \mathcal{X}$, we write

$$\begin{aligned}
&S\left(\sum_{k=0}^{\ell-5}\sum_{j=1}^{M}\lambda_{k,j}m_{k,j}\right)(x)\\
&\quad\leqslant \left\{\int_{\Gamma(x)}\left|\int_{\mathcal{X}}\sum_{k=0}^{\ell-5}\sum_{j=1}^{M}\lambda_{k,j}m_{k,j}(z)[D_t(y,z)-D_t(y,c_B)]d\mu(z)\right|^2\frac{d\mu(y)dt}{V_t(x)t}\right\}^{1/2}\\
&\qquad+\left\{\int_{\Gamma(x)}\left|\int_{\mathcal{X}}\sum_{k=0}^{\ell-5}\sum_{j=1}^{M}\lambda_{k,j}m_{k,j}(z)D_t(y,c_B)d\mu(z)\right|^2\frac{d\mu(y)dt}{V_t(x)t}\right\}^{1/2} =: \mathrm{M}_3(x)+\mathrm{M}_4(x).
\end{aligned}$$

To estimate $\mathrm{M}_3(x)$, by the Minkowski inequality, we further write, for all $x \in \mathcal{X}$,

$$\begin{aligned}
\mathrm{M}_3(x) &\leqslant \sum_{k=0}^{\ell-5}\sum_{j=1}^{M}|\lambda_{k,j}|\left\{\int_{\Gamma(x)}\left[\int_{B_{k,j}}|m_{k,j}(z)||D_t(y,z)-D_t(y,c_B)|d\mu(z)\right]^2\frac{d\mu(y)dt}{V_t(x)t}\right\}^{1/2}\\
&\leqslant \sum_{k=0}^{\ell-5}\sum_{j=1}^{M}|\lambda_{k,j}|\int_{B_{k,j}}|m_{k,j}(z)|\left\{\int_{\Gamma(x)}|D_t(y,z)-D_t(y,c_B)|^2\frac{d\mu(y)dt}{V_t(x)t}\right\}^{1/2}d\mu(z)\\
&\leqslant \sum_{k=0}^{\ell-5}\sum_{j=1}^{M}|\lambda_{k,j}|\int_{B_{k,j}}|m_{k,j}(z)|[\mathrm{M}_{3,1}(x,z)+\mathrm{M}_{3,2}(x,z)]d\mu(z),
\end{aligned}$$

where, for all $x \in \mathcal{X}$ and $z \in B_{k,j}$ with $k \in \mathbb{Z}_+ \cap [0, \ell-5]$ and $j \in \{1, \dots, M\}$,

$$\mathrm{M}_{3,1}(x,z) := \left[\int_{\substack{\Gamma(x)\cap \\ \{(y,t)\in\mathcal{X}\times(0,\infty):t\leqslant \frac{d(x,c_B)}{8}\}}} |D_t(y,z) - D_t(y,c_B)|^2 \frac{d\mu(y)dt}{V_t(x)t}\right]^{1/2}$$

and

$$\mathrm{M}_{3,2}(x,z) := \left[\int_{\substack{\Gamma(x)\cap \\ \{(y,t)\in\mathcal{X}\times(0,\infty):t> \frac{d(x,c_B)}{8}\}}} |D_t(y,z) - D_t(y,c_B)|^2 \frac{d\mu(y)dt}{V_t(x)t}\right]^{1/2}.$$

Now we give some observations. For any $z \in B_{k,j} \subset 2^{k+2}B \backslash 2^{k-2}B$, $k \in \mathbb{Z}_+ \cap [0, \ell-5]$, $j \in \{1, \dots, M\}$ and $y \in \Gamma(x)$, we have $d(z, c_B) < 2^{k+2}r_B$ and $d(y, c_B) \geqslant d(x, c_B) - d(y, x) \geqslant 2^{\ell-2}r_B - t$ and hence

$$d(y, c_B) + t \geqslant 2^{\ell-2}r_B \geqslant 2^{k+3}r_B > 2d(z, c_B).$$

Meanwhile, for any $y \in \Gamma(x) \cap \{(y,t) \in \mathcal{X} \times (0,\infty) : t \leqslant \frac{d(x,c_B)}{8}\}$, we have

$$d(y, c_B) \geqslant d(x, c_B) - d(x, y) \geqslant d(x, c_B) - \frac{d(x, c_B)}{8} = \frac{7}{8} d(x, c_B).$$

