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A family of symmetric mixed finite elements for linear elasticity on tetrahedral grids

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Abstract A family of stable mixed finite elements for the linear elasticity on tetrahedral grids are constructed, where the stress is approximated by symmetric $H(\mathrm{div})$-P_k polynomial tensors and the displacement is approximated by C^{-1}-P_{k-1} polynomial vectors, for all $k \geqslant 4$. The main ingredients for the analysis are a new basis of the space of symmetric matrices, an intrinsic $H(\mathrm{div})$ bubble function space on each element, and a new technique for establishing the discrete inf-sup condition. In particular, they enable us to prove that the divergence space of the $H(\mathrm{div})$ bubble function space is identical to the orthogonal complement space of the rigid motion space with respect to the vector-valued P_{k-1} polynomial space on each tetrahedron. The optimal error estimate is proved, verified by numerical examples.

Keywords mixed finite element, symmetric finite element, linear elasticity, conforming finite element, tetrahedral grid, inf-sup condition

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1 Introduction

In the Hellinger-Reissner mixed formulation of the linear elasticity equations, the stress is sought in $H(\mathrm{div}, \Omega, \mathbb{S})$ and the displacement in $L^2(\Omega, \mathbb{R}^3)$. It is a challenge to design stable mixed finite element spaces mainly due to the symmetric constraint of the stress tensor. To overcome this difficulty, earliest works adopted composite element techniques or weakly symmetric methods [3, 6, 7, 25, 27, 29–31]. In [9], Arnold and Winther designed the first family of mixed finite element methods in 2D, based on polynomial shape function spaces. From then on, various stable mixed elements have been constructed, see [2, 4, 5, 8–12, 17–24, 26, 32, 33].

As the displacement function is in $L^2(\Omega, \mathbb{R}^3)$, a natural discretization is the piecewise P_{k-1} polynomial without interelement continuity. It is a long-standing and challenging problem if the stress tensor can be discretized by an appropriate P_k finite element subspace of $H(\mathrm{div}, \Omega, \mathbb{S})$. Adams and Cockburn constructed such a mixed finite element in [2] where the discrete stress space is the space of $H(\mathrm{div}, \Omega, \mathbb{S})$-$P_{k+2}$ tenors whose divergence is a P_{k-1} polynomial on each tetrahedron, for $k = 2$. The method was modified and extended to a family of elements, $k \geqslant 2$, by Arnold et al. [5]. In this paper, we solve this open problem by constructing a suitable $H(\mathrm{div}, \Omega, \mathbb{S})$-$P_k$, instead of the above P_{k+2}, finite element space

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for the stress discretization, for $k \geqslant 4$. In these elements, the symmetric stress tensor is approximated by the full C^0-P_k space enriched by some so-called $H(\mathrm{div})$ bubble functions locally on each tetrahedron. A new way of proof is developed to establish the stability of the mixed elements, by characterizing the divergence of local stress space. The space of divergence of the local $H(\mathrm{div})$ bubble stress space is exactly the subspace of the P_{k-1} polynomial space orthogonal to the local rigid-motion. The optimal order error estimate is proved, verified by numerical tests of P_4 and P_5 mixed elements. Note that the P_k mixed element here has the same numbers of degrees of freedom at vertices, on edges, and faces, as that $k-2$ order mixed element in [5], while the new element promises a k order convergence in the energy norm.

The rest of the paper is organized as follows. In Section 2, we define the weak problem and the finite element method. In Section 3, we prove the well-posedness of the finite element problem, i.e., the discrete coerciveness and the discrete inf-sup condition, by which, the optimal order convergence of the new element follows. In Section 4, we provide some numerical results, using P_4 and P_5 finite elements.

2 The family of finite elements

Based on the Hellinger-Reissner principle, the linear elasticity problem within a stress-displacement (σ-u) form reads: Find $(\sigma, u) \in \Sigma \times V := H(\mathrm{div}, \Omega, \mathbb{S} = \text{symmetric } \mathbb{R}^{3\times 3}) \times L^2(\Omega, \mathbb{R}^3)$, such that

$$\begin{cases} (A\sigma, \tau) + (\mathrm{div}\tau, u) = 0, & \text{for all } \ \tau \in \Sigma, \\ (\mathrm{div}\sigma, v) = (f, v), & \text{for all } \ v \in V. \end{cases} \tag{2.1}$$

Here the symmetric tensor space for the stress Σ and the space for the vector displacement V are, respectively,

$$H(\mathrm{div}, \Omega, \mathbb{S}) := \left\{ \sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \in H(\mathrm{div}, \Omega),\ \sigma^{\mathrm{T}} = \sigma \right\}, \tag{2.2}$$

$$L^2(\Omega, \mathbb{R}^3) := \{(u_1 \ \ u_2 \ \ u_3)^{\mathrm{T}},\ u_i \in L^2(\Omega),\ i = 1, 2, 3\}. \tag{2.3}$$

This paper denotes by $H^k(T, X)$ the Sobolev space consisting of functions with domain $T \subset \mathbb{R}^3$, taking values in the finite-dimensional vector space X, and with all derivatives of order at most k square-integrable. For our purposes, the range space X will be either $\mathbb{S}$, $\mathbb{R}^3$, or $\mathbb{R}$. $\|\cdot\|_{k,T}$ is the norm of $H^k(T)$. $\mathbb{S}$ denotes the space of symmetric tensors, $H(\mathrm{div}, T, \mathbb{S})$ consists of square-integrable symmetric matrix fields with square-integrable divergence. The $H(\mathrm{div})$ norm is defined by

$$\|\tau\|^2_{H(\mathrm{div},T)} := \|\tau\|^2_{L^2(T)} + \|\mathrm{div}\tau\|^2_{L^2(T)}.$$

$L^2(T, \mathbb{R}^3)$ is the space of vector-valued functions which are square-integrable. Here, the compliance tensor $A = A(x) : \mathbb{S} \ \rightarrow \ \mathbb{S}$, characterizing the properties of the material, is bounded and symmetric positive definite uniformly for $x \in \Omega$.