From these observations, (A5) and (10.1), it follows that, for all $x \in \mathcal{X}$ and $z \in B_{k,j}$ with $k \in \mathbb{Z}_+ \cap [0, \ell-5]$ and $j \in \{1, \dots, M\}$,

$$\begin{aligned}\mathrm{M}_{3,1}(x,z) &\lesssim \frac{2^k r_B}{d(x,c_B)} \left\{\int_0^{\frac{d(x,c_B)}{8}} \int_{B(x,t)} \frac{1}{[V(c_B,x)]^2} \frac{t^{\epsilon_3 - 1}}{[d(x,c_B)]^{\epsilon_3}} \frac{d\mu(y)dt}{V_t(x)}\right\}^{1/2} \\ &\sim \frac{2^k r_B}{d(x,c_B)} \frac{1}{V(c_B,x)}\end{aligned}$$

and

$$\begin{aligned}\mathrm{M}_{3,2}(x,z) &\lesssim \frac{2^k r_B}{d(x,c_B)} \left\{\int_{\frac{d(x,c_B)}{8}}^{\infty} \int_{B(x,t)} \frac{1}{[V_t(c_B)]^2} \frac{d\mu(y)dt}{V_t(x)t}\right\}^{1/2} \\ &\lesssim \frac{2^k r_B}{d(x,c_B)} \frac{1}{V(c_B,x)} \left\{\int_{\frac{d(x,c_B)}{8}}^{\infty} \left[\frac{t}{d(x,c_B)}\right]^{-2\kappa} \frac{dt}{t}\right\}^{1/2} \\ &\sim \frac{2^k r_B}{d(x,c_B)} \frac{1}{V(c_B,x)}.\end{aligned}$$

Combining the estimates of $\mathrm{M}_{3,1}(x,z)$ and $\mathrm{M}_{3,2}(x,z)$, we find that, for all $x \in \mathcal{X}$,

$$\mathrm{M}_3(x) \lesssim \sum_{k=0}^{\ell-5} \sum_{j=1}^{M} |\lambda_{k,j}| \|m_{k,j}\|_{L^1(\mu)} \frac{2^k r_B}{d(x,c_B)} \frac{1}{V(c_B,x)}.$$

By this, Hölder's inequality, (4.1), (10.1) and $p > \frac{\nu}{\nu+1}$, we conclude that

$$\begin{aligned}&\sum_{\ell=5}^{\infty} \int_{U_\ell(B)} [\mathrm{M}_3(x)]^p d\mu(x) \\ &\quad\lesssim \sum_{\ell=5}^{\infty} \sum_{k=0}^{\ell-5} \sum_{j=1}^{M} |\lambda_{k,j}|^p \|m_{k,j}\|_{L^1(\mu)}^p \int_{U_\ell(B)} \frac{2^{kp} r_B^p}{[d(x,c_B)]^p} \frac{1}{[V(c_B,x)]^p} d\mu(x) \\ &\quad\lesssim \sum_{\ell=5}^{\infty} \sum_{k=0}^{\ell-5} \sum_{j=1}^{M} |\lambda_{k,j}|^p [\mu(B_{k,j})]^{p/q'} \|m_{k,j}\|_{L^q(\mu)}^p \frac{2^{kp} r_B^p}{2^{(\ell-2)p} r_B^p} \frac{V_{2^{\ell+2}r_B}(c_B)}{[V_{2^{\ell-2}r_B}(c_B)]^p} \\ &\quad\lesssim \sum_{\ell=5}^{\infty} \sum_{k=0}^{\ell-5} \sum_{j=1}^{M} |\lambda_{k,j}|^p 2^{-k\epsilon p} 2^{(k-\ell)p} [V_{2^{\ell+2}r_B}(c_B)]^{1-p} [\lambda(c_B, 2^{k+2}r_B)]^{p-1}\end{aligned}$$

$$\begin{aligned}&\lesssim \sum_{\ell=5}^{\infty}\sum_{k=0}^{\ell-5}\sum_{j=1}^{M}|\lambda_{k,j}|^p 2^{-k\epsilon p}2^{(k-\ell)p}2^{(\ell-k)(1-p)\nu}\\&\sim \sum_{\ell=5}^{\infty}\sum_{k=0}^{\ell-5}\sum_{j=1}^{M}|\lambda_{k,j}|^p 2^{-k\epsilon p}2^{(\ell-k)[(1-p)\nu-p]}\\&\lesssim \sum_{k=0}^{\infty}\sum_{j=1}^{M}|\lambda_{k,j}|^p \sim |b|^p_{\widetilde{H}^{p,q,1,\epsilon}_{\mathrm{mb},2}(\mu)}.\end{aligned}$$