This paper deals with a pure displacement problem (2.1) with the homogeneous boundary condition that $u \equiv 0$ on $\partial\Omega$. But the method and the analysis work for mixed boundary value problems and the pure traction boundary problem.

The domain Ω is subdivided by a family of quasi-uniform tetrahedral grids $\mathcal{T}_h$ (with the grid size h). We introduce the finite element space of order k ($k \geqslant 4$) on $\mathcal{T}_h$. The displacement space is the full C^{-1}-P_{k-1} space

$$V_h = \{v \in L^2(\Omega, \mathbb{R}^3),\ v|_K \in P_{k-1}(K, \mathbb{R}^3) \ \text{ for all } K \in \mathcal{T}_h\}. \tag{2.4}$$

Since the discrete stress space Σ_h is an $H(\mathrm{div})$ bubble enrichment of the H^1 space

$$\widetilde{\Sigma}_h = \{\sigma \in H^1(\Omega, \mathbb{S}),\ \sigma|_K \in P_k(K, \mathbb{S}) \ \text{ for all } K \in \mathcal{T}_h\}, \tag{2.5}$$

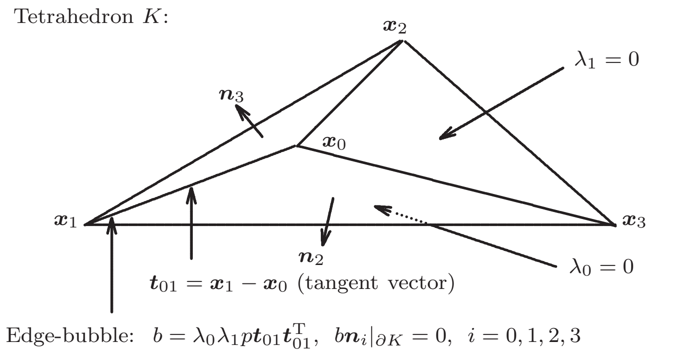

Figure 2.1 An edge-bubble function $b = \lambda_0\lambda_1 p \boldsymbol{t}_{01}\boldsymbol{t}_{01}^{\mathrm{T}}$, $p \in P_{k-2}(K, \mathbb{R})$ on an edge $\boldsymbol{x}_0\boldsymbol{x}_1$ of tetrahedron K

we first define the $H(\mathrm{div})$ bubble function space on each element. To this end, let $\boldsymbol{x}_0$, $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$ be the four vertices of a tetrahedron K, cf. Figure 2.1.

The referencing mapping is then

$$\boldsymbol{x} = F_K(\hat{\boldsymbol{x}}) = \boldsymbol{x}_0 + (\boldsymbol{x}_1 - \boldsymbol{x}_0 \ \ \boldsymbol{x}_2 - \boldsymbol{x}_0 \ \boldsymbol{x}_3 - \boldsymbol{x}_0)\hat{\boldsymbol{x}},$$

mapping the reference tetrahedron

$$\hat{K} = \{0 \leqslant \hat{x}_1, \hat{x}_2, \hat{x}_3, 1 - \hat{x}_1 - \hat{x}_2 - \hat{x}_3 \leqslant 1\}$$

to K. Then the inverse mapping is

$$\hat{\boldsymbol{x}} = \begin{pmatrix} \boldsymbol{n}_1^{\mathrm{T}} \\ \boldsymbol{n}_2^{\mathrm{T}} \\ \boldsymbol{n}_3^{\mathrm{T}} \end{pmatrix} (\boldsymbol{x} - \boldsymbol{x}_0), \tag{2.6}$$

where

$$\begin{pmatrix} \boldsymbol{n}_1^{\mathrm{T}} \\ \boldsymbol{n}_2^{\mathrm{T}} \\ \boldsymbol{n}_3^{\mathrm{T}} \end{pmatrix} = (\boldsymbol{x}_1 - \boldsymbol{x}_0 \ \ \boldsymbol{x}_2 - \boldsymbol{x}_0 \ \ \boldsymbol{x}_3 - \boldsymbol{x}_0)^{-1}. \tag{2.7}$$

By (2.6), these normal vectors are coefficients of the barycentric variables:

$$\begin{aligned} \lambda_1 &= \boldsymbol{n}_1 \cdot (\boldsymbol{x} - \boldsymbol{x}_0), \quad & \lambda_2 &= \boldsymbol{n}_2 \cdot (\boldsymbol{x} - \boldsymbol{x}_0), \\ \lambda_3 &= \boldsymbol{n}_3 \cdot (\boldsymbol{x} - \boldsymbol{x}_0), \quad & \lambda_0 &= 1 - \lambda_1 - \lambda_2 - \lambda_3. \end{aligned}$$

On each face triangle, say $\boldsymbol{x}_0\boldsymbol{x}_2\boldsymbol{x}_3$, all three edges (the tangent vectors), $\boldsymbol{x}_0\boldsymbol{x}_2$, $\boldsymbol{x}_0\boldsymbol{x}_3$ and $\boldsymbol{x}_2\boldsymbol{x}_3$, are orthogonal to the face normal vector $\boldsymbol{n}_1$. For convenience, we introduce the tangent vectors and their tensors:

$$\begin{aligned} \boldsymbol{t}_{01} &= \boldsymbol{x}_1 - \boldsymbol{x}_0, \quad & T_{01} &= \boldsymbol{t}_{01}\boldsymbol{t}_{01}^{\mathrm{T}}, \\ \boldsymbol{t}_{02} &= \boldsymbol{x}_2 - \boldsymbol{x}_0, \quad & T_{02} &= \boldsymbol{t}_{02}\boldsymbol{t}_{02}^{\mathrm{T}}, \\ \boldsymbol{t}_{03} &= \boldsymbol{x}_3 - \boldsymbol{x}_0, \quad & T_{03} &= \boldsymbol{t}_{03}\boldsymbol{t}_{03}^{\mathrm{T}}, \\ \boldsymbol{t}_{12} &= \boldsymbol{x}_2 - \boldsymbol{x}_1, \quad & T_{12} &= \boldsymbol{t}_{12}\boldsymbol{t}_{12}^{\mathrm{T}}, \\ \boldsymbol{t}_{23} &= \boldsymbol{x}_3 - \boldsymbol{x}_2, \quad & T_{23} &= \boldsymbol{t}_{23}\boldsymbol{t}_{23}^{\mathrm{T}}, \\ \boldsymbol{t}_{13} &= \boldsymbol{x}_3 - \boldsymbol{x}_1, \quad & T_{13} &= \boldsymbol{t}_{13}\boldsymbol{t}_{13}^{\mathrm{T}}. \end{aligned} \tag{2.8}$$

With them, we define a $H(\mathrm{div}, K, \mathbb{S})$ bubble function space

$$\Sigma_{K,b} = \sum_{0 \leqslant i < j \leqslant 3} \lambda_i \lambda_j P_{k-2}(K, \mathbb{R}) T_{ij}. \tag{2.9}$$

Note that each bubble function, say, a function τ in $\lambda_0\lambda_1 T_{01} P_{k-2}(K,\mathbb{R})$, vanishes on two face triangles ($\lambda_0 = 0$, $\lambda_1 = 0$) and $\tau\boldsymbol{n}_2 = \boldsymbol{0}$, $\tau\boldsymbol{n}_3 = \boldsymbol{0}$ on the other two face triangles. Then the discrete stress space of order k ($k \geqslant 4$) is defined as

$$\Sigma_h = \{\sigma \in H(\mathrm{div},\Omega,\mathbb{S}),\ \sigma = \sigma_c + \sigma_b,\ \sigma_c \in \widetilde{\Sigma}_h,\ \sigma_b|_K \in \Sigma_{K,b},\ \forall K \in \mathcal{T}_h\}. \tag{2.10}$$

Next, we define a basis for Σ_h. Given element K, let $\boldsymbol{x}_i$ and F_i, $i = 0,1,2,3$, be its vertices and faces, respectively. Given edge $E_{i,j} = \boldsymbol{x}_i\boldsymbol{x}_j$, $0 \leqslant i < j \leqslant 3$, define its $k-1$ interior nodal points

$$\boldsymbol{x}_{E_{i,j},l} = \frac{l}{k}\boldsymbol{x}_i + \left(1 - \frac{l}{k}\right)\boldsymbol{x}_j, \quad 1 \leqslant l \leqslant k-1. \tag{2.11}$$

Given F_i with three vertices $\boldsymbol{x}_0^{(i)}$, $\boldsymbol{x}_1^{(i)}$ and $\boldsymbol{x}_2^{(i)}$, define its $\frac{(k-1)(k-2)}{2}$ interior nodal points

$$\boldsymbol{x}_{F_i,j,l} = \frac{j}{k}\boldsymbol{x}_0^{(i)} + \frac{l}{k}\boldsymbol{x}_1^{(i)} + \frac{k-l-j}{k}\boldsymbol{x}_2^{(i)}, \quad 1 \leqslant j,l \ \text{ and } \ j+l \leqslant k-1. \tag{2.12}$$

We define $\frac{(k-1)(k-2)(k-3)}{6}$ interior nodal points

$$\boldsymbol{x}_{K,i,j,l} = \frac{i}{k}\boldsymbol{x}_0 + \frac{j}{k}\boldsymbol{x}_1 + \frac{l}{k}\boldsymbol{x}_2 + \frac{k-i-j-l}{k}\boldsymbol{x}_3, \tag{2.13}$$

$1 \leqslant i,j,l$ and $i+j+l \leqslant k-1$ of element K. Then the nodes for the Lagrange element of order k are

$$\begin{aligned}\mathbb{X}_K = \{\boldsymbol{x}_i,\ i = 0,\dots,3\} &\cup \{\boldsymbol{x}_{E_{i,j},l},\ 0 \leqslant i < j \leqslant 3,\ l = 1,\dots,k-1\}\\ &\cup \{\boldsymbol{x}_{F_i,j,l},\ i = 0,\dots,3,\ 1 \leqslant j,l \text{ and } j+l \leqslant k-1\}\\ &\cup \{\boldsymbol{x}_{K,i,j,l},\ 1 \leqslant i,j,l \text{ and } i+j+l \leqslant k-1\}.\end{aligned}$$

Let $\mathcal{X}_{\mathbb{E}}$ denote all interior nodes, defined in (2.11), of all the edges, $\mathcal{X}_{\mathbb{F}}$ denote all interior nodes, defined in (2.12), of all the faces, $\mathcal{X}_{\mathbb{K}}$ denote all interior nodes, defined in (2.13), of all the elements, and $\mathcal{X}_{\mathbb{V}}$ denote all the vertices of $\mathcal{T}_h$. Define the Lagrange element space of order k by

$$\mathbb{P}_h := H^1(\Omega,\mathbb{R}) \cap \{v \in L^2(\Omega),\ v|_K \in P_k(K,\mathbb{R}),\ \forall K \in \mathcal{T}_h\}.$$