By $\int_{\mathcal{X}} b(x)d\mu(x)=0$ and some arguments similar to those used in the estimate of (10.9), we see that, for all $x\in\mathcal{X}$,

$$\begin{aligned}\mathrm{M}_4(x)&=\left\{\int_{\Gamma(x)}\left|\int_{\mathcal{X}}\sum_{k=\ell-4}^{\infty}\sum_{j=1}^{M}\lambda_{k,j}m_{k,j}(z)D_t(y,c_B)d\mu(z)\right|^2\frac{d\mu(y)dt}{V_t(x)t}\right\}^{1/2}\\&\lesssim \sum_{k=\ell-4}^{\infty}\sum_{j=1}^{M}|\lambda_{k,j}|\|m_{k,j}\|_{L^1(\mu)}\frac{1}{V(c_B,x)}.\end{aligned}$$

Again, by some arguments similar to those used in the estimate of III, we know that

$$\sum_{\ell=5}^{\infty}\int_{U_\ell(B)}[\mathrm{M}_4(x)]^p d\mu(x)\lesssim |b|^p_{\widetilde{H}^{p,q,1,\epsilon}_{\mathrm{mb},2}(\mu)}.$$

Thus,

$$\mathrm{I}\lesssim \sum_{\ell=5}^{\infty}\int_{U_\ell(B)}[\mathrm{M}_3(x)]^p d\mu(x)+\sum_{\ell=5}^{\infty}\int_{U_\ell(B)}[\mathrm{M}_4(x)]^p d\mu(x)\lesssim |b|^p_{\widetilde{H}^{p,q,1,\epsilon}_{\mathrm{mb},2}(\mu)}.$$

Then we turn to estimate II. We first write

$$\begin{aligned}\mathrm{II}&\leqslant \sum_{\ell=5}^{\infty}\sum_{k=\ell-4}^{\ell+4}\sum_{j=1}^{M}|\lambda_{k,j}|^p\int_{U_\ell(B)}[S(m_{k,j})(x)]^p d\mu(x)\\&\leqslant \sum_{\ell=5}^{\infty}\sum_{k=\ell-4}^{\ell+4}\sum_{j=1}^{M}|\lambda_{k,j}|^p\int_{2B_{k,j}}[S(m_{k,j})(x)]^p d\mu(x)\\&\quad+\sum_{\ell=5}^{\infty}\sum_{k=\ell-4}^{\ell+4}\sum_{j=1}^{M}|\lambda_{k,j}|^p\int_{U_\ell(B)\backslash 2B_{k,j}}[S(m_{k,j})(x)]^p d\mu(x)=:\mathrm{II}_1+\mathrm{II}_2.\end{aligned}$$

By Hölder's inequality, the $L^q(\mu)$-boundedness ($q\in(1,\infty)$) of $S(f)$ and (4.1), we conclude that

$$\begin{aligned}\mathrm{II}_1&\lesssim \sum_{\ell=5}^{\infty}\sum_{k=\ell-4}^{\ell+4}\sum_{j=1}^{M}|\lambda_{k,j}|^p\left[\mu\left(2B_{k,j}\right)\right]^{1-\frac{p}{q}}\left\|S\left(m_{k,j}\right)\right\|^p_{L^q(\mu)}\\&\lesssim \sum_{\ell=5}^{\infty}\sum_{k=\ell-4}^{\ell+4}\sum_{j=1}^{M}|\lambda_{k,j}|^p\left[\mu\left(2B_{k,j}\right)\right]^{1-\frac{p}{q}}\left\|m_{k,j}\right\|^p_{L^q(\mu)}\\&\lesssim \sum_{\ell=5}^{\infty}\sum_{k=\ell-4}^{\ell+4}\sum_{j=1}^{M}|\lambda_{k,j}|^p 2^{-k\epsilon p}\lesssim \sum_{k=0}^{\infty}\sum_{j=1}^{M}|\lambda_{k,j}|^p\sim |b|^p_{\widetilde{H}^{p,q,1,\epsilon}_{\mathrm{mb},2}(\mu)}.\end{aligned}$$

To estimate II_2, fix $\ell\in\mathbb{N}\cap[5,\infty)$, $k\in\{\ell-4,\ldots,\ell+4\}$, $j\in\{1,\ldots,M\}$ and $x\in U_\ell(B)\backslash 2B_{k,j}$. Notice that, for any $z\in B_{k,j}$ and $x\notin 2B_{k,j}$,

$$d(x,z)\geqslant d(x,c_{B_{k,j}})-d(z,c_{B_{k,j}})\geqslant \frac{1}{2}d(x,c_{B_{k,j}}).$$