Given a node $\boldsymbol{x} \in \mathcal{X}_{\mathbb{V}} \cup \mathcal{X}_{\mathbb{E}} \cup \mathcal{X}_{\mathbb{F}} \cup \mathcal{X}_{\mathbb{K}}$, let $\varphi_{\boldsymbol{x}} \in \mathbb{P}_h$ be its associated nodal basis function, which is defined as

$$\varphi_{\boldsymbol{x}}(\boldsymbol{x}) = 1 \text{ and } \varphi_{\boldsymbol{x}}(\boldsymbol{x}') = 0 \text{ for any } \boldsymbol{x}' \in \mathcal{X}_{\mathbb{V}} \cup \mathcal{X}_{\mathbb{E}} \cup \mathcal{X}_{\mathbb{F}} \cup \mathcal{X}_{\mathbb{K}} \text{ other than } \boldsymbol{x}.$$

Given edge E, let T_E be a matrix of rank one defined similarly to that in (2.8). We need the orthogonal complement matrices $T_{E,j}^{\perp} \in \mathbb{S}$, $j = 1,\dots,5$, of matrix T_E, which are defined by

$$T_{E,j}^{\perp} : T_E = 0, \quad T_{E,j}^{\perp} : T_{E,j}^{\perp} = 1, \quad \text{and} \quad T_{E,i}^{\perp} : T_{E,j}^{\perp} = 0 \quad \text{for } i \neq j, \tag{2.14}$$

where the inner product $A : B = \sum_{i,j=1}^{3} a_{ij}b_{ij}$ for two matrices $A = \{a_{ij}\}_{i,j=1}^{3}$ and $B = \{b_{ij}\}_{i,j=1}^{3}$.

Given face F, let $\boldsymbol{t}_{F,j}$, $j = 1,2,3$, be unit tangential vectors of its three edges, which allow for defining

$$T_{F,j} = \boldsymbol{t}_{F,j}\boldsymbol{t}_{F,j}^{\mathrm{T}}, \quad j = 1,2,3. \tag{2.15}$$

Define their orthogonal complement matrices $T_{F,m}^{\perp}, m = 1,2,3$, such that

$$T_{F,j} : T_{F,m}^{\perp} = 0, \quad \text{and} \quad T_{F,j}^{\perp} : T_{F,m}^{\perp} = \delta_{j,m}, \quad j,m = 1,2,3. \tag{2.16}$$

A canonical basis of $\mathbb{S}$ reads

$$\begin{aligned}&\mathbb{T}_1 = \begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}, \quad \mathbb{T}_2 = \begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}, \quad \text{and} \quad \mathbb{T}_3 = \begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix},\\ &\mathbb{T}_4 = \begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}, \quad \mathbb{T}_5 = \begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}, \quad \text{and} \quad \mathbb{T}_6 = \begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}.\end{aligned} \tag{2.17}$$

With these preparations, the basis functions of Σ_h can be classified into six classes:

(1) Vertex-based basis functions: given node $\boldsymbol{x} \in \mathcal{X}_{\mathbb{V}}$, its six associated basis functions of Σ_h read

$$\tau_{\boldsymbol{x},i} = \varphi_{\boldsymbol{x}} \mathbb{T}_i, \quad i = 1, \ldots, 6.$$

(2) Edge-based basis functions with nonzero flux: given node $\boldsymbol{x} \in \mathcal{X}_{\mathbb{E}}$ on edge E, its five associated basis functions with nonzero flux of Σ_h read

$$\tau^{(nb)}_{E,\boldsymbol{x},i} = \varphi_{\boldsymbol{x}} T^{\perp}_{E,i}, \quad i = 1, \ldots, 5.$$

(3) Edge-based basis functions with zero flux: given node $\boldsymbol{x} \in \mathcal{X}_{\mathbb{E}}$ on edge E, letting $K_1, \ldots, K_{\ell_E}$ be elements which share the common edge E, its associated basis functions with zero flux of Σ_h read

$$\tau^{(b)}_{E,K_i,\boldsymbol{x}} = \varphi_{\boldsymbol{x}}|_{K_i} T_E, \quad i = 1, \ldots, \ell_E.$$

(4) Face-based basis functions with nonzero flux: given node $\boldsymbol{x} \in \mathcal{X}_{\mathbb{F}}$ on face F, its three associated basis functions with nonzero flux of Σ_h read

$$\tau^{(nb)}_{F,\boldsymbol{x},i} = \varphi_{\boldsymbol{x}} T^{\perp}_{F,i}, \quad i = 1, 2, 3.$$

(5) Face-based basis functions with zero flux: given node $\boldsymbol{x} \in \mathcal{X}_{\mathbb{F}}$ on face F, letting K_1 and K_2 be two elements which share the common face F, its associated basis functions with zero flux of Σ_h read

$$\tau^{(b)}_{F,K_i,\boldsymbol{x},j} = \varphi_{\boldsymbol{x}}|_{K_i} T_{F,j}, \quad i = 1, 2, \quad j = 1, 2, 3.$$

(6) Volume-based basis functions: given node $\boldsymbol{x} \in \mathcal{X}_{\mathbb{K}}$ inside K, its six associated basis functions of Σ_h read

$$\tau_{K,\boldsymbol{x},i} = \varphi_{\boldsymbol{x}} \mathbb{T}_i, \quad i = 1, \ldots, 6.$$

To characterize the bubble space $\Sigma_{K,b}$, we need the following lemma.

Lemma 2.1. *The six symmetric tensors T_{ij} in* (2.8) *are linearly independent, and form a basis of* $\mathbb{S}$.

Proof. Each tensor $T_{ij} = \boldsymbol{t}_{ij}\boldsymbol{t}_{ij}^{\mathrm{T}}$ is a positive semi-definite matrix, on a tetrahedron K. We would show that the constants c_{ij} are all equal to zero in

$$T = c_{01}T_{01} + c_{02}T_{02} + c_{03}T_{03} + c_{12}T_{12} + c_{23}T_{23} + c_{13}T_{13} = 0.$$

First, we compute the bilinear form (cf. Figure 2.1), by (2.7),

$$\boldsymbol{n}_1^{\mathrm{T}} T \boldsymbol{n}_1 = c_{01} 1 \cdot 1 + c_{02} 0 + c_{03} 0 + c_{12}(-1)(-1) + c_{23} 0 + c_{13}(-1)(-1) = 0.$$

Here, by (2.7) and (2.8),

$$\begin{aligned} &\boldsymbol{t}_{01}^{\mathrm{T}} \boldsymbol{n}_1 = 1, \\ &\boldsymbol{t}_{12}^{\mathrm{T}} \boldsymbol{n}_1 = (\boldsymbol{t}_{02}^{\mathrm{T}} - \boldsymbol{t}_{01}^{\mathrm{T}}) \boldsymbol{n}_1 = 0 - 1, \\ &\boldsymbol{t}_{13}^{\mathrm{T}} \boldsymbol{n}_1 = (\boldsymbol{t}_{03}^{\mathrm{T}} - \boldsymbol{t}_{01}^{\mathrm{T}}) \boldsymbol{n}_1 = 0 - 1. \end{aligned}$$

Symmetrically, by evaluating $\boldsymbol{n}_i^{\mathrm{T}} T \boldsymbol{n}_i$ for $i = 0, 1, 2, 3$, where $\boldsymbol{n}_0 = -\boldsymbol{n}_1 - \boldsymbol{n}_2 - \boldsymbol{n}_3$, we have

$$\begin{cases} c_{01} + c_{02} + c_{03} = 0, \\ c_{01} + c_{12} + c_{13} = 0, \\ c_{02} + c_{12} + c_{23} = 0, \\ c_{03} + c_{13} + c_{23} = 0. \end{cases} \tag{2.18}$$

Note that $\boldsymbol{n}_0 \neq \boldsymbol{0}$ as K is a non-singular tetrahedron. Next, we introduce three (non-unit) vectors $\boldsymbol{s}_i$ orthogonal to the three pairs of skew edges, $\overline{\boldsymbol{x}_0\boldsymbol{x}_1}$ and $\overline{\boldsymbol{x}_2\boldsymbol{x}_3}$, $\overline{\boldsymbol{x}_0\boldsymbol{x}_2}$ and $\overline{\boldsymbol{x}_1\boldsymbol{x}_3}$, $\overline{\boldsymbol{x}_0\boldsymbol{x}_3}$ and $\overline{\boldsymbol{x}_1\boldsymbol{x}_2}$, respectively (cf. Figure 2.1), i.e.,

$$\boldsymbol{s}_1 = \frac{\boldsymbol{t}_{01} \times \boldsymbol{t}_{23}}{6|K|},$$

because $|K| \neq 0$ and consequently $|\boldsymbol{t}_{01} \times \boldsymbol{t}_{23}| \neq 0$. Thus $\boldsymbol{s}_1 \cdot \boldsymbol{t}_{01} = 0$, $\boldsymbol{s}_1 \cdot \boldsymbol{t}_{02} = -1$, $\boldsymbol{s}_1 \cdot \boldsymbol{t}_{03} = -1$, $\boldsymbol{s}_1 \cdot \boldsymbol{t}_{12} = -1$, $\boldsymbol{s}_1 \cdot \boldsymbol{t}_{13} = -1$, and $\boldsymbol{s}_1 \cdot \boldsymbol{t}_{23} = 0$. By evaluating $\boldsymbol{s}_i^{\mathrm{T}} T \boldsymbol{s}_i$, it follows that

$$\begin{cases} c_{02} + c_{03} + c_{12} + c_{13} = 0, \\ c_{01} + c_{03} + c_{12} + c_{23} = 0, \\ c_{01} + c_{02} + c_{13} + c_{23} = 0. \end{cases} \tag{2.19}$$

By the first two equations in (2.18) and the first equation in (2.19), we get $2c_{01} = 0$. Symmetrically, we find all $c_{ij} = 0$. Thus $\{T_{ij}\}$ is a linearly independent set of tensors. As $\dim \mathbb{S} = 6$, $\{T_{ij}\}$ is a basis. □

It follows from the definition of V_h (P_{k-1} polynomials) and Σ_h (P_k polynomials) that $\operatorname{div} \Sigma_h \subset V_h$. This, in turn, leads to a strong divergence-free space:

$$Z_h := \{\tau_h \in \Sigma_h \mid (\operatorname{div} \tau_h, v) = 0 \text{ for all } v \in V_h\} = \{\tau_h \in \Sigma_h \mid \operatorname{div} \tau_h = 0 \text{ pointwise }\}. \tag{2.20}$$

The mixed finite element approximation of Problem (1.1) reads: Find $(\sigma_h,\ u_h) \in \Sigma_h \times V_h$ such that

$$\begin{cases} (A\sigma_h, \tau) + (\operatorname{div}\tau, u_h) = 0, & \text{for all } \ \tau \in \Sigma_h, \\ (\operatorname{div} \sigma_h, v) = (f, v), & \text{for all } \ v \in V_h. \end{cases} \tag{2.21}$$

3 Stability and convergence

The convergence of the finite element solutions follows the stability and the standard approximation property. So we consider first the well-posedness of the discrete problem (2.21). By the standard theory, we only need to prove the following two conditions, based on their counterpart at the continuous level.

(1) K-ellipticity. There exists a constant $C > 0$, independent of the meshsize h such that

$$(A\tau, \tau) \geqslant C\|\tau\|^2_{H(\operatorname{div})} \quad \text{for all } \tau \in Z_h, \tag{3.1}$$

where Z_h is the divergence-free space defined in (2.20).

(2) Discrete B-B condition. There exists a positive constant $C > 0$ independent of the meshsize h, such that

$$\inf_{0 \neq v \in V_h} \sup_{0 \neq \tau \in \Sigma_h} \frac{(\operatorname{div}\tau, v)}{\|\tau\|_{H(\operatorname{div})} \|v\|_{L^2(\Omega)}} \geqslant C. \tag{3.2}$$

It follows from $\operatorname{div} \Sigma_h \subset V_h$ that $\operatorname{div} \tau = 0$ for any $\tau \in Z_h$. This implies the above K-ellipticity condition (3.1). It remains to show the discrete B-B condition (3.2), in the following two lemmas.