By this, the Minkowski inequality and an argument similar to that used in the estimate of (10.9), we further obtain

$$
\begin{aligned}
S(m_{k,j})(x) &\leqslant \left\{ \int_{\Gamma(x)} \left[\int_{B_{k,j}} |m_{k,j}(z)||D_t(y,z)|d\mu(z) \right]^2 \frac{d\mu(y)dt}{V_t(x)t} \right\}^{1/2} \\
&\leqslant \int_{B_{k,j}} |m_{k,j}(z)| \left[\int_{\Gamma(x)} |D_t(y,z)|^2 \frac{d\mu(y)dt}{V_t(x)t} \right]^{1/2} d\mu(z) \\
&\lesssim \frac{1}{V(x, c_{B_{k,j}})} \|m_{k,j}\|_{L^1(\mu)}.
\end{aligned}
$$

From this, Hölder's inequality, (4.1) and (10.1), we deduce that

$$
\begin{aligned}
\mathrm{II}_2 &\lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\ell+4} \sum_{j=1}^{M} |\lambda_{k,j}|^p \int_{U_\ell(B)\setminus 2B_{k,j}} \frac{1}{[V(x, c_{B_{k,j}})]^p} d\mu(x) \|m_{k,j}\|_{L^1(\mu)}^p \\
&\lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\ell+4} \sum_{j=1}^{M} |\lambda_{k,j}|^p \left[\int_{2^{\ell+7}B\setminus 2B_{k,j}} \frac{1}{V(x, c_{B_{k,j}})} d\mu(x) \right]^p [\mu(2^{\ell+2}B)]^{1-p} \\
&\quad \times [\mu(B_{k,j})]^{p/q'} \|m_{k,j}\|_{L^q(\mu)}^p \\
&\lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\ell+4} \sum_{j=1}^{M} |\lambda_{k,j}|^p [\widetilde{K}_{B_{k,j},\, 2^{\ell+6}B}^{(\rho),\, p}]^p [\mu(2^{\ell+2}B)]^{1-p} \\
&\quad \times 2^{-k\epsilon p} [\lambda(c_B, 2^{k+2} r_B)]^{p-1} [\widetilde{K}_{B_{k,j},\, 2^{k+2}B}^{(\rho),\, p}]^{-p} \\
&\lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\ell+4} \sum_{j=1}^{M} |\lambda_{k,j}|^p 2^{-k\epsilon p} \sim |b|_{\widetilde{H}_{\mathrm{mb},\,2}^{p,\,q,\,1,\,\epsilon}(\mu)}^p,
\end{aligned}
$$

which, together with the estimate for II_1, implies that $\mathrm{II} \lesssim |b|_{\widetilde{H}_{\mathrm{mb},\,2}^{p,\,q,\,1,\,\epsilon}(\mu)}^p$.

To estimate IV, observe that

$$
\begin{aligned}
\mathrm{IV} &\leqslant \sum_{\ell=0}^{4} \int_{U_\ell(B)} \left| S\left(\sum_{k=0}^{\ell+4} \sum_{j=1}^{M} \lambda_{k,j} m_{k,j} \right)(x) \right|^p d\mu(x) \\
&\quad + \sum_{\ell=0}^{4} \int_{U_\ell(B)} \left| S\left(\sum_{k=\ell+5}^{\infty} \sum_{j=1}^{M} \lambda_{k,j} m_{k,j} \right)(x) \right|^p d\mu(x) =: \mathrm{IV}_1 + \mathrm{IV}_2.
\end{aligned}
$$

By some arguments similar to those used in the estimates for II_1 and III, we respectively obtain

$$
\mathrm{IV}_1 \lesssim |b|_{\widetilde{H}_{\mathrm{mb},\,2}^{p,\,q,\,1,\,\epsilon}(\mu)}^p \quad \text{and} \quad \mathrm{IV}_2 \lesssim |b|_{\widetilde{H}_{\mathrm{mb},\,2}^{p,\,q,\,1,\,\epsilon}(\mu)}^p,
$$

which, together with the estimates for I–III, completes the proof of Step 2 and hence Theorem 10.10. □

Remark 10.11. (i) Let $\rho \in (1,\infty)$, $\gamma \in [1,\infty)$ and $\frac{\nu}{\nu+1} < p \leqslant 1 < q \leqslant 2$. Combining Propositions 8.1 and 8.2, and Theorems 10.10 and 9.4, we finally obtain

$$
\widehat{H}_{\mathrm{atb},\,\rho}^{p,\,q,\,\gamma}(\mu) = H_{\mathrm{at}}^p(\mu) = \widetilde{H}_{\mathrm{atb},\,\rho}^{p,\,q,\,\gamma}(\mu) = \widetilde{H}_{\mathrm{mb},\,\rho}^{p,\,q,\,\gamma,\,\epsilon}(\mu)
$$

over an RD-space $(\mathcal{X}, d, \mu)$ with $\mu(\mathcal{X}) = \infty$.