Lemma 3.1. *For any $v_h \in V_h$, there is a $\tau_h \in \widetilde{\Sigma}_h \subset \Sigma_h$ such that, for any polynomial $p \in P_{k-3}(K, \mathbb{R}^3)$, $K \in \mathcal{T}_h$,*

$$\int_K (\operatorname{div} \tau_h - v_h) \cdot p \, d\boldsymbol{x} = 0 \quad \textit{and} \quad \|\tau_h\|_{H(\operatorname{div})} \leqslant C\|v_h\|_{L^2(\Omega)}. \tag{3.3}$$

Proof. Let $v_h \in V_h$. By the stability of the continuous formulation, there is a $\tau \in H^1(\Omega, \mathbb{S})$ such that,

$$\operatorname{div} \tau = v_h \quad \text{and} \quad \|\tau\|_{H^1(\Omega)} \leqslant C\|v_h\|_{L^2(\Omega)}.$$

As $\tau \in H^1(\Omega, \mathbb{S})$, we modify the Scott-Zhang [28] interpolation operator slightly to define a flux preserving interpolation,

$$I_h \ : \ H^1(\Omega, \mathbb{S}) \to \Sigma_h \cap H^1(\Omega, \mathbb{S}) = \widetilde{\Sigma}_h,$$

$$\tau \mapsto \tau_h := I_h\tau.$$

Here the interpolation is done inside a subspace, the continuous finite element subspace $\Sigma_h \cap H^1(\Omega, \mathbb{S})$. $I_h\tau$ is defined by its values at the Lagrange nodes.

At a vertex node or a node inside an edge, $\boldsymbol{x}_i$, $I_h\tau(\boldsymbol{x}_i)$ is defined as the nodal value of τ at the point if τ is continuous, but in general, $I_h\tau(\boldsymbol{x}_i)$ is defined as an average value on a face triangle, on whose edge the node is, as in [28]. After defining the nodal values at edges of tetrahedra, the nodal values of τ_h at the nodes inside each face triangle F of a tetrahedron are defined by the L^2-orthogonal projection on the triangle F:

$$\int_F \tau_{h,ij} p dS = \int_F \tau_{ij} p dS, \quad \forall\, p \in P_{k-3}(F, \mathbb{R}), \quad i, j = 1, 2, 3, \tag{3.4}$$

where $\tau_{h,ij}$ and τ_{ij} are the (i,j)-th components of τ_h and τ, respectively, and F is a face triangle of a tetrahedron in the tetrahedral triangulation $\mathcal{T}_h$. The number of equations in (3.4) is the same as the number of internal degrees of freedom of P_k polynomials, $\dim P_{k-3}$. At the Lagrange nodes inside a tetrahedron, $I_h\tau(\boldsymbol{x}_i)$ is defined by the L^2-orthogonal projection on the tetrahedron, satisfying

$$\int_K \tau_{h,ij} p d\boldsymbol{x} = \int_K \tau_{ij} p d\boldsymbol{x}, \quad \forall\, p \in P_{k-4}(K, \mathbb{R}), \tag{3.5}$$

where K is an element of $\mathcal{T}_h$. It follows by the stability of the Scott-Zhang operator that

$$\|I_h\tau\|_{H^1(\Omega)} \leqslant C\|\tau\|_{H^1(\Omega)} \leqslant C\|v_h\|_{L^2(\Omega)}.$$

In particular,

$$\|I_h\tau\|_{H(\mathrm{div})} \leqslant \|I_h\tau\|_{H^1(\Omega)} \leqslant C\|v_h\|_{L^2(\Omega)}.$$

By (3.4) and (3.5), we get a partial-divergence matching property of I_h: for any $p \in P_{k-3}(K, \mathbb{R}^3)$, as the symmetric gradient $\epsilon(p) \in P_{k-4}(K, \mathbb{S})$,

$$\begin{aligned} \int_K (\mathrm{div}\, \tau_h - v_h) \cdot p\, d\boldsymbol{x} &= \int_{\partial K} (\tau_h \boldsymbol{n}) \cdot p\, ds - \int_K \tau_h : \epsilon(p)\, d\boldsymbol{x} - \int_K v_h \cdot p\, d\boldsymbol{x} \\ &= \int_{\partial K} (\tau \boldsymbol{n}) \cdot p\, ds - \int_K \tau : \epsilon(p)\, d\boldsymbol{x} - \int_K v_h \cdot p\, d\boldsymbol{x} \\ &= \int_K (\mathrm{div}\, \tau - v_h) \cdot p\, d\boldsymbol{x} = 0. \end{aligned}$$

The proof is complete. □

Given element K, let $R(K)$ be the space of 6-dimensional, local rigid motions:

$$R(K) = \left\{ \left. \begin{pmatrix} a_1 - a_4 y - a_5 z \\ a_2 + a_4 x - a_6 z \\ a_3 + a_5 x + a_6 y \end{pmatrix} \right| a_1, a_2, a_3, a_4, a_5, a_6 \in \mathbb{R} \right\}. \tag{3.6}$$

Let $R^\perp(K)$ be the orthogonal complement of $R(K)$ with respect to $P_{k-1}(K, \mathbb{R}^3)$. We have the following key result.