(ii) It is still unclear whether $\widehat{H}_{\mathrm{atb},\,\rho}^{p,\,q,\,\gamma}(\mu)$ (or $H_{\mathrm{at}}^p(\mu)$) and $\widetilde{H}_{\mathrm{atb},\,\rho}^{p,\,q,\,\gamma}(\mu)$ (or $\widetilde{H}_{\mathrm{mb},\,\rho}^{p,\,q,\,\gamma,\,\epsilon}(\mu)$) coincide or not for any $q \in (2,\infty]$ over RD-spaces $(\mathcal{X}, d, \mu)$ with $\mu(\mathcal{X}) = \infty$.

(iii) Let $(\mathcal{X}, d, \mu) := (\mathbb{R}^D, |\cdot|, dx)$ with the D-dimensional Lebesgue measure dx, $\rho \in (1,\infty)$, $\gamma \in [1,\infty)$, $\frac{D}{D+1} < p \leqslant 1 < q < \infty$ and $\epsilon \in (0,\infty)$. By Theorem 9.4, we see that $\widehat{H}_{\mathrm{atb},\,\rho}^{p,\,q,\,\gamma}(\mu) = H^p(\mathbb{R}^D)$. Now we deal with the relation between $\widetilde{H}_{\mathrm{atb},\,\rho}^{p,\,q,\,\gamma}(\mu)$ and $H^p(\mathbb{R}^D)$. To this end, consider [2, Theorem 5.4] with $\varphi(x,t) := t^p$ ($p \in (0,1]$) and $L = -\Delta$, we notice that $H_{-\Delta}^p(\mathbb{R}^D) = H^p(\mathbb{R}^D)$ (see [12]), $q(\varphi) = 1$,

$r(\varphi)=\infty$, $\ell(\varphi)=p=i(\varphi)$, $q\in(1,\infty)$ and $p_{-\Delta}=1$ therein. By $\mathrm{e}^{t\Delta}1=1$, $-\Delta$ satisfying [2, (H1) and (H2)] and [2, p. 107, (6.16)] (see also [31, Remark 5.1]), we conclude that, for any $(p,q,M)_L$-atom a defined in [2, Definition 5.2],

$$\int_{\mathbb{R}^D} a(x)dx=0.$$

Thus, a is a (p,q)-atom. From this, Step 2 of the proof of Theorem 10.10 and the proof of [2, Theorem 5.4], we deduce that

$$(H^p(\mathbb{R}^D)\cap L^2(\mathbb{R}^D))=\widetilde{\mathbb{H}}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu). \tag{10.10}$$

Thus, $H^p(\mathbb{R}^D)=\widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)$.

Moveover, by Step 2 of the proof of Theorem 10.10 and (10.10), we know that

$$\widetilde{\mathbb{H}}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)\subset\widetilde{\mathbb{H}}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)\subset H^p(\mathbb{R}^D)\cap L^2(\mathbb{R}^D)=\widetilde{\mathbb{H}}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu).$$

Thus, by this and Theorems 9.4 and 7.9, we have

$$\widetilde{H}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu)=\widetilde{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)=H^p(\mathbb{R}^D)=\widehat{H}^{p,\,q,\,\gamma}_{\mathrm{atb},\,\rho}(\mu)=\widehat{H}^{p,\,q,\,\gamma,\,\epsilon}_{\mathrm{mb},\,\rho}(\mu).$$

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant Nos. 11301534, 11171027, 11361020 and 11101339) and Da Bei Nong Education Fund (Grant No. 1101-2413002), the Specialized Research Fund for the Doctoral Program of Higher Education of China (Grant No. 20120003110003) and the Fundamental Research Funds for Central Universities of China (Grant Nos. 2012LYB26, 2012CXQT09, 2013YB60 and 2014KJJCA10). The authors express their deep thanks to Professor Tuomas Hytönen for indicating them an important example (see Example 6.3 above). Dachun Yang also wishes to express his sincere thanks to Professor Yasuo Komori-Furuya for a suggestive conversation on the topic of this article in a conference held in Tokyo Metropolitan University of Japan in 2012.

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