Lemma 3.2. *It holds that*

$$\mathrm{div}\, \Sigma_{K,b} = R^\perp(K). \tag{3.7}$$

Proof. It is immediate that

$$\mathrm{div}\, \Sigma_{K,b} \subset R^\perp(K).$$

If $\mathrm{div}\, \Sigma_{K,b} \neq R^\perp(K)$, there is a nonzero $v_h \in R^\perp(K)$ such that

$$\int_K \mathrm{div}\, \tau_h \cdot v_h\, d\boldsymbol{x} = 0, \quad \forall\, \tau_h \in \Sigma_{K,b}.$$

By integration by parts, for $\tau_h \in \Sigma_{K,b}$, we have

$$\int_K \operatorname{div} \tau_h \cdot v_h d\boldsymbol{x} = -\int_K \tau_h : \epsilon(v_h) d\boldsymbol{x} = 0, \tag{3.8}$$

where $\epsilon(v_h)$ is the symmetric gradient, $(\nabla v_h + \nabla^{\mathrm{T}} v_h)/2$.

Let $\{M_{ij}, i = 0, 1, 2, j = i+1, \dots, 3\}$ be the dual basis of the symmetric matrix space, of $\{T_{ij}\}$, defined in (2.8), i.e.,

$$M_{ij} = M_{ij}^{\mathrm{T}}, \quad M_{ij} : T_{i'j'} = \delta_{ij,i'j'}. \tag{3.9}$$

Under the dual basis, we have a unique expansion, as $\epsilon(v_h) \in P_{k-2}(K, \mathbb{S})$,

$$\epsilon(v_h) = q_1 M_{01} + q_2 M_{02} + q_3 M_{03} + q_4 M_{12} + q_5 M_{23} + q_6 M_{13}, \tag{3.10}$$

for some $q_i \in P_{k-2}(K, \mathbb{R})$. Selecting $\tau_1 = \lambda_0 \lambda_1 q_1 T_{01} \in \Sigma_{K,b}$, we have, by (3.9),

$$0 = \int_K \tau_1 : \epsilon(v_h) d\boldsymbol{x} = \int_K \lambda_0 \lambda_1 q_1^2(\boldsymbol{x}) d\boldsymbol{x}.$$

As $\lambda_0 \lambda_1 > 0$ on K, we conclude that $q_1 \equiv 0$. Similarly, the other five q_i in (3.10) are zero, which implies that v_h is a rigid motion. On the other hand, $v_h \in R^{\perp}(K)$ which indicates that it cannot be a nonzero local rigid motion. Thus, $v_h \equiv 0$ and $\operatorname{div} \Sigma_{K,b} = R^{\perp}(K)$. □

Lemma 3.3. *For any $v_h \in V_h$, if*

$$\int_K v_h \cdot p \, d\boldsymbol{x} = 0 \quad \text{for all } p \in P_{k-3}(K, \mathbb{R}^3) \text{ and all } K \in \mathcal{T}_h, \tag{3.11}$$

then there is a $\tau_h \in \Sigma_h$ such that

$$\operatorname{div} \tau_h = v_h \quad \text{and} \quad \|\tau_h\|_{H(\operatorname{div})} \leqslant C \|v_h\|_{L^2(\Omega)}. \tag{3.12}$$

Proof. As we assume polynomial degree $k \geqslant 4$ in V_h,

$$p \in P_{k-3}(K, \mathbb{R}^3) \supset P_1(K, \mathbb{R}^3) \supset R(K).$$

So if v_h satisfies (3.11), $v_h|_K \in R^{\perp}(K)$ for any element K. Then it follows from Lemma 3.2 that there exists a $\tau_K \in \Sigma_{K,b}$ such that

$$\operatorname{div} \tau_K = v_h|_K, \quad \|\tau_K\|_{L^2(K)} = \{\min \|\tau\|_{L^2(K)}, \operatorname{div} \tau = v_h|_K, \tau \in \Sigma_{K,b}\}.$$

Let $\tau_h|_K = \tau_K$ for any $K \in \mathcal{T}_h$. As the matching $\operatorname{div} \tau_h = v_h$ is independently done on each element K, by affine mapping and scaling argument, (3.12) holds. □

We are in the position to show the well-posedness of the discrete problem.

Lemma 3.4. *For the discrete problem* (2.21)*, the K-ellipticity* (3.1) *and the discrete B-B condition* (3.2) *hold uniformly. Consequently, the discrete mixed problem* (2.21) *has a unique solution* $(\sigma_h, u_h) \in \Sigma_h \times V_h$.

Proof. The K-ellipticity immediately follows from the fact that $\operatorname{div} \Sigma_h \subset V_h$. To prove the discrete B-B condition (3.2), for any $v_h \in V_h$, it follows from Lemma 3.1 that there exists a $\tau_1 \in \Sigma_h$ such that, for any polynomial $p \in P_{k-3}(K, \mathbb{R}^3)$,

$$\int_K (\operatorname{div} \tau_1 - v_h) \cdot p d\boldsymbol{x} = 0 \quad \text{and} \quad \|\tau_1\|_{H(\operatorname{div})} \leqslant C \|v_h\|_{L^2(\Omega)}. \tag{3.13}$$

Then it follows from Lemma 3.3 that there is a $\tau_2 \in \Sigma_h$ such that

$$\operatorname{div} \tau_2 = v_h - \operatorname{div} \tau_1 \quad \text{and} \quad \|\tau_2\|_{H(\operatorname{div})} \leqslant C \|\operatorname{div} \tau_1 - v_h\|_{L^2(\Omega)}. \tag{3.14}$$

Let $\tau = \tau_1 + \tau_2$, which implies that

$$\operatorname{div} \tau = v_h \text{ and } \|\tau\|_{H(\operatorname{div})} \leqslant C \|v_h\|_{L^2(\Omega)}. \tag{3.15}$$

This proves the discrete B-B condition (3.2). □

Theorem 3.5. *Let $(\sigma, u) \in \Sigma \times V$ be the exact solution of problem* (2.1) *and $(\tau_h, u_h) \in \Sigma_h \times V_h$ the finite element solution of* (2.21)*. Then, for $k \geqslant 4$,*

$$\|\sigma - \sigma_h\|_{H(\mathrm{div})} + \|u - u_h\|_{L^2(\Omega)} \leqslant Ch^k(\|\sigma\|_{H^{k+1}(\Omega)} + \|u\|_{H^k(\Omega)}). \tag{3.16}$$

Proof. The stability of the elements and the standard theory of mixed finite element methods [13, 14] give the following quasioptimal error estimate immediately,

$$\|\sigma - \sigma_h\|_{H(\mathrm{div})} + \|u - u_h\|_{L^2(\Omega)} \leqslant C \inf_{\tau_h \in \Sigma_h, v_h \in V_h} (\|\sigma - \tau_h\|_{H(\mathrm{div})} + \|u - v_h\|_{L^2(\Omega)}). \tag{3.17}$$

Let P_h denote the local L^2 projection operator, or triangle-wise interpolation operator, from V to V_h, satisfying the error estimate

$$\|v - P_h v\|_{L^2(\Omega)} \leqslant Ch^k \|v\|_{H^k(\Omega)} \quad \text{for any } v \in H^k(\Omega, \mathbb{R}^3). \tag{3.18}$$

Choosing $\tau_h = I_h\sigma \in \Sigma_h$, where I_h is defined in (3.4) and (3.5), we have [28], as I_h preserves symmetric P_k functions locally,

$$\|\sigma - \tau_h\|_{L^2(\Omega)} + h|\sigma - \tau_h|_{H(\mathrm{div})} \leqslant Ch^{k+1}\|\sigma\|_{H^{k+1}(\Omega)}. \tag{3.19}$$

Let $v_h = P_h v$ and $\tau_h = I_h\sigma$ in (3.17), by (3.18) and (3.19), we obtain (3.16). □

Remark 3.6. By using a mesh dependent norm technique, see for example [29], we can prove the following optimal error estimate,

$$\|\sigma - \sigma_h\|_{L^2(\Omega)} \leqslant Ch^{k+1}\|\sigma\|_{H^{k+1}(\Omega)},$$

provided that $\sigma \in H^{k+1}(\Omega, \mathbb{S})$.

4 Numerical tests

We compute one example in 3D, by P_4 and by P_5 mixed finite element methods. It is a pure displacement problem on the unit cube $\Omega = (0, 1)^3$ with a homogeneous boundary condition that $u \equiv 0$ on $\partial\Omega$. In the computation, we let

$$A\sigma = \frac{1}{2\mu}\left(\sigma - \frac{\lambda}{2\mu + n\lambda}\,\mathrm{tr}(\sigma)\delta\right), \quad n = 3, \quad \text{where } \delta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and $\mu = 1/2$ and $\lambda = 1$ are the Lamé constants.

Let the exact solution on the unit square $[0, 1]^3$ be

$$u = \begin{pmatrix} 2^4 \\ 2^5 \\ 2^6 \end{pmatrix} x(1-x)y(1-y)z(1-z). \tag{4.1}$$

Then, the true stress function σ and the load function f are defined by the equations in (2.1), for the given solution u.

In the computation, the level one grid is the given domain with a diagonal line shown in Figure 4.1. Each grid is refined into a half-sized grid uniformly, to get a higher level grid, shown in Figure 4.1. In all the computation, the discrete systems of equations are solved by Matlab backslash solver. In Table 4.1, the errors and the convergence order in various norms are listed for the true solution (4.1), by the P_4 mixed finite element in (2.10) and (2.4), with $k = 4$ there. The optimal order of convergence is achieved in Table 4.1, confirming Theorem 3.5.

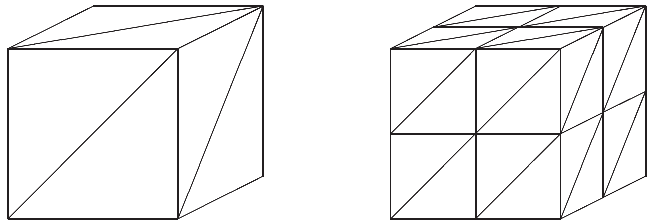

Figure 4.1 The initial grid for (4.1), and its level 2 refinement

Table 4.1 The error and the order of convergence by the P_4 finite element, $k = 4$ in (2.4) and (2.10), for (4.1)

	$\|\sigma - \sigma_h\|_{L^2(\Omega)}$	h^n	$\|u - u_h\|_{L^2(\Omega)}$	h^n	$\|\operatorname{div}(\sigma - \sigma_h)\|_{L^2(\Omega)}$	h^n
1	0.19929801	0.0	0.06133241	0.0	1.47254873	0.0
2	0.00804695	4.6	0.00714869	3.1	0.09203430	4.0
3	0.00029057	4.8	0.00049143	3.9	0.00575214	4.0

Table 4.2 The error and the order of convergence by the P_5 finite element, $k = 5$ in (2.4) and (2.10), for (4.1)

	$\|I_h\sigma - \sigma_h\|_{L^2(\Omega)}$	h^n	$\|I_h u - u_h\|_{L^2(\Omega)}$	h^n	$\|\operatorname{div}(I_h\sigma - \sigma_h)\|_{L^2(\Omega)}$	h^n
1	0.00000002	0.0	0.01937914	0.0	0.00000011	0.0
2	0.00000002	0.0	0.00089726	4.4	0.00000031	0.0

In Table 4.2, the errors and the convergence order in various norms are listed for the true solution (4.1), by the P_5 mixed finite element in (2.10) and (2.4), with $k = 5$ there. Here the exact solution σ is a polynomial tensor of degree 5. Thus, it is in the stress finite element space Σ_h and the finite element solution σ_h is exact. It is computed so, shown in the second column and the six column in Table 4.2. The optimal order of convergence is achieved for the displacement u in Table 4.2 (up to the computer accuracy), confirming Theorem 3.5.

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